Interaction Notes
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Single Port Equivalent Circuits for Antennas and Scatterers

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Abstract

This note takes the singularity expansion of the response of antennas and scatterers in free space and converts it into equivalent circuit representations. The object is assumed to have a gap (port) which is treated from both short circuit and open circuit points of view. Considering the solutions for the response to an incident wave as well as for excitation at the gap equivalent circuits with both impedance elements and sources are developed which represent the solution to both problems. From the short circuit point of view this is done by the parallel combination of voltage sources in series with pole admittances. From the open circuit point of view this is done by the series combination of current sources in parallel with pole impedances. Modified forms of these basic types are also considered and realizability and approximation problems are discussed.

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I. Introduction

In the complex frequency (s) plane there can be various equivalent representations of the same electromagnetic problem. The singularity expansion method (SEM) expands the solution in terms of s plane singularities such as poles, branch points, essential singularities, and entire functions.[4,5] Power series (Taylor, Laurant, or Rayleigh series) expansion around s = 0 is another approach if there are no singularities other than poles at s = 0. In a more general case one can have an asymptotic series around s = 0 with more general types of terms. One can also have an asymptotic series for $s \to \infty$ with its argument appropriately restricted. All of these s plane expansions represent the same physical problem.

It is possible that two physical problems can have the same solutions if the corresponding variables are appropriately interpreted. One problem is said to be an analog of the other. Equivalent electrical circuits are often used to represent various physical problems.

Considering a general electromagnetic scattering or antenna problem one would like to have various ways of representing pertinent electromagnetic quantities as electrical parameters in an equivalent circuit. Such representations could be helpful for several reasons including (but not necessarily limited to):

1. physical insight
2. computational convenience
3. capability of using established circuit transformation techniques
4. combination of the electromagnetic analysis with physical circuit elements, transmission lines, etc. which are constructed as part of an antenna or scatterer
5. use of existing computerized circuit analysis programs

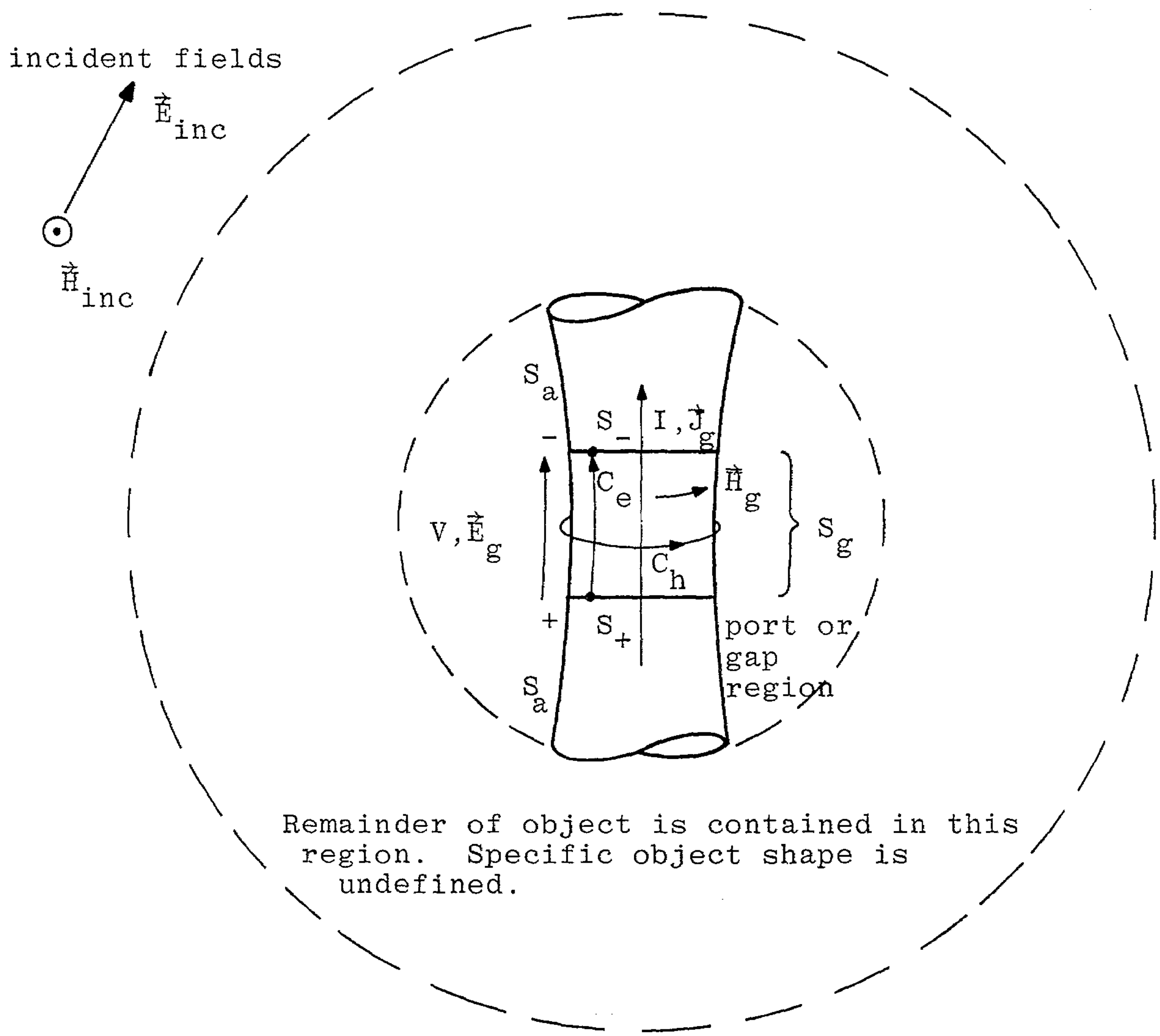
This note dwells on the use of SEM representations for constructing equivalent circuits of antennas and scatterers at some "gap" or "port." While the present note considers representations for a single port, some future notes could extend this to multiport circuits for multiport antennas and scatterers.

In this note let us concentrate on objects (such as finite size objects with sufficiently simple media in free space) whose delta function response to an appropriately defined source (such as an incident wave) has only poles as singularities in the finite s plane. Complex conjugate pole pairs can be thought of as resonant circuits and this observation forms the basis of the circuit representations used in this note. For convenience only first order poles are considered.

For purposes of forming equivalent circuits certain electromagnetic quantities are used which have direct electrical circuit interpretation. Having defined some port for the antenna or scatterer then one can calculate open circuit voltage, short circuit current, and driving point admittance or impedance. A circuit can be constructed to have the same port properties, matching the above quantities for frequency and time domain uses.

One approach to equivalent circuits has been to construct separate circuits to match the impedance and open circuit voltage.[9] As will be seen the present approach puts all the terminal properties in a single network involving lumped elements (resistors, inductors, capacitors) and sources (voltage or current).

To state the problem more precisely consider some scatterer or antenna as illustrated in figure 1.1. Let there be a single port for consideration which will be characterized by a current and a voltage. For this purpose the gap region comprising the port is assumed small compared to radian wavelengths of interest. This allows the electromagnetic fields to be locally quasistatic (i.e., in the vicinity of the gap). Note that the gap is defined in figure 1.1 such that there are two isolated sides or terminals separated by the gap region with surface S_g. The antenna or scatterer surface is designated by S_a (not including the gap) so that $S_a \cup S_g$ designates the surface of the entire object including the gap.



The gap tangential electric field, voltage, and current are shown with a convention to directly give the antenna impedance as $\tilde{V}_{oc}/\tilde{I}_{sc}$.

Figure 1.1. Antenna or Scatterer with Single Port

The gap volume is referred to as V_g and the antenna or scatterer volume as V_a. The entire object including the gap has volume $V_a \cup V_g$. Where needed there are the small surfaces S_+ and S_- separating V_a and V_g where S_+ is the side of the gap with positive convention in figure 1.1, and similarly for S_-.

One can define a current I through the gap and a voltage V across the gap as indicated in figure 1.1. The detailed definitions of the quantities are considered in later sections. The electromagnetic field quantities will be reduced to such circuit type quantities as voltage, current, admittance, and impedance for use in defining equivalent circuits.

In the presence of some set of incident fields $\vec{E}_{inc}$, $\vec{H}_{inc}$ and utilizing the above approximations for defining the port quantities one can define the open circuit voltage as

$$V_{oc}(t) \equiv V(t)\Big|_{I=0} \tag{1.1}$$

Physically this corresponds to removal of any electrical connections across the gap region. The gap surface and the volume interior to the gap surface in its immediate vicinity is free space. Note in this approximation the displacement current across the gap is assumed negligible.

Simiarly under incident field excitation the short circuit current is

$$I_{sc}(t) \equiv I(t)\Big|_{V=0} \tag{1.2}$$

This condition corresponds to making the gap region perfectly conducting.

The driving point impedance at the gap is defined by setting the incident field equal to zero as

$$\tilde{Z}_a(s) = - \left. \frac{\tilde{V}(s)}{\tilde{I}(s)} \right|_{\text{zero incident field}} \tag{1.3}$$

where a tilde ~ over a quantity indicates the Laplace transform (two sided) over time t, making the quantities functions of the complex frequency s. Note that polarity convention for $\tilde{V}$ and $\tilde{I}$ in figure 1.1 is chosen so that Z_a is the antenna impedance in the usual sense, representing power flow out from the gap onto the antenna and radiated into the surrounding space. For this purpose the port current or voltage ($\tilde{I}$ or $\tilde{V}$) must be specified in the sense of a source, and then the other of the two quantities calculated or measured to determine the port impedance $\tilde{Z}_a$.

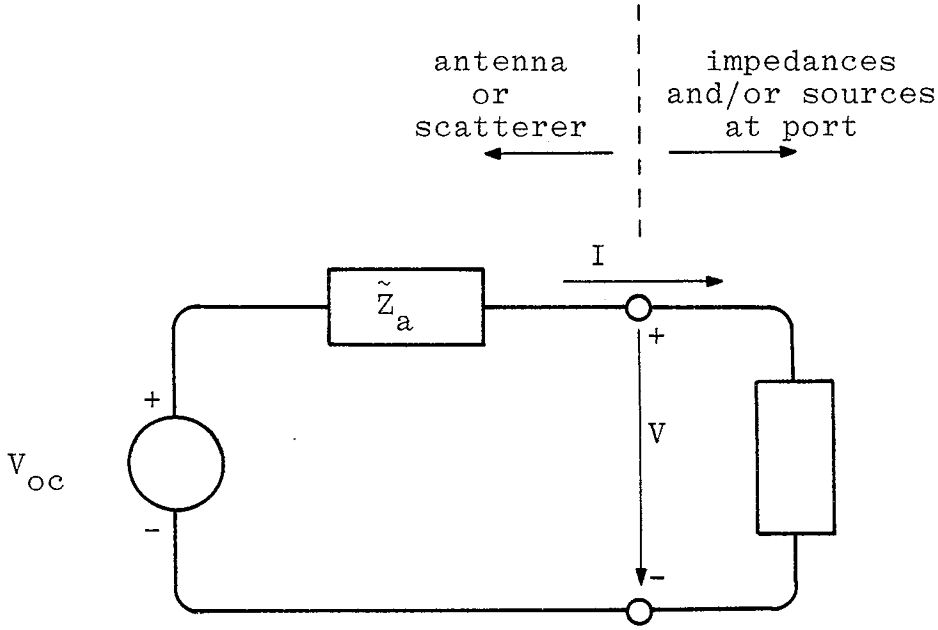
These port quantities are summarized in the Thevenin and Norton equivalent circuits illustrated in figure 1.2. For this purpose one can have some sort of load impedance (say $\tilde{Z}_L$) and also sources (say $\tilde{V}_p$ or $\tilde{I}_p$) attached to the port terminals. Our concern is with the circuit representation of the antenna or scatterer at the port terminals. With the convention chosen for V and I looking into the port from the exterior we have the relation

$$\frac{\tilde{V}_{oc}(s)}{\tilde{I}_{sc}(s)} = \tilde{Z}_a(s) \tag{1.4}$$

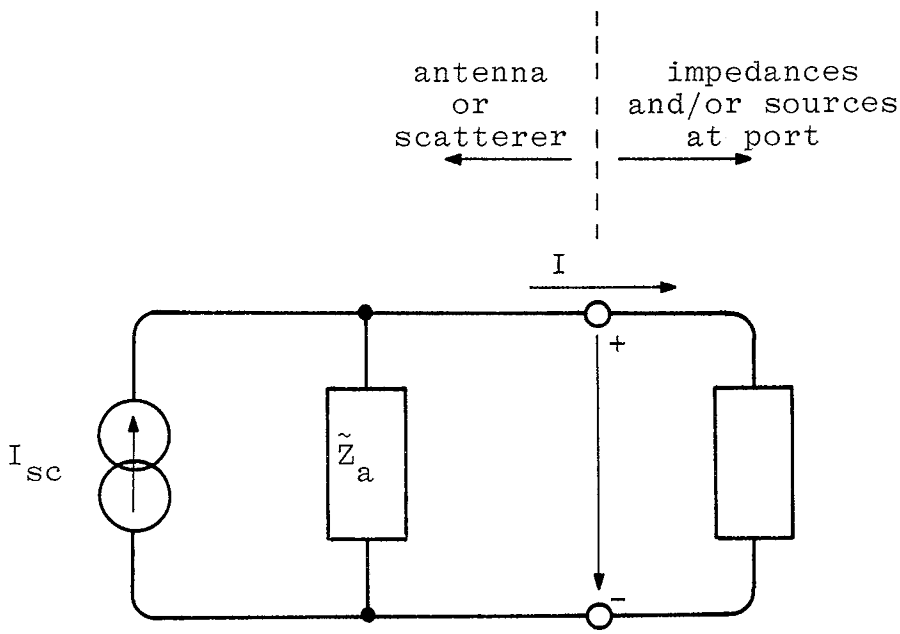
with a change in sign from that in equation 1.3. For convenience define the admittance at the port as

$$\tilde{Y}_a(s) \equiv \tilde{Z}_a(s)^{-1} \tag{1.5}$$

The four quantities $\tilde{V}_{oc}$, $\tilde{I}_{sc}$, $\tilde{Z}_a$, and $\tilde{Y}_a$ (only two of which are independent) are the basic quantities to be considered in constructing an equivalent circuit for the antenna or scatterer as seen from the port of interest.



A. Thevenin equivalent circuit



B. Norton equivalent circuit

Figure 1.2. Thevenin and Norton Equivalent Circuits of Single Port Antenna or Scatterer

There are some convenient constants for normalizing the various results. These are based on the permittivity ε_o and permeability μ_o of free space. We have the free space impedance and speed of light

$$Z_o \equiv \sqrt{\frac{\mu_o}{\varepsilon_o}} \quad , \quad c \equiv \frac{1}{\sqrt{\mu_o \varepsilon_o}} \tag{1.6}$$

The propagation constant or complex wave number is

$$\gamma \equiv \frac{s}{c} \tag{1.7}$$

II. Specified Gap Electric Field for Admittance

In calculating the admittance as part of the short circuit boundary value problem let us specify a gap electric field $\vec{E}_g$ which is defined as conservative on the gap. It can be derived from a potential function Φ_g as

$$\vec{E}_g(\vec{r},t) = -\nabla_s \Phi_g(\vec{r},t) \ , \ \vec{r} \in S_g$$
$$\vec{E}_g(\vec{r},t) = -\nabla \Phi_g(\vec{r},t) \ , \ \vec{r} \in V_g \tag{2.1}$$

depending on whether a surface or volume formulation, respectively, is being used. This potential function assumes two different constant values on S_+ and on S_- (opposite "sides" of the gap). The gap voltage for circuit purposes is taken as

$$V(t) = \Phi_g(\vec{r},t)\Big|_{\vec{r}\in S_+} - \Phi_g(\vec{r},t)\Big|_{\vec{r}\in S_-} \tag{2.2}$$

Since this gap field is conservative then we have the line-integral relation for the voltage

$$V(t) = -\int_{C_e} \vec{E}_g(\vec{r},t) \cdot \vec{1}_e(\vec{r}) \, d\ell \tag{2.3}$$

where C_e is a contour from S_+ to S_-. $\vec{1}_e$ is the unit tangent vector to C_e. This contour is indicated in figure 1.1 on S_g, but it could also be in V_g (inside $S_g \cup S_- \cup S_+$).

Besides choosing the gap electric field as the gradient of a potential function one could also choose it divergenceless (at least in V_g or on S_g depending on the form of source field) as

$$\nabla \cdot \vec{E}_g(\vec{r},t) = \vec{0} \ , \ \nabla^2 \Phi_g(\vec{r},t) = \vec{0} \ , \ \vec{r} \in V_g$$
$$\nabla_s \cdot \vec{E}_g(\vec{r},t) = \vec{0} \ , \ \nabla_s^2 \Phi_g(\vec{r},t) = \vec{0} \ , \ \vec{r} \in S_g \tag{2.4}$$

This is a chargeless gap source field.

This gap electric field is specified as a source electric field in the sense as is used as a forcing function in an integral equation to be discussed later. In normalized form we can define our source field as

$$\vec{E}_g(\vec{r},t) = -V(t)\vec{e}_g(\vec{r})$$
$$\vec{E}_s(\vec{r},t) = \begin{cases} \vec{E}_g(\vec{r},t) & \text{for } \vec{r} \in S_g \text{ (or } V_g) \\ 0 & \text{for } \vec{r} \notin S_g \text{ (or } V_g) \end{cases} \tag{2.5}$$

where

$$\int_{C_e} \vec{e}_g(\vec{r}) \cdot \vec{1}_e(\vec{r})\, d\ell = 1 \tag{2.6}$$

Note that V is a function of time V(t), but will be Laplace transformed into the complex frequency plane as $\tilde{V}(s)$ for admittance purposes.

For a gap of length Δ as a cylinder with generators parallel to the z axis let us choose a convenient example that $\vec{E}_g$ is uniform and parallel to the z axis as

$$\vec{E}_g(\vec{r},t) = -\frac{1}{\Delta}V(t)\vec{1}_z$$
$$\vec{e}_g(\vec{r}) = -\frac{1}{\Delta}\vec{1}_z \tag{2.7}$$

This is a particularly simple form of gap field, yet useful in practical cases.

The resulting current through the gap can be calculated one way as

$$I(t) \equiv \oint_{C_h} \vec{J}_s(\vec{r},t) \cdot \vec{1}_{h_n}(\vec{r})\, d\ell$$
$$I(t) \equiv \int_{S_h} \vec{J}(\vec{r},t) \cdot \vec{1}_{S_h}(\vec{r})\, dS \tag{2.8}$$

for surface and volume formulations respectively. For the case of surface current density the contour C_h lies on S_g and goes around the gap once in a right hand sense (with respect to I) as shown in figure 1.1. $\vec{1}_h$ is parallel to C_h in the right hand sense as in figure 1.1; $\vec{1}_g$ is the outward pointing normal to S_g; $\vec{1}_{h_n}$ is perpendicular to C_h and parallel to S_g with

$$\begin{aligned}
\vec{1}_h(\vec{r}) \times \vec{1}_{h_n}(\vec{r}) &= \vec{1}_g(\vec{r}) \\
\vec{1}_{h_n}(\vec{r}) \times \vec{1}_g(\vec{r}) &= \vec{1}_h(\vec{r}) \\
\vec{1}_g(\vec{r}) \times \vec{1}_h(\vec{r}) &= \vec{1}_{h_n}(\vec{r}) \\
\vec{r} &\in S_g
\end{aligned} \tag{2.9}$$

For the case of volume current density S_h is a surface inside V_g which divides V_g in two. S_h is bounded by C_h. $\vec{1}_{S_h}$ is the unit normal to S_h in the direction of the gap electric field. For convenience we can let S_h be a surface of constant V_g. This makes $\vec{1}_{S_h}$ parallel to $\vec{e}_g$. It also makes $\vec{1}_{h_n}$ parallel to $\vec{e}_g$ on C_h and S_g.

Another way one might define the resulting current is based on the magnetic field around the gap as

$$I(t) \equiv \oint_{C_h} \vec{H}(\vec{r},t) \cdot \vec{1}_h(\vec{r})\, d\ell \tag{2.10}$$

where C_h is taken just outside S_g. However, equation 2.8 is just the surface integral of $\nabla \times \vec{H}$ over S_h so that

$$\oint_{C_h} \vec{H}(\vec{r},t) \cdot \vec{1}_h(\vec{r})d\ell = \int_{S_h} \left[\vec{J}(\vec{r},t) + \epsilon_o \frac{\partial}{\partial t} \vec{E}(\vec{r},t)\right] \cdot \vec{1}_{S_h}(\vec{r})dS \tag{2.11}$$

Note the addition of the displacement current density $\epsilon_o \partial\vec{E}/\partial t$ in addition to the current density $\vec{J}$. The use of the current density

$\vec{J}$ describing all the charge motion (including polarization current density) would seem a more natural definition. However, in many cases the displacement current at the gap is negligible compared to the "charge motion" current; in such cases the two definitions are approximately equivalent.

Assuming that radian wavelengths are large compared to the gap dimensions, and that the admittance for current flow into the remainder of the object (scatterer/antenna) is large compared to admittances associated with energy stored in the immediate gap vicinity, then the current I is approximately a constant through the gap. In other words I in equations 2.8 and 2.10 is approximately independent of the particular choice of the path C_h and associated surface S_h. This is to be expected in that the contrary (significant variation of I through the gap) would imply large charge buildup in the gap region. We are assuming in effect that most of the charge is out on the antenna or scatterer.

A better definition of the current I through the gap might be made by averaging the current through one surface S_h through the gap (equations 2.6) over all such surfaces S_h in some sense. As a simple example if the gap region were a cylinder of length Δ with S_h as planes orthogonal to the cylinder axis (the z axis) then one might define I as

$$
\begin{aligned}
I(t) &\equiv \frac{1}{\Delta}\int_{z_+}^{z_-}\left\{\int_{S_h} \vec{J}(\vec{r},t)\cdot\vec{1}_{S_h}(\vec{r})\, dS\right\}dz \\
&= \frac{1}{\Delta}\int_{V_g} \vec{J}(\vec{r},t)\cdot\vec{1}_z\, dV \\
&\equiv \frac{1}{\Delta}\left\langle \vec{J}(\vec{r},t) \;\dot{,}\; \vec{1}_z \right\rangle_g \\
&\equiv \left\langle \vec{J}(\vec{r},t) \;\dot{,}\; \vec{e}_g \right\rangle_g
\end{aligned}
\qquad (2.12)
$$

The last form brings the current definition into a form which is a symmetric product with range of integration over the gap. The form shown is for a current density with integration over V_g. For a surface current density on S_g a symmetric product with integration over S_g is appropriate.

A subscript g is used to indicate integration over S_g or V_g as appropriate. Similarly a subscript a is used to indicate integration over the antenna or scatterer excluding the gap. A subscript a+g indicates integration over the object including the gap region.

In defining the resulting current in the gap one can look at the form of the resulting admittance

$$\tilde{Y}_a(s) = - \frac{\tilde{I}(s)}{\tilde{V}(s)} \tag{2.13}$$

In another form this is

$$-\tilde{I}(s) = \tilde{Y}_a(s)\, \tilde{V}(s) \tag{2.14}$$

which leads to two power types of relations as

$$\begin{aligned} -\tilde{I}(s)\, \tilde{V}(s) &= \tilde{Y}_a(s)\, \tilde{V}^2(s) \\ -\tilde{I}(s)\, \tilde{V}(-s) &= \tilde{Y}_a(s)\, \tilde{V}(s)\, \tilde{V}(-s) \end{aligned} \tag{2.15}$$

Note the use of $-s$ in the second case. For $s = i\omega$ we have (where a bar — over a quantity indicates complex conjugate)

$$\begin{aligned} \tilde{V}(-s)\Big|_{s=i\omega} &= \tilde{V}(\overline{s})\Big|_{s=i\omega} \\ &= \overline{\tilde{V}}(s)\Big|_{s=i\omega} \end{aligned} \tag{2.16}$$

which results from the voltage being the Laplace transform (two sided) of a real valued V(t).

One way to compute such power expressions on a microscopic basis is to consider certain volume integrals. First we have for "power" generated by the gap field

$$\left\langle \tilde{\vec{J}}(\vec{r},s) \; ; \; \tilde{\vec{E}}_g(\vec{r},s) \right\rangle_g = \left\langle \tilde{\vec{J}}(\vec{r},s) \; ; \; (-\tilde{V}(s)\, \vec{e}_g(\vec{r})) \right\rangle_g$$

$$= -\tilde{V}(s) \left\langle \tilde{\vec{J}}(\vec{r},s) \; ; \; \vec{e}_g(\vec{r}) \right\rangle_g$$

$$= -\tilde{V}(s)\, \tilde{I}(s) = \tilde{Y}_a(s)\, \tilde{V}^2(s) \tag{2.17}$$

where we have taken as our generalized definition of $\tilde{I}(s)$ the expression

$$\tilde{I}(s) \equiv \left\langle \tilde{\vec{J}}(\vec{r},s) \; ; \; \vec{e}_g(\vec{r}) \right\rangle_g \tag{2.18}$$

This definition is consistent with the special case of averaging in equation 2.12. The admittance is then

$$\tilde{Y}_a(s) = -\frac{\tilde{I}(s)}{\tilde{V}(s)}$$

$$= -\frac{\left\langle \tilde{\vec{J}}(\vec{r},s) \; ; \; \tilde{\vec{E}}_g(\vec{r},s) \right\rangle_g}{\tilde{V}^2(s)}$$

$$= -\frac{\left\langle \tilde{\vec{J}}(\vec{r},s) \; ; \; \vec{e}_g(\vec{r}) \right\rangle_g}{\tilde{V}(s)} \tag{2.19}$$

where $\tilde{V}(s)$ and $\vec{e}_g(\vec{r})$ are fundamental properties of the defined gap electric field.

An alternate approach to the same result is to use a power relation with one term conjugated (or better, s replaced by -s to preserve analyticity in the complex frequency plane) as

$$\left\langle \tilde{\vec{J}}(\vec{r},s) \stackrel{\bullet}{,} \tilde{\vec{E}}_g(\vec{r},-s) \right\rangle_g = \left\langle \tilde{\vec{J}}(\vec{r},s) \stackrel{\bullet}{,} (-\tilde{V}(-s)\, \vec{e}_g(\vec{r})) \right\rangle_g$$

$$= -\tilde{V}(-s) \left\langle \tilde{\vec{J}}(\vec{r},s) \stackrel{\bullet}{,} \vec{e}_g(\vec{r}) \right\rangle_g$$

$$= -\tilde{V}(-s)\, \tilde{I}(s) = \tilde{Y}_a(s)\, \tilde{V}(s)\, \tilde{V}(-s) \tag{2.20}$$

where our current definition is the same as in equation 2.18. This gives an admittance as

$$\tilde{Y}_a(s) = -\frac{\tilde{I}(s)}{\tilde{V}(s)}$$

$$= -\frac{\left\langle \tilde{\vec{J}}(\vec{r},s) \stackrel{\bullet}{,} \tilde{\vec{E}}_g(\vec{r},-s) \right\rangle_g}{\tilde{V}(s)\, \tilde{V}(-s)}$$

$$= -\frac{\left\langle \tilde{\vec{J}}(\vec{r},s) \stackrel{\bullet}{,} \vec{e}_g(\vec{r}) \right\rangle_g}{\tilde{V}(s)} \tag{2.21}$$

which is the same as in equation 2.19.

Conveniently the current definition as an average in equation 2.18, together with the chosen general gap field as in equations 2.1 through 2.6 making the field conservative and factorable into a space part times a frequency or time part, gives an admittance which is independent of whether or not one term in the "power" expression is conjugated. Let us take this average type of definition as standard for our purposes. In time domain this is

$$I(t) \equiv \left\langle \tilde{\vec{J}}(\vec{r},t) \stackrel{\bullet}{,} \vec{e}_g(\vec{r}) \right\rangle_g \tag{2.22}$$

Later formulas in some cases will not be dependent on the exact form of the definition. However, some interesting special results will be obtained for which the definition simplifies matters.

III. Specified Gap Surface or Volume Current Density for Impedance

In calculating the impedance as part of the open circuit boundary value problem let us specify a gap current density $\vec{J}_g$ or surface current density $\vec{J}_{s_g}$. Letting this be a divergenceless (solenoidal) function we can define it as the curl of an appropriate vector potential as

$$\begin{aligned} \vec{J}_{s_g}(\vec{r},t) &= \nabla_s \times \vec{\Psi}_g(\vec{r},t) \quad , \quad \vec{r}\varepsilon S_g \\ \vec{J}_g(\vec{r},t) &= \nabla \times \vec{\Psi}_g(\vec{r},t) \quad , \quad \vec{r}\varepsilon V_g \end{aligned} \tag{3.1}$$

with the property

$$\begin{aligned} \nabla_s \cdot \vec{J}_{s_g}(\vec{r},t) &= 0 \quad , \quad \vec{r}\varepsilon S_g \\ \nabla \cdot \vec{J}_g(\vec{r},t) &= 0 \quad , \quad \vec{r}\varepsilon V_g \end{aligned} \tag{3.2}$$

In addition we require that none of this source current in the gap cross S_g, i.e.

$$\begin{aligned} \vec{1}_g(\vec{r}) \cdot \vec{J}_{s_g}(\vec{r},t) &= 0 \quad , \quad \vec{r}\varepsilon S_g \\ \vec{1}_g(\vec{r}) \cdot \vec{J}_g(\vec{r},t) &= 0 \quad , \quad \vec{r}\varepsilon S_g \\ \vec{J}_g(\vec{r},t) &= \vec{0} \quad , \quad \vec{r}\not\varepsilon V_g \end{aligned} \tag{3.3}$$

Write the current density as

$$\begin{aligned} \vec{J}_{s_g}(\vec{r},t) &= I(t)\, \vec{j}_{s_g}(\vec{r}) \quad , \quad \vec{r}\varepsilon S_g \\ \vec{J}_g(\vec{r},t) &= I(t)\, \vec{j}_g(\vec{r}) \quad , \quad \vec{r}\varepsilon V_g \end{aligned} \tag{3.4}$$

so that just as in the case of a source electric field (section II) the source current density is specified as a function of space times a function of time. The normalized gap current density satisfies the condition

$$\oint_{C_h} \vec{j}_{s_g}(\vec{r}) \cdot \vec{1}_{h_n}(\vec{r})\, d\ell = 1$$
$$\int_{S_h} \vec{j}_g(\vec{r}) \cdot \vec{1}_{S_h}(\vec{r})\, d\ell = 1 \tag{3.5}$$

and the source current through the gap is

$$I(t) = \oint_{C_h} \vec{J}_{s_g}(\vec{r},t) \cdot \vec{1}_{h_n}(\vec{r})\, d\ell$$
$$I(t) = \int_{S_h} \vec{J}_g(\vec{r},t) \cdot \vec{1}_{S_h}(\vec{r})\, d\ell \tag{3.6}$$

where the contour C_h around S_g and bounding a surface S_h in V_g are discussed previously.

The current can be also related to the vector potential $\vec{\Psi}_g$ by an integral transformation as

$$I(t) = \int_{S_h} [\nabla \times \vec{\Psi}_g(\vec{r},t)] \cdot \vec{1}_{S_h}\, d\ell$$
$$= \oint_{C_h} \vec{\Psi}_g(\vec{r},t) \cdot \vec{1}_h\, d\ell \tag{3.7}$$

and similarly for the limiting form of surface current density as

$$I(t) = \oint_{C_h} [\nabla_s \times \vec{\Psi}_g(\vec{r},t)] \cdot \vec{1}_{h_n} \, d\ell$$

$$= \oint_{C_{h_+}} \vec{\Psi}_g(\vec{r},t) \cdot \vec{1}_h \, d\ell \qquad (3.8)$$

where C_{h_+} consists of points taken as the limit $\vec{r} \to S_g$ from the outside.

For use in integral equations the gap current density discussed above is used to define a source current density as

$$\vec{J}_{s_g}(\vec{r},t) \equiv \begin{cases} I(t)\, \vec{j}_{s_g}(\vec{r}) & \text{for } \vec{r}\varepsilon S_g \\ \vec{0} & \text{for } \vec{r}\not\varepsilon S_g \end{cases}$$

$$\vec{J}_g(\vec{r},t) = \begin{cases} I(t)\, \vec{j}_g(\vec{r}) & \text{for } \vec{r}\varepsilon V_g \\ \vec{0} & \text{for } \vec{r}\not\varepsilon V_g \end{cases} \qquad (3.9)$$

$$\oint_{C_h} \vec{j}_{s_g}(\vec{r}) \cdot \vec{1}_{h_n} \, d\ell = 1$$

$$\int_{S_h} \vec{j}_g(\vec{r}) \cdot \vec{1}_{S_h} \, dS = 1$$

This general form of $\vec{J}_{s_g}$ will be used to specify a gap source current with integration over S_g or V_g as appropriate.

The resulting voltage across the gap can be calculated from

$$V(t) \equiv -\int_{C_e} \vec{E}(\vec{r},t) \cdot \vec{1}_e(\vec{r}) \, d\ell \qquad (3.10)$$

where, as before, C_e is a contour in V_g or on S_g from S_+ to S_- (two equipotentials at the ends of the gap). Note that there is an infinite number of possible choices for C_e. Assuming that radian wavelengths are large compared to gap dimensions, and that admittance for current flow into the rest of the antenna is large compared to those in the gap region then V(t) as defined by equation 3.10 is approximately independent of the specific choice of C_e. The electric field is locally quasi-static (local to the gap) and is hence locally approximately conservative.

A better definition of the voltage V across the gap can be found by averaging the form in equation 3.10 over all paths C_e in some sense. Suppose the gap region were a cylinder of length Δ and cross section area A with S_h as planes orthogonal to the cylinder axis (the z axis) with all C_e taken parallel to the z axis. Let the current density be uniform in this volume. Then we have

$$V(t) \equiv -\frac{1}{A}\int_{S_h}\left\{\int_{z_+}^{z_-} \vec{E}(\vec{r},t) \cdot \vec{1}_z \, dz\right\} dS$$

$$= -\int_{V_g} \vec{E}(\vec{r},t) \cdot \left[\frac{1}{A}\vec{1}_z\right] dV$$

$$\equiv -\int_{V_g} \vec{E}(\vec{r},t) \cdot \vec{j}_g \, dV$$

$$\equiv -\left\langle \vec{E}(\vec{r},t) \, ; \, \vec{j}_g \right\rangle_g \qquad (3.11)$$

with integration over the gap volume. Similarly if the current is concentrated on the surface S_g of perimeter P and is uniform we have

$$V(t) \equiv -\frac{1}{P} \oint_{C_h} \left\{ \int_{z_+}^{z_-} \vec{E}(\vec{r},t) \cdot \vec{1}_z \, dz \right\} d\ell$$

$$= -\int_{S_g} \vec{E}(\vec{r},t) \cdot \left[\frac{1}{P} \vec{1}_z \right] dS$$

$$\equiv -\int_{S_g} \vec{E}(\vec{r},t) \cdot \vec{J}_{s_g} \, dS$$

$$\equiv -\left\langle \vec{E}(\vec{r},t) \, ; \, \vec{J}_{s_g} \right\rangle_g \tag{3.12}$$

with integration over the gap surface.

Corresponding to the admittance discussion in the previous section we have the impedance

$$\tilde{Z}_a(s) = -\frac{\tilde{V}(s)}{\tilde{I}(s)} \tag{3.13}$$

Writing this as

$$\tilde{V}(s) = -\tilde{Z}_a(s)\,\tilde{I}(s) \tag{3.14}$$

we have power types of relations as

$$\begin{aligned} \tilde{V}(s)\,\tilde{I}(s) &= -\tilde{Z}_a(s)\,\tilde{I}^2(s) \\ \tilde{V}(s)\,\tilde{I}(-s) &= -\tilde{Z}_a(s)\,\tilde{I}(s)\,\tilde{I}(-s) \end{aligned} \tag{3.15}$$

For comparison to these macroscopic power formulas one can consider the local quantities integrated over the gap region. Corresponding to the first of equations 3.15 we have

$$\left\langle \tilde{\vec{E}}(\vec{r},s) \,;\, \tilde{\vec{J}}_g(\vec{r},s) \right\rangle_g = \left\langle \tilde{\vec{E}}(\vec{r},s) \,;\, \tilde{I}(s)\, \vec{j}_g(\vec{r}) \right\rangle_g$$

$$= \left\langle \tilde{\vec{E}}(\vec{r},s) \,;\, \vec{j}_g(\vec{r}) \right\rangle_g \tilde{I}(s)$$

$$= -\tilde{V}(s)\, \tilde{I}(s) = \tilde{Z}_a(s)\, \tilde{I}^2(s) \qquad (3.16)$$

which gives the impedance as

$$\tilde{Z}_a(s) = -\frac{\tilde{V}(s)}{\tilde{I}(s)}$$

$$= \frac{\left\langle \tilde{\vec{E}}(\vec{r},s) \,;\, \tilde{\vec{J}}_g(\vec{r},s) \right\rangle_g}{\tilde{I}^2(s)}$$

$$= \frac{\left\langle \tilde{\vec{E}}(\vec{r},s) \,;\, \vec{j}_g(\vec{r}) \right\rangle_g}{\tilde{I}(s)} \qquad (3.17)$$

Using the power relationship with one term having s replaced by -s we have

$$\left\langle \tilde{\vec{E}}(\vec{r},s) \,;\, \tilde{\vec{J}}_g(\vec{r},-s) \right\rangle_g = \left\langle \tilde{\vec{E}}(\vec{r},s) \,;\, \tilde{I}(-s)\, \vec{j}_g(\vec{r}) \right\rangle_g$$

$$= \left\langle \tilde{\vec{E}}(\vec{r},s) \,;\, \vec{j}_g(\vec{r}) \right\rangle_g \tilde{I}(-s)$$

$$= -\tilde{V}(s)\, \tilde{I}(-s) = \tilde{Z}_a(s)\, \tilde{I}(s)\, \tilde{I}(-s) \qquad (3.18)$$

which gives the impedance as

$$\tilde{Z}_a(s) = -\frac{\tilde{V}(s)}{\tilde{I}(s)}$$

$$= \frac{\left\langle \tilde{\vec{E}}(\vec{r},s) \,;\, \tilde{\vec{J}}_g(\vec{r},-s) \right\rangle_g}{\tilde{I}(s)\, \tilde{I}(-s)}$$

$$= \frac{\left\langle \tilde{\vec{E}}(\vec{r},s) \,;\, \vec{j}_g(\vec{r}) \right\rangle_g}{\tilde{I}(s)} \qquad (3.19)$$

which is the same as in equation 3.17.

The voltage definition as a gap average in equations 3.11 and 3.12 together with the factored divergenceless chosen form for the current density in equations 3.1 through 3.6 gives an impedance independent of whether or not one term in the "power" expression is conjugated (for $s = i\omega$). Let us take this average type of definition for present purposes. For a general gap and time domain this is

$$V(t) \equi -\left\langle \vec{E}(\vec{r},t) \;\dot{,}\; \vec{J}_g(\vec{r}) \right\rangle_g \tag{3.18}$$

with integration over volume or surface as appropriate, and where $\vec{J}_g$ can be replaced by $\vec{J}_{s_g}$ as required. Note the similarity of this definition of voltage for impedance purposes to the definition of current for admittance purposes in equation 2.22.

IV. Short and Open Circuit Boundary Value Problems

The short and open circuit boundary value problems are defined by setting any impedance loading the gap of the antenna or scatterer to be respectively zero and infinity (or zero admittance). The formal solution of such boundary value problems is representable by integral equations where the domain of integration is the antenna or scatterer of interest, including or excluding the gap depending on which problem is being solved.

A. Integral equations

For the developments in this note consider an impedance integral equation[8,12] which we write in the general form

$$\tilde{\vec{E}}_s(\vec{r},s) = \left\langle \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s) \;\dot{,}\; \tilde{\vec{J}}_s(\vec{r}\,',s) \right\rangle$$
$$\tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s) = \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) + \tilde{\vec{\vec{Z}}}_\ell(\vec{r},s)\,\delta(\vec{r}-\vec{r}\,') \tag{4.1}$$

where

$$\tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) = s\mu_o \, \tilde{\vec{\vec{G}}}_o(\vec{r},\vec{r}\,';s)$$
$$\tilde{\vec{\vec{G}}}_o(\vec{r},\vec{r}\,';s) = \left[\vec{\vec{1}} - \frac{1}{\gamma^2}\nabla\nabla\right] \tilde{G}_o(\vec{r},\vec{r}\,';s)$$
$$\tilde{G}_o(\vec{r},\vec{r}\,';s) = \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \tag{4.2}$$
$$\vec{\vec{1}} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \vec{1}_x\vec{1}_x + \vec{1}_y\vec{1}_y + \vec{1}_z\vec{1}_z = \text{identity dyad}$$

with care taken to properly evaluate the integrals near $\vec{r} = \vec{r}\,'$ and where $\tilde{\vec{\vec{Z}}}_\ell(\vec{r},s)$ is any added load impedance in the object. Equation 4.1 is written for surface current densities but applies equally to volume current densities (and volume load impedances) as well as mixed volume and surface forms.

For our short and open circuit boundary value problem we have denoted the antenna or scatterer (less the gap region) by S_a, or merely by a subscript a. We have denoted the gap region by S_g or subscript g. Let our general integral equation be of the form

$$\tilde{\vec{E}}_s(\vec{r},s) = \left\langle \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s) \; ; \; \tilde{\vec{J}}_s(\vec{r}\,',s) \right\rangle \tag{4.3}$$

Then the short circuit boundary value problem has the form

$$\begin{aligned} \tilde{\vec{E}}_s(\vec{r},s) &= \left\langle \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s) \; \dot{;} \; \tilde{\vec{J}}_s(\vec{r}\,',s) \right\rangle_{a+g} , \quad \vec{r}\varepsilon S_a \cup S_g \\ &= \left\langle \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s) \; \dot{;} \; \tilde{\vec{J}}_s(\vec{r}\,',s) \right\rangle_{a} \\ &\quad + \left\langle \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s) \; \dot{;} \; \tilde{\vec{J}}_s(\vec{r}\,';s) \right\rangle_{g} , \quad \vec{r}\varepsilon S_a \cup S_g \end{aligned} \tag{4.4}$$

where $S_a \cup S_g$ is the entire antenna or scatterer, considered as surfaces here, but not necessarily so.

The open circuit boundary value problem has the form

$$\tilde{\vec{E}}_s(\vec{r},s) = \left\langle \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s) \; \dot{;} \; \tilde{\vec{J}}_s(\vec{r}\,',s) \right\rangle_{a} , \quad \vec{r}\varepsilon S_a \tag{4.5}$$

The resulting electric field $\tilde{\vec{E}}_g$ and voltage V in the gap region is evaluatable as an integral over the current on S_a as

$$\tilde{\vec{E}}_g(\vec{r},s) = \tilde{\vec{E}}_s(\vec{r},s) - \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) \; \dot{;} \; \tilde{\vec{J}}_s(\vec{r}\,',s) \right\rangle_{a} , \quad \vec{r}\varepsilon S_g \tag{4.6}$$

where $\tilde{\vec{E}}_s(\vec{r},s)$ is any source electric field (such as an incident electric field) which may be present.

If there is a source current density $\tilde{\vec{J}}_{s_g}(\vec{r},s)$ present in the gap region then one would write

$$\tilde{\vec{E}}(\vec{r},s) = \tilde{\vec{E}}_s(\vec{r},s) - \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) \;\dot{,}\; \tilde{\vec{J}}_s(\vec{r},s) + \tilde{\vec{J}}_{s_g}(\vec{r},s) \right\rangle_{a+g} \;,\; \vec{r}\in S_a \cup S_g$$

$$= \left\langle \tilde{\vec{\vec{Z}}}_\ell(\vec{r};s)\;\delta(\vec{r} - \vec{r}\,') \;\dot{,}\; \tilde{\vec{J}}_s(\vec{r},s) \right\rangle_{a+g} \;,\; \vec{r}\in S_a \cup S_g$$

$$= \tilde{\vec{E}}_g(\vec{r},s) \;,\; \vec{r}\in S_g \tag{4.7}$$

This formulation allows for the presence of both source and response currents and for possible impedance loading in the gap region. The details depend on the specifics of how the gap region is driven.

B. Incident field

Sections II and III have discussed manners of specifying the gap electric field or current density so as to define the source electric field in the integral equations (section IV.A) for calculating admittance and impedance. For determining the short circuit current and open circuit voltage one needs some sort of incident field for the source field.

A source electric field in the form of a general incident electric field can be defined as

$$\begin{aligned} \tilde{\vec{E}}_s(\vec{r},s) &= \tilde{\vec{E}}_{inc}(\vec{r},s) = E_o \sum_p \tilde{f}_p(s)\; \tilde{\vec{\delta}}_p(\vec{r},s) \\ \vec{E}_s(\vec{r},t) &= \vec{E}_{inc}(\vec{r},t) = E_o \sum_p f_p(t) * \vec{\delta}_p(\vec{r},t) \end{aligned} \tag{4.8}$$

where the subscript p indicates different incident waves characterized by various polarizations, angles of incidence, and more general spatial forms consistent with Maxwell's equations. In a generalized sense $f_p(t)$ is the incident waveform and $\vec{\delta}_p(\vec{r},t)$ is the general spatial form.

The summations in equation 4.8 might represent a single incident wave, some finite number of incident waves, or even an infinite number of incident waves. Such a sum can even be replaced by an integral over a continuous spectrum of incident waves if desired.

For a plane wave propagating in the direction $\vec{1}_{1,p}$ with polarization $\vec{1}_{2,p}$ with

$$
\begin{aligned}
\vec{1}_{1,p} \times \vec{1}_{2,p} &\equiv \vec{1}_{3,p} \\
\vec{1}_{2,p} \times \vec{1}_{3,p} &= \vec{1}_{1,p} \\
\vec{1}_{3,p} \times \vec{1}_{1,p} &= \vec{1}_{2,p} \\
\vec{1}_{1,p} \cdot \vec{1}_{2,p} = \vec{1}_{2,p} \cdot \vec{1}_{3,p} &= \vec{1}_{3,p} \cdot \vec{1}_{1,p} = 0
\end{aligned}
\qquad (4.9)
$$

then $\vec{1}_{2,p}$ gives the electric field orientation and $\vec{1}_{3,p}$ gives the magnetic field orientation. The general spatial form becomes

$$
\begin{aligned}
\tilde{\vec{\delta}}_p(\vec{r},s) &= \vec{1}_{2,p}\, e^{-\gamma \vec{1}_{1,p}\cdot\vec{r}} \\
\vec{\delta}_p(\vec{r},t) &= \vec{1}_{2,p}\, \delta\left(t - \frac{\vec{1}_{1,p}\cdot\vec{r}}{c}\right)
\end{aligned}
\qquad (4.10)
$$

which is a propagating delta function. In such cases $f_p(t)$ is the incident waveform at all spatial points except for a simple time delay.

Not just any spatial form $\tilde{\vec{\delta}}_p(\vec{r},s)$ of the incident electric field can be used. It must be consistent with Maxwell's equations. Plane waves in free space (equations 4.10) form one type which are consistent. Many other types are also possible.

C. Notation for short circuit quantities

The various electromagnetic quantities to be used in constructing equivalent circuits such as natural frequencies, eigenimpedances, etc. will be distinguished as to which of the two types of boundary value problems they pertain. For the short circuit quantities let us use a subscript sc if the quantity is used, for both short circuit current and admittance. For those pertaining only to the short circuit current use subscript ssc; for those pertaining to admittance use subscript y. Some of these are listed in table 4.1.

$\alpha_{sc} = (\beta,\beta')_{sc} = (\beta_{sc},\beta'_{sc})$	index set for short circuit natural frequencies
$\vec{\nu}_{\alpha_{sc}}$	short circuit natural modes
$\tilde{\eta}'_{\alpha_{sc}}$	admittance coupling coefficient
$\tilde{\eta}_{\alpha_{sc}}$	short circuit current coupling coefficients
β_{sc}	index set for short circuit eigenmodes

Table 4.1 Short circuit quantities

D. Notation for open circuit quantities

Similarly use a subscript oc if the quantity is used for both open circuit voltage and impedance resulting from the open circuit boundary value problem. For those pertaining to the open circuit voltage use a subscript ocv; for those pertaining to the impedance use a subscript z.

$\alpha_{oc} = (\beta,\beta')_{oc} = (\beta_{oc},\beta'_{oc})$	index set for open circuit natural frequencies
$\vec{\nu}_{\alpha_{oc}}$	open circuit natural modes
$\tilde{\eta}'_{\alpha_{oc}}$	impedance coupling coefficients
$\tilde{\eta}_{\alpha_{oc}}$	open circuit voltage coupling coefficients
β_{oc}	index set for open circuit eigenmodes

Table 4.2 Open circuit quantities

V. N-Port Representation of Boundary Value Problems

In discussing equivalent circuits for antennas and scatterers it is instructive to consider an N-port representation of such objects. This can be defined from the moment method (MoM) numerical representation of the object in a certain format. As in the previous section let our integral equation be an impedance integral equation of the general form

$$\begin{aligned}
\tilde{\vec{E}}_s(\vec{r},s) &= \left\langle \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s) \; ; \; \tilde{\vec{J}}(\vec{r}\,'s) \right\rangle \\
\tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s) &= \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) + \tilde{\vec{\vec{Z}}}_\ell(\vec{r},s)\,\delta(\vec{r}-\vec{r}\,') \\
\tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) &= s\mu_o\,\tilde{\vec{\vec{G}}}_o(\vec{r},\vec{r}\,';s)
\end{aligned} \tag{5.1}$$

where $\tilde{\vec{\vec{Z}}}_\ell(\vec{r},s)$ represents impedance loading of the object.

Following Harrington[12] we expand the response $\tilde{\vec{J}}_s$ in a set of basis functions with coefficients $\tilde{J}_n$ and the excitation $\tilde{\vec{E}}_s$ in a set of testing functions with coefficients $\tilde{V}_n$. This gives a matrix equation

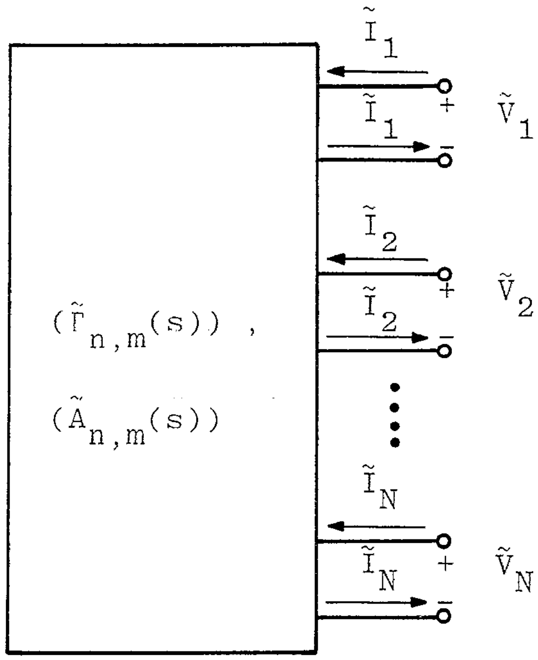
$$(\tilde{V}_n(s)) = (\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{I}_n(s)) \;,\; n,m = 1,2,3,\ldots,N \tag{5.2}$$

and we can say that we have matricized the integral operator of equations 5.1. The matrix elements $\tilde{\Gamma}_{n,m}(s)$ are the generalized impedances as defined by Harrington.

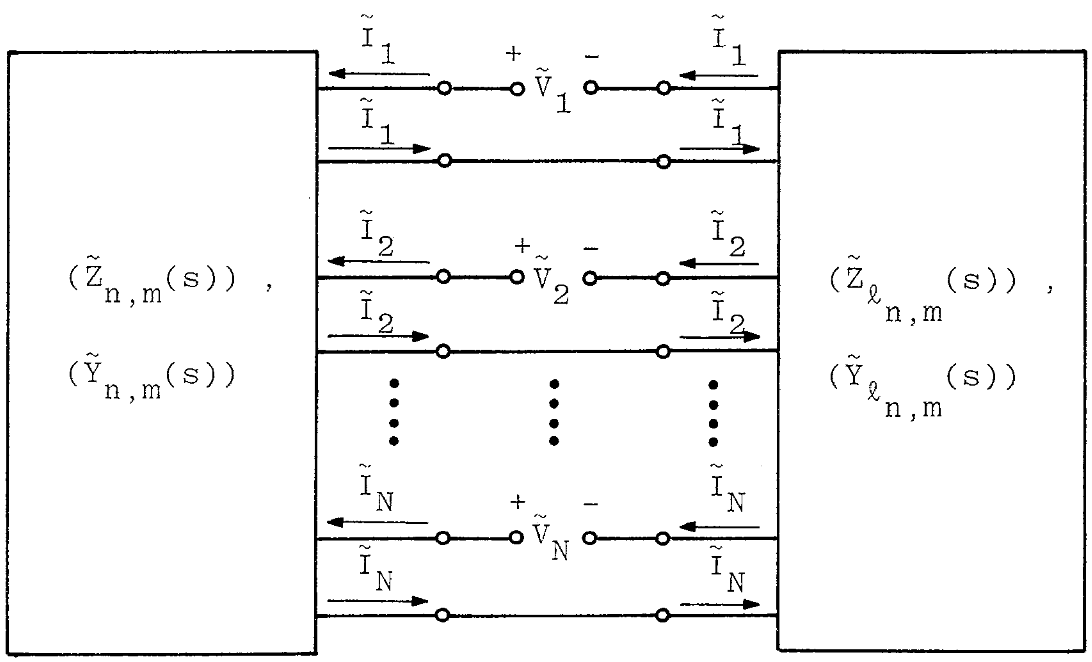
Figure 5.1 shows the N-port network representation of the antenna or scatterer. The $\tilde{I}_n$ are the port currents and the $\tilde{V}_n$ are the port voltages. A slightly more general form of the kernel of our integral equation

$$\tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s) = \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) + \tilde{\vec{\vec{Z}}}_\ell(\vec{r},\vec{r}\,';s) \tag{5.3}$$

with corresponding impedance matrices



A. Current and voltage conventions



B. Two N-port network conventions

Figure 5.1. N-Port Circuit Representation of Antenna or Scatterer Including Loading

$$(\tilde{\Gamma}_{n,m}(s)) = (\tilde{Z}_{n,m}(s)) + (\tilde{Z}_{\ell_{n,m}}(s)) \tag{5.4}$$

which separates the scattered (or radiated) field part of the generalized impedances from the loading part. Figure 5.1A shows the single N-port representation of antenna or scatterer. Figure 5.1B shows the representation as two N-port networks with the $(\tilde{Z}_{n,m}(s))$ and $(\tilde{Z}_{\ell_{n,m}}(s))$ separated. Note that the case of only local effect of the impedance loading as in equations 5.1 has the matrix $(\tilde{Z}_{\ell_{n,m}}(s))$ as a diagonal matrix; with the $\tilde{Z}_{\ell_{n,m}}$ appearing as series elements to the input of the $(\tilde{Z}_{n,m}(s))$ network; this assumes subsectional basis and testing functions have been used so as to have the $\tilde{I}_n(s)$ and $\tilde{V}_n(s)$ represent the local currents and electric fields on the body.

There is a matrix of admittances

$$(\tilde{A}_{n,m}(s)) \equiv (\tilde{\Gamma}_{n,m}(s))^{-1} = \left[(\tilde{Z}_{n,m}(s)) + (\tilde{Z}_{\ell_{n,m}}(s))\right]^{-1} \tag{5.5}$$

where the matrix elements $\tilde{A}_{n,m}$ are generalized admittances. We also have

$$(\tilde{Y}_{n,m}(s)) \equiv (\tilde{Z}_{n,m}(s))^{-1} \tag{5.6}$$

as the admittance matrix corresponding to the perfectly conducting body (i.e., without loading). For the loading we can also define

$$(\tilde{Y}_{\ell_{n,m}}(s)) \equiv (\tilde{Z}_{\ell_{n,m}}(s))^{-1} \tag{5.7}$$

With the identity matrix as

$$\begin{aligned} (\delta_{n,m}) &= (\tilde{Z}_{n,m}(s)) \cdot (\tilde{Y}_{n,m}(s)) \\ \delta_{n,m} &= \begin{cases} 1 \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases} \end{aligned} \tag{5.8}$$

we also have

$$
\begin{aligned}
(\tilde{A}_{n,m}(s)) &= (\tilde{Y}_{n,m}(s)) \cdot (\tilde{Z}_{n,m}(s)) \cdot \left[(\tilde{Z}_{n,m}(s)) + (\tilde{Z}_{\ell_{n,m}}(s))\right]^{-1} \\
&= (\tilde{Y}_{n,m}(s)) \cdot \left\{\left[(\tilde{Z}_{n,m}(s)) + (\tilde{Z}_{\ell_{n,m}}(s))\right] \cdot (\tilde{Y}_{n,m}(s))\right\}^{-1} \\
&= (\tilde{Y}_{n,m}(s)) \cdot \left\{\delta_{n,m} + (\tilde{Z}_{\ell_{n,m}}(s)) \cdot (\tilde{Y}_{n,m}(s))\right\}^{-1}
\end{aligned}
\tag{5.9}
$$

or in another form

$$
(\tilde{A}_{n,m}(s)) = \left\{\delta_{n,m} + (\tilde{Y}_{n,m}(s)) \cdot (\tilde{Z}_{\ell_{n,m}}(s))\right\}^{-1} \cdot (\tilde{Y}_{n,m}(s)) \tag{5.10}
$$

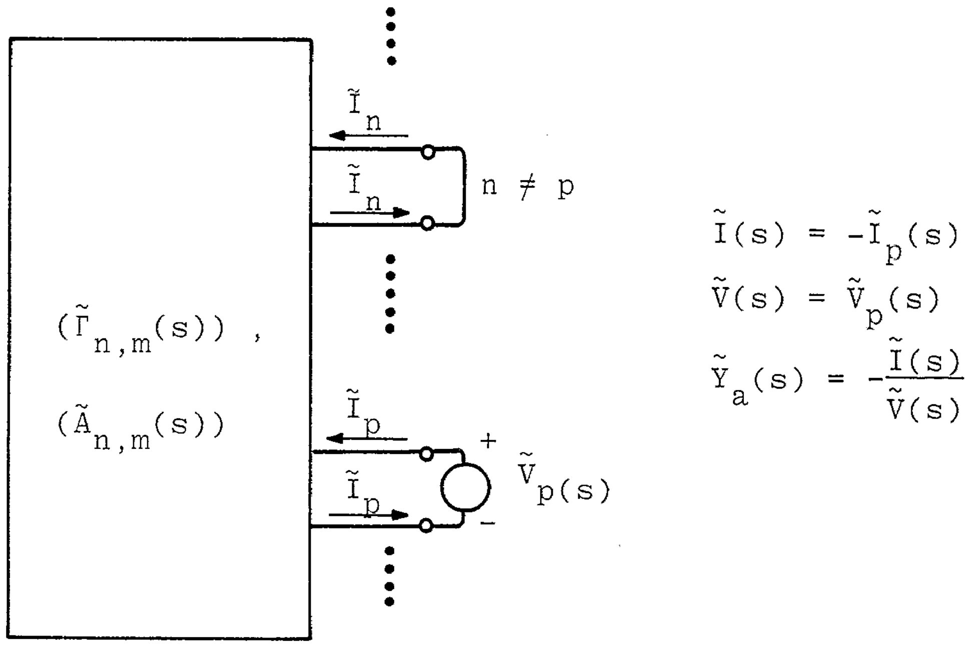
Note that when using the admittance form the diagonal element $\tilde{A}_{n,n}$ appears in parallel at the nth port, whereas $\tilde{\Gamma}_{n,n} = \tilde{Z}_{n,n} + \tilde{Z}_{\ell_{n,n}}$ appears in series at the nth port.

In terms of the equivalent N-port network one can now consider the open circuit and short circuit boundary value problems discussed in the previous section. For this purpose let one of the ports $n = 1,2,\ldots,N$ denoted by $n = p$ correspond to our antenna or scatterer port of interest. The convention for $\tilde{I}_n$ is that for power into the antenna or scatterer while for $\tilde{I}$ we are considering power into the load. Hence we have

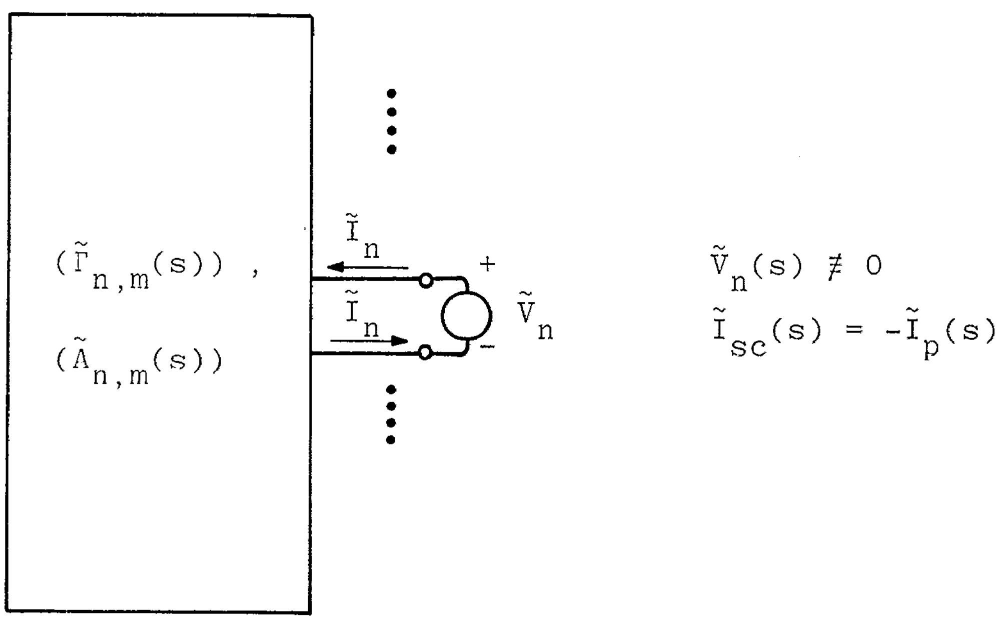
$$
\begin{aligned}
\tilde{I}(s) &= -\tilde{I}_p(s) \\
\tilde{V}(s) &= \tilde{V}_p(s)
\end{aligned}
\tag{5.11}
$$

For the short circuit boundary value problem we have the configurations illustrated in figure 5.2. For the admittance we have

$$
\begin{aligned}
\tilde{V}_n(s) &= 0 \quad , \quad n \neq p \\
\tilde{Y}_a(s) &= \frac{\tilde{I}_p(s)}{\tilde{V}_p(s)} = -\frac{\tilde{I}(s)}{\tilde{V}(s)}
\end{aligned}
\tag{5.12}
$$



A. Admittance



B. Short circuit current

Figure 5.2. N-Port Configurations Corresponding to Short Circuit Boundary Value Problem

For this situation one might wish there to be no impedance loading across the gap so that "all" the current from the antenna/scatterer is available to the load. Actually the capacitance across the gap will still remain but the loading is thereby minimized in a restricted sense. Stated another way one may wish that the loading impedance $\tilde{\overleftrightarrow{Z}}_\ell$ be infinite in the gap. For the short circuit current we have

$$\begin{aligned} &\tilde{V}_n(s) \neq 0 \quad , \quad \text{all } n \\ &\tilde{I}_{sc}(s) = -\tilde{I}_p(s) \end{aligned} \tag{5.13}$$

where the $\tilde{V}_n(s)$ are sources given by the incident field.

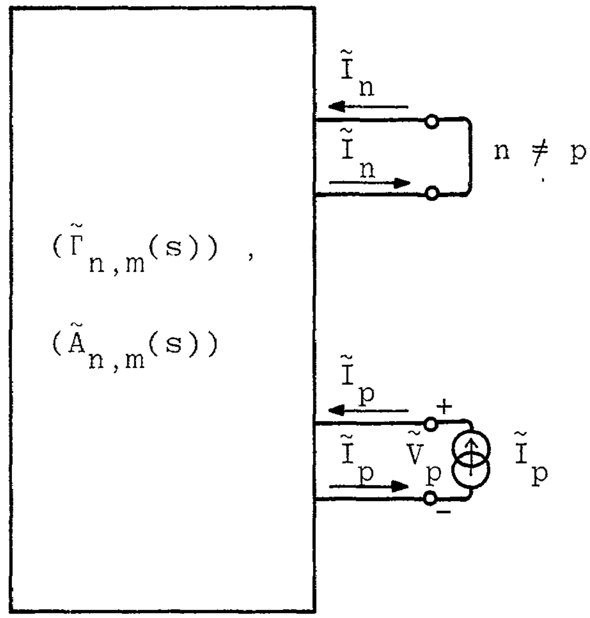
For the open circuit boundary value problem we have the configurations illustrated in figure 5.3. For the impedance we have

$$\begin{aligned} &\tilde{V}_n = 0 \quad , \quad n \neq p \\ &\tilde{Z}_a(s) = \frac{\tilde{V}_p(s)}{\tilde{I}_p(s)} = -\frac{\tilde{V}(s)}{\tilde{I}(s)} \end{aligned} \tag{5.14}$$

For the open circuit voltage we have

$$\begin{aligned} &\tilde{V}_n(s) \neq 0 \quad , \quad \text{all } n \\ &\tilde{I}_p(s) = 0 \\ &\tilde{V}_{oc}(s) = \tilde{V}_p(s) \end{aligned} \tag{5.15}$$

where the $\tilde{V}_n(s)$ for $n \neq p$ are sources given by the incident field and the open circuit voltage is $\tilde{V}_p(s)$ which is the voltage across the nth zone including both the incident field and the mutual terms from the remainder of the network.

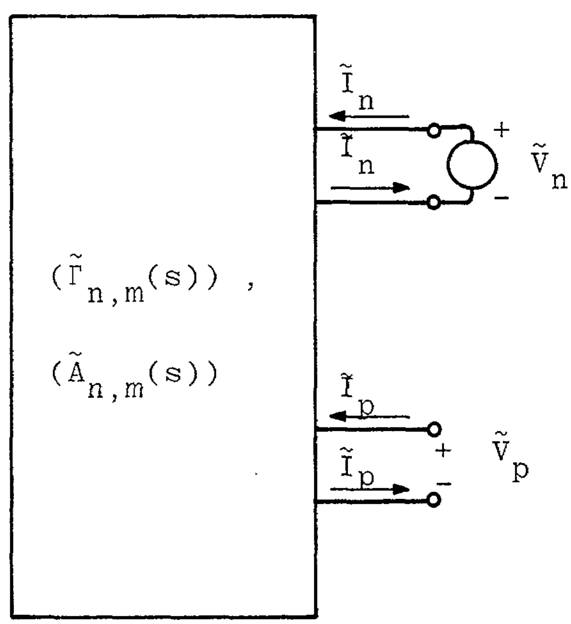


$$\tilde{I}(s) = -\tilde{I}_p(s)$$

$$\tilde{V}(s) = \tilde{V}_p(s)$$

$$\tilde{Z}_a(s) = -\frac{\tilde{V}(s)}{\tilde{I}(s)}$$

A. Impedance



$$\tilde{V}_n(s) \neq 0$$

$$\tilde{I}_p(s) \equiv 0$$

$$\tilde{V}_{oc}(s) = \tilde{V}_p(s)$$

B. Open circuit voltage

Figure 5.3. N-Port Configuration Corresponding to Open Circuit Boundary Value Problem

Concerning the pth port and its relation to the antenna port there are some subtle points regarding the way the current is brought from the antenna/scatterer to the load. The detailed design of the port (gap) needs to be considered in treating the local pth contributions to the port voltage, current, and admittance/impedance.

VI. Some Limiting Admittance or Impedance Properties of Port

In constructing equivalent circuits for antennas/scatterers at a port it is useful to consider some of the physical properties of the port in limiting frequency or time regimes. These results may in turn be used to help define the short circuit current, open circuit voltage, and admittance/impedance of the antenna/scatterer at the port of interest. The high and low frequency properties establish constraints on these functions.

A. High frequency impedance

In the limit of high frequencies (or early time) one can consider the antenna/scatterer response according to the characteristics of the gap geometry. As illustrated in figure 6.1A there are various specific gap geometries one can consider.

In calculating the impedance for high frequencies let $s \to \infty$ in the right half s plane with $|\arg(s)| \leq \pi/2 - \delta$ with $\delta > 0$. This corresponds to early time after turning on the source field or current density in the gap. Only the immediate gap vicinity influences this early-time result.

As an example consider first the circular biconical gap geometry in figure 6.1A. Let the port shrink toward the conical apex. Then the impedance has the limiting form[1]

$$\lim_{\substack{s \to \infty \\ |\arg(s)| \leq \frac{\pi}{2} - \delta}} \tilde{Z}_a(s)\Big|_{\text{cone}} = \frac{Z_o}{2\pi} \ln\left[\frac{\cot\left(\frac{\theta_1}{2}\right)}{\cot\left(\frac{\theta_2}{2}\right)}\right] \tag{6.1}$$

where θ_1 and θ_2 are the two polar angles specifying the two conducting circular conical surfaces.

Another example is the distributed source gap with a uniform axial source electric field on a circular cylinder as illustrated in figure 6.1A. With a source band of length Δ and radius a the high frequency limiting impedance considering only fields exterior to the cylinder is

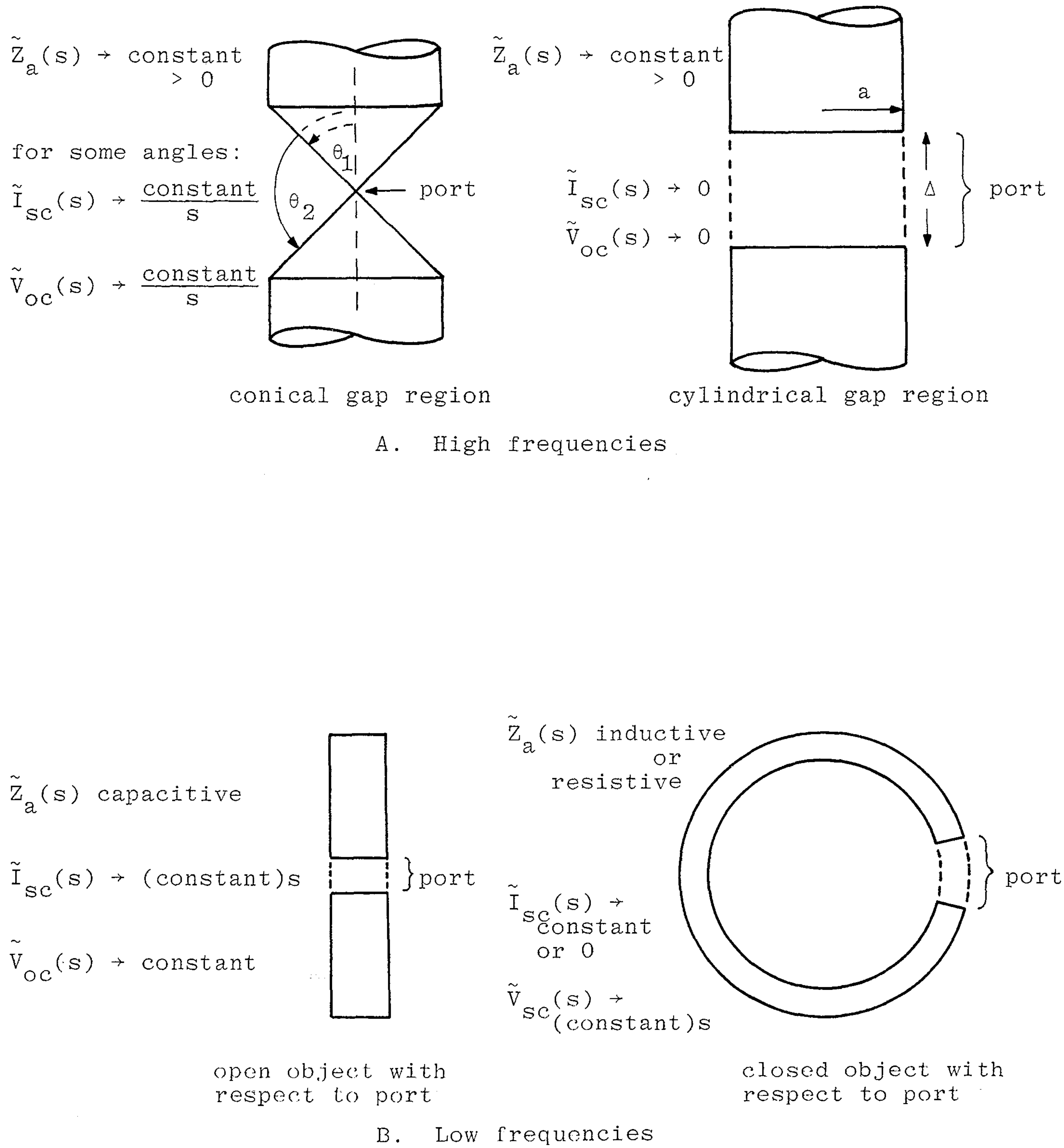


Figure 6.1. Some Limiting Properties of Port

$$\lim_{\substack{s \to \infty \\ |\arg(s)| \leq \frac{\pi}{2}-\delta}} \tilde{Z}_a(s)\Bigg|_{\substack{\text{cylinder} \\ \text{exterior}}} = Z_o \frac{\Delta}{2\pi a} \tag{6.2}$$

If the interior fields are also included then the appropriate high frequency impedance driven by the source distribution on the cylindrical surface is

$$\lim_{\substack{s \to \infty \\ |\arg(s)| \leq \frac{\pi}{2}-\delta}} \tilde{Z}_a(s)\Bigg|_{\substack{\text{cylinder} \\ \text{exterior} \\ \text{and} \\ \text{interior}}} = \frac{Z_o}{2} \frac{\Delta}{2\pi a} = Z_o \frac{\Delta}{4\pi a} \tag{6.3}$$

Viewed in a more general manner a surface gap has a surface electric field $\tilde{\vec{E}}_s(\vec{r},s)$ which results in a surface current density $\tilde{\vec{J}}_s(\vec{r},s)$. At high frequencies this becomes

$$\lim_{\substack{s \to \infty \\ |\arg(s)| \leq \frac{\pi}{2}-\delta}} \tilde{\vec{J}}_s(\vec{r},s) \to \frac{2}{Z_o} \tilde{\vec{E}}_s(\vec{r},s) \tag{6.4}$$

Using the specified electric field from section II as

$$\tilde{\vec{E}}_s(\vec{r},s) = \tilde{\vec{E}}_g(\vec{r},s) = -\tilde{V}(s)\, \vec{e}_g(\vec{r}) \tag{6.5}$$

we have the high frequency admittance

$$\lim_{\substack{s \to \infty \\ |\arg(s)| \leq \frac{\pi}{2}-\delta}} \tilde{Y}_a(s) = \frac{2}{Z_o} \left\langle \vec{e}_g(\vec{r}) \; ; \; \vec{e}_g(\vec{r}) \right\rangle_g \tag{6.6}$$

Similarly one can regard the surface current density as the source as in section III as

$$\tilde{\vec{J}}_s(\vec{r},s) = \tilde{I}(s)\, \vec{j}_{s_g}(\vec{r}) \tag{6.7}$$

We have the high frequency impedance

$$\lim_{\substack{s \to \infty \\ |\arg(s)| \leq \frac{\pi}{2} - \delta}} \tilde{Z}_a(s) = \frac{Z_o}{2} \left\langle \vec{j}_{s_g}(\vec{r}) \; ; \; \vec{j}_{s_g}(\vec{r}) \right\rangle_g \qquad (6.8)$$

Here the inside and outside fields are both included from equation 6.4. If only exterior fields are to be included then one can reduce the high frequency admittance (equation 6.6) by a factor of 2 and increase the high frequency impedance (equation 6.8) by a factor of 2.

These considerations indicate that in general the high frequency impedance is a positive constant, i.e. resistive. One can postulate an inductive or capacitive high frequency impedance by the addition of a lumped ideal series inductor or parallel capacitor at the gap. However, real physical structures are better described as distributed at high frequencies than as lumped. Except in special cases then one can regard $\tilde{Z}_a(s)$ as a constant resistance at high frequencies (in the right half s plane).

The distributed gaps considered here have been those with source surfaces. With source volumes the considerations may be somewhat different, but one should be careful regarding the physical characteristics of the gap distributions chosen. Various types of gaps could be considered and their high frequency impedances tabulated. In turn these results can be used to impose contraints on the high frequency gap impedance.

B. High frequency short circuit current and open circuit voltage

The high frequency short circuit current and open circuit voltage are related by the corresponding impedance or admittance via equations 1.4 through 1.5. Since, for the cases we are considering, the high frequency impedance is a positive constant (resistance) the short circuit current and open circuit voltage have the same high frequency behavior.

Consider a step function plane wave in time domain incident on our antenna/scatterer of interest. Assume that there is no focusing of scattered high frequency fields onto the gap from other parts of the object, i.e. the fields incident on the gap region are bounded in an early time sense without unbounded steeply rising transient fields arriving later in time. For the conical gap region in figure 6.1A let the incident wave arrive with a propagation direction between the two cones. Just after the incident wave reaches the apex (say a time Δt after) the apex can collect energy from a volume with linear dimensions Δt into some resistive load or back into the high frequency resistive impedance. This energy proportional to $(\Delta t)^3$ can be equated to an energy given by V and I both proportional to Δt. This gives a response in Laplace form proportional to s^{-2}. The response of such a gap to a delta function plane wave then falls off like s^{-1}. Faster falloffs at high frequency are also possible if the direction of incidence is not between the cones or if the polarization of the incident wave is appropriately orthogonal to the gap region.

Another example to consider is the circular cylindrical gap region shown in figure 6.1A. An incident step function plane wave can give an early time gap voltage for a surface type gap proportional to $(\Delta t)^{1/2}$ from a surface integral as in equation 3.11, provided the direction of incidence is perpendicular to the cylinder axis. This corresponds to $s^{-3/2}$ in complex frequency form. For a delta function incident plane wave this is an $s^{-1/2}$ dependence. For other directions of incidence the response (open circuit voltage and short circuit current) fall faster to zero at high frequencies.

Here we see that different types of high frequency gap response (short circuit current and open circuit voltage) are possible. However, their high frequency responses to delta function plane waves all go to zero, except in special limiting cases applying to special gap designs.

Using high frequency techniques such as GTD (geometrical theory of diffraction) the high frequency response characteristics of various gap designs can be tabulated. These results can be used as constraints on the representations of the short circuit current and open circuit voltage.

C. Low frequency impedance

At low frequencies one can consider the antenna/scatterer response as a quasi-static problem involving the total object geometry. As illustrated in figure 6.1B there are some basic object topologies one can consider for this purpose.

In this note we have restricted our attention to finite size objects in free space. Such objects are known to have delta function responses with only pole singularities in the complex frequency plane.[4,5] Note that for our impedance/admittance problem the gap region must be either of finite non-zero dimensions (no delta gaps which would give infinite admittance allowed) or of special limiting zero dimensions preserving a finite non-zero admittance for frequencies near (but not necessarily at) zero.

In addition when we are dealing with an impedance/admittance problem for a passive antenna/scatterer we have a description as a positive real function. This implies that any pole at zero frequency is at most of first order. We can then expand such a function as

$$\begin{aligned} \tilde{Z}_a(s) &= Z_{a_{-1}} s^{-1} + Z_{a_o} s^{o} + Z_{a_1} s^{1} + \cdots \\ \tilde{Y}_a(s) &= Y_{a_{-1}} s^{-1} + Y_{a_o} s^{o} + Y_{a_1} s^{1} \end{aligned} \qquad (6.9)$$

which are of course related. Either $\tilde{Z}_a$ has a first order pole at $s = 0$, is a non-zero constant at $s = 0$, or has a first order zero at $s = 0$. These are summarized as

1. impedance pole (admittance zero) at s = 0

$$Y_{a_{-1}} = Y_{a_o} = 0$$
$$Z_{a_{-1}} = Y_{a_1}^{-1} > 0 \qquad (6.10)$$

2. impedance constant (admittance constant) at s = 0

$$Y_{a_{-1}} = 0$$
$$Z_{a_{-1}} = 0 \qquad (6.11)$$
$$Z_{a_o} = Y_{a_o}^{-1} > 0$$

3. impedance zero (admittance pole) at s = 0

$$Z_{a_{-1}} = Z_{a_o} = 0$$
$$Z_{a_1} = Y_{a_{-1}}^{-1} > 0 \qquad (6.12)$$

Note that these results still apply with the reduced assumption of stability for the antenna/scatterer instead of passivity. This is seen by noting that under open circuit or short circuit conditions the object would be unstable if higher than a first order zero or pole, or non-positive leading coefficients were allowed.

Divide antennas/scatterers into two classes, those which allow no current to flow at DC (s = 0) in response to a voltage at the gap, and those which do allow such a current. The former can be considered topologically as two objects joined only through the gap, while the latter can be considered as a single object which has two small portions of its surface additionally joined through the gap region. The former has current paths through the gap never closed in the object, while the latter has all current paths through the gap closed in the object (in a low frequency sense). The former has a capacitive impedance as $s \to 0$ and the latter has an inductive or resistive impedance as $s \to 0$.

Considering the former object as illustrated in figure 6.1B as an open object with respect to the port we have the case given by equations 6.10. The object has a capacitance C_a (as measured at the port) and we have

$$\tilde{Y}_a(s) = \tilde{Z}_a^{-1}(s) = sC_a + \cdots$$
$$Y_{a_1} = Z_{a_{-1}}^{-1} = C_a > 0 \qquad (6.13)$$

Considering the latter object as illustrated in figure 6.1B as a closed object with respect to the port we have the cases given by equations 6.11 and 6.12. The object has inductance L_a and perhaps resistance R_a (if none of the closed current paths is perfectly conducting along its entire circuit of the object) and we have

$$\tilde{Z}_a(s) = \tilde{Y}_a^{-1}(s) = R_a + sL_a + \cdots$$
$$Z_{a_o} = Y_{a_o}^{-1} = R_a > 0 \quad \text{for } R_a \neq 0 \qquad (6.14)$$
$$Z_{a_1} = Y_{a_{-1}}^{-1} = L_a > 0 \quad \text{for } R_a = 0$$

D. Low frequency short circuit current and open circuit voltage

The low frequency short circuit current and open circuit voltage are related by the low frequency impedance just discussed. Now since the low frequency impedance can be capacitive, resistive, or inductive the short circuit current and open circuit voltage can have different low frequency behavior.

For the open type of object in figure 6.1B the open circuit voltage at low frequencies is proportional to the incident electric field. Considered as an electromagnetic field sensor this is an electric field sensor characterized by an equivalent height $\vec{h}_{eq}$ with[2]

$$\tilde{V}_{oc}(s) = -\vec{h}_{eq} \cdot \tilde{\vec{E}}_s(s) \quad , \quad s \to 0 \qquad (6.15)$$

Note that this has no singularity as $s \to 0$ but may have a zero if $\tilde{\vec{E}}_{inc}$ is orthogonal to $\vec{h}_{eq}$; here $\tilde{\vec{E}}_s(s)$ is assumed to be a uniform, constant, non-zero vector for $s \to 0$. The short circuit current on the other hand is given by an equivalent area as

$$
\begin{aligned}
\tilde{I}_{sc}(s) &= -s\vec{A}_{eq} \cdot \tilde{\vec{D}}_s(s) \quad , \quad s \to 0 \\
&= \tilde{Z}_a^{-1}(s)\, \tilde{V}_{oc}(s) \\
&= sC_a\, \tilde{V}_{oc}(s) \quad , \quad s \to 0 \qquad (6.16) \\
&= -sC_a\vec{h}_{eq} \cdot \tilde{\vec{E}}_s(s) \quad , \quad s \to 0 \\
\epsilon_o\vec{A}_{eq} &= C_a\vec{h}_{eq}
\end{aligned}
$$

The open type of object then has its delta function response to an external incident field (electric) as a constant for open circuit voltage, but a zero for short circuit current. This type of object is characterized as an electric dipole at low frequencies both in transmission (driven at the gap) and in reception (by reciprocity).

For the closed type of object in figure 6.1B the open circuit voltage is proportional to the time rate of change of the incident magnetic field via an equivalent area as[2]

$$\tilde{V}_{oc}(s) = s\vec{A}_{eq} \cdot \tilde{\vec{B}}_s(s) \quad , \quad s \to 0 \qquad (6.17)$$

The open circuit voltage response to a delta function incident field then has a zero at $s = 0$. The short circuit current is given by

$$\tilde{I}_{sc}(s) = \tilde{Z}_a^{-1}(s)\ \tilde{V}_{oc}(s)$$

$$= (R_a + sL_a)^{-1}\ \tilde{V}_{oc}(s) \quad , \quad s \to 0$$

$$= s(R_a + sL_a)^{-1}\ \vec{A}_{eq} \cdot \tilde{\vec{B}}_s(s) \quad , \quad s \to 0 \tag{6.18}$$

For non-zero resistance we have

$$\tilde{I}_{sc}(s) = sR_a^{-1}\ \vec{A}_{eq} \cdot \tilde{\vec{B}}_s(s) \quad , \quad s \to 0 \tag{6.19}$$

which gives a zero for the delta function response at $s = 0$. For zero resistance the response is characterized by an equivalent length as

$$\tilde{I}_{sc}(s) = L_a^{-1}\ \vec{A}_{eq} \cdot \tilde{\vec{B}}_s(s) \quad , \quad s \to 0$$

$$= \vec{\ell}_{eq} \cdot \tilde{\vec{H}}_s(s) \quad , \quad s \to 0 \tag{6.20}$$

$$\mu_o \vec{A}_{eq} = L_a \vec{\ell}_{eq}$$

The short circuit current for a closed object then has its delta function response to an external incident field (magnetic) as a zero for the open circuit voltage, a zero for the short circuit current for a resistive object ($R_a > 0$), and a constant for the short circuit current for an inductive object ($R_a = 0$). This type of closed object is characterized as a magnetic dipole (loop) at low frequencies both in transmission (driven at the gap) and in reception (by reciprocity).

VII. Analytic Properties of Impedance and Admittance in the Complex Frequency Plane

Assuming that we have a passive linear object (antenna/scatterer) there are certain useful properties of the associated impedances and admittances. Specifically these are positive real (PR) functions, much as used in circuit theory except applied to distributed electromagnetic systems. Let us here consider some of the implications of the PR property. These can be applied to the antenna/scatterer impedance and admittance at the port (equations 1.3 through 1.5) and to the eigenimpedances and eigenadmittances[8] of the scatterer for both the short circuit and open circuit boundary value problems (as well as any specific case of passive linear load at the antenna gap).

In this context let us restrict consideration to impedances which are ratios of the Laplace transforms of voltages to currents or electric fields to current densities, where such voltages, currents, etc. are real valued time functions. Then since the Laplace transform $\tilde{g}(s)$ of a real valued time function $g(t)$ is conjugate symmetric, i.e.

$$\tilde{g}(\overline{s}) = \overline{\tilde{g}}(s) \tag{7.1}$$

where the Laplace transform exists. The ratio of two such transforms is also conjugate symmetric so that we have for the impedance and admittance

$$\begin{aligned} \tilde{Z}_a(\overline{s}) &= \overline{\tilde{Z}}_a(s) \\ \tilde{Y}_a(\overline{s}) &= \overline{\tilde{Y}}_a(s) \end{aligned} \tag{7.2}$$

Similarly for the eigenimpedances and eigenadmittances (eigenvalues of impedance or admittance operators) we have the same conjugate symmetric property. This particular characteristic of conjugate symmetry applies not only to impedances and admittances, but to any Laplace transform of a real valued time function.

This property of conjugate symmetry is often stated in another form as $\tilde{g}(s)$ is real for real values of s. Conjugate symmetry immediately implies this as can be seen by setting s real in equation 7.1. One should be careful in applying this conjugate symmetric property to consider only those s for which the function is uniquely defined. For cases with only pole singularities in the finite s plane only the precise pole locations need be avoided.

The passive property of the object is reflected in the non-negative real part on the $i\omega$ axis of the s plane as

$$\begin{aligned} \mathrm{Re}\left[\tilde{Z}_a(i\omega)\right] &\geq 0 \\ \mathrm{Re}\left[\tilde{Y}_a(i\omega)\right] &\geq 0 \end{aligned} \tag{7.3}$$

for the impedance and admittance at the port. This is seen by requiring for CW excitation of the gap that real power flow from the source at the gap into the object and surrounding space, or at least not flow back into the source if the object is purely reactive (such as in the case of a perfectly conducting cavity).

If at time $t = t_o$ one drives the gap with a current source which is bounded in amplitude then the gap voltage grows slower than an exponential at late time, otherwise the object would be unstable and hence not passive. The voltage (the response) also commences no sooner than the source current. Hence the Laplace transform of the voltage is analytic throughout the half plane. Letting the current be simple, say a step function, its Laplace transform is analytic with no zeros in the right half plane. Hence the impedance is analytic throughout the right half plane. Interchanging the roles of voltage and current by driving the gap with a suitable voltage source we find that the admittance is also analytic throughout the right half plane. Since the impedance and admittance are mutually reciprocal and since each is analytic and therefore finite in the right half plane for $|s|$ finite, then the reciprocal of each has no zeros in the right half plane (possibly excluding at infinity). Hence we have

$$\left.\begin{array}{l}\tilde{Z}_a(s) \\ \tilde{Y}_a(s)\end{array}\right\} \begin{array}{c}\text{analytic (no singularities) and } \neq 0 \\ \text{for } \mathrm{Re}[s] > 0\end{array} \tag{7.4}$$

For $s \to \infty$ in the right half plane the restriction of equations 7.3 implies that such functions be no more singular than a positive constant times s or go to zero no faster than a positive constant divided by s. Note that an exponential function, for example, representing a delay in the impedance would represent an advance in the admittance (and conversely) giving a non-causal and therefore non-passive object. The gap current or voltage must immediately respond to the other quantity with a time behavior lying between first derivative and first integral (or s and s^{-1}) for early times. Stated another way

$$\begin{array}{l} \left.\begin{array}{l}\tilde{Z}_a(s) \sim As^q \\ \tilde{Y}_a(s) \sim A^{-1}s^{-q}\end{array}\right\} \begin{array}{l} s \to \infty \\ |\arg(s)| \leq \frac{\pi}{2} - \delta \end{array} \\ A > 0 \\ -1 \leq q \leq 1 \\ \delta > 0 \end{array} \tag{7.5}$$

where we restrict the result away from the $i\omega$ axis to allow for cases of alternating zeros and poles on the $i\omega$ axis as in cavities. Recall the discussion in section VI.A that for antennas and scatterers with finite, non-zero gap dimensions characterized by surface source distributions the impedance tends to a constant, i.e.

$$\left.\begin{aligned}\tilde{Z}_a &\sim A > 0 \\ \tilde{Y}_a &\sim A^{-1} > 0\end{aligned}\right\} \quad \begin{aligned}&s \to \infty \\ &|\arg(s)| \leq \frac{\pi}{2} - \delta\end{aligned} \tag{7.6}$$

Combining these results noting that $Re[\tilde{Z}_a]$ and $Re[\tilde{Y}_a]$ are positive on the $i\omega$ axis and on a semicircle near infinity in the right half plane with no singularities (points of non-analyticity of $\tilde{Z}_a$ or $\tilde{Y}_a$ within the right half plane) we conclude

$$\begin{aligned}Re\left[\tilde{Z}_a(s)\right] &\geq 0 \qquad \text{for } Re[s] \geq 0 \\ Re\left[\tilde{Y}_a(s)\right] &\geq 0 \qquad \text{for } Re[s] \geq 0\end{aligned} \tag{7.7}$$

This results from the well known property that the maximum and minimum values of the real and imaginary parts of a function, analytic within a contour, are achieved on the contour.

On the $i\omega$ axis we may have zeros and poles. However, they must be simple with real coefficients. Suppose we have

$$\begin{aligned}\tilde{Z}_a(s) &= \tilde{Y}_a^{-1}(s) = B_p(s - s_\alpha)^p + B_{p+1}(s - s_\alpha)^{p+1} + \cdots \\ s_\alpha &= i\omega_\alpha \quad , \quad Re\left[s_\alpha\right] = 0\end{aligned} \tag{7.8}$$

Then $p > 1$ and $p < -1$ would violate equations 7.3 For integer p the acceptable cases are

$$\begin{aligned}&p = \pm 1 \text{ with } B_p > 0 \\ &p = 0 \quad \text{with } Re\left[B_p\right] \geq 0 \quad , \quad |B_p| \neq 0\end{aligned} \tag{7.9}$$

For cases of brach points at $s = s_\alpha$ in equations 7.7 one might allow non-integer p with $-1 \leq p \leq 1$ with suitable restrictions on B_p to insure compliance with equations 7.3.

While equations 7.8 and 7.9 are for poles and zeros on the $i\omega$ axis they can be applied in an approximate sense to poles and zeros near the $i\omega$ axis. Consider a zero with small negative real part. Then if the nearest zeros and poles are far away compared to the distance of the zero from the $i\omega$ axis the coefficient B_p would have small imaginary part or small $|\arg(B_p)|$ compared to $\pi/2$ where arg is defined to range as $-\pi < \arg \leq \pi$ for its principal value. Under such circumstances the zero would also have to be simple (first order). Similarly for a pole with small negative real part isolated away from other poles and zeros the pole would have to be first order and have a residue with small imaginary part or small argument.

The requirement of non-negative real part in equations 7.3 and 7.4 can be put in another format as

$$
\begin{aligned}
&-\frac{\pi}{2} \leq \arg(\tilde{Z}_a(s)) \leq \frac{\pi}{2} \quad , \quad \mathrm{Re}[s] > 0 \\
&-\frac{\pi}{2} \leq \arg(\tilde{Y}_a(s)) \leq \frac{\pi}{2} \quad , \quad \mathrm{Re}[s] > 0 \\
&\arg(\tilde{Z}_a(s)) + \arg(\tilde{Y}_a(s)) = 0
\end{aligned}
\tag{7.10}
$$

where the $i\omega$ axis is included except at zeros and singularities at which points arg is undefined. Defining the principal branch of the logarithm such that arg is zero on the positive real axis of the s plane (where $\tilde{Z}_a$ and $\tilde{Y}_a$ are positive) we have

$$
\begin{aligned}
&\left.\begin{aligned}
\ln(\tilde{Z}_a(s)) &= \ln(|\tilde{Z}_a(s)|) + i \arg(\tilde{Z}_a(s)) \\
\ln(\tilde{Y}_a(s)) &= \ln(|\tilde{Y}_a(s)|) + i \arg(\tilde{Y}_a(s))
\end{aligned}\right\} \text{analytic for } \mathrm{Re}[s] > 0 \\
&\ln(\tilde{Z}_a(s)) + \ln(\tilde{Y}_a(s)) = 0
\end{aligned}
\tag{7.11}
$$

Hence the logarithms of PR functions are analytic throughout the right half plane.

Consider the logarithms of PR functions as functions of ln(s) instead of s. The right half of the s plane is described by

$$\ln(s) = \ln(|s|) + i\arg(s)$$
$$-\frac{\pi}{2} \le \arg(s) \le \frac{\pi}{2} \qquad (7.12)$$

Now s is here an analytic function of ln(s) with no zeros provided we exclude s = 0. Hence in the right half s plane $\ln(\tilde{Z}_a)$ and $\ln(\tilde{Y}_a)$ are analytic functions of ln(s). Consider the PR function $\tilde{g}(s)$ and form the new function

$$\tilde{g}'(s) \equiv \ln(\tilde{g}(s)) - \ln(s) \qquad (7.13)$$

which is also an analytic function of ln(s) in the right half s plane. Restrict our attention to the first quadrant of the s plane ($0 \le \arg(s) \le \pi/2$). On the positive real axis we have

$$\mathrm{Im}[\tilde{g}'(s)]\Big|_{s>0} = [\arg(\tilde{g}(s)) - \arg(s)]\Big|_{s>0} = 0 \qquad (7.14)$$

On the positive imaginary axis we have

$$\mathrm{Im}[\tilde{g}'(s)]\Big|_{\substack{s=i\omega \\ \omega>0}} = [\arg(\tilde{g}(s)) - \arg(s)]\Big|_{\substack{s=i\omega \\ \omega>0}} \qquad (7.15)$$

but

$$\arg(s)\Big|_{\substack{s=i\omega \\ \omega>0}} = \frac{\pi}{2} \quad , \quad -\frac{\pi}{2} \le \arg(\tilde{g}(s))\Big|_{\substack{s=i\omega \\ \omega>0}} \le \frac{\pi}{2} \qquad (7.16)$$

giving

$$-\pi \le \mathrm{Im}(\tilde{g}'(s))\Big|_{\substack{s=i\omega \\ \omega>0}} \le 0 \qquad (7.17)$$

Since $\mathrm{Im}(\tilde{g}'(s))$ is zero on the real axis and non-positive on the positive imaginary axis, it is non-positive in the first quadrant. Hence in the first quadrant

$$\arg(\tilde{g}'(s)) - \arg(s) \leq 0 \tag{7.18}$$

or

$$\arg(\tilde{g}(s)) \leq \arg(s) \tag{7.19}$$

Similarly defining

$$\tilde{g}''(s) \equiv \ln(\tilde{g}(s)) + \ln(s) \tag{7.20}$$

we find that $\mathrm{Im}[\tilde{g}''(s)]$ is non-negative in the first quadrant or

$$\arg(\tilde{g}(s)) + \arg(s) \geq 0 \tag{7.21}$$

or

$$-\arg(\tilde{g}(s)) \leq \arg(s) \tag{7.22}$$

Combining equations 7.19 and 7.22 gives for the first quadrant

$$|\arg(\tilde{g}(s))| \leq \arg(s) \tag{7.23}$$

Repeating this procedure for the lower half of the right half plane (the fourth quadrant) we find

$$|\arg(\tilde{g}(s))| \leq -\arg(s) \tag{7.24}$$

This is also directly obtainable from equation 7.23 using the conjugate symmetry of $\tilde{g}(s)$. Combining the results gives

$$|\arg(\tilde{g}(s))| \leq |\arg(s)| \quad \text{for } |\arg(s)| \leq \frac{\pi}{2} \tag{7.25}$$

except for zeros or poles on the $i\omega$ axis and for $s = 0$ where arg is undefined.

Applying this result to our object of interest we have

$$|\arg(\tilde{Z}_a(s))| \le |\arg(s)| \quad \text{for } |\arg(s)| \le \frac{\pi}{2}$$
$$|\arg(\tilde{Y}_a(s))| \le |\arg(s)| \quad \text{for } |\arg(s)| \le \frac{\pi}{2} \qquad (7.26)$$

This is a somewhat tighter result than that in equations 7.10, although they can be derived from each other.[10]

Following Guillemin[11] we can say that a function of s which is real for s real is PR if:

a) It is analytic in the right half plane.
b) Its real part is non-negative on the imaginary axis.
c) Any imaginary axis poles are simple and have positive real residues.

Here we have applied this concept based on physical principles (passivity) to distributed systems such as impedance/admittance at a port of an antenna/scatterer. It can also be applied to eigenimpedances and eigenadmittances[8] but a detailed discussion of this is reserved for a future note.

In approximating a PR function corresponding to the impedance or admittance of a distributed object one has the problem of an infinite number of poles or zeros. Make a pole expansion of our general PR function as

$$\tilde{g}(s) = \sum_{\alpha} g_\alpha (s - s_\alpha)^{-n_\alpha} + \tilde{g}_e(s) \qquad (7.27)$$

where $\tilde{g}_e(s)$ is an entire function and n_α is a positive integer indicating pole order. Since we are considering finite size objects in free space there are no branch singularities which would have to be included also for more general objects.

Let us try to approximate $\tilde{g}(s)$ by a finite sum of poles plus an entire function which we write as a truncated power series about the origin. Assuming first order poles we have

$$\tilde{g}(s) \simeq \sum_{\alpha \varepsilon A_\alpha} g_\alpha (s - s_\alpha)^{-1} + \sum_{n=0}^{n'} g_n' s^n \qquad g_n' \text{ real} \tag{7.28}$$

where A_α represents some set of natural frequencies s_α taken in conjugate pairs for those s_α not on the real axis. This set A_α may for example include all the poles within some specified radius of the origin of the s plane.

As discussed previously a PR function can behave no more singularly than s^1 as $s \to \infty$ in the right half plane. One might use this property to restrict $n' = 1$, i.e. restrict the entire function approximation to two terms. If one further restricts the behavior to a positive constant for $s \to \infty$ in the right half plane as discussed in section VI.A, then one would have

$$\tilde{g}(s) \simeq \sum_{\alpha \varepsilon A_\alpha} g_\alpha (s - s_\alpha)^{-1} + g_o' \quad , \quad g_o' > 0 \tag{7.29}$$

In order to determine the value of g_o' one might set

$$\begin{aligned} g_o' &\equiv \lim_{\substack{s \to \infty \\ |\arg(s)| \le \frac{\pi}{2} - \delta}} \tilde{g}(s) \\ &\equiv \tilde{g}(\infty) > 0 \end{aligned} \tag{7.30}$$

giving the proper high frequency behavior. An alternate procedure has a match made near s = 0 by rewriting equation 7.29 as

$$\begin{aligned} \tilde{g}(s) &= \sum_{\alpha \varepsilon A_\alpha} g_\alpha [(s - s_\alpha)^{-1} + s_\alpha^{-1}] + \tilde{g}(0) \\ &= \sum_{\alpha \varepsilon A_\alpha} g_\alpha \frac{s}{s_\alpha} (s - s_\alpha)^{-1} + \tilde{g}(0) \end{aligned} \tag{7.31}$$

provided $\tilde{g}(s)$ has no pole at $s = 0$. This latter form allows one to match $\tilde{g}(0)$ from the low frequency properties of the scatterer.

These procedures allow one to approximate $\tilde{Z}_a(s)$ and $\tilde{Y}_a(s)$ in a sense which has a finite number of "exact" poles and residues, and is "exact" in either the high or low frequency limit. In addition as more poles are added to the approximation the previous s_α and g_α as well as g_o' or $\tilde{g}(0)$ will not have to be recomputed, assuming their exact determination from $\tilde{g}(s)$.

VIII. Admittance

In finding an equivalent circuit based on the short circuit properties of the antenna or scatterer let us first consider the admittance at the port. For this purpose the gap electric field is specified as discussed in section II. This gap electric field is introduced into the short circuit boundary value problem discussed in section IV. The integral equation is solved in singularity expansion method (SEM) and eigenmode expansion method (EEM) forms. This solution is substituted into the admittance formulas of sections I and II.

A. SEM form

With the source field as $\vec{E}_g$ we have the surface current density (assuming first order poles) as[3,4,5,6,7]

$$\tilde{\vec{J}}_s(\vec{r},s) = \frac{\tilde{V}(s)}{Z_o}\left\{\sum_{\alpha_{sc}} \tilde{\eta}'_{\alpha_{sc}} \vec{\nu}^{(\vec{J}_s)}_{\alpha_{sc}}(\vec{r})\,(s - s_{\alpha_{sc}})^{-1} + \text{entire function}\right\} \tag{8.1}$$

In this form $\tilde{V}(s)$ and Z_o are first factored out of the surface current density before making the singularity expansion. The remainder is then of a normalized delta function response for the current density (Laplace transformed).

The natural frequencies, modes, and coupling vectors satisfy

$$\left\langle \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s_{\alpha_{sc}}) \;\dot{,}\; \vec{\nu}^{(\vec{J}_s)}_{\alpha_{sc}}(\vec{r}\,')\right\rangle_{a+g} = \vec{0}, \quad \vec{r}\in S_a \cup S_g$$

$$\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \;\dot{,}\; \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s_{\alpha_{sc}})\right\rangle_{a+g} = \vec{0}, \quad \vec{r}\,'\in S_a \cup S_g \tag{8.2}$$

where for symmetric operators we can set

$$\vec{\mu}_{\alpha_{sc}}(\vec{r}) \equiv \vec{\nu}^{(\vec{J}_s)}_{\alpha_{sc}}(\vec{r}) \tag{8.3}$$

However, let us use the $\vec{\mu}_{\alpha_{sc}}$ in the formulas to allow for cases that equation 8.3 does not apply. The coupling coefficients are given by

$$\tilde{\eta}'_{\alpha_{sc}} = -Z_o \frac{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \ ;\cdot \ \vec{e}_g(\vec{r}) \right\rangle_g}{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \ ;\cdot \ \frac{\partial}{\partial s} \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\Big|_{s=s_{\alpha_{sc}}} \ ;\cdot \ \vec{\nu}^{(\vec{J}_s)}_{\alpha_{sc}}(\vec{r}\,') \right\rangle_{a+g}} \tag{8.4}$$

where the integral in the numerator is only over S_g since the gap electric field is non-zero only there.

From section II we have the admittance as

$$\tilde{Y}_a(s) = -\frac{\tilde{I}(s)}{\tilde{V}(s)} = -\frac{\left\langle \tilde{\vec{J}}_s(\vec{r},s) \ ;\cdot \ \vec{e}_g(\vec{r}) \right\rangle_g}{\tilde{V}(s)} \tag{8.5}$$

Thus we can write the admittance as

$$\tilde{Y}_a(s) = \frac{1}{Z_o} \sum_{\alpha_{sc}} a_{\alpha_{sc}} (s - s_{\alpha_{sc}})^{-1} + \tilde{Y}_{sce}(s) \tag{8.6}$$

$$a_{\alpha_{sc}} = Z_o \frac{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \ ;\cdot \ \vec{e}_g(\vec{r}) \right\rangle_g \left\langle \vec{\nu}^{(\vec{J}_s)}_{\alpha_{sc}}(\vec{r}) \ ;\cdot \ \vec{e}_g(\vec{r}) \right\rangle_g}{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \ ;\cdot \ \frac{\partial}{\partial s} \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\Big|_{s=s_{\alpha_{sc}}} \ ;\cdot \ \vec{\nu}^{(\vec{J}_s)}_{\alpha_{sc}}(\vec{r}\,') \right\rangle_{a+g}}$$

where $\tilde{Y}_{sce}(s)$ is an entire function admittance. The normalized admittance residues $a_{\alpha_{sc}}$ have dimensions s^{-1}. Note the symmetrical expression for the $a_{\alpha_{sc}}$; for symmetrical operators we have

$$\vec{\mu}_{\alpha_{sc}}(\vec{r}) \equiv \vec{\nu}_{\alpha_{sc}}^{(\vec{J}_s)}(\vec{r})$$

$$a_{\alpha_{sc}} = Z_o \frac{\left\langle \vec{\nu}_{\alpha_{sc}}^{(\vec{J}_s)}(\vec{r}) \; ; \; \vec{e}_g(\vec{r}) \right\rangle_g^2}{\left\langle \vec{\nu}_{\alpha_{sc}}^{(\vec{J}_s)}(\vec{r}) \; ; \; \frac{\partial}{\partial s} \tilde{\vec{\Gamma}}(\vec{r},\vec{r}\,';s)\Big|_{s=s_{\alpha_{sc}}} \; ; \; \vec{\nu}_{\alpha_{sc}}^{(\vec{J}_s)}(\vec{r}\,') \right\rangle_{a+g}} \tag{8.7}$$

Note that the formulas in equations 8.6 and 8.7 are of such a form that they can be applied to line and volume integral equation formulations directly, as well as to surface integral equation formulations.

Viewed another way the admittance can be written as

$$\tilde{Y}_a(s) = \sum_{\alpha_{sc}} \tilde{Y}_{\alpha_{sc}}(s) + \tilde{Y}_{sce}(s)$$

$$\tilde{Y}_{\alpha_{sc}}(s) \equiv \frac{1}{Z_o} a_{\alpha_{sc}} (s - s_{\alpha_{sc}})^{-1} \tag{8.8}$$

The $\tilde{Y}_{\alpha_{sc}}$ can be referred to as pole admittances; these are elementary terms for use in constructing equivalent circuits.

For certain types of objects, in particular those with zero admittance at $s = 0$, it is convenient to write the admittance in another form as

$$\tilde{Y}_a(s) = \frac{1}{Z_o} \sum_{\alpha_{sc}} a_{\alpha_{sc}} \left[(s - s_{\alpha_{sc}})^{-1} + s_{\alpha_{sc}}^{-1} \right] + \tilde{Y}'_{sce}(s)$$

$$= \sum_{\alpha_{sc}} \tilde{Y}'_{\alpha_{sc}}(s) + \tilde{Y}'_{sce}(s)$$

$$\tilde{Y}'_{\alpha_{sc}}(s) \equiv \frac{1}{Z_o} a_{\alpha_{sc}} \left[(s - s_{\alpha_{sc}})^{-1} + s_{\alpha_{sc}}^{-1} \right] = \frac{1}{Z_o} \frac{a_{\alpha_{sc}}}{s_{\alpha_{sc}}} \frac{s}{s - s_{\alpha_{sc}}}$$

$$= \tilde{Y}_{\alpha_{sc}}(s) + \frac{1}{Z_o} a_{\alpha_{sc}} s_{\alpha_{sc}}^{-1} \qquad (8.9)$$

$$\tilde{Y}'_{sce}(s) = \tilde{Y}_{sce}(s) - \frac{1}{Z_o} \sum_{\alpha_{sc}} a_{\alpha_{sc}} s_{\alpha_{sc}}^{-1}$$

The $\tilde{Y}'_{\alpha_{sc}}(s)$ can be referred to as modified pole admittances or zero-subtracted pole admittances; these can also be used in constructing equivalent circuits. In using this modified form the explicit entire function admittance is changed to $\tilde{Y}'_{sce}(s)$ which can be referred to as a modified entire function. The two forms used here differ only by a constant in the case of the admittance (assuming a convergent sum).

Both $\tilde{Y}_{\alpha_{sc}}(s)$ and $\tilde{Y}'_{\alpha_{sc}}(s)$ will be used later in considering finite sums of poles together with finite polynomial entire functions in approximating the admittance and short circuit current in circuit form.

Note that the modified form of the admittance poles can be used only if there is no $s_{\alpha_{sc}} = 0$ (no short circuit natural frequency at the origin in the s plane) with non-zero residue. Hence such a form is inappropriate for the admittance of a perfectly conducting loop. It is, however, quite appropriate for the admittance of an electric dipole antenna consisting of two separate parts joined only through the gap region and surrounded by a non-conducting medium.

B. EEM form

Consider now the eigenmode expansion form of the admittance.[8] The surface current density associated with the source electric field $\vec{E}_g$ in the gap is

$$\tilde{\vec{J}}_s(\vec{r},s) = -\tilde{V}(s)\left\{\sum_{\beta_{sc}} \tilde{\lambda}_{\beta_{sc}}(s)^{-1}\,\tilde{\vec{R}}_{\beta_{sc}}(\vec{r},s)\,\frac{\left\langle \tilde{\vec{L}}_{\beta_{sc}}(\vec{r},s)\,;\,\vec{e}_g(\vec{r})\right\rangle_g}{\left\langle \tilde{\vec{R}}_{\beta_{sc}}(\vec{r},s)\,;\,\tilde{\vec{L}}_{\beta_{sc}}(\vec{r},s)\right\rangle_{a+g}}\right\} \tag{8.10}$$

where the operator with kernel $\tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)$ is assumed diagonalizable. Here the eigenvalues, right eigenmodes, and left eigenmodes satisfy

$$\left\langle \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\,;\,\tilde{\vec{R}}_{\beta_{sc}}(\vec{r}\,',s)\right\rangle_{a+g} = \tilde{\lambda}_{\beta_{sc}}(s)\,\tilde{\vec{R}}_{\beta_{sc}}(\vec{r},s),\quad \vec{r}\in S_a \cup S_g$$

$$\left\langle \tilde{\vec{L}}_{\beta_{sc}}(\vec{r},s)\,;\,\tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\right\rangle_{a+g} = \tilde{\lambda}_{\beta_{sc}}(s)\,\tilde{\vec{L}}_{\beta_{sc}}(\vec{r}\,',s),\quad \vec{r}\,'\in S_a \cup S_g \tag{8.11}$$

and where for symmetric operators we can set

$$\tilde{\vec{L}}_{\beta_{sc}}(\vec{r},s) \equiv \tilde{\vec{R}}_{\beta_{sc}}(\vec{r},s) \tag{8.12}$$

The admittance is then

$$\tilde{Y}_a(s) = -\frac{\tilde{I}(s)}{\tilde{V}(s)} = -\frac{\left\langle \tilde{\vec{J}}_s(\vec{r},s)\,;\,\vec{e}_g(\vec{r})\right\rangle_g}{\tilde{V}(s)}$$

$$= \sum_{\beta_{sc}} \tilde{Y}_{\beta_{sc}}(s) \tag{8.13}$$

$$\tilde{Y}_{\beta_{sc}}(s) = \tilde{\lambda}_{\beta_{sc}}(s)^{-1}\,\frac{\left\langle \tilde{\vec{L}}_{\beta_{sc}}(\vec{r},s)\,;\,\vec{e}_g(\vec{r})\right\rangle_g \left\langle \tilde{\vec{R}}_{\beta_{sc}}(\vec{r},s)\,;\,\vec{e}_g(\vec{r},s)\right\rangle_g}{\left\langle \tilde{\vec{R}}_{\beta_{sc}}(\vec{r},s)\,;\,\tilde{\vec{L}}_{\beta_{sc}}(\vec{r},s)\right\rangle_{a+g}}$$

which for symmetric operators reduces to

$$\tilde{Y}_{\beta_{sc}}(s) = \tilde{\lambda}_{\beta_{sc}}(s)^{-1} \frac{\left\langle \tilde{\vec{\vec{R}}}_{\beta_{sc}}(\vec{r},s) \; ; \; \vec{e}_g(\vec{r}) \right\rangle_g^2}{\left\langle \tilde{\vec{\vec{R}}}_{\beta_{sc}}(\vec{r},s) \; ; \; \tilde{\vec{\vec{R}}}_{\beta_{sc}}(\vec{r},s) \right\rangle_{a+g}} \tag{8.14}$$

The operator can be conveniently taken as an impedance operator as discussed in section IV.A. In this form the eigenvalues $\tilde{\lambda}_{\beta_{sc}}(s)$ are eigenimpedances $\tilde{Z}_{\beta_{sc}}(s)$ with PR properties for cases of passive loading impedance.

Comparing the EEM form to the SEM form of the admittance there are certain similarities, particularly in the symmetric forms of the terms in both expansions. Setting

$$\alpha_{sc} = (\beta_{sc}, \beta'_{sc}) \tag{8.15}$$

as discussed in a previous note[8] we have

$$\tilde{Y}_{\beta_{sc}}(s) = \sum_{\beta'_{sc}} \tilde{Y}_{\beta_{sc},\beta'_{sc}}(s) + \tilde{Y}_{\beta_{sc},sce}(s) \tag{8.16}$$

where $\tilde{Y}_{\beta_{sc},sce}(s)$ accounts for any additional entire function required in expanding the eigenvalue associated admittances $\tilde{Y}_{\beta_{sc}}(s)$ in terms of the appropriate pole admittances $\tilde{Y}_{\beta_{sc},\beta'_{sc}}(s)$. Note that we have

$$\tilde{Y}_{\beta_{sc},\beta'_{sc}}(s) \equiv \frac{1}{Z_o} a_{\beta_{sc},\beta'_{sc}} (s - s_{\beta_{sc},\beta'_{sc}})^{-1}$$

$$a_{\beta_{sc},\beta'_{sc}} = Z_o \frac{\left\langle \vec{\mu}_{\beta_{sc},\beta'_{sc}}(\vec{r}) \; ; \; \vec{e}_g(\vec{r}) \right\rangle_g \left\langle \vec{\nu}^{(\vec{J}_s)}_{\beta_{sc},\beta'_{sc}}(\vec{r}) \; ; \; \vec{e}_g(\vec{r}) \right\rangle_g}{\left\langle \vec{\mu}_{\beta_{sc},\beta'_{sc}}(\vec{r}) \; ; \; \frac{\partial}{\partial s} \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\Big|_{s=s_{\beta_{sc},\beta'_{sc}}} \; ; \; \vec{\nu}^{(\vec{J}_s)}_{\beta_{sc},\beta'_{sc}}(\vec{r}\,') \right\rangle_{a+g}} \tag{8.17}$$

$$= Z_o \left[\frac{\partial}{\partial s} \lambda_{\beta_{sc}}(s) \Big|_{s=s_{\beta_{sc},\beta'_{sc}}} \right]^{-1} \frac{\left\langle \vec{\mu}_{\beta_{sc},\beta'_{sc}}(\vec{r}) ; \vec{e}_g(\vec{r}) \right\rangle_g \left\langle \vec{\nu}^{(\vec{J}_s)}_{\beta_{sc},\beta'_{sc}} ; \vec{e}_g(\vec{r}) \right\rangle_g}{\left\langle \vec{\mu}_{\beta_{sc},\beta'_{sc}}(\vec{r}) ; \vec{\nu}^{(\vec{J}_s)}_{\beta_{sc},\beta'_{sc}}(\vec{r}) \right\rangle_{a+g}}$$

where a relation for the kernel derivative in terms of the eigenvalue derivative (with respect to s) discussed in a previous note is used.[8]

The EEM form of the admittance involves terms which are more complicated functions of s than in the SEM form. However, each EEM term can be divided down into pole admittances (SEM) plus a possible entire function. The eigenmode form gives an ordering or grouping of the SEM terms which can be used in the ordering of the circuit elements and sources in equivalent circuit representations.

The eigenvalue associated admittances can be also expanded in terms of modified pole admittances $\tilde{Y}'_{\beta_{sc},\beta'_{sc}}(s)$ using

$$\tilde{Y}_{\beta_{sc}}(s) = \sum_{\beta'_{sc}} \tilde{Y}'_{\beta_{sc},\beta'_{sc}}(s) + \tilde{Y}'_{\beta_{sc},sce}(s)$$

$$\tilde{Y}'_{\beta_{sc},\beta'_{sc}}(s) = \frac{1}{Z_o} a_{\beta_{sc},\beta'_{sc}} \left[\left(s - s_{\beta_{sc},\beta'_{sc}} \right)^{-1} + s^{-1}_{\beta_{sc},\beta'_{sc}} \right]$$

$$= \frac{1}{Z_o} \frac{a_{\beta_{sc},\beta'_{sc}}}{s_{\beta_{sc},\beta'_{sc}}} \frac{s}{s - s_{\beta_{sc},\beta'_{sc}}} \tag{8.18}$$

$$\tilde{Y}'_{\beta_{sc},sce}(s) = \tilde{Y}_{\beta_{sc},sce}(s) - \frac{1}{Z_o} \sum_{\beta'_{sc}} a_{\beta_{sc},\beta'_{sc}} \, s^{-1}_{\beta_{sc},\beta'_{sc}}$$

This form is appropriate for cases that $\tilde{Y}_{\beta_{sc}}(s)$ has no pole at $s = 0$.

IX. Short Circuit Current

Consider now the short circuit current induced by some incident or other source field. The source field can be an incident plane wave or other spatial form. The various spatial dependences appear as part of the integrand in determining the coupling coefficients in the SEM solution or mode coefficients in the EEM solution. The short circuit current is compared to the admittance in the sense of a ratio of coupling coefficients or mode coefficients. This shows how the admittance and short circuit current fit together in a single representation which can be used for equivalent circuits.

Let the incident (source) electric field have the general form as specified in section IV with

$$\tilde{\vec{E}}_s(\vec{r},s) = \tilde{\vec{E}}_{inc}(\vec{r},s) = E_o \sum_p \tilde{f}_p(s)\, \tilde{\vec{\delta}}_p(\vec{r},s) \tag{9.1}$$

Here the index p is for different incident waves which may be planar or of other type as described by $\tilde{\vec{\delta}}_p(\vec{r},s)$. The incident waveform is $\tilde{f}_p(s)$ and E_o is a convenient normalization constant.

A. SEM form

The surface current density (assuming first order poles) is

$$\tilde{\vec{J}}_s(\vec{r},s) = \frac{E_o}{Z_o} \sum_p \tilde{f}_p(s) \left\{ \sum_{\alpha_{sc}} \tilde{\eta}_{\alpha_{sc},p}\, \vec{\nu}_{\alpha_{sc}}^{(\vec{J}_s)}(\vec{r})\, (s - s_{\alpha_{sc}})^{-1} + \text{entire function} \right\} \tag{9.2}$$

In this form E_o, Z_o, and $\tilde{f}_p(s)$ are factored out of the surface current density before making the singularity expansion. The portion in braces is the delta function response to the pth incident wave.

The natural frequencies, modes, and coupling vectors are the same as in the admittance calculations (equations 8.2 and 8.3).

The coupling coefficients for the case of the incident field with shorted antenna gap are

$$\tilde{\eta}_{\alpha_{sc},p} = Z_o \frac{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \tilde{\vec{\delta}}_p(\vec{r}, s_{\alpha_{sc}}) \right\rangle_{a+g}}{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \frac{\partial}{\partial s} \tilde{\vec{\Gamma}}(\vec{r}, \vec{r}\,'; s) \Big|_{s=s_{\alpha_{sc}}} \; ; \; \vec{\nu}^{(\vec{J}_s)}_{\alpha_{sc}}(\vec{r}\,') \right\rangle_{a+g}} \tag{9.3}$$

The short circuit current is found as an appropriate average over the surface current density. As discussed in section II there are various ways to define this average. The position in the gap region where the short circuit current is calculated is avoided to some extent by averaging over the current density as in equation 2.18 in the form

$$\tilde{I}_{sc}(s) = \left\langle \tilde{\vec{J}}_s(\vec{r}, s) \; ; \; \vec{e}_g(\vec{r}) \right\rangle_g \tag{9.4}$$

where the weighting function $\vec{e}_g$ is chosen the same as for the gap electric field for the admittance calculations.

The short circuit current is then

$$\begin{aligned} \tilde{I}_{sc}(s) &= \sum_p \tilde{f}_p(s) \left\{ \frac{E_o}{Z_o} \sum_{\alpha_{sc}} b_{\alpha_{sc},p}(s - s_{\alpha_{sc}})^{-1} + \tilde{I}_{sce,p}(s) \right\} \\ &= \sum_p \tilde{f}_p(s) \left\{ \frac{1}{Z_o} \sum_{\alpha_{sc}} V_{\alpha_{sc},p} \, a_{\alpha_{sc}} (s - s_{\alpha_{sc}})^{-1} + \tilde{V}_{sce,p}(s) \, \tilde{Y}_{sce}(s) \right\} \end{aligned} \tag{9.5}$$

where $\tilde{I}_{sce,p}(s)$ denotes a set of entire functions for the short circuit current. The normalized short circuit current residues are

$$b_{\alpha_{sc},p} = \tilde{\eta}_{\alpha_{sc},p} \left\langle \vec{\nu}_{\alpha_{sc}}^{(\vec{J}_s)}(\vec{r}) \; ; \; \vec{e}_g(\vec{r}) \right\rangle_g$$

$$= Z_o \frac{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \tilde{\vec{\delta}}_p(\vec{r}, s_{\alpha_{sc}}) \right\rangle_{a+g} \left\langle \vec{\nu}_{\alpha_{sc}}^{(\vec{J}_s)}(\vec{r}) \; ; \; \vec{e}_g(\vec{r}) \right\rangle_g}{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \frac{\partial}{\partial s} \tilde{\vec{\Gamma}}(\vec{r}, \vec{r}\,'; s) \Big|_{s=s_{\alpha_{sc}}} \; ; \; \vec{\nu}_{\alpha_{sc}}^{(\vec{J}_s)}(\vec{r}\,') \right\rangle_{a+g}} \tag{9.6}$$

The normalized short circuit current residues are written in the form

$$E_o b_{\alpha_{sc},p} \equiv V_{\alpha_{sc},p} \; a_{\alpha_{sc}} \tag{9.7}$$

where $a_{\alpha_{sc}}$ is the normalized admittance residue introduced in section VIII. This defines what we will call the voltage source coefficients $V_{\alpha_{sc},p}$ which can be written as

$$V_{\alpha_{sc},p} = E_o \frac{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \tilde{\vec{\delta}}_p(\vec{r}, s_{\alpha_{sc}}) \right\rangle_{a+g}}{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \vec{e}_g(\vec{r}) \right\rangle_g} \tag{9.8}$$

Note that $\tilde{\vec{\delta}}_p$ is dimensionless while $\vec{e}_g$ has dimensions of m^{-1}; E_o has dimensions V/m. Thus $V_{\alpha_{sc},p}$ has dimensions of volts as required. Also note that $V_{\alpha_{sc},p}$ is simpler in form than $a_{\alpha_{sc}}$ and $b_{\alpha_{sc},p}$. The kernel derivative cancels as does the integral used in defining the current; only the integrals over the source electric field remain. An addition elementary voltage source coefficient is that associated with the entire function as

$$\tilde{V}_{sce,p}(s) = \frac{\tilde{I}_{sce,p}(s)}{\tilde{Y}_{sce}(s)} \tag{9.9}$$

assuming the denominator is not identically zero. Here the voltage coefficient is written as a function of s for purposes of generality. In this note it will find some use as a constant when considering finite expansions for circuit approximations.

In terms of the pole admittances the short circuit current is written as

$$\tilde{I}_{sc}(s) = \sum_{\alpha_{sc}} \left\{ \sum_p V_{\alpha_{sc},p} \; \tilde{f}_p(s) \right\} \tilde{Y}_{\alpha_{sc}}(s) + \left\{ \sum_p \tilde{V}_{sce,p}(s) \; \tilde{f}_p(s) \right\} \tilde{Y}_{sce}(s) \tag{9.10}$$

If $\sum_p V_{\alpha_{sc},p}\tilde{f}_p(s)$ is interpreted as a voltage source or a series combination of voltage sources which are placed in series with the pole admittance $\tilde{Y}_{\alpha_{sc}}(s)$, and if such series combinations of admittances and voltage sources (including entire function part) are placed in parallel the short circuit current and admittance of the circuit will both be that required. This observation forms the basis for the equivalent circuits to be developed.

For the case that there is a short circuit natural frequency $s_{\alpha_{sc}}$ (an admittance pole) at s = 0 one expects in general that the corresponding voltage source coefficients $V_{o,p}$ are zero. This is so as to give a finite response for CW excitation in the static limit. For plane wave incidence this result is required by reciprocity together with no radiated power at zero frequency.[4] In a more general sense as long as the incident electric and magnetic fields are bounded as $s \to 0$ and satisfy source free Maxwell's equations in the vicinity of the object, there is no short circuit current pole at s = 0.

As was done in the case of the admittance the short circuit current can also be written in the form of modified poles as

$$\tilde{I}_{sc}(s) = \sum_p \tilde{f}_p(s) \left\{ \frac{E_o}{Z_o} \sum_{\alpha_{sc}} b_{\alpha_{sc},p} \left[(s - s_{\alpha_{sc}})^{-1} + s_{\alpha_{sc}}^{-1} \right] + \tilde{I}'_{sce,p}(s) \right\}$$

$$= \sum_p \tilde{f}_p(s) \left\{ \frac{1}{Z_o} \sum_{\alpha_{sc}} V_{\alpha_{sc},p} \, a_{\alpha_{sc}} \left[(s - s_{\alpha_{sc}})^{-1} + s_{\alpha_{sc}}^{-1} \right] \right.$$

$$\left. + \tilde{V}'_{sce,p}(s) \, \tilde{Y}'_{sce}(s) \right\} \tag{9.11}$$

In this form the voltage source coefficient associated with the explicit entire function is

$$\tilde{V}_{sce,p}(s) = \frac{\tilde{I}'_{sce,p}(s)}{\tilde{Y}'_{sce}(s)} \tag{9.12}$$

which can be considered as a function of s for generality.

The entire functions in this modified form can be written in terms of the unmodified entire functions to give

$$\tilde{I}'_{sce,p}(s) = \tilde{I}_{sce,p}(s) - \frac{E_o}{Z_o} \sum_{\alpha_{sc}} b_{\alpha_{sc},p} \, s_{\alpha_{sc}}^{-1}$$

$$= \tilde{V}_{sce,p}(s) \, \tilde{Y}_{sce}(s) - \frac{1}{Z_o} \sum_{\alpha_{sc}} V_{\alpha_{sc},p} \, a_{\alpha_{sc}} \, s_{\alpha_{sc}}^{-1}$$

$$= \tilde{V}'_{sce,p}(s) \, \tilde{Y}'_{sce}(s)$$

$$= \tilde{V}'_{sce,p}(s) \left\{ \tilde{Y}_{sce}(s) - \frac{1}{Z_o} \sum_{\alpha_{sc}} a_{\alpha_{sc}} \, s_{\alpha_{sc}}^{-1} \right\} \tag{9.13}$$

Hence the voltage source coefficient for the modified entire function is related to the unmodified one as

$$\tilde{V}'_{sce,p}(s) = \frac{\tilde{V}_{sce,p}(s)\ \tilde{Y}_{sce}(s) - \frac{1}{Z_o}\sum_{\alpha_{sc}} V_{\alpha_{sc},p}\ a_{\alpha_{sc}}\ s_{\alpha_{sc}}^{-1}}{\tilde{Y}_{sce}(s) - \frac{1}{Z_o}\sum_{\alpha_{sc}} a_{\alpha_{sc}}\ s_{\alpha_{sc}}^{-1}} \tag{9.14}$$

assuming the denominator is not identically zero.

In terms of the modified pole admittances the short circuit current is

$$\tilde{I}_{sc}(s) = \sum_{\alpha_{sc}} \left\{ \sum_p V_{\alpha_{sc},p}\ \tilde{f}_p(s) \right\} \tilde{Y}'_{\alpha_{sc}}(s) + \left\{ \sum_p \tilde{V}'_{sce,p}(s)\ \tilde{f}_p(s) \right\} \tilde{Y}'_{sce}(s) \tag{9.15}$$

Just as in equation 9.10 where unmodified pole admittances are used the same voltage sources $\sum_p V_{\alpha_{sc},p}\tilde{f}_p(s)$ appear, except in series with (multiply) the modified pole admittances $\tilde{Y}'_{\alpha_{sc}}(s)$. Parallel combination of such voltage sources in series with modified pole admittances can also be used for constructing equivalent circuits including modified entire function terms.

B. EEM form

The eigenmode expansion of the surface current density is

$$\tilde{\vec{J}}_s(\vec{r},s) = E_o \sum_p \tilde{f}_p(s) \left\{ \sum_{\beta_{sc}} \tilde{\lambda}_{\beta_{sc}}^{-1}(s)\ \tilde{\vec{R}}_{\beta_{sc}}(\vec{r},s) \frac{\left\langle \tilde{\vec{L}}_{\beta_{sc}}(\vec{r},s)\ ;\ \tilde{\vec{\delta}}_p(\vec{r},s) \right\rangle_{a+g}}{\left\langle \tilde{\vec{R}}_{\beta_{sc}}(\vec{r},s)\ ;\ \tilde{\vec{L}}_{\beta_{sc}}(\vec{r},s) \right\rangle_{a+g}} \right\} \tag{9.16}$$

where E_0 and $\tilde{f}_p(s)$ are factored out for convenience.

The short circuit current is

$$\tilde{I}_{sc}(s) = \left\langle \tilde{\vec{J}}_s(\vec{r},s) \,;\, \vec{e}_g(\vec{r}) \right\rangle_g$$

$$= E_0 \sum_p \tilde{f}_p(s) \left\{ \sum_{\beta_{sc}} \tilde{\lambda}^{-1}_{\beta_{sc}}(s) \frac{\left\langle \tilde{\vec{L}}_{\beta_{sc}}(\vec{r},s) \,;\, \tilde{\vec{\delta}}_p(\vec{r},s) \right\rangle_{a+g} \left\langle \tilde{\vec{R}}_\beta(\vec{r},s) \,;\, \vec{e}_g(\vec{r}) \right\rangle_g}{\left\langle \tilde{\vec{R}}_\beta(\vec{r},s) \,;\, \tilde{\vec{L}}_\beta(\vec{r},s) \right\rangle_{a+g}} \right\}$$

$$= \sum_{\beta_{sc}} \left\{ \sum_p \tilde{V}_{\beta_{sc},p}(s)\, \tilde{f}_p(s) \right\} \tilde{Y}_{\beta_{sc}}(s)$$

$$\tilde{V}_{\beta_{sc},p}(s) = E_0 \frac{\left\langle \tilde{\vec{L}}_{\beta_{sc}}(\vec{r},s) \,;\, \tilde{\vec{\delta}}_p(\vec{r},s) \right\rangle_{a+g}}{\left\langle \tilde{\vec{L}}_{\beta_{sc}}(\vec{r},s) \,;\, \vec{e}_g(\vec{r}) \right\rangle_g} \qquad (9.17)$$

where $\tilde{Y}_{\beta_{sc}}(s)$ is the admittance associated with each eigenmode as discussed in section VII. The voltage source coefficients are $\tilde{V}_{\beta_{sc},p}(s)$; these together with the incident waveform(s) $\tilde{f}_p(s)$ convert the admittances associated with each eigenmode to short circuit currents associated with each eigenmode.

In a circuit sense $\sum_p \tilde{V}_{\beta_{sc},p}(s)\, \tilde{f}_p(s)$ can be interpreted as a voltage source or a series combination of voltage sources placed in series with the eigenmode-associated admittance $\tilde{Y}_{\beta_{sc}}(s)$. Such series combinations of voltage sources and admittances can then be placed in parallel to give both the admittance and short circuit current required.

As discussed in section VII the admittance associated with each eigenvalue $\tilde{Y}_{\beta_{sc}}(s)$ can be written as an SEM expansion in terms of the pole admittances $\tilde{Y}_{\beta_{sc},\beta'_{sc}}(s)$ or of the modified pole admittances $\tilde{Y}'_{\beta_{sc},\beta'_{sc}}(s)$.

Note that the voltage source coefficients from the eigenmode expansion (equations 9.17) equal those from the singularity expansion (equation 9.8) at the appropriate natural frequencies as

$$\tilde{V}_{\beta_{sc},p}(s_{\beta_{sc},\beta'_{sc}}) = V_{\beta_{sc},\beta'_{sc},p} \tag{9.18}$$

This property gives a direct connection between the eigenmode and singularity expansions of the short circuit current. The residues of the poles for the short circuit current are now their correct values on an eigenmode by eigenmode basis.

The individual eigenmode expansion terms for the short circuit current can be singularity expanded as

$$E_o \sum_p \tilde{f}_p(s)\, \tilde{\lambda}^{-1}_{\beta_{sc}}(s) \frac{\left\langle \tilde{\vec{\vec{L}}}_{\beta_{sc}}(\vec{r},s) \; \dot{;} \; \tilde{\vec{\vec{\delta}}}_p(\vec{r},s)\right\rangle_{a+g} \left\langle \tilde{\vec{\vec{R}}}_{\beta_{sc}}(\vec{r},s) \; \dot{;} \; \vec{e}_g(\vec{r})\right\rangle_g}{\left\langle \tilde{\vec{\vec{R}}}_{\beta_{sc}}(\vec{r},s) \; \dot{;} \; \tilde{\vec{\vec{L}}}_{\beta_{sc}}(\vec{r},s)\right\rangle_{a+g}}$$

$$= \sum_{\beta'_{sc}} \left\{ \sum_p V_{\beta_{sc},\beta'_{sc},p}\, \tilde{f}_p(s) \right\} \tilde{Y}_{\beta_{sc},\beta'_{sc}}(s)$$

$$+ \left\{ \sum_p \tilde{V}_{\beta_{sc},sce,p}(s)\, \tilde{f}_p(s) \right\} \tilde{Y}_{\beta_{sc},sce}(s)$$

$$= \sum_{\beta'_{sc}} \left\{ \sum_p V_{\beta_{sc},\beta'_{sc},p}\, \tilde{f}_p(s) \right\} \tilde{Y}'_{\beta_{sc},\beta'_{sc}}(s)$$

$$+ \left\{ \sum_p \tilde{V}'_{\beta_{sc},sce,p}(s)\, \tilde{f}_p(s) \right\} \tilde{Y}'_{\beta_{sc},sce}(s) \tag{9.19}$$

where the pole admittances $\tilde{Y}_{\beta_{sc},\beta'_{sc}}(s)$ are given by equations 8.17 and the modified pole admittances $\tilde{Y}'_{\beta_{sc},\beta'_{sc}}(s)$ are given by equations 8.18 with equations 8.17.

The associated entire function portions of the eigenmode expansion terms for the voltage sources are given by

$$\tilde{V}_{\beta_{sc},sce,p}(s) = \frac{\tilde{I}_{\beta_{sc},sce,p}(s)}{\tilde{Y}_{\beta_{sc},sce}(s)}$$

$$\tilde{V}'_{\beta_{sc},sce,p}(s) = \frac{\tilde{I}'_{\beta_{sc},sce,p}(s)}{\tilde{Y}'_{\beta_{sc},sce}(s)} \tag{9.20}$$

Here $\tilde{I}_{\beta_{sc},sce,p}(s)$ is the possible entire function associated with a particular term (denoted by β_{sc}) in the eigenmode expansion; it corresponds to $\tilde{I}_{sce,p}(s)$ in equations 9.5 and is obtained by writing α_{sc} as β_{sc},β'_{sc} to form a double summation. Similarly $\tilde{I}'_{\beta_{sc},sce,p}(s)$ is the possible entire function associated with a particular term in the eigenmode expansion where the individual terms have been expanded in terms of modified pole admittances; it corresponds to $\tilde{I}'_{sce,p}(s)$ in equations 9.11 with α_{sc} split as β_{sc},β'_{sc}. The entire function portions of the singularity expansion of the eigenvalue associated admittances $\tilde{Y}_{\beta_{sc},sce}(s)$ and $\tilde{Y}'_{\beta_{sc},sce}(s)$ have been previously discussed (equations 8.16 and 8.18).

The entire function voltage source for the eigenmode expansion terms can be related for modified and unmodified pole admittance terms through

$$\begin{aligned}
\tilde{I}'_{\beta_{sc},sce,p}(s) &= \tilde{I}_{\beta_{sc},sce,p}(s) - \frac{E_o}{Z_o}\sum_{\beta'_{sc}} b_{\beta_{sc},\beta'_{sc},p}\, s^{-1}_{\beta_{sc},\beta'_{sc}} \\
&= \tilde{V}_{\beta_{sc},sce,p}(s)\tilde{Y}_{\beta_{sc},sce}(s) - \frac{1}{Z_o}\sum_{\beta'_{sc}} V_{\beta_{sc},\beta'_{sc},p}\, a_{\beta_{sc},\beta'_{sc}}\, s^{-1}_{\beta_{sc},\beta'_{sc}} \\
&= \tilde{V}'_{\beta_{sc},sce,p}(s)\ \tilde{Y}'_{\beta_{sc},sce}(s) \\
&= \tilde{V}'_{\beta_{sc},sce,p}(s)\left\{\tilde{Y}_{\beta_{sc},sce}(s) - \frac{1}{Z_o}\sum_{\beta'_{sc}} a_{\beta_{sc},\beta'_{sc}}\ s^{-1}_{\beta_{sc},\beta'_{sc}}\right\}
\end{aligned} \tag{9.21}$$

or

$$\tilde{V}'_{\beta_{sc},sce,p}(s) = \frac{\tilde{V}_{\beta_{sc},sce,p}(s)\tilde{Y}_{\beta_{sc},sce}(s) - \frac{1}{Z_o}\sum_{\beta'_{sc}} V_{\beta_{sc},\beta'_{sc}p}\, a_{\beta_{sc},\beta'_{sc}}\, s^{-1}_{\beta_{sc},\beta'_{sc}}}{\tilde{Y}_{\beta_{sc},sce}(s) - \frac{1}{Z_o}\sum_{\beta'_{sc}} a_{\beta_{sc},\beta'_{sc}}\, s^{-1}_{\beta_{sc},\beta'_{sc}}} \tag{9.22}$$

X. Equivalent Circuits for Admittance and Short Circuit Current

From the previous two sections we can now construct equivalent-circuit representations to give both the admittance and short circuit current, i.e. the complete equivalent circuit at the port. These equivalent circuits involve both admittances and sources, taken for convenience as voltage sources. Both SEM and EEM forms can be produced, and the two related. The individual pole terms give what can be termed elementary circuit modules with formulas for individual circuit elements. Elementary circuit modules can be combined by conjugate pairs to form individual resonant (or tank) circuits corresponding to pole pairs, although this complicates the form of the circuit modules somewhat.

The reader should note the equivalent circuits developed here are in general formal circuits in that the individual impedance elements (resistors, inductors, capacitors) exhibited are not necessarily realizable as lumped, passive elements. The element values may even be complex. While certain aspects of these circuits have realizability properties associated with positive real functions, this does not mean that any form of circuit with same mathematical response is realizable element by element. Certainly these aspects need further development.

A. EEM form

Consider first the form of the equivalent circuit based on the eigenmode expansion as illustrated in figure 10.1. This consists of an infinite set of parallel subcircuits, each subcircuit being associated with a different eigenmode with index β_{sc}. Each subcircuit consists of the series combination of an eigenvalue associated admittance $\tilde{Y}_{\beta_{sc}}(s)$ and a corresponding voltage source $\sum_{p} \tilde{V}_{\beta_{sc},p}(s)\ \tilde{f}_p(s)$ combining the eigenmode voltage source coefficients $\tilde{V}_{\beta_{sc},p}(s)$ with the incident waveforms $\tilde{f}_p(s)$. Note that the voltage source coefficients are in general frequency dependent, and that the eigenvalue associated admittances, having some number of poles, are in general a complicated circuit.

The circuit of figure 10.1 is rather simple in its general form. Note the ordering of the subcircuits with $\beta_{sc} = 1$ nearest the port. The corresponding eigenvalue is taken as that with its zeros (natural frequencies) clustered nearest $s = 0$ in some sense in the complex frequency plane.

B. SEM form

Using the pole admittances and voltage sources we have the equivalent circuit as indicated in figure 10.2. Note the parallel combination of circuit modules, each module consisting of the series combination of an admittance and a voltage source. Corresponding to each admittance pole there is an admittance $\tilde{Y}_{\alpha_{sc}}(s)$ and a voltage source $\sum_p V_{\alpha_{sc},p}\, \tilde{f}_p(s)$. The voltage source coefficients $V_{\alpha_{sc},p}$ are scalar constants (in general complex); the source waveforms are $\tilde{f}_p(s)$ in complex frequency domain and the voltage sources have direct time domain representations by use of $f_p(t)$, the given transient source waveforms. When listing specific pole terms say with $(\beta_{sc},\beta'_{sc}) = (1,1)$ an additional subscript of sc is added for clarity.

For convenience the circuit modules corresponding to poles are grouped together according to which eigenvalue they belong. Furthermore if one superimposes a complex frequency plane on figure 10.2 with the origin at about the port location, then the circuit modules can be envisioned as being located at the pole locations $s_{\beta_{sc},\beta'_{sc}}$ in this s plane. Note the conjugate arrangement of the circuit modules corresponding to conjugate pole pairs. Each of the sets of circuit modules corresponding to an eigenvalue is grouped in what can be referred to as arcs designated by $\beta = 1,2,\ldots$ as discussed in a previous note.[8]

The entire function contributions associated with each eigenvalue are also indicated by the series combination of the admittance $\tilde{Y}_{\beta_{sce}}(s)$ in series with the voltage source $\sum_p \tilde{V}_{\beta_{sc},sce,p}(s)\, \tilde{f}_p(s)$. These are also grouped in their appropriate arcs. Alternatively it may be more convenient to group all the entire function

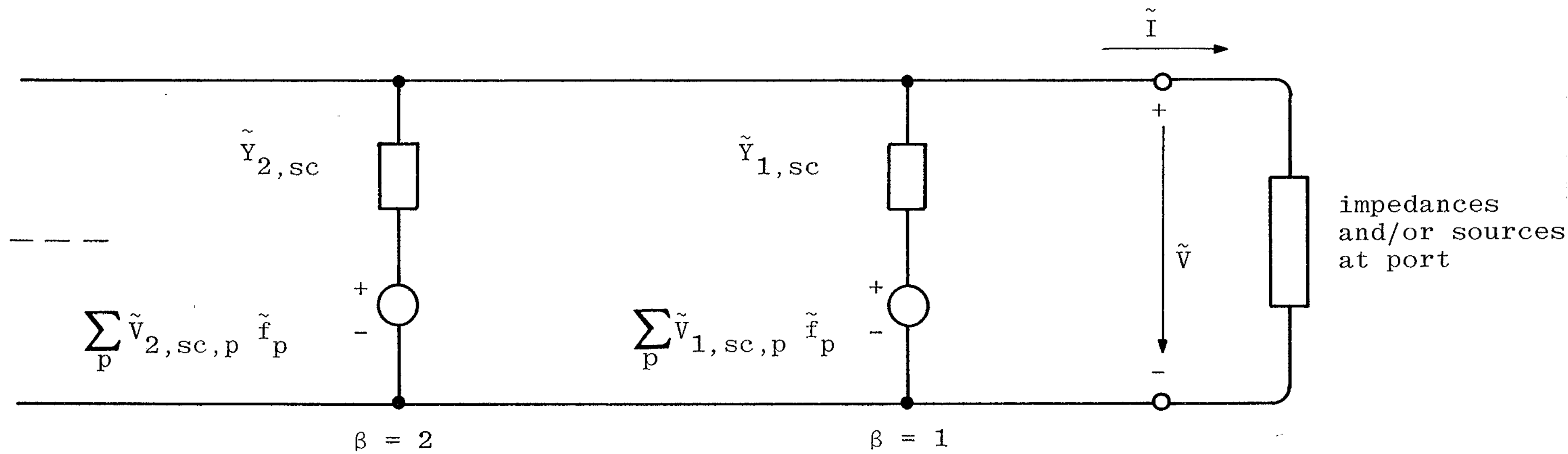


Figure 10.1. Equivalent Circuit for Admittance and Short Circuit Current Based on Eigenmode Expansion

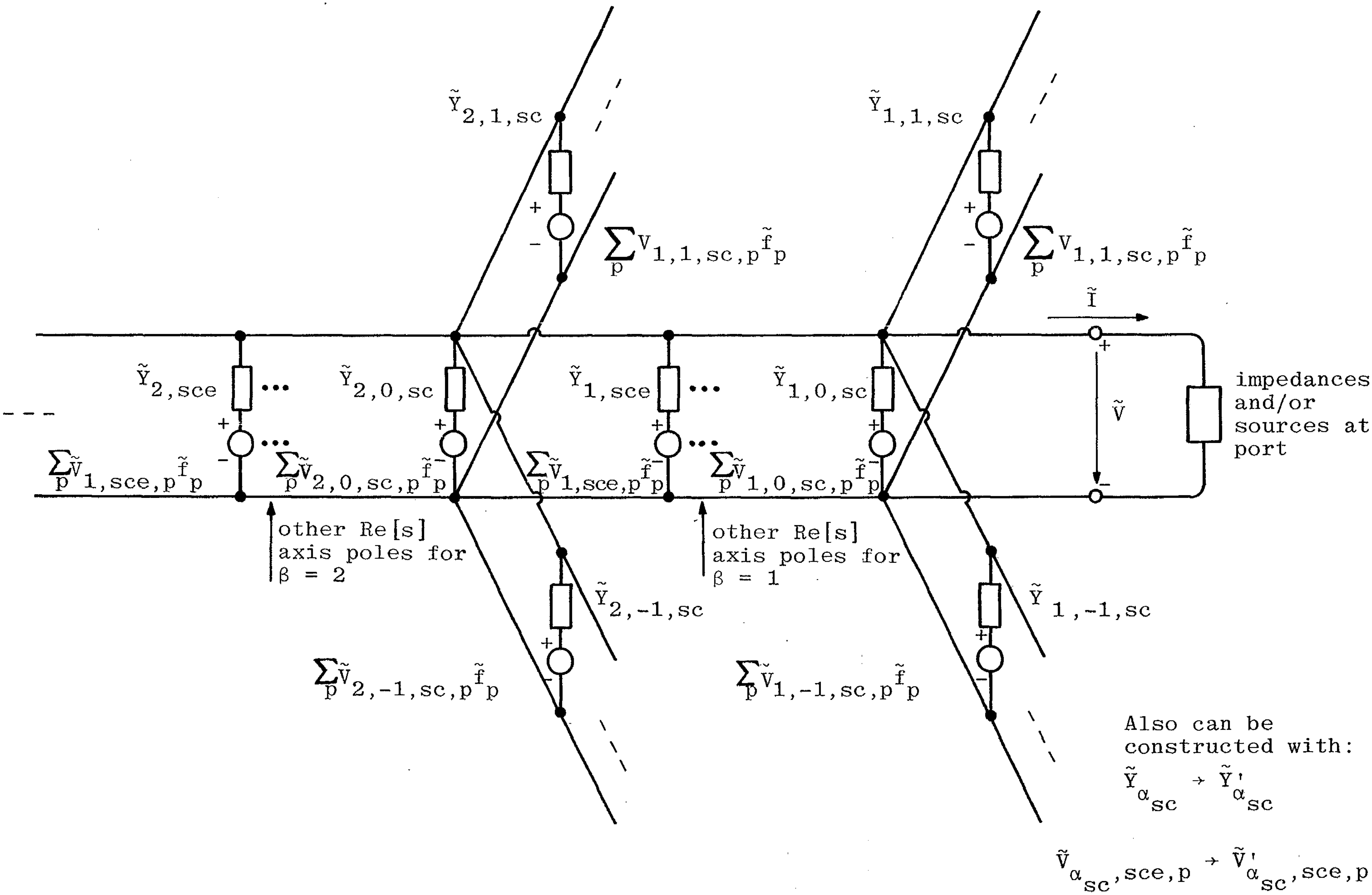


Figure 10.2. Equivalent Circuit for Admittance and Short Circuit Current Based on Singularity Expansion

contributions together in a single circuit module with admittance $\tilde{Y}_{sce}(s)$ and voltage source $\sum_p \tilde{V}_{sce,p}(s)\ \tilde{f}_p(s)$. For this case collect the entire function circuit modules in figure 10.2 into one such module located near the port terminals.

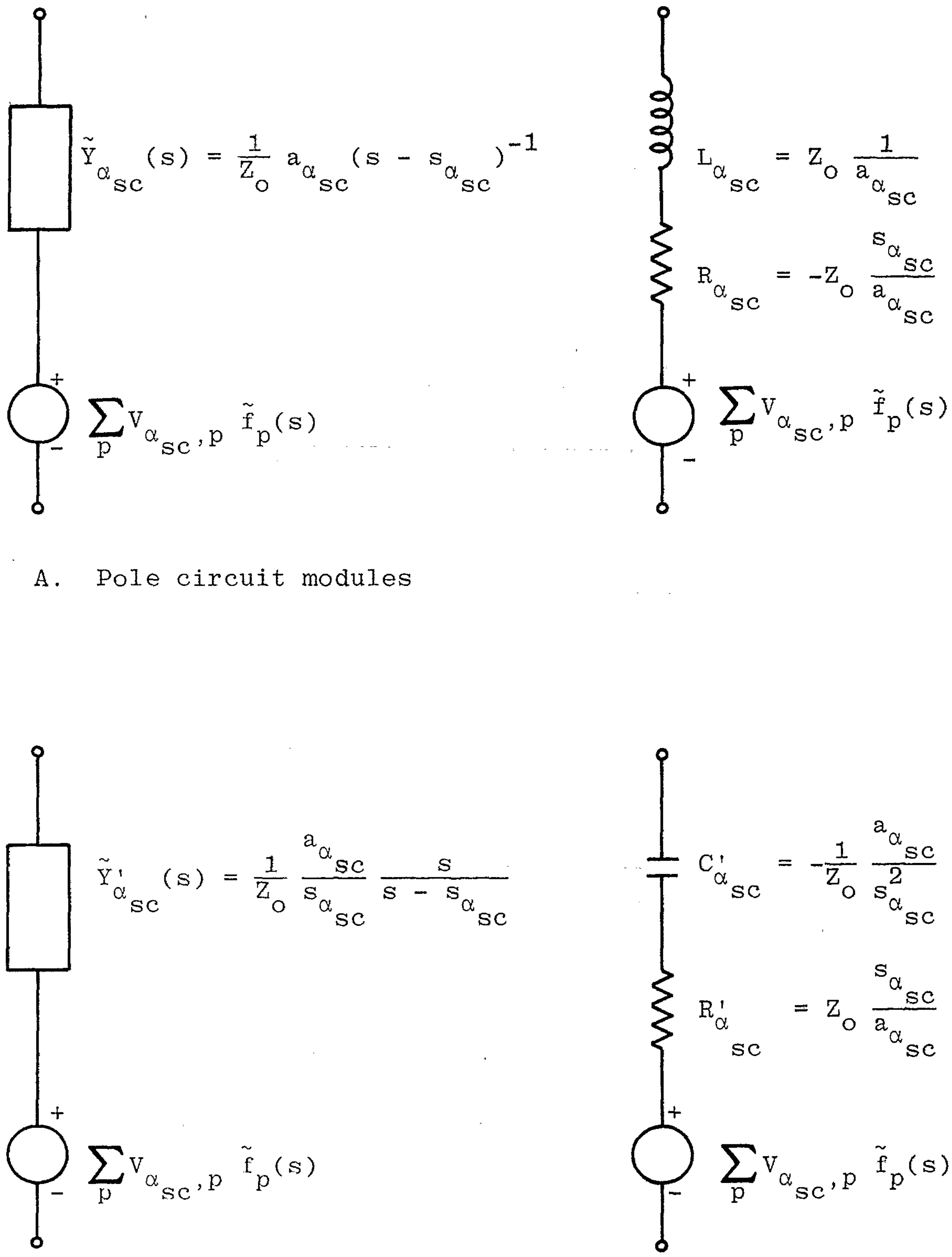
This particular circuit format can also be used with the "modified" circuit quantities by replacing the pole admittances by modified pole admittances $\tilde{Y}'_{\alpha_{sc}}(s)$, and by substituting the modified entire function admittances and voltage sources for the unmodified ones.

C. Elementary circuit modules

The admittance associated with each pole can be cast in the form of circuit elements as

$$\begin{aligned}
\tilde{Y}_{\alpha_{sc}}(s) &= \frac{1}{Z_o}\, a_{\alpha_{sc}} (s - s_{\alpha_{sc}})^{-1} \\
&= \left[sL_{\alpha_{sc}} + R_{\alpha_{sc}} \right]^{-1} \\
L_{\alpha_{sc}} &= Z_o \frac{1}{a_{\alpha_{sc}}} \\
R_{\alpha_{sc}} &= -Z_o \frac{s_{\alpha_{sc}}}{a_{\alpha_{sc}}} = -s_{\alpha_{sc}} L_{\alpha_{sc}}
\end{aligned} \tag{10.1}$$

where $a_{\alpha_{sc}}$ is a complex constant characteristic of the admittance residues as given in equations 8.6 and 8.7. This gives a circuit form for the pole circuits as illustrated in figure 10.3A. The modified pole admittances can be written in terms of circuit elements as



A. Pole circuit modules

B. Modified pole circuit modules

Figure 10.3. Elementary Circuit Modules for Pole Terms for Admittance and Short Circuit Current

$$\tilde{Y}'_\alpha(s) = \frac{1}{Z_o} a_{\alpha_{sc}} \left[(s - s_{\alpha_{sc}})^{-1} + s_{\alpha_{sc}}^{-1} \right]$$

$$= \frac{1}{Z_o} \frac{a_{\alpha_{sc}}}{s_{\alpha_{sc}}} \frac{s}{s - s_{\alpha_{sc}}}$$

$$= \frac{1}{Z_o} a_{\alpha_{sc}} \left[s_{\alpha_{sc}} - s_{\alpha_{sc}}^2 s^{-1} \right]^{-1}$$

$$= \left[R'_{\alpha_{sc}} + \frac{1}{sC'_{\alpha_{sc}}} \right]^{-1} \qquad (10.2)$$

$$C'_{\alpha_{sc}} = - \frac{1}{Z_o} \frac{a_{\alpha_{sc}}}{s_{\alpha_{sc}}^2}$$

$$R'_{\alpha_{sc}} = Z_o \frac{s_{\alpha_{sc}}}{a_{\alpha_{sc}}} = - \frac{1}{s_{\alpha_{sc}} C'_{\alpha_{sc}}} = -R_{\alpha_{sc}}$$

Combined with the voltage source this gives the circuit representation in figure 10.3B.

While these circuits are simple in form the circuit element values are in general complex numbers. For computer purposes these formal circuit elements should still be useful. For realization as actual circuit elements these are somewhat limited, although they may be useful in some limiting cases such as $s_{\alpha_{sc}}$ near the real or imaginary axes. The consideration of specific examples should be enlightening in this regard.

D. Entire-function circuit modules

The admittance associated with the entire function contribution is somewhat more problematical. If we use the considerations of section VI the entire function admittance may be approximated by a constant term based on high and low frequency considerations.

If the admittance $\tilde{Y}_a(s)$ is capacitive for $s \to 0$ then one might use the modified pole admittances $\tilde{Y}'_{\alpha_{sc}}(s)$ which are capacitive for $s \to 0$ and thereby avoid the use of an entire function admittance (restricted to a constant, i.e. a conductance). For cases that the $\tilde{Y}_{\alpha_{sc}}(s)$ (unmodified pole admittances) are used the appropriate admittance is found from equations 8.9 as

$$\tilde{Y}'_{sce}(s) \simeq G'_{sce} = 0$$

$$\tilde{Y}_{sce}(s) \simeq G_{sce} = \frac{1}{Z_o} \sum_{\alpha_{sc}} a_{\alpha_{sc}} s^{-1}_{\alpha_{sc}} \tag{10.3}$$

$$= -\sum_{\alpha_{sc}} \tilde{Y}_{\alpha_{sc}}(0)$$

where the entire function admittances are approximated as constant conductances. The summation, instead of extending over all α_{sc} might be restricted to those α_{sc} which are used in a finite approximation to the circuit of figure 10.2.

If the admittance is inductive or resistive as $s \to 0$ (as in a loop) then one can evaluate the admittance as $s \to 0$, subtract off the pole terms (or only those pole terms used), and match the remainder with a constant conductance near $s = 0$. This gives

$$\tilde{Y}_{sce}(s) \simeq G_{sce}$$

$$\tilde{Y}'_{sce}(s) \simeq G'_{sce} \tag{10.4}$$

$$G'_{sce} = G_{sce} - \frac{1}{Z_o} \sum_{\alpha_{sc}} a_{\alpha_{sc}} s^{-1}_{\alpha_{sc}}$$

$$= G_{sce} + \sum_{\alpha_{sc}} \tilde{Y}_{\alpha_{sc}}(0)$$

In the case of a resistive loop the conductance at s = 0 can be used to establish

$$G'_{sce} = \tilde{Y}_a(0) \tag{10.5}$$

However, for the case of a perfectly conducting loop we have

$$\tilde{Y}_a(s) = \frac{1}{sL} + c_0 + c_1 s + \cdots \tag{10.6}$$

The first term is a pole which is subtracted. If the pole at zero frequency is left in unmodified form because the s = 0 value cannot be subtracted, and if the other poles are put in modified form, then the entire function admittance corresponds to c_o.

The entire function voltage sources can be approximated by considering the short circuit current as $s \to 0$. It is assumed that the incident waves have, for the delta function response, zero time adjusted so that the short circuit current begins exactly at t = 0. Then as discussed in section VI the short circuit current has simple behavior for both $s \to 0$ and $s \to \infty$.

For a capacitive antenna as $s \to 0$ we have $\tilde{I}_{sc}(s) \to 0$ for a delta function incident wave. If the modified pole admittances are used and if the entire function voltage source is approximated as a constant then we have to still make the short circuit current zero at s = 0. These requirements give

$$\tilde{V}'_{sce,p}(s) \simeq V'_{sce,p} \qquad \text{(not used with zero admittance)}$$

$$\tilde{V}_{sce,p}(s) \simeq V_{sce,p} = \frac{\tilde{I}_{sce,p}(0)}{\tilde{Y}_{sce}(0)} = \frac{\tilde{I}_{sce,p}(0)}{G_{sce}} \tag{10.7}$$

$$= \frac{\sum_{\alpha_{sc}} V_{\alpha_{sc},p} \tilde{Y}_{\alpha_{sc}}(0)}{\sum_{\alpha_{sc}} \tilde{Y}_{\alpha_{sc}}(0)} = \frac{\sum_{\alpha_{sc}} V_{\alpha_{sc},p} a_{\alpha_{sc}} s_{\alpha_{sc}}^{-1}}{\sum_{\alpha_{sc}} a_{\alpha_{sc}} s_{\alpha_{sc}}^{-1}}$$

For a capacitive antenna the use of modified pole admittances would seem to simplify matters.

For inductive or resistive admittance as $s \to 0$ consider the behavior of $\tilde{I}_{sc}(s)$ as $s \to 0$. For a resistive loop $\tilde{I}_{sc}(s) \to 0$ as $s \to 0$ for a delta function incident wave. In this case if one uses modified pole admittances then this condition is met with no additional entire function voltage source to go with G'_{sce} in equation 10.5. For a perfectly conducting loop we can have $\tilde{I}_{sc}(s) \to$ constant as $s \to 0$ for a delta function incident wave. For use with modified pole admittances one could use

$$\tilde{I}'_{sce,p} \simeq \tilde{I}'_{sce,p}(0) \tag{10.8}$$

summed with $\tilde{f}_p(s)$ as a current source noting that the pole admittance of the form $1/(sL)$ is not modified. This gives a voltage source for this one pole as

$$\tilde{V}'_{sce,p}(s) \simeq sL\,\tilde{I}_{sce,p}(0) \tag{10.9}$$

which is frequency dependent in a simple form. Note that this prevents the short circuit current from unphysically growing as $s \to 0$ for a delta function incident wave.

For use with resistive admittance as $s \to 0$ (as in equations 10.4) then we have for unmodified pole admittances

$$\begin{aligned}\tilde{V}_{sce,p}(s) \simeq V_{sce,p} &= \frac{\tilde{I}_{sce,p}(0)}{\tilde{Y}_{sce}(0)} \\ &= \frac{I_{sce,p}}{G_{sce}} \\ &= \frac{-\sum_{\alpha_{sc}} V_{\alpha_{sc,p}}\,\tilde{Y}_{\alpha_{sc}}(0)}{G'_{sce} - \sum_{\alpha_{sc}} \tilde{Y}_{\alpha_{sc}}(0)}\end{aligned} \tag{10.10}$$

where G'_{sce} is the admittance for $s \to 0$, and where as before the summation over α_{sc} may be a finite approximation.

Considering the various forms of possible circuits based on the short circuit current and admittance it would appear that this type of circuit might be more readily useful for some types of antennas/scatterers than for other types. Note in particular how convenient is the use of modified pole admittances with appropriate voltage sources for capacitive objects, since the entire function admittance is approximated as zero and the corresponding voltage source is ignored.

E. Conjugate pair circuit modules

In attempting to realize circuits to give the correct admittance and short circuit current by the procedure in this note one is faced with the difficulty of the presence of complex element values. Corresponding to natural frequencies on the negative real axis of the s plane ($\mathrm{Im}[s_{\alpha_{sc}}] = 0$) the element values are inherently real as long as our original integral equation deals only with the Laplace transform of real-valued time functions. Likewise one can combine conjugate pole pairs to produce something with the same properties, although the circuit form be more complicated thereby. Note that while the circuit elements produced are then real valued they may not be always positive. Similar comments apply to the entire function admittances (real but not necessarily positive).

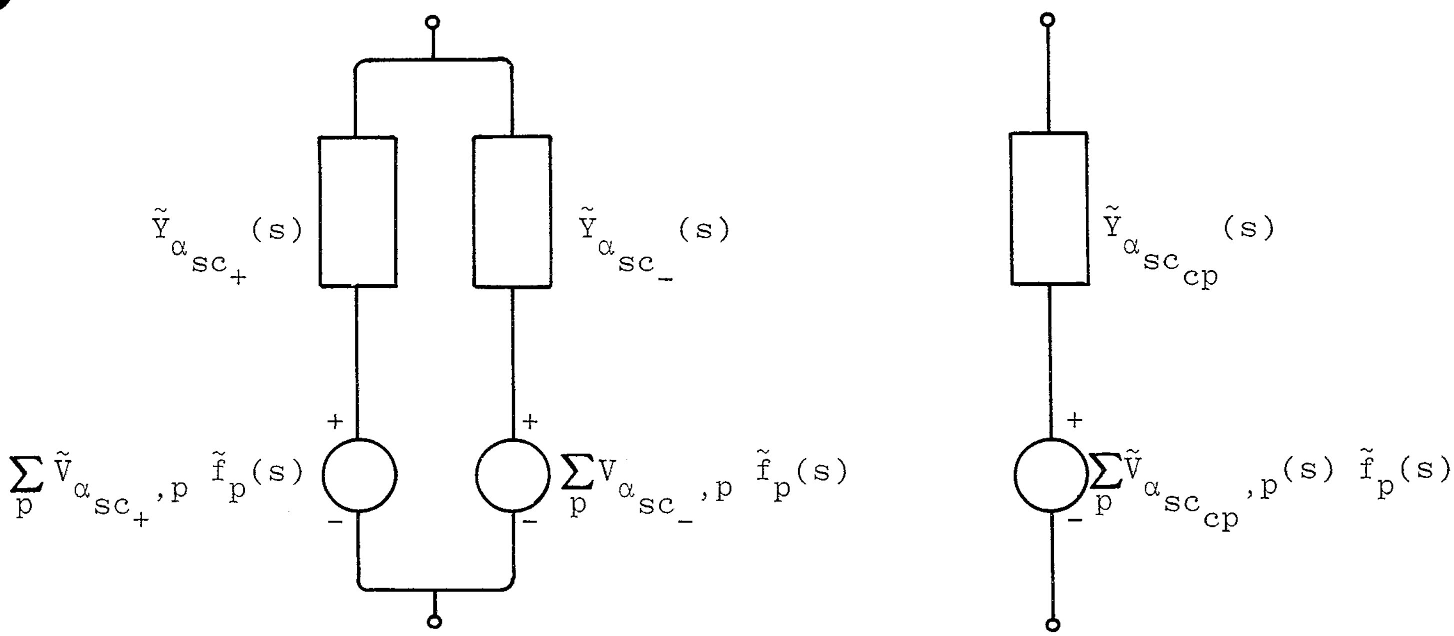
To begin define $s_{\alpha_{sc_+}}$ and $s_{\alpha_{sc_-}}$ such that

$$s_{\alpha_{sc_-}} = \overline{s}_{\alpha_{sc_+}} \qquad (10.11)$$

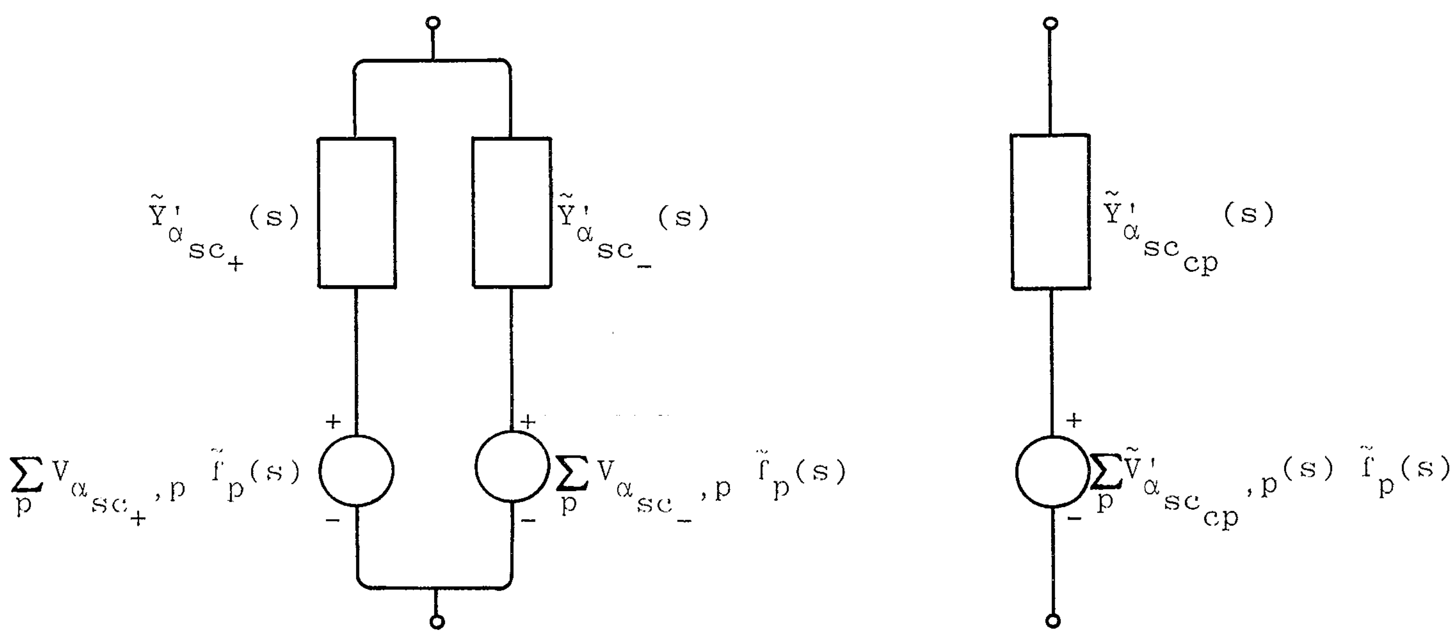
$$\mathrm{Im}\left[s_{\alpha_{sc_+}}\right] > 0$$

forming a pole pair. As indicated in figure 10.4 one can form thevenin equivalent circuits of such conjugate pairs of pole circuit modules. The conjugate pair (cp) admittances are

$$\begin{aligned}
\tilde{Y}_{\alpha_{sc_{cp}}}(s) &= \tilde{Y}_{\alpha_{sc_+}}(s) + \tilde{Y}_{\alpha_{sc_-}}(s) \\
&= \frac{1}{Z_o}\left\{a_{\alpha_{sc_+}}(s - s_{\alpha_{sc_+}})^{-1} + a_{\alpha_{sc_-}}(s - s_{\alpha_{sc_-}})^{-1}\right\} \\
&= \frac{1}{Z_o}\left\{a_{\alpha_{sc_+}}(s - s_{\alpha_{sc_+}})^{-1} + \bar{a}_{\alpha_{sc_+}}(s - \bar{s}_{\alpha_{sc_-}})^{-1}\right\} \\
&= \frac{1}{Z_o}\,\frac{a_{\alpha_{sc_+}}(s - \bar{s}_{\alpha_{sc_+}}) + \bar{a}_{\alpha_{sc_+}}(s - s_{\alpha_{sc_+}})}{(s - s_{\alpha_{sc_+}})(s - \bar{s}_{\alpha_{sc_+}})} \\
&= \frac{1}{Z_o}\,\frac{2\,\mathrm{Re}\left[a_{\alpha_{sc_+}}\right]s - 2\,\mathrm{Re}\left[a_{\alpha_{sc_+}}\bar{s}_{\alpha_{sc_+}}\right]}{s^2 - 2\,\mathrm{Re}\left[s_{\alpha_{sc_+}}\right]s + \left|s_{\alpha_{sc_+}}\right|^2} \\
&\equiv \frac{1}{Z_o}\,\frac{c_1 s + c_2}{c_3 s^2 + c_4 s + c_5}
\end{aligned} \qquad (10.12)$$



A. Equivalent for pole circuit module conjugate pair



B. Equivalent for modified pole circuit module conjugate pair

Figure 10.4. Thevenin Equivalent Circuits for Pole Pairs

$$\tilde{Y}'_{\alpha_{sc_{cp}}}(s) = \tilde{Y}'_{\alpha_{sc_+}}(s) + \tilde{Y}'_{\alpha_{sc_-}}(s)$$

$$= \frac{1}{Z_o}\left\{\frac{a_{\alpha_{sc_+}}}{s_{\alpha_{sc_+}}}\frac{s}{s - s_{\alpha_{sc_+}}} + \frac{a_{\alpha_{sc_-}}}{s_{\alpha_{sc_-}}}\frac{s}{s - s_{\alpha_{sc_-}}}\right\}$$

$$= \frac{1}{Z_o}\left\{\frac{a_{\alpha_{sc_+}}}{s_{\alpha_{sc_+}}}\frac{s}{s - s_{\alpha_{sc_+}}} + \frac{\bar{a}_{\alpha_{sc_+}}}{\bar{s}_{\alpha_{sc_+}}}\frac{s}{s - \bar{s}_{\alpha_{sc_+}}}\right\}$$

$$= \frac{1}{Z_o}\frac{\frac{a_{\alpha_{sc_+}}}{s_{\alpha_{sc_+}}}(s - \bar{s}_{\alpha_{sc_+}})s + \frac{\bar{a}_{\alpha_{sc_+}}}{\bar{s}_{\alpha_{sc_+}}}(s - s_{\alpha_{sc_+}})s}{(s - s_{\alpha_{sc_+}})(s - \bar{s}_{\alpha_{sc_+}})}$$

$$= \frac{1}{Z_o}\frac{2\,\mathrm{Re}\left[\frac{a_{\alpha_{sc_+}}}{s_{\alpha_{sc_+}}}\right]s^2 - 2\,\mathrm{Re}\left[\frac{a_{\alpha_{sc_+}}\bar{s}_{\alpha_{sc_+}}}{s_{\alpha_{sc_+}}}\right]s}{s^2 - 2\,\mathrm{Re}\left[s_{\alpha_{sc_+}}\right]s + \left|s_{\alpha_{sc_+}}\right|^2}$$

$$\equiv \frac{1}{Z_o}\frac{c'_2 s^2 + c'_1 s}{c'_5 s^2 + c'_4 s + c'_3} = \frac{1}{Z_o}\frac{c'_1 s^{-1} + c'_2}{c'_3 s^{-2} + c'_4 s^{-1} + c'_5}$$

for pole pair admittances and modified pole pair admittances respectively. Note that all coefficients of powers of s are real numbers.

The short circuit current associated with a conjugate pole pair can be written as

$$\tilde{I}_{sc_{\alpha_{sc_{cp}}}}(s) = \tilde{I}_{sc_{\alpha_{sc_+}}}(s) + \tilde{I}_{sc_{\alpha_{sc_-}}}(s)$$

$$= \sum_p \tilde{f}_p(s) \left[V_{\alpha_{sc_+},p} \tilde{Y}_{\alpha_{sc_+}}(s) + V_{\alpha_{sc_-},p} \tilde{Y}_{\alpha_{sc_-}}(s) \right] \quad (10.13)$$

which is quite frequency dependent due to the pole admittances. The open circuit voltage associated with such a pair can be written as

$$\tilde{V}_{oc_{\alpha_{sc_{cp}}}}(s) = \frac{\tilde{I}_{sc_{\alpha_{sc_+}}}(s) + \tilde{I}_{sc_{\alpha_{sc_-}}}(s)}{\tilde{Y}_{\alpha_{sc_+}}(s) + \tilde{Y}_{\alpha_{sc_-}}(s)}$$

$$= \sum_p \tilde{V}_{\alpha_{sc_{cp}},p}(s)\, \tilde{f}_p(s) \quad (10.14)$$

$$\tilde{V}_{\alpha_{sc_{cp}},p}(s) = \frac{V_{\alpha_{sc_+},p} \tilde{Y}_{\alpha_{sc_+}}(s) + V_{\alpha_{sc_-},p} \tilde{Y}_{\alpha_{sc_-}}(s)}{\tilde{Y}_{\alpha_{sc_+}}(s) + \tilde{Y}_{\alpha_{sc_-}}(s)}$$

$$= \mathrm{Re}\left[V_{\alpha_{sc_+},p}\right] + i\, \mathrm{Im}\left[V_{\alpha_{sc_+},p}\right] \frac{\tilde{Y}_{\alpha_{sc_+}}(s) - \tilde{Y}_{\alpha_{sc_-}}(s)}{\tilde{Y}_{\alpha_{sc_+}}(s) + \tilde{Y}_{\alpha_{sc_-}}(s)}$$

If the pole voltage source coefficients $V_{\alpha_{sc},p}$ are real numbers then the thevenin voltage source for the pole pair simplifies somewhat to involving only the same simple coefficients. However, for general complex $V_{\alpha_{sc},p}$ the resulting voltage source coefficients $\tilde{V}_{\alpha_{sc_{cp}},p}(s)$ are frequency dependent. For practical circuit realization then one might prefer real or almost real pole voltage source coefficients.

The short circuit current associated with a modified conjugate pole pair is

$$\tilde{I}'_{sc_{\alpha_{sc_{cp}}}}(s) = \tilde{I}'_{sc_{\alpha_{sc_+}}}(s) + \tilde{I}'_{sc_{\alpha_{sc_-}}}(s)$$

$$= \sum_p \tilde{f}_p(s) \left[V_{\alpha_{sc_+},p} \, \tilde{Y}'_{\alpha_{sc_+}}(s) + V_{\alpha_{sc_-},p} \, \tilde{Y}'_{\alpha_{sc_-}}(s) \right] \tag{10.15}$$

The open circuit voltage associated with a modified pole admittance pair is

$$\tilde{V}'_{oc_{\alpha_{sc_{cp}}}}(s) = \frac{\tilde{I}'_{sc_{\alpha_{sc_+}}}(s) + \tilde{I}'_{sc_{\alpha_{sc_-}}}(s)}{\tilde{Y}'_{\alpha_{sc_+}}(s) + \tilde{Y}_{\alpha_{sc_-}}(s)}$$

$$= \sum_p \tilde{V}'_{\alpha_{sc_{cp}},p}(s) \, \tilde{f}_p(s) \tag{10.16}$$

$$\tilde{V}'_{\alpha_{sc_{cp}},p}(s) = \frac{V_{\alpha_{sc_+},p} \, \tilde{Y}'_{\alpha_{sc_+}}(s) + V_{\alpha_{sc_-},p} \, \tilde{Y}'_{\alpha_{sc_-}}(s)}{\tilde{Y}'_{\alpha_{sc_+}}(s) + \tilde{Y}_{\alpha_{sc_-}}(s)}$$

$$= \mathrm{Re}\left[V_{\alpha_{sc_+},p} \right] + i \; \mathrm{Im}\left[V_{\alpha_{sc_+},p} \right] \frac{\tilde{Y}'_{\alpha_{sc_+}}(s) - \tilde{Y}'_{\alpha_{sc_-}}(s)}{\tilde{Y}_{\alpha_{sc_+}}(s) + \tilde{Y}'_{\alpha_{sc_-}}(s)}$$

Again real valued $V_{\alpha_{sc},p}$ lead to simpler equivalent circuits for the modified conjugate pole pairs.

The admittance of a conjugate pole pair can be put into the form of circuit elements. Write the admittance in the form .

$$\tilde{Y}^{-1}_{\alpha_{sc_{cp}}}(s) = Z_o \frac{c_3 s^2 + c_4 s + c_5}{c_1 s + c_2}$$

$$= Z_o \left\{ \frac{c_3}{c_1} s + \frac{\left(c_4 - \frac{c_3}{c_1} c_2 - \frac{c_5}{c_2} c_1\right) s}{c_1 s + c_2} + \frac{c_5}{c_2} \right\}$$

$$= sL_{\alpha_{sc_{cp}}} + \left[\frac{1}{R^{(p)}_{\alpha_{sc_{cp}}}} + \frac{1}{sL^{(p)}_{\alpha_{sc_{cp}}}}\right]^{-1} + R_{\alpha_{sc_{cp}}}$$

$$L_{\alpha_{sc_{cp}}} = Z_o \frac{c_3}{c_2} = \frac{Z_o}{2\ \mathrm{Re}\left[a_{\alpha_{sc_+}}\right]}$$

$$= \left[L^{-1}_{\alpha_{sc_+}} + L^{-1}_{\alpha_{sc_-}}\right]^{-1} = \frac{\left|L_{\alpha_{sc_+}}\right|^2}{2\ \mathrm{Re}\left[L_{\alpha_{sc_+}}\right]}$$

$$R_{\alpha_{sc_{cp}}} = Z_o \frac{c_5}{c_2} = -\frac{Z_o \left|s_{\alpha_{sc_+}}\right|^2}{2\ \mathrm{Re}\left[a_{\alpha_{sc_+}} \bar{s}_{\alpha_{sc_+}}\right]}$$

$$= -\frac{Z_o}{2\ \mathrm{Re}\left[\frac{a_{\alpha_{sc_+}}}{s_{\alpha_{sc_+}}}\right]}$$

$$= \left[R^{-1}_{\alpha_{sc_+}} + R^{-1}_{\alpha_{sc_-}}\right]^{-1} = \frac{\left|R_{\alpha_{sc_+}}\right|^2}{2\ \mathrm{Re}\left[R_{\alpha_{sc_+}}\right]} \qquad (10.17)$$

$$R^{(p)}_{\alpha_{sc_{cp}}} = Z_o \left\{ \frac{c_4}{c_1} - \frac{c_2 c_3}{c_1^2} - \frac{c_5}{c_2} \right\}$$

$$L^{(p)}_{\alpha_{sc_{cp}}} = Z_o \left\{ \frac{c_4}{c_2} - \frac{c_3}{c_1} - \frac{c_1 c_5}{c_2^2} \right\}$$

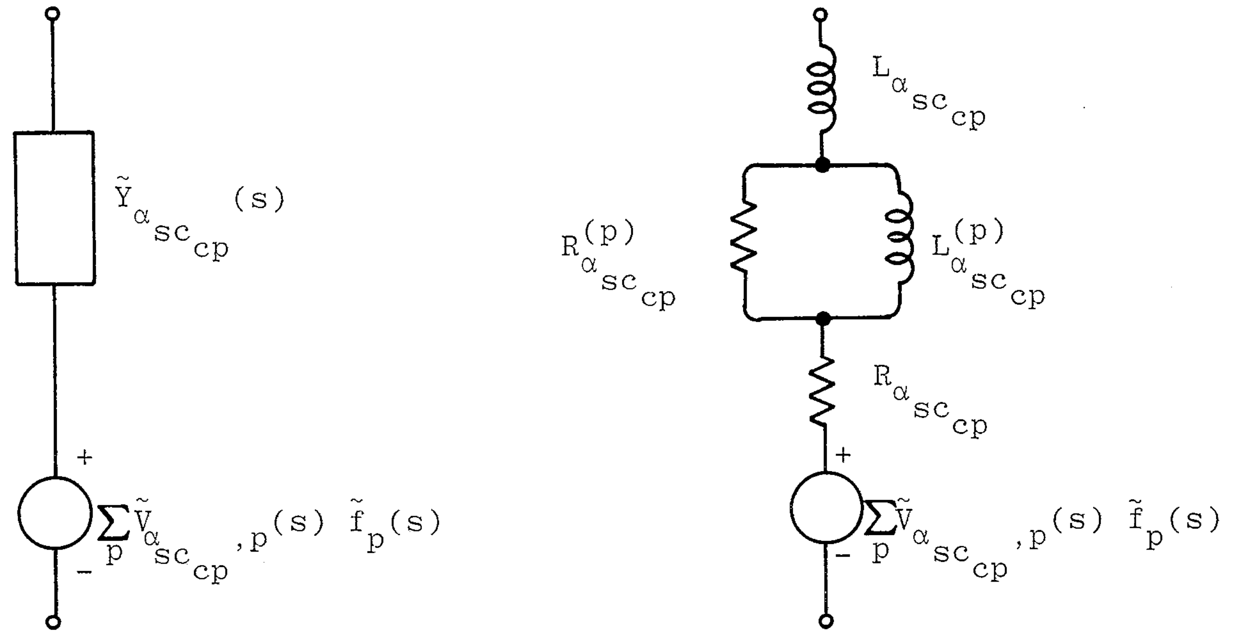
where these terms can be written out in terms of $s_{\alpha_{sc_+}}$ and $a_{\alpha_{sc_+}}$ using equations 10.12. This circuit form is shown in figure 10.5A. Since it only involves inductances and resistances then for highly resonant cases (poles near the $i\omega$ axis) one or more negative elements must be involved. However there are various ways to make a formal circuit which gives the proper pole-pair admittance. The above one is merely illustrative.

The admittance of a conjugate modified pole pair can be written as

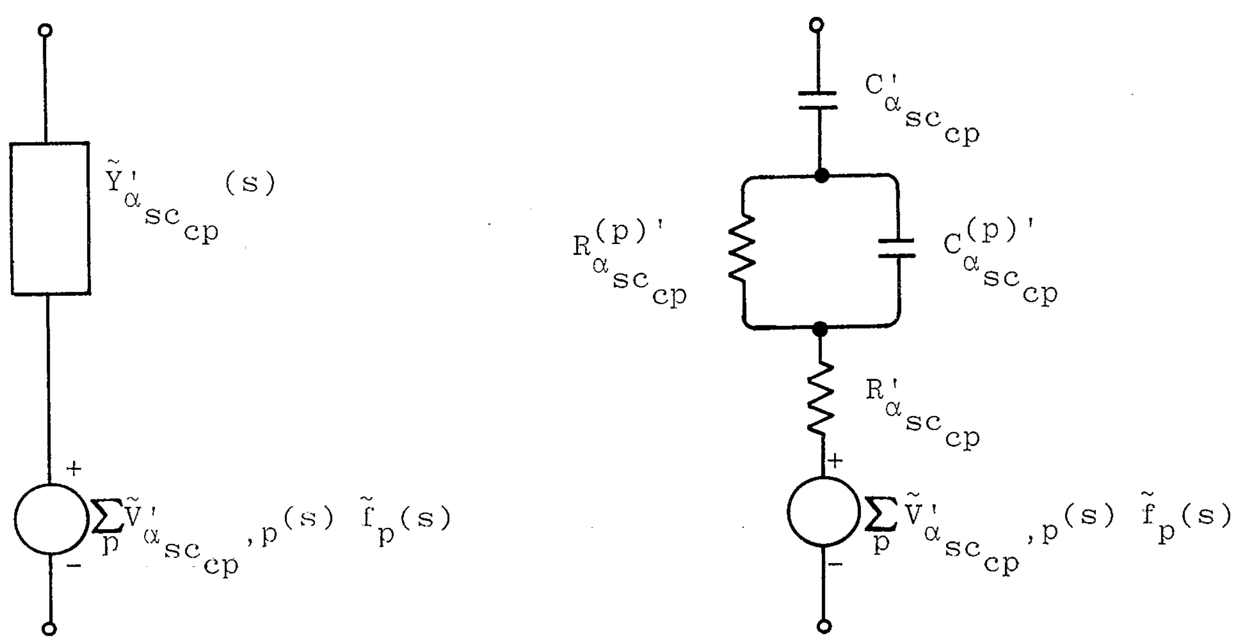
$$\tilde{Y}'^{-1}_{\alpha_{sc_{cp}}}(s) = Z_o \frac{c_3' s^{-2} + c_4' s^{-1} + c_5'}{c_1' s^{-1} + c_2'}$$

$$= Z_o \left\{ \frac{c_3'}{c_1'} s^{-1} + \frac{\left(c_4' - \frac{c_3'}{c_1'} c_2' - \frac{c_5'}{c_2'} c_1' \right) s^{-1}}{c_1' s^{-1} + c_2'} + \frac{c_5'}{c_2'} \right\}$$

$$= \frac{1}{sC'_{\alpha_{sc_{cp}}}} + \left[\frac{1}{R^{(p)'}_{\alpha_{sc_{cp}}}} + sC^{(p)'}_{\alpha_{sc_{cp}}} \right]^{-1} + R'_{\alpha_{sc_{cp}}} \tag{10.18}$$



A. Conjugate pole pair circuit



B. Conjugate modified pole pair circuit

Figure 10.5. Circuit Element Possibilities for Pole Pair Equivalent Circuits

$$C'_{\alpha_{sc_{cp}}} = \frac{1}{Z_o} \frac{c'_1}{c'_3} = - \frac{2}{Z_o} \frac{\mathrm{Re}\left[\dfrac{a_{\alpha_{sc_+}} \bar{s}_{\alpha_{sc_+}}}{s_{\alpha_{sc_+}}}\right]}{\left|s_{\alpha_{sc_+}}\right|^2}$$

$$= - \frac{2}{Z_o} \mathrm{Re}\left[\frac{a_{\alpha_{sc_+}}}{s^2_{\alpha_{sc_+}}}\right]$$

$$= C'_{\alpha_{sc_+}} + C'_{\alpha_{sc_-}} = 2\ \mathrm{Re}\left[C_{\alpha_{sc_+}}\right]$$

$$R'_{\alpha_{sc_{cp}}} = Z_o \frac{c'_5}{c'_2} = \frac{Z_o}{2\ \mathrm{Re}\left[\dfrac{a_{\alpha_{sc_+}}}{s_{\alpha_{sc_+}}}\right]}$$

$$= \left[R'^{-1}_{\alpha_{sc_+}} + R'^{-1}_{\alpha_{sc_-}}\right]^{-1} = \frac{\left|R'_{\alpha_{sc_+}}\right|^2}{2\ \mathrm{Re}\left[R'_{\alpha_{sc_+}}\right]}$$

$$R^{(p)'}_{\alpha_{sc_{cp}}} = Z_o \left\{\frac{c'_4}{c'_1} - \frac{c'_2 c'_3}{c'^2_1} - \frac{c'_5}{c'_2}\right\}$$

$$C^{(p)'}_{\alpha_{sc_{cp}}} = \frac{1}{Z_o} \left\{\frac{c'_4}{c'_2} - \frac{c'_3}{c'_1} - \frac{c'_1 c'_5}{c'^2_2}\right\}^{-1}$$

which is illustrated in figure 10.5B. Note in comparing equations 10.18 to 10.17 that the various constants c'_n correspond to c_n so that the modified pole circuits can be more readily compared to the pole circuits. This relationship between the two types of circuits is reflected in the interchange between inductances and capacitances with resistances remaining as resistances. Again since only resistances and capacitances are involved in the circuit illustrated here negative elements are involved for poles near the $i\omega$ axis. This circuit is then merely illustrative.

In converting from pole or modified pole circuits to conjugate pair circuits the element values for the admittances can be made real. However this does not necessarily make the element values positive. The foregoing give two kinds of conjugate pair circuits. Which one is more useful likely depends on the application, i.e. on what specific type of antenna/scatterer is being considered.

XI. Impedance

Consider now an alternate way to construct equivalent circuits based on the open circuit properties of the antenna or scatterer. Beginning with the impedance the gap current density is specified as in section III. This is introduced into the open circuit boundary value problem discussed in section IV. Using the SEM and EEM forms of the open circuit response the impedance is found from the formulas of sections I and II.

A. SEM form

Assume a source current density $\vec{J}_{s_s}$ on the gap surface (or perhaps in the gap volume). Let there be no impedance loading in the gap region. We have the general form of the surface current density on the object as

$$\tilde{\vec{J}}_s(\vec{r},s) = \tilde{I}(s)\left\{ \sum_{\alpha_{oc}} \tilde{\eta}'_{\alpha_{oc}} \vec{\nu}^{(\vec{J}_s)}_{\alpha_{oc}}(\vec{r})(s - s_{\alpha_{oc}})^{-1} + \text{entire function} \right\} \tag{11.1}$$

For the open circuit problem the natural frequencies, modes, and coupling vectors satisfy

$$\left\langle \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s_{\alpha_{oc}}) \; ; \; \vec{\nu}^{(\vec{J}_s)}_{\alpha_{oc}}(\vec{r}\,') \right\rangle_a = \vec{0} \quad , \quad \vec{r} \in S_a$$

$$\left\langle \vec{\mu}_{\alpha_{oc}}(\vec{r}) \; ; \; \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s_{\alpha_{oc}}) \right\rangle_a = \vec{0} \quad , \quad \vec{r}\,' \in S_a \tag{11.2}$$

where for symmetric operators we can set

$$\vec{\mu}_{\alpha_{oc}}(\vec{r}) \equiv \vec{\nu}^{(\vec{J}_s)}_{\alpha_{oc}}(\vec{r}) \tag{11.3}$$

which is not used in many of the formulas to provide for greater generality.

In order to find the coupling coefficients let us first find an appropriate source electric field incident on the object (with no other incident field) as

$$\tilde{\vec{E}}_s(\vec{r},s) = -\left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) \;\dot{,}\; \tilde{\vec{J}}_{s_g}(\vec{r}\,',s) \right\rangle_g \tag{11.4}$$

where there has been assumed no impedance loading in the gap region. The coupling coefficients for the current density on the object are then

$$\tilde{\eta}'_{\alpha_{oc}} = \frac{1}{\tilde{I}(s)} \frac{\left\langle \vec{\mu}_{\alpha_{oc}}(\vec{r}) \;\dot{,}\; \tilde{\vec{E}}_s(\vec{r},s) \right\rangle_a}{\left\langle \vec{\mu}_{\alpha_{oc}}(\vec{r}) \;\dot{,}\; \frac{\partial}{\partial s} \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\Big|_{s=s_{\alpha_{oc}}} \;\dot{,}\; \vec{\nu}^{(\vec{J}_s)}_{\alpha_{oc}}(\vec{r}\,') \right\rangle_a}$$

$$= -\frac{\left\langle \vec{\mu}_{\alpha_{oc}}(\vec{r}) \;\dot{,}\; \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s_{\alpha_{oc}}) \;\dot{,}\; \tilde{\vec{J}}_{s_g}(\vec{r}\,') \right\rangle_g \right\rangle_a}{\left\langle \vec{\mu}_{\alpha_{oc}}(\vec{r}) \;\dot{,}\; \frac{\partial}{\partial s} \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\Big|_{s=s_{\alpha_{oc}}} \;\dot{,}\; \vec{\nu}^{(\vec{J}_s)}_{\alpha_{oc}}(\vec{r}\,') \right\rangle_a} \tag{11.5}$$

Note that the gap source current density is discontinuous at the two ends of the gap, implying an infinite gap charge density there with corresponding influence on $\tilde{\vec{E}}_s$. This needs to be compensated by a discontinuity in $\tilde{J}_s$ at the object ends just outside the gap.

From section III we have the impedance as

$$\tilde{Z}_a(s) = -\frac{\tilde{V}(s)}{\tilde{I}(s)} = \frac{\left\langle \tilde{\vec{E}}(\vec{r},s) \;\dot{,}\; \tilde{\vec{J}}_{s_g}(\vec{r}) \right\rangle_g}{\tilde{I}(s)} \tag{11.6}$$

The electric field is obtained from the current on the object as well as the source current in the gap form

$$\tilde{\vec{E}}(\vec{r},s) = -\left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) \,\dot{;}\, \tilde{\vec{J}}_s(\vec{r}\,',s)\right\rangle_a - \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) \,\dot{;}\, \tilde{\vec{J}}_{s_g}(\vec{r}\,',s)\right\rangle_g$$

$$= -\tilde{I}(s)\left\{\sum_{\alpha_{oc}} \eta'_{\alpha_{oc}} \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) \,\dot{;}\, \vec{\nu}^{(\vec{J}_s)}_{\alpha_{oc}}(\vec{r}\,')\right\rangle_a (s - s_{\alpha_{oc}})^{-1} \right.$$

$$+ \text{ entire function}$$

$$\left. + \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) \,\dot{;}\, \vec{j}_{s_g}(\vec{r}\,')\right\rangle \right\}$$

$$= -\tilde{I}(s)\left\{\sum_{\alpha_{oc}} \eta'_{\alpha_{oc}} \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s_{\alpha_{oc}}) \,\dot{;}\, \vec{\nu}^{(\vec{J}_s)}_{\alpha_{oc}}(\vec{r}\,')\right\rangle_a (s - s_{\alpha_{oc}})^{-1} \right.$$

$$\left. + \vec{Z}'(\vec{r})s^{-1} + \text{ entire function} \right\} \tag{11.7}$$

$$\vec{Z}'(r) = \lim_{s \to 0} s \left\{\sum_{\alpha_{oc}} \eta'_{\alpha_{oc}} \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) \,\dot{;}\, \vec{\nu}^{(\vec{J}_s)}_{\alpha_{oc}}(\vec{r}\,')\right\rangle_a \left(-\frac{1}{s_{\alpha_{oc}}}\right) \right.$$

$$\left. + \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) \,\dot{;}\, \vec{j}_{s_g}(\vec{r}\,')\right\rangle \right\}$$

Here the entire functions have all been lumped into one. Note the pole at zero frequency introduced by the impedance operator; this allows a pole in the impedance at s = 0 such as would occur for an open (capacitive) object.

The impedance can now be written as

$$\tilde{Z}_a(s) = Z_o \left\{ \sum_{\alpha_{oc}} a_{\alpha_{oc}} (s - s_{\alpha_{oc}})^{-1} + a_{o_{oc}} s^{-1} \right\} + \tilde{Z}_{sce}(s)$$

(11.8)

$$a_{\alpha_{oc}} = \frac{1}{Z_o} \frac{\left\langle \vec{\mu}_{\alpha_{oc}}(\vec{r}) ; \left\langle \tilde{\overset{\leftrightarrow}{Z}}(\vec{r},\vec{r}\,';s_{\alpha_{oc}}) ; \vec{j}_{s_g}(\vec{r}\,') \right\rangle_g \right\rangle_a \left\langle \left\langle \tilde{\overset{\leftrightarrow}{Z}}(\vec{r},\vec{r}\,';s_{\alpha_{oc}}) ; \vec{\nu}_{\alpha_{oc}}^{(\vec{J}_s)}(\vec{r}\,') \right\rangle_a ; \vec{j}_{s_g}(\vec{r}) \right\rangle_g}{\left\langle \vec{\mu}_{\alpha_{oc}}(\vec{r}) ; \frac{\partial}{\partial s} \tilde{\overset{\leftrightarrow}{\Gamma}}(\vec{r},\vec{r}\,';s) \Big|_{s=s_{\alpha_{oc}}} ; \vec{\nu}_{\alpha_{oc}}^{(\vec{J}_s)}(\vec{r}\,') \right\rangle_a}$$

$$= \frac{1}{Z_o} \frac{\left\langle \vec{\mu}_{\alpha_{oc}}(\vec{r}) ; \left\langle \tilde{\overset{\leftrightarrow}{Z}}(\vec{r},\vec{r}\,';s_{\alpha_{oc}}) ; \vec{j}_{s_g}(\vec{r}\,') \right\rangle_g \right\rangle_a \left\langle \vec{\nu}_{\alpha_{oc}}^{(\vec{J}_s)}(\vec{r}) ; \left\langle \tilde{\overset{\leftrightarrow}{Z}}(\vec{r},\vec{r}\,';s_{\alpha_{oc}}) ; \vec{j}_{s_g}(\vec{r}) \right\rangle_g \right\rangle_a}{\left\langle \vec{\mu}_{\alpha_{oc}}(\vec{r}) ; \frac{\partial}{\partial s} \tilde{\overset{\leftrightarrow}{\Gamma}}(\vec{r},\vec{r}\,';s) \Big|_{s=s_{\alpha_{oc}}} ; \vec{\nu}_{\alpha_{oc}}^{(\vec{J}_s)}(\vec{r}\,') \right\rangle_a}$$

$$a_{o_{oc}} = -\frac{1}{Z_o} \left\langle \vec{Z}'(\vec{r}) ; \vec{j}_{s_g}(\vec{r}) \right\rangle_g$$

where the symmetric property of the impedance kernel $\tilde{\overset{\leftrightarrow}{Z}}(\vec{r},\vec{r}\,';s)$ has been used and where $\tilde{Z}_{sce}(s)$ is an entire function impedance. For symmetrical integral equation kernels $\tilde{\overset{\leftrightarrow}{\Gamma}}(\vec{r},\vec{r}\,';s)$ we have

$$\vec{\mu}_{\alpha_{oc}}(\vec{r}) \equiv \vec{\nu}_{\alpha_{oc}}^{(\vec{J}_s)}(\vec{r})$$

(11.9)

$$a_{\alpha_{oc}} = \frac{1}{Z_o} \frac{\left\langle \vec{\nu}_{\alpha_{oc}}^{(\vec{J}_s)}(\vec{r}) ; \left\langle \tilde{\overset{\leftrightarrow}{Z}}(\vec{r},\vec{r}\,';s_{\alpha_{oc}}) ; \vec{j}_{s_g}(\vec{r}\,') \right\rangle_g \right\rangle_a^2}{\left\langle \vec{\nu}_{\alpha_{oc}}^{(\vec{J}_s)}(\vec{r}) ; \frac{\partial}{\partial s} \tilde{\overset{\leftrightarrow}{\Gamma}}(\vec{r},\vec{r}\,';s) \Big|_{s-s_{\alpha_{oc}}} ; \vec{\nu}_{\alpha_{oc}}^{(\vec{J}_s)}(\vec{r}\,') \right\rangle_a}$$

While the formulas in equations 11.8 and 11.9 are in the form of surface integrals, they can be readily interpreted in terms of line and volume integral formulations as well.

In another form let us write

$$\tilde{Z}_a(s) = \sum_{\alpha_{oc}} \tilde{Z}_{\alpha_{oc}}(s) + \tilde{Z}_{sce}(s)$$

$$\tilde{Z}_{\alpha_{oc}}(s) \equiv Z_o \, a_{\alpha_{oc}} (s - s_{\alpha_{oc}})^{-1} \tag{11.10}$$

where the $a_{o_{oc}}$ term is absorbed into the α_{oc} index for convenience of notation. Note that the possible pole at $s = 0$ still needs special treatment. The $\tilde{Z}_{\alpha_{oc}}$ can be referred to as pole impedances.

For certain types of objects, in particular those with zero impedance at $s = 0$, one can write the impedance in another convenient form as

$$\tilde{Z}_a(s) = Z_o \sum_{\alpha_{oc}} a_{\alpha_{oc}} \left[(s - s_{\alpha_{oc}})^{-1} + s_{\alpha_{oc}}^{-1}\right] + \tilde{Z}'_{sce}(s)$$

$$= \sum_{\alpha_{oc}} \tilde{Z}'_{\alpha_{oc}}(s) + \tilde{Z}'_{sce}(s)$$

$$\tilde{Z}'_{\alpha_{oc}}(s) \equiv Z_o \, a_{\alpha_{oc}} \left[(s - s_{\alpha_{oc}})^{-1} + s_{\alpha_{oc}}^{-1}\right] = Z_o \frac{a_{\alpha_{oc}}}{s_{\alpha_{oc}}} \frac{s}{s - s_{\alpha_{oc}}}$$

$$= \tilde{Z}_{\alpha_{oc}}(s) + Z_o \, a_{\alpha_{oc}} s_{\alpha_{oc}}^{-1} \tag{11.11}$$

$$\tilde{Z}'_{sce}(s) = \tilde{Z}_{sce}(s) - Z_o \sum_{\alpha_{oc}} a_{\alpha_{oc}} s_{\alpha_{oc}}^{-1}$$

The $\tilde{Z}'_a$ are referred to as modified pole impedances, or zero subtracted pole impedances.

The modified pole impedances are appropriate for cases of a zero in the impedance at s = 0, say for a perfectly conducting loop. For a resistive loop with a finite non-zero impedance at s = 0 such a form is also useful. Such a modified pole impedance form is inappropriate, however, in the case of an object with an impedance pole at s = 0, such as a capacitive electric dipole.

B. EEM form

The eigenmode form of the impedance is derived from the surface current density on the object associated with the source current density in the gap as

$$\tilde{\vec{J}}_s(\vec{r},s) = \sum_{\beta_{oc}} \tilde{\lambda}_{\beta_{oc}}^{-1}(s)\ \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s)\ \frac{\left\langle \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s)\ ;\ \tilde{\vec{E}}_s(\vec{r},s)\right\rangle_a}{\left\langle \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s)\ ;\ \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s)\right\rangle_a} \tag{11.12}$$

where the eigenvalues, right eigenmodes, and left eigenmodes satisfy

$$\left\langle \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\ ;\ \tilde{\vec{R}}_{\beta_{oc}}(\vec{r}\,',s)\right\rangle_a = \tilde{\lambda}_{\beta_{oc}}(s)\ \tilde{\vec{R}}_{\beta_{oc}}(s)\ ,\ \vec{r}\varepsilon S_a$$

$$\left\langle \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s)\ ;\ \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\right\rangle_a = \tilde{\lambda}_{\beta_{oc}}(s)\ \tilde{\vec{L}}_{\beta_{oc}}(\vec{r}\,'s,)\ ,\ \vec{r}\,'\varepsilon S_a \tag{11.13}$$

where for symmetric operators we can set

$$\tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) \equiv \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) \tag{11.14}$$

and where the appropriate source electric field in equation 11.12 is given by equation 11.4. The response current density can then be written for $\vec{r}\varepsilon S_a$ as

$$\tilde{\vec{J}}_s(\vec{r},s) = -\sum_{\beta_{oc}} \tilde{\lambda}_{\beta_{oc}}^{-1}(s)\tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) \frac{\left\langle \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) ; \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) ; \tilde{\vec{J}}_{s_g}(\vec{r}\,',s)\right\rangle_g \right\rangle_a}{\left\langle \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) ; \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s)\right\rangle_a}$$

$$= -\tilde{I}(s) \sum_{\beta_{oc}} \tilde{\lambda}_{\beta_{oc}}^{-1}(s)\tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) \frac{\left\langle \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) ; \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) ; \vec{j}_{s_g}(\vec{r})\right\rangle_g \right\rangle_a}{\left\langle \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) ; \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s)\right\rangle_a}$$

(11.15)

The electric field at the gap is obtained from the current on the object plus the source current in the gap as

$$\tilde{\vec{E}}(\vec{r},s) = -\left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) ; \tilde{\vec{J}}_s(\vec{r}\,',s)\right\rangle_a - \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) ; \tilde{\vec{J}}_{s_g}(\vec{r}\,',s)\right\rangle_g$$

$$= \tilde{I}(s)\left\{\sum_{\beta_{oc}} \tilde{\lambda}_{\beta_{oc}}^{-1}(s) \frac{\left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) ; \tilde{\vec{R}}_{\beta_{oc}}(r',s)\right\rangle_a \left\langle \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) ; \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) ; \vec{j}_{s_g}(\vec{r}\,')\right\rangle_g\right.}{\left\langle \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) ; \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s)\right\rangle_a} - \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) ; \vec{j}_{s_g}(\vec{r}\,',s)\right\rangle_g \right\}$$

(11.16)

The impedance is then

$$\tilde{Z}_a(s) = -\frac{\tilde{V}(s)}{\tilde{I}(s)} = \frac{\left\langle \tilde{\vec{E}}(\vec{r},s) \; ; \; \vec{J}_{s_g}(\vec{r}) \right\rangle_g}{\tilde{I}(s)}$$

$$= \sum_{\beta_{oc}} \tilde{Z}_{\beta_{oc}}(s) + \tilde{Z}_g(s) \tag{11.17}$$

$$\tilde{Z}_{\beta_{oc}}(s) = \tilde{\lambda}^{-1}_{\beta_{oc}}(s) \frac{\left\langle \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) ; \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) ; \vec{J}_{s_g}(\vec{r}\,') \right\rangle_g \right\rangle_a \left\langle \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) ; \tilde{\vec{R}}_{\beta_{oc}}(\vec{r}\,',s) \right\rangle_a ; \vec{J}_{s_g}(\vec{r}) \right\rangle_g}{\left\langle \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) \; ; \; \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) \right\rangle_a}$$

$$= \tilde{\lambda}^{-1}_{\beta_{oc}}(s) \frac{\left\langle \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) ; \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) ; \vec{J}_{s_g}(\vec{r}\,') \right\rangle_g \right\rangle_a \left\langle \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) ; \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) ; \vec{J}_{s_g}(\vec{r}\,') \right\rangle_g \right\rangle_a}{\left\langle \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) \; ; \; \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) \right\rangle_a}$$

$$\tilde{Z}_g(s) = -\left\langle \vec{J}_{s_g}(\vec{r}) \; ; \; \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) \; ; \; \vec{J}_{s_g}(\vec{r}\,') \right\rangle_g$$

where the symmetric properties of $\tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s)$ have been used. Note the appearance of a gap impedance term $\tilde{Z}_g(s)$ associated with the electric field from the gap current. For symmetrical integral equation kernels $\tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)$ we have

$$\tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) \equiv \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s)$$

$$\tilde{Z}_{\beta_{oc}}(s) = \tilde{\lambda}^{-1}_{\beta_{oc}}(s) \frac{\left\langle \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) \; ; \; \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) \; ; \; \vec{J}_{s_g}(\vec{r}\,') \right\rangle_g \right\rangle_a^2}{\left\langle \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) \; ; \; \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) \right\rangle_a} \tag{11.18}$$

If the operator of the integral equation is an impedance operator as discussed in section IV.A then the eigenvalues $\tilde{\lambda}_{\beta_{oc}}(s)$ are eigenimpedances $\tilde{Z}_{\beta_{oc}}(s)$ with PR properties for cases of passive loading impedance.

Comparing the EEM and SEM forms of the impedance we first set

$$\alpha_{oc} = (\beta_{oc}, \beta'_{oc}) \tag{11.19}$$

and then expand the eigenmode associated impedances as

$$\tilde{Z}_{\beta_{oc}}(s) = \sum_{\beta'_{oc}} \tilde{Z}_{\beta_{oc},\beta'_{oc}}(s) + \tilde{Z}_{\beta_{oc},sce}(s) \tag{11.20}$$

where $\tilde{Z}_{\beta_{oc},sce}(s)$ accounts for any additional entire function resulting from expanding the eigenvalue associated impedances $\tilde{Z}_{\beta_{oc}}(s)$ in terms of the associated pole impedances $\tilde{Z}_{\beta_{oc},\beta'_{oc}}(s)$. Note that we have

$$\tilde{Z}_{\beta_{sc},\beta'_{sc}}(s) = Z_o a_{\beta_{oc},\beta'_{oc}} (s - s_{\beta_{oc},\beta'_{oc}})^{-1} \tag{11.21}$$

$$a_{\beta_{oc},\beta'_{oc}} = \frac{1}{Z_o} \frac{\left\langle \vec{\mu}_{\beta_{oc},\beta'_{oc}}(\vec{r}) ; \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s_{\beta_{oc},\beta'_{oc}}) ; \vec{J}_{s_g}(\vec{r}\,') \right\rangle_g \right\rangle_a \left\langle \vec{\nu}^{(\vec{J}_s)}_{\beta_{oc},\beta'_{oc}}(\vec{r}) ; \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s_{\beta_{oc},\beta'_{oc}}) ; \vec{J}_{s_g}(\vec{r}) \right\rangle_g \right\rangle_a}{\left\langle \vec{\mu}_{\beta_{oc},\beta'_{oc}}(\vec{r}) ; \frac{\partial}{\partial s} \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s) \Big|_{s=s_{\beta_{oc},\beta'_{oc}}} ; \vec{\nu}^{(\vec{J}_s)}_{\beta_{oc},\beta'_{oc}}(\vec{r}\,') \right\rangle_a}$$

$$= \frac{1}{Z_o} \left[\frac{\partial}{\partial s} \tilde{\lambda}_{\beta_{oc}}(s) \Big|_{s=s_{\beta_{oc},\beta'_{oc}}} \right]^{-1} \frac{\left\langle \vec{\mu}_{\beta_{oc},\beta'_{oc}}(\vec{r}) ; \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s_{\beta_{oc},\beta'_{oc}}) ; \vec{J}_{s_g}(\vec{r}) \right\rangle_g \right\rangle_a \left\langle \vec{\nu}^{(\vec{J}_s)}_{\beta_{oc},\beta'_{oc}}(\vec{r}) ; \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s_{\beta_{oc},\beta'_{oc}}) ; \vec{J}_{s_g}(\vec{r}\,') \right\rangle_g \right\rangle_a}{\left\langle \vec{\mu}_{\beta_{oc},\beta'_{oc}}(\vec{r}) ; \vec{\nu}^{(\vec{J}_s)}_{\beta_{oc},\beta'_{oc}}(\vec{r}) \right\rangle}$$

where the kernel derivative is converted to the eigenvalue derivative (with respect to s) as discussed in a previous note.[8]

The eigenvalue associated impedances can also be expanded in terms of modified pole impedances $\tilde{Z}'_{\beta_{oc},\beta'_{oc}}(s)$ as

$$\tilde{Z}_{\beta_{oc}}(s) = \sum_{\beta'_{oc}} \tilde{Z}'_{\beta_{oc},\beta'_{oc}}(s) + \tilde{Z}_{\beta_{oc},sce}(s)$$

$$\tilde{Z}'_{\beta_{oc},\beta'_{oc}}(s) = Z_o\, a_{\beta_{oc},\beta'_{oc}} \left[(s - s_{\beta_{oc},\beta'_{oc}})^{-1} + s^{-1}_{\beta_{oc},\beta'_{oc}} \right]$$

$$= Z_o \frac{a_{\beta_{oc},\beta'_{oc}}}{s_{\beta_{oc},\beta'_{oc}}} \frac{s}{s - s_{\beta_{oc},\beta'_{oc}}} \qquad (11.22)$$

$$\tilde{Z}'_{\beta_{oc},sce}(s) = \tilde{Z}_{\beta_{oc},sce}(s) - Z_o \sum_{\beta'_{oc}} a_{\beta_{oc},\beta'_{oc}}\, s^{-1}_{\beta_{oc},\beta'_{oc}}$$

This form is appropriate for cases that $\tilde{Z}_{\beta_{oc}}(s)$ has no pole at $s = 0$.

Recall the appearance of the gap impedance term $\tilde{Z}_g(s)$ in equations 11.17 this is considered as an extra term in the eigenmode expansion. Near $s = 0$ it can have a significant effect. Recalling the problem discussed earlier of no current crossing the gap ends for each of the natural modes or eigenmodes, appropriate limits of the current on each side of the gap "ends" need to be used to have the solution correspond to the physical situation. Specifically one should avoid a capacitive open circuit in series with the gap so that the correct low frequency impedance (an inductance or resistance in the case of a loop) is attainable. One might leave $\tilde{Z}_g(s)$ in its present form, or perhaps apportion it amongst the eigenvalue associated impedances.

XII. Open Circuit Voltage

In the presence of an incident field with no current in the gap (no loading and no source current in the gap) let us now consider the open circuit voltage at the port (gap). As before consider an incident wave of the general form specified in section IV as

$$\tilde{\vec{E}}_s(\vec{r},s) = \tilde{\vec{E}}_{inc}(\vec{r},s) = E_o \sum_p \tilde{f}_p(s)\tilde{\vec{\delta}}_p(\vec{r},s) \tag{12.1}$$

where again $\tilde{f}_p(s)$ is the waveform of the pth wave with spatial dependence contained in $\tilde{\vec{\delta}}_p(\vec{r},s)$ (typically a delta-function plane wave), and E_o is a normalizing constant.

The open circuit voltage and impedance are compared in that they have different coupling coefficients or eigenmode coefficients for the same mode sets. This allows one to combine the impedance and open circuit voltage in a single circuit representation.

A. SEM form

The surface current density on the object (assuming first order poles) is

$$\tilde{\vec{J}}_s(\vec{r},s) = \frac{E_o}{Z_o} \sum_p \tilde{f}_p(s) \left\{ \sum_{\alpha_{oc}} \tilde{\eta}_{\alpha_{oc},p} \vec{\nu}_{\alpha_{oc}}^{(\vec{J}_s)}(\vec{r})(s - s_{\alpha_{oc}})^{-1} + \text{entire function} \right\} \tag{12.2}$$

The natural frequencies, modes, and coupling vectors are the same as in the impedance calculations (equations 11.2 and 11.3). The coupling coefficients for the case of incident field with open gap are

$$\tilde{\eta}_{\alpha_{oc},p} = Z_o \frac{\left\langle \vec{\mu}_{\alpha_{oc}}(\vec{r}) \; ; \; \tilde{\vec{\delta}}_p(\vec{r}, s_{\alpha_{oc}}) \right\rangle_a}{\left\langle \vec{\mu}_{\alpha_{oc}}(\vec{r}) \; ; \; \frac{\partial}{\partial s} \tilde{\vec{\Gamma}}(\vec{r}, \vec{r}\,'; s) \Big|_{s=s_{\alpha_{oc}}} \; ; \; \vec{\nu}^{(\vec{J}_s)}_{\alpha_{oc}}(\vec{r}) \right\rangle_a} \tag{12.3}$$

As discussed in section III the open circuit voltage is found as an appropriate average over the electric field as

$$\tilde{V}_{oc}(s) = -\left\langle \tilde{\vec{E}}(\vec{r}, s) \; ; \; \vec{j}_{s_g}(\vec{r}) \right\rangle_g \tag{12.4}$$

where the weighting function $\vec{j}_{s_g}$ is chosen the same as for the source current density in the gap for impedance calculations. Then we need the electric field in the gap which we obtain from

$$\tilde{\vec{E}}(\vec{r}, s) = -\left\langle \tilde{\vec{Z}}(\vec{r}, \vec{r}\,'; s) \; ; \; \tilde{\vec{J}}_s(\vec{r}\,', s) \right\rangle + \tilde{\vec{E}}_s(\vec{r}, s) \tag{12.5}$$

giving

$$\tilde{\vec{E}}(\vec{r}, s) = \frac{E_o}{Z_o} \sum_p \tilde{f}_p(s) \left\{ \sum_{\alpha_{oc}} -\tilde{\eta}_{\alpha_{oc},p} \left\langle \tilde{\vec{Z}}(\vec{r}, \vec{r}\,'; s_{\alpha_{oc}}) \; ; \; \vec{\nu}^{(\vec{J}_s)}_{\alpha_{oc}}(\vec{r}\,') \right\rangle_a (s - s_{\alpha_{oc}})^{-1} \right.$$
$$\left. + \text{ entire function} \right\} \tag{12.6}$$

where the $\tilde{\vec{E}}_s$ term is part of the entire function

The open circuit voltage is now

$$\tilde{V}_{oc}(s) = \sum_p \tilde{f}_p(s) \left\{ E_o \sum_{\alpha_{oc}} b_{\alpha_{oc},p} (s - s_{\alpha_{oc}})^{-1} + \tilde{V}_{oce,p}(s) \right\}$$
$$= \sum_p \tilde{f}_p(s) \left\{ Z_o \sum_{\alpha_{oc}} I_{\alpha_{oc},p} a_{\alpha_{oc}} (s - s_{\alpha_{oc}})^{-1} + \tilde{I}_{oce,p}(s) \, \tilde{Z}_{oce}(s) \right\} \tag{12.7}$$

where $\tilde{V}_{oce,p}(s)$ denotes a set of entire functions for the open circuit voltage. The normalized open circuit voltage residues are

$$b_{\alpha_{oc},p} = \frac{1}{Z_o} \tilde{\eta}_{\alpha_{oc},p} \left\langle\left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s_{\alpha_{oc}}) \; ; \; \vec{\nu}_{\alpha_{oc}}^{(\vec{J}_s)}(\vec{r}\,') \right\rangle_a \; ; \; \vec{j}_{s_g}(\vec{r}) \right\rangle_g$$

$$= \frac{\left\langle \vec{\mu}_{\alpha_{oc}}(\vec{r}) ; \tilde{\vec{\vec{\delta}}}_p(\vec{r},s_{\alpha_{oc}}) \right\rangle_a \left\langle \vec{\nu}_{\alpha_{oc}}^{(\vec{J}_s)}(\vec{r}) ; \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s_{\alpha_{oc}}) ; \vec{j}_{s_g}(\vec{r}\,') \right\rangle_g \right\rangle_a}{\left\langle \vec{\mu}_{\alpha_{oc}}(\vec{r}) ; \frac{\partial}{\partial s} \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s) \Big|_{s=s_{\alpha_{oc}}} ; \vec{\nu}_{\alpha_{oc}}^{(\vec{J}_s)}(\vec{r}) \right\rangle_a} \tag{12.8}$$

The normalized open circuit voltage residues are written in the form

$$E_o b_{\alpha_{oc},p} = Z_o I_{\alpha_{oc},p} a_{\alpha_{oc}} \tag{12.9}$$

where $a_{\alpha_{oc}}$ is the normalized impedance residue introduced in section XI. This defines the current source coefficients $I_{\alpha_{oc},p}$ as

$$I_{\alpha_{oc},p} = E_o \frac{\left\langle \vec{\mu}_{\alpha_{oc}}'(\vec{r}) \; ; \; \tilde{\vec{\vec{\delta}}}_p(\vec{r},s_{\alpha_{oc}}) \right\rangle_a}{\left\langle \vec{\mu}_{\alpha_{oc}}(\vec{r}) \; ; \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s_{\alpha_{oc}}) \; ; \; \vec{j}_{s_g}(\vec{r}\,') \right\rangle_g \right\rangle_a} \tag{12.10}$$

Note that $\tilde{\vec{\vec{Z}}}$ integrated over S_g has dimensions of ohms, $\tilde{\vec{\vec{\delta}}}_p$ is dimensionless and $\vec{j}_{s_g}$ has dimensions of m^{-1}, giving $I_{\alpha_{oc},p}$ dimensions of amperes as required. Also note that $I_{\alpha_{oc},p}$ is simpler in form than $a_{\alpha_{oc}}$ and $b_{\alpha_{oc},p}$. The integrals over the derivative of the integral equation kernel cancel as does the integral used in defining the response current. Only the integrals over the source electric field from the incident wave and from the gap source current remain. An additional elementary current source coefficient is that associated with the entire function as

$$\tilde{I}_{oce,p}(s) = \frac{\tilde{V}_{oce,p}(s)}{\tilde{Z}_{oce}(s)} \tag{12.11}$$

assuming the denominator is not identically zero. The entire function current source coefficient is here written as a function of s for generality. However, for some purposes it is later approximated as a constant when considering finite expansions for circuit approximations.

In terms of the pole impedances the open circuit voltage is written as

$$\tilde{V}_{oc}(s) = \sum_{\alpha_{oc}} \left\{ \sum_p I_{\alpha_{oc},p} \; \tilde{f}_p(s) \right\} \tilde{Z}_{\alpha_{oc}}(s) + \left\{ \sum_p \tilde{I}_{sce,p}(s) \; \tilde{f}_p(s) \right\} \tilde{Z}_{sce}(s) \tag{12.12}$$

Here $\sum_p I_{\alpha_{oc},p}\tilde{f}_p(s)$ can be interpreted as a current source or a parallel combination of current sources in parallel with the pole impedance $\tilde{Z}_{\alpha_{oc}}(s)$. Such a representation gives both the impedance and open circuit voltage.

For the case of an open circuit natural frequency $s_{\alpha_{oc}}$ (an impedance pole) at $s = 0$ the corresponding current source coefficients $I_{o,p}$ are in general zero, as long as electric and magnetic fields in the incident wave are bounded for $s \to 0$.

In terms of modified pole impedances one can also obtain the open circuit voltage. Rearranging equation 12.7 gives

$$\begin{aligned} \tilde{V}_{oc}(s) &= \sum_p \tilde{f}_p(s) \left\{ E_o \sum_{\alpha_{oc}} b_{\alpha_{oc},p} \left[(s - s_{\alpha_{oc}})^{-1} + s_{\alpha_{oc}}^{-1} \right] + \tilde{V}'_{oce,p}(s) \right\} \\ &= \sum_p \tilde{f}_p(s) \left\{ Z_o \sum_{\alpha_{oc}} I_{\alpha_{oc},p} a_{\alpha_{oc}} \left[(s - s_{\alpha_{oc}})^{-1} + s_{\alpha_{oc}}^{-1} \right] \right. \\ &\quad \left. + \tilde{I}'_{sce,p}(s) \tilde{Z}'_{oce}(s) \right\} \end{aligned} \tag{12.13}$$

The current source coefficient associated with the explicit entire function in this new form is

$$\tilde{I}'_{oce,p}(s) = \frac{\tilde{V}'_{oce,p}(s)}{\tilde{Z}'_{oce}(s)} \tag{12.14}$$

The modified entire functions can be written in terms of the original ones as

$$\begin{aligned}
\tilde{V}'_{oce,p}(s) &= \tilde{V}_{oce,p}(s) - E_o \sum_{\alpha_{oc}} b_{\alpha_{oc},p} \, s_{\alpha_{oc}}^{-1} \\
&= \tilde{I}_{oce,p}(s) \, \tilde{Z}_{oce}(s) - Z_o \sum_{\alpha_{oc}} I_{\alpha_{oc},p} \, a_{\alpha_{oc}} \, s_{\alpha_{oc}}^{-1} \\
&= \tilde{I}'_{oce,p}(s) \, \tilde{Z}'_{oce}(s) \\
&= \tilde{I}'_{oce,p}(s) \left\{ \tilde{Z}_{oce}(s) - Z_o \sum_{\alpha_{oc}} a_{\alpha_{oc}} \, s_{\alpha_{oc}}^{-1} \right\}
\end{aligned} \tag{12.15}$$

Hence the current source coefficient for the modified entire function is related to the unmodified one as

$$\tilde{I}'_{oce,p}(s) = \frac{\tilde{I}_{oce,p}(s) \, \tilde{Z}_{oce}(s) - Z_o \sum_{\alpha_{oc}} I_{\alpha_{oc},p} \, a_{\alpha_{oc}} \, s_{\alpha_{oc}}^{-1}}{\tilde{Z}_{oce}(s) - Z_o \sum_{\alpha_{oc}} a_{\alpha_{oc}} \, s_{\alpha_{oc}}^{-1}} \tag{12.16}$$

assuming the denominator is not identically zero.

In terms of the modified pole impedances the open circuit voltage is written

$$\tilde{V}_{oc}(s) = \sum_{\alpha_{oc}} \left\{ I_{\alpha_{oc},p} \, \tilde{f}_p(s) \right\} \tilde{Z}'_{\alpha_{oc}}(s)$$

$$+ \left\{ \sum_p I'_{oce,p}(s) \, \tilde{f}_p(s) \right\} \tilde{Z}'_{oce}(s) \tag{12.17}$$

Just as in equation 12.12 where unmodified pole impedances are used the same current sources $\sum_p I_{\alpha_{oc},p}\tilde{f}_p(s)$ appear, except in parallel with (multiply) the modified pole impedances $\tilde{Z}'_{\alpha_{oc}}(s)$. Series combinations of such current sources in parallel with modified pole impedances can also be used to construct equivalent circuits along with modified entire function terms.

B. EEM form

The eigenmode expansion of the surface current density on the object associated with the incident field is

$$\tilde{\vec{J}}_s(\vec{r},s) = E_o \sum_p \tilde{f}_p(s) \left\{ \sum_{\beta_{oc}} \tilde{\lambda}^{-1}_{\beta_{oc}}(s) \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) \frac{\left\langle \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) \,;\, \tilde{\vec{\delta}}_p(\vec{r},s) \right\rangle_a}{\left\langle \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) \,;\, \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) \right\rangle_a} \right\} \tag{12.18}$$

The resulting electric field at the open gap is

$$\tilde{\vec{E}}(\vec{r},s) = -\left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) \,;\, \tilde{\vec{J}}_s(\vec{r},s) \right\rangle_a + \tilde{\vec{E}}_s(\vec{r},s)$$

$$= E_o \sum_p \tilde{f}_p(s) \left\{ \sum_{\beta_{oc}} -\tilde{\lambda}^{-1}_{\beta_{oc}} \frac{\left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s) ; \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) \right\rangle_a \left\langle \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) ; \tilde{\vec{\delta}}_p(\vec{r},s) \right\rangle_a}{\left\langle \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) ; \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) \right\rangle_a} + \tilde{\vec{\delta}}_p(\vec{r},s) \right\} \tag{12.19}$$

The open circuit voltage is

$$\tilde{V}_{oc}(s) = -\left\langle \tilde{\vec{E}}(\vec{r},s) \,;\, \vec{J}_{s_g}(\vec{r}) \right\rangle_g$$

$$= E_o \sum_p \tilde{f}_p(s) \left\{ \sum_{\beta_{oc}} \tilde{\lambda}^{-1}_{\beta_{oc}}(s) \frac{\left\langle \tilde{\vec{L}}_{\beta_{oc}}(\vec{r}) \,;\, \tilde{\vec{\delta}}_p(\vec{r},s) \right\rangle_a \left\langle\left\langle \tilde{\vec{\vec{Z}}}(r,r';s) \cdot \tilde{\vec{R}}_{\beta_{oc}}(\vec{r}',s) \right\rangle_a ; \vec{J}_{s_g}(\vec{r}') \right\rangle_g}{\left\langle \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) \,;\, \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) \right\rangle_a} - \left\langle \tilde{\vec{\delta}}_p(\vec{r},s) \,;\, \vec{J}_{s_g}(\vec{r}) \right\rangle_g \right\}$$

$$= E_o \sum_p \tilde{f}_p(s) \left\{ \sum_{\beta_{oc}} \tilde{\lambda}^{-1}_{\beta_{oc}}(s) \frac{\left\langle \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) \,;\, \tilde{\vec{\delta}}_p(\vec{r},s) \right\rangle_a \left\langle \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) \,;\, \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}';s) \cdot \vec{J}_{s_g}(\vec{r}') \right\rangle_g \right\rangle_a}{\left\langle \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s) \,;\, \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) \right\rangle_a} - \left\langle \tilde{\vec{\delta}}_p(\vec{r},s) \,;\, \vec{J}_{s_g}(\vec{r}) \right\rangle_g \right\}$$

$$= \sum_{\beta_{oc}} \left\{ \sum_p \tilde{I}_{\beta_{oc},p}(s) \tilde{f}_p(s) \right\} \tilde{Z}_{\beta_{oc}}(s) + \left\{ \sum_p \tilde{I}_{g,p}(s) \tilde{f}_p(s) \right\} \tilde{Z}_g(s)$$

(12.20)

$$\tilde{I}_{\beta_{oc},p}(s) = E_o \frac{\left\langle \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) \,;\, \tilde{\vec{\delta}}_p(\vec{r},s) \right\rangle_a}{\left\langle \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s) \,;\, \left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}';s) \cdot \vec{J}_{s_g}(\vec{r}') \right\rangle_g \right\rangle_a}$$

$$\tilde{I}_{g,p}(s) = E_o \frac{\left\langle \tilde{\vec{\delta}}_p(\vec{r},s) \,;\, \vec{J}_{s_g}(\vec{r}) \right\rangle_g}{\left\langle \vec{J}_{s_g}(\vec{r}) \cdot \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}';s) \cdot \vec{J}_{s_g}(\vec{r}') \right\rangle_g}$$

where $\tilde{Z}_{\beta_{oc}}(s)$ is the impedance associated with each open circuit eigenmode as discussed in section XI. The corresponding current cource coefficients are $\tilde{I}_{\beta_{oc},p}(s)$; these together with the incident waveforms $\tilde{f}_p(s)$ convert the impedance associated with each eigenmode into a corresponding open circuit voltage. Note the presence of a "gap" current source with coefficients $\tilde{I}_{g,p}(s)$ and associated impedance $\tilde{Z}_g(s)$. In a circuit sense $\sum_p \tilde{I}_{\beta_{oc},p}(s)\tilde{f}_p(s)$ can be interpreted as a current source or parallel combination of current sources placed in parallel with the eigenmode associated impedance $\tilde{Z}_{\beta_{oc}}(s)$. The gap current source $\sum_p \tilde{I}_{g,p}(s)\tilde{f}_p(s)$ and impedance $Z_g(s)$ fit into the circuit in the same manner. Note that if $\tilde{Z}_g(s)$ has a pole at $s = 0$ then $\tilde{I}_{g,p}(s)$ has a zero there (for bounded incident fields at $s = 0$).

In relating the EEM form of the open circuit voltage to the SEM form the current source coefficients relate at the natural frequencies as

$$\tilde{I}_{\beta_{oc},p}(s_{\beta_{oc},\beta'_{oc}}) = I_{\beta_{oc},\beta'_{oc},p} \tag{12.21}$$

The gap current source coefficients are more problematical in this regard. Note that wavelengths are generally assumed large compared to the gap.

The individual eigenmode expansion terms for the open circuit voltage can be expanded as

$$\begin{aligned}
E_o \sum_p \tilde{f}_p(s)\tilde{\lambda}^{-1}_{\beta_{oc}}(s) & \frac{\left\langle \tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s);\tilde{\vec{\delta}}_p(\vec{r},s)\right\rangle_a \left\langle \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s);\left\langle \tilde{\vec{\vec{Z}}}(\vec{r},\vec{r}\,';s);\vec{J}_{s_g}(\vec{r}\,')\right\rangle_g\right\rangle_a}{\left\langle \tilde{\vec{R}}_{\beta_{oc}}(\vec{r},s);\tilde{\vec{L}}_{\beta_{oc}}(\vec{r},s)\right\rangle_a} \\
&= \sum_{\beta'_{oc}} \left\{\sum_p I_{\beta_{oc},\beta'_{oc},p}\tilde{f}_p(s)\right\} \tilde{Z}_{\beta_{sc},\beta'_{sc}}(s) \\
&\quad + \left\{\sum_p \tilde{I}_{\beta_{oc},sce,p}(s)\tilde{f}_p(s)\right\}\tilde{Z}_{\beta_{sc},oce}(s) \\
&= \sum_{\beta'_{oc}} \left\{\sum_p I_{\beta_{oc},\beta'_{oc},p}\tilde{f}_p(s)\right\} \tilde{Z}'_{\beta_{sc},\beta'_{sc}}(s) \\
&\quad + \left\{\sum_p \tilde{I}'_{\beta_{sc},sce,p}(s)\ \tilde{f}_p(s)\right\}\tilde{Z}'_{\beta_{sc},oce}(s)
\end{aligned} \tag{12.22}$$

where the pole impedances $\tilde{Z}_{\beta_{sc},oce}(s)$ are given by equations 11.21 and the modified pole impedances $\tilde{Z}'_{\beta_{sc},\beta'_{sc}}(s)$ are given by equations 11.22 with equations 11.21.

The entire function portions of the eigenmode expansion terms for the current sources are given by

$$
\begin{aligned}
\tilde{I}_{\beta_{oc},oce,p}(s) &= \frac{\tilde{V}_{\beta_{oc},oce,p}(s)}{\tilde{Z}_{\beta_{oc},oce}(s)} \\
\tilde{I}'_{\beta_{oc},oce,p}(s) &= \frac{\tilde{V}'_{\beta_{oc},oce,p}(s)}{\tilde{Z}_{\beta_{oc},oce}(s)}
\end{aligned}
\tag{12.23}
$$

Here $\tilde{V}_{\beta_{oc},oce,p}(s)$ is the possible entire function associated with a particular term (denoted by β_{oc}) in the eigenmode expansion; it corresponds to $\tilde{V}_{oce,p}(s)$ in equations 12.7. Similarly $\tilde{V}'_{\beta_{oc},oce,p}(s)$ is the possible remaining entire function after an eigenmode term has been expanded in terms of modified pole impedances; it corresponds to $\tilde{V}'_{oce,p}(s)$ in equations 12.13.

The entire function current source for the eigenmode expansion terms can be related for modified and unmodified pole impedance terms through

$$
\begin{aligned}
\tilde{V}'_{\beta_{oc},oce,p}(s) &= \tilde{V}_{\beta_{oc},oce,p}(s) - E_o \sum_{\beta'_{oc}} b_{\beta_{oc},\beta'_{oc},p} s^{-1}_{\beta_{oc},\beta'_{oc}} \\
&= \tilde{I}_{\beta_{oc},oce,p}(s)\tilde{Z}_{\beta_{oc},oce}(s) \\
&\qquad - Z_o \sum_{\beta'_{oc}} I_{\beta_{oc},\beta'_{oc},p} a_{\beta_{oc},\beta'_{oc}} s^{-1}_{\beta_{oc},\beta'_{oc}} \\
&= \tilde{I}'_{\beta_{oc},oce,p}(s)\tilde{Z}'_{\beta_{oc},oce}(s) \\
&= \tilde{I}'_{\beta_{oc},oce,p}(s)\left\{\tilde{Z}_{\beta_{oc},oce}(s) - Z_o \sum_{\beta'_{oc}} a_{\beta_{oc},\beta'_{oc}} s^{-1}_{\beta_{oc},\beta'_{oc}}\right\}
\end{aligned}
\tag{12.24}
$$

or

$$\tilde{I}'_{\beta_{oc},oce,p}(s) = \frac{\tilde{I}_{\beta_{oc},oce,p}(s)\tilde{Z}_{\beta_{oc},oce}(s) - Z_o \sum_{\beta'_{oc}} I_{\beta_{oc},\beta'_{oc},p} a_{\beta_{oc},\beta'_{oc}} s^{-1}_{\beta_{oc},\beta'_{oc}}}{\tilde{Z}_{\beta_{oc},oce}(s) - Z_o \sum_{\beta'_{oc}} a_{\beta_{oc},\beta'_{oc}} s^{-1}_{\beta_{oc},\beta'_{oc}}} \tag{12.25}$$

In comparing the EEM to the SEM form for the open circuit voltage one should consider the contribution of the additional "gap" term in the EEM solution. In the SEM solution this term is naturally included in the form of any pole at $s = 0$, or entire function, it may contain. In the EEM solution it is a separate term but should be considered when expanding the EEM solution in SEM form so that all the contributions are present.

XIII. Equivalent Circuits for Impedance and Open Circuit Voltage

From the discussion of the previous two sections let us now consider the equivalent circuit at the gap for the open circuit boundary value problem. This equivalent circuit gives both the open circuit voltage and impedance based on a series combination of parallel impedances with current sources. The individual open circuit pole modules give what can be referred to as elementary circuit modules with formulas for individual circuit elements. Conjugate pairs of pole circuits can be combined as well to form circuits with real valued circuit elements (inductances, capacitances, and resistances).

As in the case of the short-circuit based equivalent circuits, those of the open-circuit type still have the problem of realizability of the individual impedance elements (resistors, inductors, capacitors). The element values may be complex numbers and their nearness to being positive real numbers may be of significant interest.

A. EEM form

The form of the equivalent circuit based on the eigenmode expansion is shown in figure 13.1. It consists of an infinite set of series subcircuits, each subcircuit being associated with a different eigenmode with index β_{oc}, or with the gap term with subscript g which might be considered to correspond to a special value of β_{oc} (say $\beta_{oc} = 0$). Each subcircuit is a parallel combination of an impedance, $\tilde{Z}_{\beta_{oc}}(s)$ or $\tilde{Z}_g(s)$, and a current source, $\sum_p \tilde{I}_{\beta_{oc},p}(s)\tilde{f}_p(s)$ or $\sum_p \tilde{I}_{g,p}(s)\tilde{f}_p(s)$, combining the eigenmode current source coefficients $\tilde{I}_{\beta_{oc},p}(s)$ with the incident waveforms $\tilde{f}_p(s)$. The current source coefficients are in general frequency dependent and the associated impedances are complicated circuits since they have a number of poles in each such impedance.

In the circuit of figure 13.1, after the gap portion, the subcircuits are ordered with $\beta_{oc} = 1$ nearest the port. The

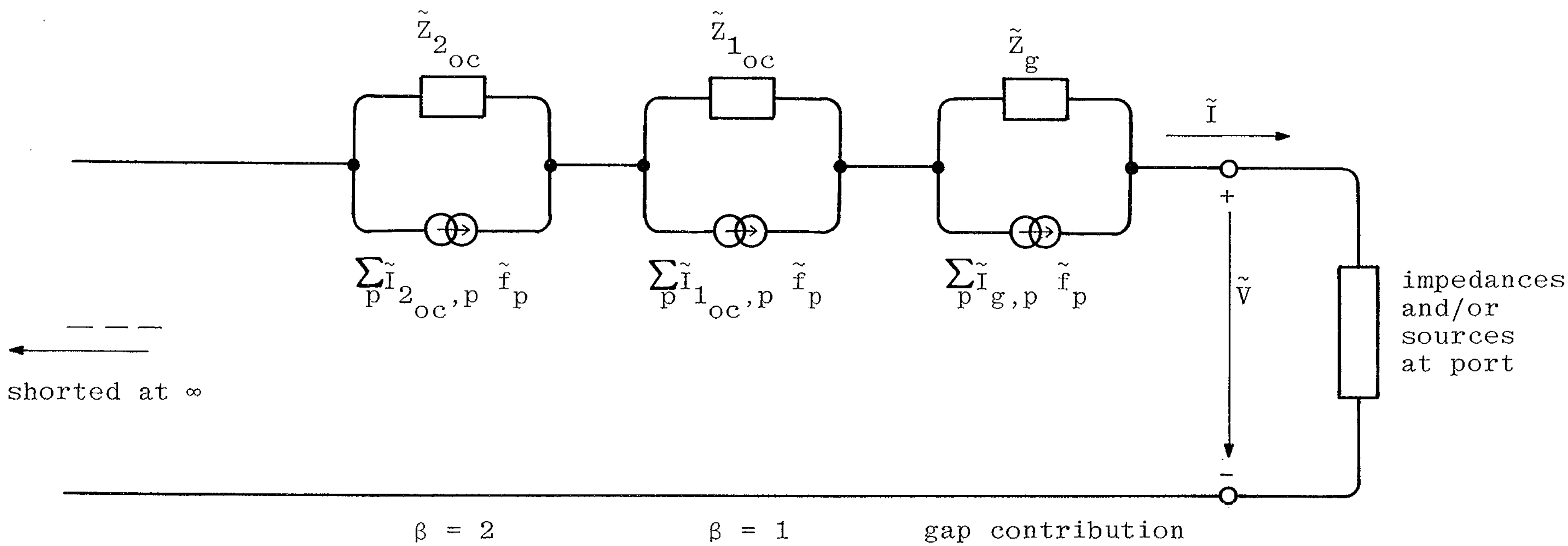


Figure 13.1. Equivalent Circuit for Impedance and Open Circuit Voltage Based on Eigenmode Expansion

corresponding eigenvalue is taken as that with its zeros (natural frequencies) clustered nearest $s = 0$ in some sense in the complex frequency plane.

In approximating this type of circuit with series modules, note that after truncating the series the circuit must be closed (shorted) to complete the current path to the load.

B. SEM form

The pole impedances and current sources are included in the equivalent circuit in figure 13.2. The gap impedance and current source is also listed for completeness, although one may prefer to only include the pole impedances associated with this term (such as at $s = 0$) and include any entire function contribution with entire functions from other eigenimpedance terms. Corresponding to each impedance pole there is an impedance $\tilde{Z}_{\alpha_{oc}}(s)$ and a current source $\sum_p I_{\alpha_{oc},p}\tilde{f}_p(s)$. The current source coefficients $I_{\alpha_{oc},p}$ are scalar constants (in general complex); the source waveforms are $\tilde{f}_p(s)$ in complex frequency domain and the current sources can be directly represented in time domain by use of $f_p(t)$. A specific pole is listed by $(\beta_{oc},\beta'_{oc}) = (3,1)$, for example; in this case an additional subscript of oc is added for clarity.

The circuit modules corresponding to individual poles are grouped together according to which open circuit eigenvalue they belong. If one superimposes a complex frequency plane on figure 13.2 with the origin at the port location, then the circuit modules can be envisioned as at the pole locations $s_{\beta_{oc},\beta'_{oc}}$ in this s plane.

The entire function contributions associated with each eigenvalue are also indicated by the parallel combination of the impedance $\tilde{Z}_{\beta,oce}(s)$ with the current source $\sum_p \tilde{I}_{\beta_{oc},oce,p}(s)\tilde{f}_p(s)$. These are grouped in their appropriate eigenvalue arcs. Alternatively one might more conveniently group all the entire function contributions together in a single circuit module with impedance $\tilde{Z}_{oce}(s)$ and current source $\sum_p \tilde{I}_{oce,p}(s)\tilde{f}_p(s)$. For this case collect

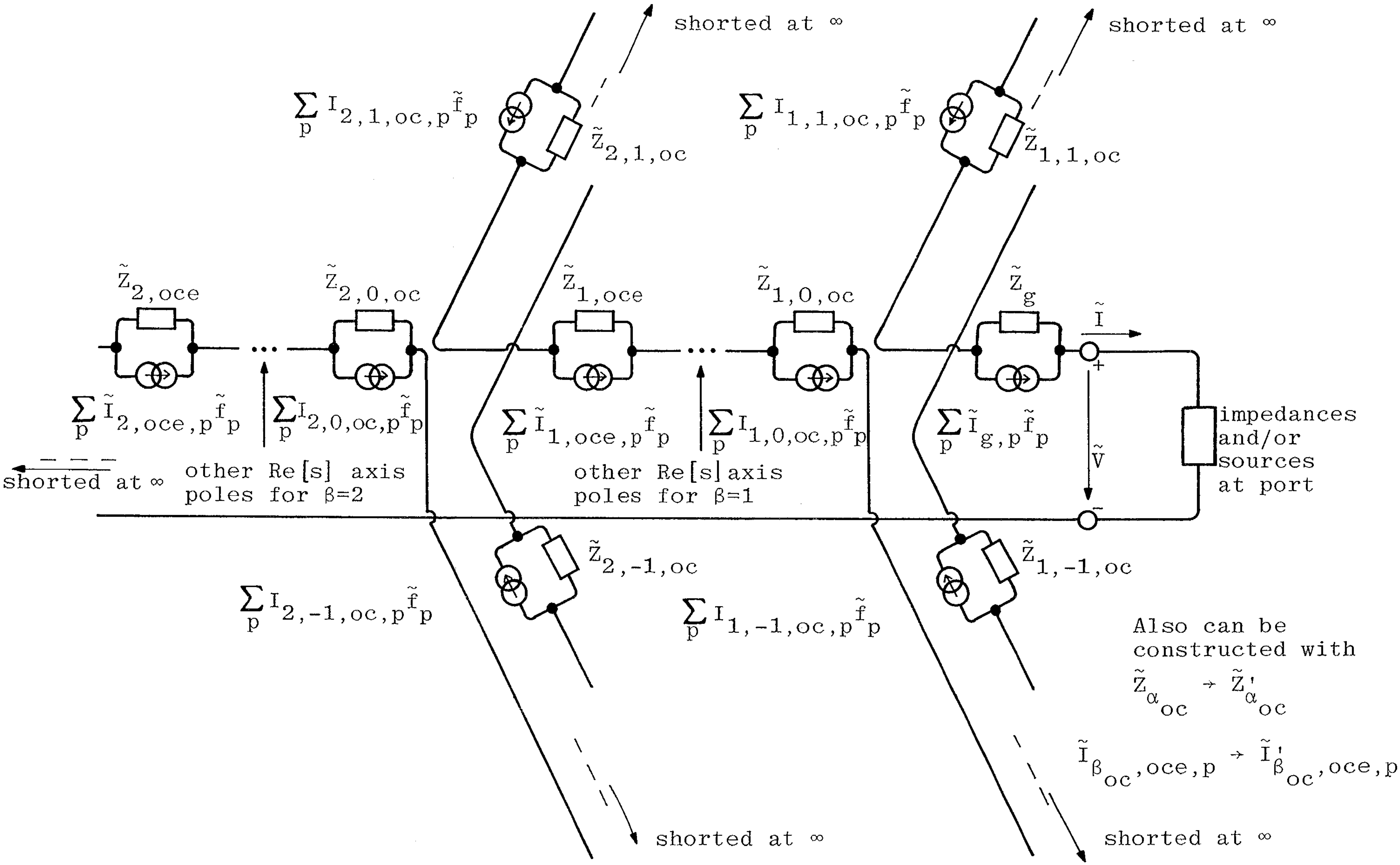


Figure 13.2. Equivalent Circuit for Impedance and Open Circuit Voltage Based on Singularity Expansion

the entire function circuit modules into one such module located near the port terminals.

This particular circuit format can also be used with the "modified" circuit quantities by replacing the pole impedances by modified pole impedances $\tilde{Z}'_{\alpha_{oc}}(s)$, and by substituting the modified entire function impedances and current sources for the unmodified ones.

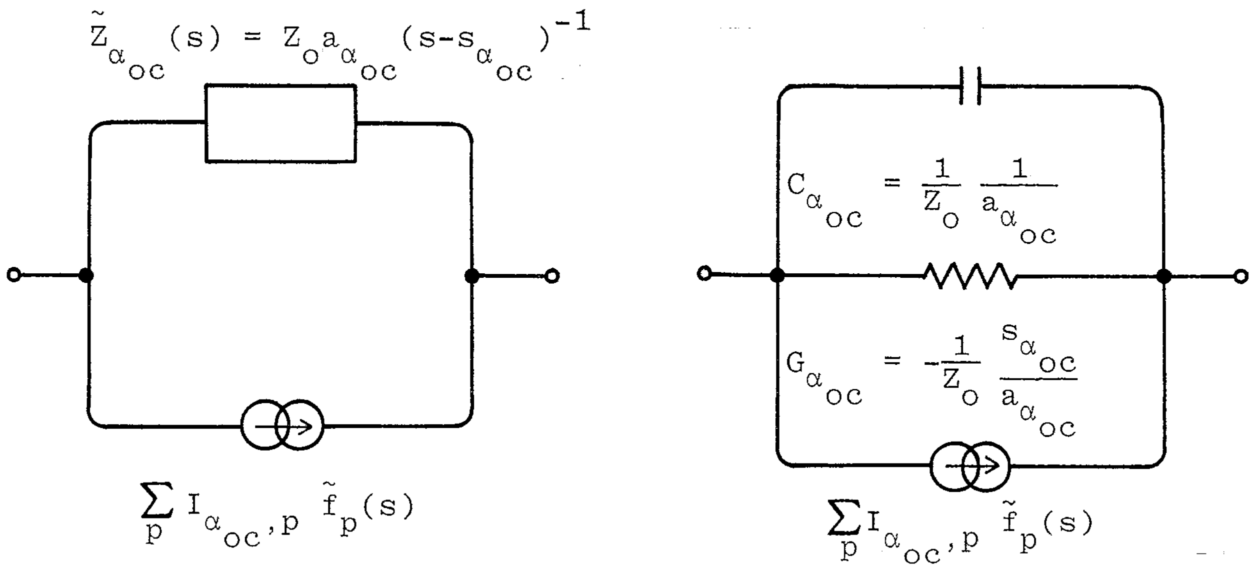
Again note that at the truncation of the sets of circuit modules associated with each eigenvalue, as well as with the entire circuit, the current path must be closed (shorted).

C. Elementary circuit modules

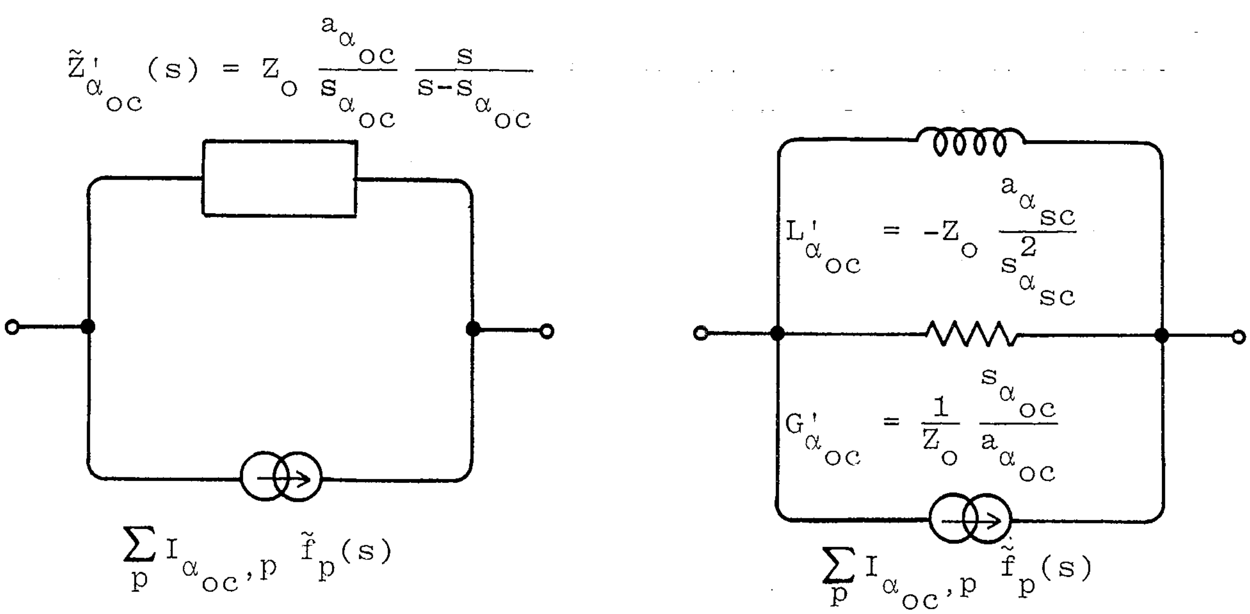
The pole impedances are written as

$$\begin{aligned}
\tilde{Z}_{\alpha_{oc}}(s) &= Z_o a_{\alpha_{oc}} (s - s_\alpha)^{-1} \\
&= \left[sC_{\alpha_{oc}} + G_{\alpha_{oc}} \right]^{-1} \\
C_{\alpha_{oc}} &= \frac{1}{Z_o a_{\alpha_{oc}}} \\
G_{\alpha_{oc}} &= -\frac{s_{\alpha_{oc}}}{Z_o a_{\alpha_{oc}}} = -s_{\alpha_{oc}} C_{\alpha_{oc}}
\end{aligned} \tag{13.1}$$

where $a_{\alpha_{oc}}$ is a complex constant characteristic of the impedance residues as given in equations 11.8 and 11.9. The corresponding circuit form is given in figure 13.3A. The modified pole impedances are written in terms of circuit elements as



A. Pole circuit modules



B. Modified pole circuit modules

Figure 13.3. Elementary Circuit Modules for Pole Terms for Impedance and Open Circuit Voltage

$$\tilde{Z}'_{\alpha_{oc}}(s) = Z_o a_{\alpha_{oc}} \left[(s - s_{\alpha_{oc}})^{-1} + s_{\alpha_{oc}}^{-1} \right]$$

$$= Z_o \frac{a_{\alpha_{oc}}}{s_{\alpha_{oc}}} \frac{s}{s - s_{\alpha_{oc}}}$$

$$= Z_o a_{\alpha_{oc}} \left[s_{\alpha_{oc}} - s_{\alpha_{oc}}^2 s^{-1} \right]^{-1}$$

$$= \left[G'_{\alpha_{oc}} + \frac{1}{sL'_{\alpha_{oc}}} \right]^{-1} \tag{13.2}$$

$$L'_{\alpha_{oc}} = -Z_o \frac{a_{\alpha_{oc}}}{s_{\alpha_{oc}}^2}$$

$$G'_{\alpha_{oc}} = \frac{1}{Z_o} \frac{s_{\alpha_{oc}}}{a_{\alpha_{oc}}} = -\frac{1}{s_{\alpha_{oc}} L'_{\alpha_{oc}}} = -G_{\alpha_{oc}}$$

Combined with the current source this gives the circuit representation in figure 13.3B.

As in the admittance case the circuit elements for the pole impedances and modified pole impedances are in general complex numbers. For computer purposes such formal circuit elements are still quite useful. In trying to realize actual circuit elements the present complex ones are somewhat limited, except they may be almost real for limiting cases of $s_{\alpha_{oc}}$ near the real or imaginary axes. The consideration of specific examples should help clarify these issues.

D. Entire-function circuit modules

Again using the considerations of section VI let us approximate the entire function impedance as a constant based on high and low frequency considerations.

If the impedance is inductive for $s \to 0$ then the modified pole impedances $\tilde{Z}_{\alpha_{oc}}(s)$ which are inductive for $s \to 0$ would be appropriate and avoid the use of an additional entire function (restricted to a constant resistance). For this case we have

$$\tilde{Z}'_{oce}(s) \simeq \tilde{R}'_{oce} = 0$$

$$\tilde{Z}_{oce}(s) \simeq R_{oce} = Z_o \sum_{\alpha_{oc}} a_{\alpha_{oc}} s_{\alpha_{oc}}^{-1} \tag{13.3}$$

$$= -\sum_{\alpha_{oc}} \tilde{Z}_{\alpha_{oc}}(0)$$

where the entire function impedances are approximated by constant resistances. The summation, instead of extending over all α_{oc} might be restricted to those α_{oc} which are used in a finite approximation to the circuit of figure 13.2.

If the impedance is capacitive as $s \to 0$ or resistive (as in a loop) as $s \to 0$, then one can calculate the impedance as $s \to 0$, subtract off the pole terms (or only those pole terms used), and match the remainder with a constant resistance near $s = 0$. This gives

$$\tilde{Z}_{oce}(s) \simeq R_{oce}$$

$$\tilde{Z}'_{oce}(s) \simeq R'_{oce}$$

$$R'_{oce} = R_{oce} - Z_o \sum_{\alpha_{oc}} a_{\alpha_{oc}} s_{\alpha_{oc}}^{-1} \tag{13.4}$$

$$= R_{oce} + \sum_{\alpha_{oc}} \tilde{Z}_{\alpha_{oc}}(0)$$

In the case of a resistive loop the resistance at s = 0 is

$$R'_{oce} = \tilde{Z}_a(0) \tag{13.5}$$

However, for the case of a capacitive object we have

$$\tilde{Z}_a(s) = \frac{1}{sC} + c_0 + c_1 s + \cdots \tag{13.6}$$

The first term is a pole which is subtracted. If the pole at zero frequency is left in unmodified form, because the s = 0 value cannot be subtracted, and if the other poles are put in modified form, then the entire function impedance corresponds to c_0.

The entire function current sources can be approximated by considering the open circuit voltage as $s \to 0$. It is assumed that the incident waves have, for the delta function response, zero time adjusted for short circuit current beginning at t = 0. As discussed in section VI the short circuit current has simple behavior for both $s \to 0$ and $s \to \infty$.

For an inductive antenna as $s \to 0$ we have $\tilde{V}_{oc}(s) \to 0$ for a delta function incident wave. If the unmodified pole impedances are used and if the entire function current source is approximated as a constant then we still have to make the open circuit voltage zero at s = 0. These requirements give

$$\tilde{I}'_{oce,p}(s) \simeq I'_{oce,p} \quad \text{(not used with zero impedance)}$$

$$\tilde{I}_{oce,p}(s) \simeq I_{oce,p} = \frac{\tilde{V}_{oce,p}(0)}{\tilde{Z}_{oce}(0)} = \frac{\tilde{V}_{oce,p}(0)}{R_{oce}} \tag{13.7}$$

$$= \frac{\sum\limits_{\alpha_{oc}} I_{\alpha_{oc},p} \tilde{Z}_{\alpha_{oc}}(0)}{\sum\limits_{\alpha_{oc}} \tilde{Z}_{\alpha_{oc}}(0)} = \frac{\sum\limits_{\alpha_{oc}} I_{\alpha_{oc},p} a_{\alpha_{oc}} s_{\alpha_{oc}}^{-1}}{\sum\limits_{\alpha_{oc}} a_{\alpha_{oc}} s_{\alpha_{oc}}^{-1}}$$

For an inductive antenna (for $s \to 0$) the use of modified pole impedances would seem to be simpler.

For capacitive or resistive impedance as $s \to 0$ consider the behavior of $\tilde{V}_{oc}(s)$ as $s \to 0$. For a resistive loop (closed object) $\tilde{V}_{oc}(s) \to 0$ as $s \to 0$ for a delta function incident wave. For such a case modified pole impedances can be used with no additional entire function current source included with R'_{oce} in equation 13.5. For a capacitive (open) object we can have $\tilde{V}_{oc}(s) \to$ constant as $s \to 0$ for a delta function incident wave. For use with modified pole impedances one could use

$$V'_{oce,p} \simeq \tilde{V}'_{oce,p}(0) \tag{13.8}$$

summed with $\tilde{f}_p(s)$ as a voltage source, noting that the pole impedance of the form $1/s()$ is not modified. This is expressed as a current source for this particular pole as

$$\tilde{I}'_{oce,p}(s) \simeq sC\,\tilde{V}_{oce,p}(0) \tag{13.9}$$

which is a special frequency dependent form. This prevents the open circuit voltage from unphysically growing as $s \to 0$ for a delta function incident wave.

For resistive impedance as $s \to 0$ (as in equations 13.4) then for unmodified pole impedances we have

$$\tilde{I}_{oce,p}(s) \simeq I_{oce,p} = \frac{\tilde{V}_{oce,p}(0)}{\tilde{Z}_{oce}(0)}$$

$$= \frac{V_{oce,p}}{R_{oce}}$$

$$= \frac{-\sum_{\alpha_{oc}} I_{\alpha_{oc},p}\tilde{Z}_{\alpha_{oc}}(0)}{R'_{oce} - \sum_{\alpha_{oc}} \tilde{Z}_{\alpha_{oc}}(0)} \tag{13.10}$$

where R'_{oce} is the impedance for $s \to 0$, and where the summation over α_{oc} may be approximated as a finite summation.

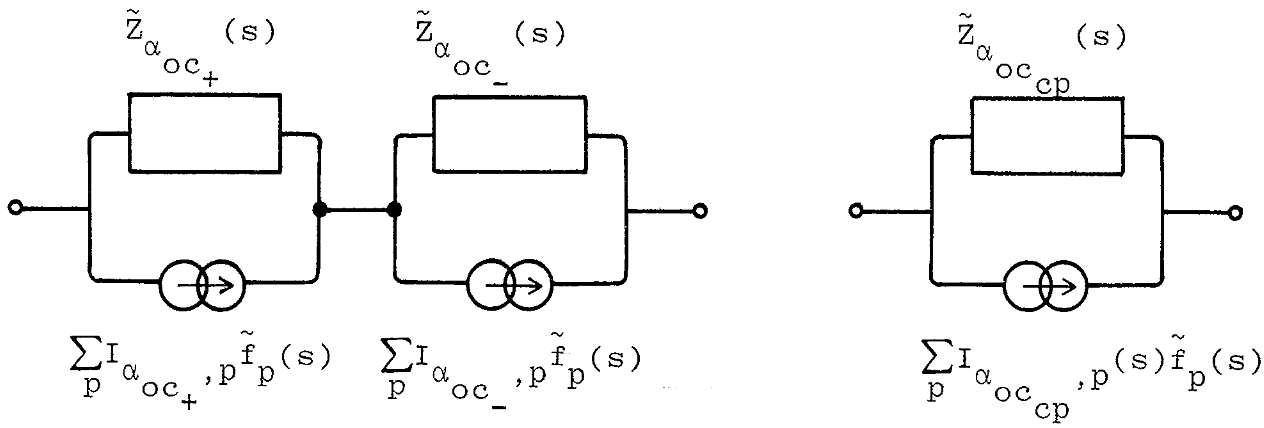
As in the short circuit case the open circuit case leads to various types of possible circuits. For the case of inductive objects (closed with zero resistance as $s \to 0$) the use of modified pole impedances would seem to simplify matters somewhat by eliminating the constant entire function impedance and current source, at least as an approximation.

E. Conjugate pair circuit modules

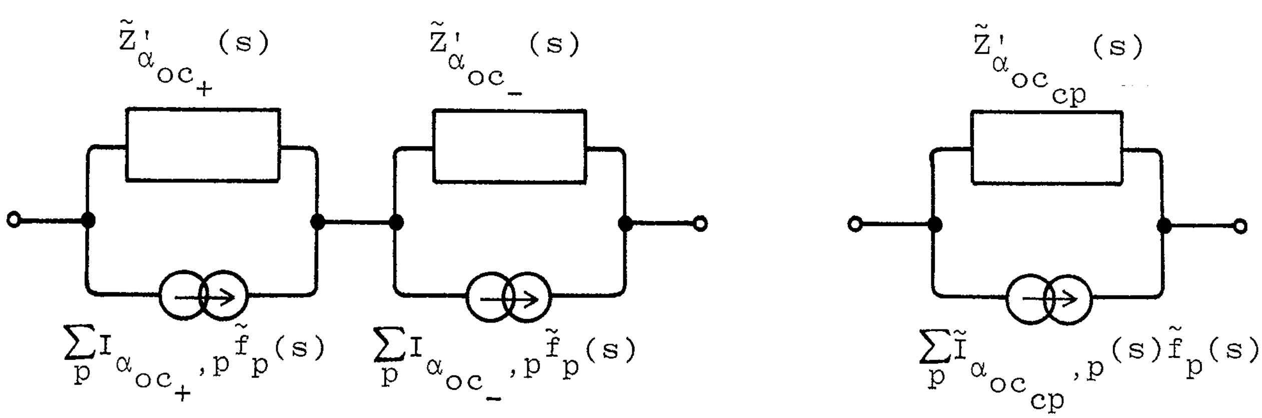
As in the case of the short-circuit-based circuits, the open-circuit-based circuits have the problem of complex element values. For special cases such as real axis poles and entire function contributions the elements (inductances, resistances, and capacitances) are real but not necessarily positive. Considering the case of conjugate pole pairs let us define

$$\begin{aligned} s_{\alpha_{oc_-}} &= \bar{s}_{\alpha_{oc_+}} \\ \mathrm{Im}\left[s_{\alpha_{oc_+}}\right] &> 0 \end{aligned} \tag{13.11}$$

forming a pole pair. As indicated in figure 13.4 one can form norton equivalent circuits of such conjugate pairs of pole circuit modules. The conjugate pair (cp) impedances are



A. Equivalent for pole circuit module conjugate pair



B. Equivalent for modified pole circuit module conjugate pair

Figure 13.4. Norton Equivalent Circuits for Pole Pairs

$$\tilde{Z}_{\alpha_{oc_{cp}}}(s) = \tilde{Z}_{\alpha_{oc_+}}(s) + \tilde{Z}_{\alpha_{oc_-}}(s)$$

$$= Z_o \left\{ a_{\alpha_{oc_+}} (s - s_{\alpha_{oc_+}})^{-1} + a_{\alpha_{oc_-}} (s - s_{\alpha_{oc_-}})^{-1} \right\}$$

$$= Z_o \left\{ a_{\alpha_{oc_+}} (s - s_{\alpha_{oc_+}})^{-1} + \bar{a}_{\alpha_{oc_+}} (s - \bar{s}_{\alpha_{oc_+}})^{-1} \right\}$$

$$= Z_o \frac{a_{\alpha_{oc_+}} (s - \bar{s}_{\alpha_{oc_+}}) + \bar{a}_{\alpha_{oc_+}} (s - s_{\alpha_{oc_+}})}{(s - s_{\alpha_{oc_+}})(s - \bar{s}_{\alpha_{oc_+}})}$$

$$= Z_o \frac{2 \operatorname{Re}\left[a_{\alpha_{oc_+}}\right] s - 2 \operatorname{Re}\left[a_{\alpha_{oc_+}} \bar{s}_{\alpha_{oc_+}}\right]}{s^2 - 2 \operatorname{Re}\left[s_{\alpha_{oc_+}}\right] s + \left|s_{\alpha_{oc_+}}\right|^2}$$

$$\equiv Z_o \frac{d_1 s + d_2}{d_3 s^2 + d_4 s + d_5} \qquad (13.12)$$

$$\tilde{Z}'_{\alpha_{oc_{cp}}}(s) = \tilde{Z}'_{\alpha_{oc_+}}(s) + \tilde{Z}'_{\alpha_{oc_-}}(s)$$

$$= Z_o \left\{ \frac{a_{\alpha_{oc_+}}}{s_{\alpha_{oc_+}}} \frac{s}{s - s_{\alpha_{oc_+}}} + \frac{a_{\alpha_{oc_-}}}{s_{\alpha_{oc_-}}} \frac{s}{s - s_{\alpha_{oc_-}}} \right\}$$

$$= Z_o \left\{ \frac{a_{\alpha_{oc_+}}}{s_{\alpha_{oc_+}}} \frac{s}{s - s_{\alpha_{oc_+}}} + \frac{\bar{a}_{\alpha_{oc_+}}}{\bar{s}_{\alpha_{oc_+}}} \frac{s}{s - \bar{s}_{\alpha_{oc_+}}} \right\}$$

$$= Z_o \frac{\dfrac{a_{\alpha_{oc_+}}}{s_{\alpha_{oc_+}}}(s - \bar{s}_{\alpha_{oc_+}})s + \dfrac{\bar{a}_{\alpha_{oc_+}}}{\bar{s}_{\alpha_{oc_+}}}(s - s_{\alpha_{oc_+}})s}{(s - s_{\alpha_{oc_+}})(s - \bar{s}_{\alpha_{oc_+}})}$$

$$= Z_o \frac{2 \operatorname{Re}\left[\dfrac{a_{\alpha_{oc_+}}}{s_{\alpha_{oc_+}}}\right]s^2 - 2 \operatorname{Re}\left[\dfrac{a_{\alpha_{oc_+}}\bar{s}_{\alpha_{oc_+}}}{s_{\alpha_{oc_+}}}\right]s}{s^2 - 2 \operatorname{Re}\left[s_{\alpha_{oc_+}}\right]s + \left|s_{\alpha_{oc_+}}\right|^2}$$

$$\equiv Z_o \frac{d'_2 s^2 + d'_1 s}{d'_5 s^2 + d'_4 s + d'_3} = Z_o \frac{d'_1 s^{-1} + d'_2}{d'_3 s^{-2} + d'_4 s^{-1} + d'_5}$$

for pole pair impedances and modified pole pair impedances respectively. Note that all coefficients of powers of s are real numbers.

The open circuit voltage associated with a conjugate pole pair can be written as

$$\tilde{V}_{oc_{\alpha_{oc_{cp}}}}(s) = \tilde{V}_{oc_{\alpha_{oc_+}}}(s) + \tilde{V}_{oc_{\alpha_{oc_-}}}(s)$$

$$= \sum_p \tilde{f}_p(s)\left[I_{\alpha_{oc_+},p}\tilde{Z}_{\alpha_{oc_+}}(s) + I_{\alpha_{oc_-},p}\tilde{Z}_{\alpha_{oc_-}}(s)\right] \tag{13.13}$$

which gives a short circuit current

$$\tilde{I}_{sc_{\alpha_{oc_{cp}}}}(s) = \frac{\tilde{V}_{oc_{\alpha_{oc_+}}}(s) + \tilde{V}_{oc_{\alpha_{oc_-}}}(s)}{\tilde{Z}_{\alpha_{oc_+}}(s) + \tilde{Z}_{\alpha_{oc_-}}(s)}$$

$$= \sum_p \tilde{I}_{\alpha_{oc_{cp}},p}(s)\tilde{f}_p(s) \tag{13.14}$$

$$\tilde{I}_{\alpha_{oc_{cp}},p}(s) = \frac{I_{\alpha_{oc_+},p}\tilde{Z}_{\alpha_{oc_+}}(s) + I_{\alpha_{oc_-},p}\tilde{Z}_{\alpha_{oc_-}}(s)}{\tilde{Z}_{\alpha_{oc_+}}(s) + \tilde{Z}_{\alpha_{oc_-}}(s)}$$

$$= \mathrm{Re}\left[I_{\alpha_{oc_+},p}\right] + i\ \mathrm{Im}\left[I_{\alpha_{oc_+},p}\right] \frac{\tilde{Z}_{\alpha_{oc_+}}(s) - \tilde{Z}_{\alpha_{oc_-}}(s)}{\tilde{Z}_{\alpha_{oc_+}}(s) + \tilde{Z}_{\alpha_{oc_-}}(s)}$$

Again, as in the short circuit case, if the pole current source coefficients $I_{\alpha_{oc},p}$ are real numbers then the norton current source for the open circuit pole pair simplifies somewhat. However, for general complex $\tilde{I}_{\alpha_{oc},p}(s)$ the resulting current source coefficients $\tilde{I}_{\alpha_{oc_{cp}},p}(s)$ are frequency dependent.

The open circuit voltage associated with a modified conjugate pole pair is

$$\tilde{V}'_{oc_{\alpha_{oc_{cp}}}}(s) = \tilde{V}'_{oc_{\alpha_{oc_+}}}(s) + \tilde{V}_{oc_{\alpha_{oc_-}}}(s)$$

$$= \sum_p \tilde{f}_p(s)\left[I_{\alpha_{oc_+},p}\tilde{Z}'_{\alpha_{oc_+}}(s) + I_{\alpha_{oc_-},p}(s)\tilde{Z}'_{\alpha_{oc_-}}(s)\right] \tag{13.15}$$

which gives a short circuit current

$$\tilde{I}'_{sc_{\alpha_{oc_{cp}}}}(s) = \frac{\tilde{V}'_{oc_{\alpha_{oc_+}}}(s) + \tilde{V}_{oc_{\alpha_{oc_-}}}(s)}{\tilde{Z}_{\alpha_{oc_+}}(s) + \tilde{Z}_{\alpha_{oc_-}}(s)}$$

$$= \sum_p \tilde{I}'_{\alpha_{oc_{cp}},p}(s)\tilde{f}_p(s) \tag{13.16}$$

$$\tilde{I}'_{\alpha_{oc_{cp}},p}(s) = \frac{I_{\alpha_{oc_+},p}\tilde{Z}'_{\alpha_{oc_+}}(s) + I_{\alpha_{oc_-},p}\tilde{Z}'_{\alpha_{oc_-}}(s)}{\tilde{Z}'_{\alpha_{oc_+}}(s) + \tilde{Z}'_{\alpha_{oc_-}}(s)}$$

$$= \mathrm{Re}\left[I_{\alpha_{oc_+},p}\right] + i\ \mathrm{Im}\left[I_{\alpha_{oc_+},p}\right]\frac{\tilde{Z}'_{\alpha_{oc_+}}(s) - \tilde{Z}'_{\alpha_{oc_-}}(s)}{\tilde{Z}'_{\alpha_{oc_+}}(s) + \tilde{Z}'_{\alpha_{oc_-}}(s)}$$

Again real valued $I_{\alpha_{sc},p}$ lead to simpler equivalent circuits for the modified conjugate pole pairs.

The impedance of a conjugate pole pair can be put into the form of circuit elements. Write the impedance in the form

$$\tilde{Z}^{-1}_{\alpha_{oc_{cp}}}(s) = \frac{1}{Z_o} \frac{d_3 s^2 + d_4 s + d_5}{d_1 s + d_2}$$

$$= \frac{1}{Z_o} \left\{ \frac{d_3}{d_1} s + \frac{\left(d_4 - \frac{d_3}{d_1} d_2 - \frac{d_5}{d_2} d_1 \right) s}{d_1 s + d_2} + \frac{d_5}{d_2} \right\}$$

$$= sC_{\alpha_{oc_{cp}}} + \left[\frac{1}{G^{(s)}_{\alpha_{oc_{cp}}}} + \frac{1}{sC^{(s)}_{\alpha_{oc_{cp}}}} \right]^{-1} + G_{\alpha_{oc_{cp}}}$$

$$C_{\alpha_{oc_{cp}}} = \frac{1}{Z_o} \frac{d_3}{d_1} = \frac{1}{Z_o} \frac{1}{2 \ \mathrm{Re}\left[a_{\alpha_{oc_+}} \right]}$$

$$= \left[C^{-1}_{\alpha_{oc_+}} + C^{-1}_{\alpha_{oc_-}} \right]^{-1} = \frac{\left| C_{\alpha_{oc_+}} \right|^2}{2 \ \mathrm{Re}\left[C_{\alpha_{oc_+}} \right]}$$

$$G_{\alpha_{oc_{cp}}} = \frac{1}{Z_o} \frac{d_5}{d_2} = - \frac{1}{Z_o} \frac{\left| s_{\alpha_{oc_+}} \right|^2}{2 \ \mathrm{Re}\left[a_{\alpha_{oc_+}} \bar{s}_{\alpha_{oc_+}} \right]}$$

$$= - \frac{1}{Z_o} \frac{1}{2 \ \mathrm{Re}\left[\frac{a_{\alpha_{oc_+}}}{s_{\alpha_{oc_+}}} \right]}$$

$$= \left[G^{-1}_{\alpha_{oc_+}} + G^{-1}_{\alpha_{oc_-}} \right]^{-1} = \frac{\left| G_{\alpha_{oc_+}} \right|^2}{2 \ \mathrm{Re}\left[G_{\alpha_{oc_+}} \right]} \qquad (13.17)$$

$$G_{\alpha_{oc_{cp}}}^{(s)} = \frac{1}{Z_o}\left\{\frac{d_4}{d_1} - \frac{d_2 d_3}{d_1^2} - \frac{d_5}{d_2}\right\}$$

$$C_{\alpha_{oc_{cp}}}^{(s)} = \frac{1}{Z_o}\left\{\frac{d_4}{d_2} - \frac{d_3}{d_1} - \frac{d_1 d_5}{d_2^2}\right\}$$

where these terms can be written out in terms of $s_{\alpha_{oc+}}$ and $a_{\alpha_{oc+}}$ using equations 13.12. This circuit form is shown in figure 13.5A. Again conductances and capacitances require negative elements for high resonant cases. However the above choice is but one of many.

The impedance of a conjugate modified pole pair can be written as

$$\tilde{Z}'^{-1}_{\alpha_{oc_{cp}}}(s) = \frac{1}{Z_o}\,\frac{d_3' s^{-2} + d_4' s^{-1} + d_5'}{d_1' s^{-1} + d_2'}$$

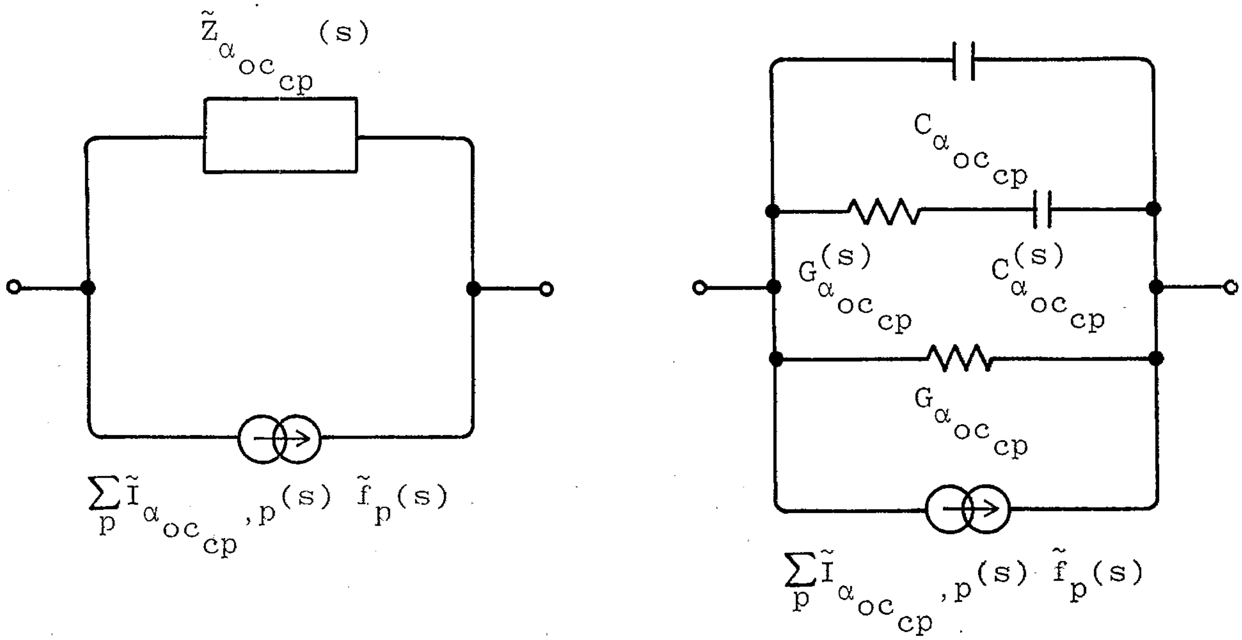
$$= \frac{1}{Z_o}\left\{\frac{d_3'}{d_1'} s^{-1} + \frac{\left(d_4' - \frac{d_3'}{d_1'} d_2' - \frac{d_5'}{d_2'}\right) s^{-1}}{d_1' s^{-1} + d_2'} + \frac{d_5'}{d_2'}\right\}$$

$$= \frac{1}{sL'_{\alpha_{oc_{cp}}}} + \left[\frac{1}{G_{\alpha_{oc_{cp}}}^{(s)'}} + sL_{\alpha_{oc_{cp}}}^{(s)'}\right]^{-1} + G'_{\alpha_{oc_{cp}}}$$

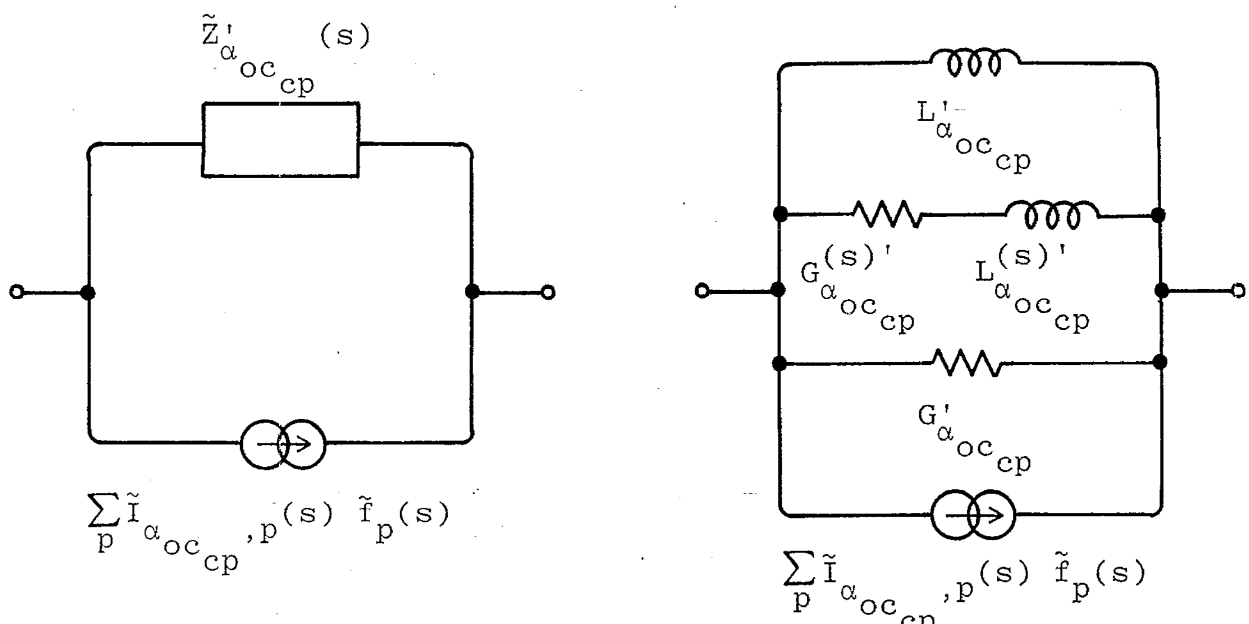
$$L'_{\alpha_{oc_{cp}}} = Z_o \frac{d_1'}{d_3'} = -2Z_o \frac{\mathrm{Re}\left[\dfrac{a_{\alpha_{oc_+}} \bar{s}_{\alpha_{oc_+}}}{s_{\alpha_{oc_+}}}\right]}{\left|s_{\alpha_{oc_+}}\right|^2}$$

$$= -2Z_o\, \mathrm{Re}\left[\frac{a_{\alpha_{oc_+}}}{s^2_{\alpha_{oc_+}}}\right]$$

$$= L'_{\alpha_{oc_+}} + L'_{\alpha_{oc_-}} = 2\mathrm{Re}\left[L_{\alpha_{oc_+}}\right]$$



A. Conjugate pole pair circuit



B. Conjugate modified pole pair circuit

Figure 13.5. Circuit Element Possibilities for Pole Pair Equivalent Circuits

$$(13.18)$$

$$G'_{\alpha_{oc_{cp}}} = \frac{1}{Z_o}\frac{d_5'}{d_2'} = \frac{1}{Z_o}\frac{1}{2\mathrm{Re}\left[\frac{a_{\alpha_{oc_+}}}{s_{\alpha_{oc_+}}}\right]}$$

$$= \left[G'^{-1}_{\alpha_{oc_+}} + G'^{-1}_{\alpha_{oc_-}}\right]^{-1} = \frac{\left|G_{\alpha_{oc_+}}\right|^2}{2\mathrm{Re}\left[G_{\alpha_{oc_+}}\right]^2}$$

$$G^{(s)'}_{\alpha_{oc_{cp}}}(s) = \frac{1}{Z_o}\left\{\frac{d_4'}{d_1'} - \frac{d_2'd_3'}{d_1'^2} - \frac{d_5'}{d_2'}\right\}$$

$$L^{(s)'}_{\alpha_{oc_{cp}}}(s) = Z_o\left\{\frac{d_4'}{d_2'} - \frac{d_3'}{d_1'} - \frac{d_1'd_5'}{d_2'^2}\right\}^{-1}$$

which is illustrated in figure 13.5B. Note in comparing equations 13.18 to 13.17 that the various constants d_n' correspond to d_n so that modified and unmodified pole circuits can be directly compared. These can also be compared to the results for the short circuit pole pairs in section X which have dual form to the results of this section. The use of only conductances and inductances requires one or more negative elements for highly resonant cases, but other more general forms are possible.

XIV. Admittance and Impedance from Short Circuit and Open Circuit Quantities

Having considered the equivalent circuits resulting from the short circuit and open circuit boundary value problems, let us consider some results obtained by combining the results of those two problems. Specifically consider the admittance $\tilde{Y}_a(s)$ and impedance $\tilde{Z}_a(s)$ and construct these from both sets of natural frequencies.

Hence let us try a form as

$$\tilde{Z}_a(s) = \tilde{Y}_a^{-1}(s) = Ze^{\tilde{h}(s)} \frac{\prod_{\alpha_{sc}} (s - s_{\alpha_{sc}})}{\prod_{\alpha_{oc}} (s - s_{\alpha_{oc}})} \tag{14.1}$$

where Z is some scaling constant with dimensions of impedance (ohms). Of course there is some interpretation of the infinite products required to make the ratio converge to the proper answer. Here $\tilde{h}(s)$ is an entire function included for generality; $e^{\tilde{h}(s)}$ has no zeros in the finite plane as well.

In line with the approximations discussed in section VII, together with the high and low frequency limiting forms discussed in section VI, let us consider only a finite portion of the natural frequencies clustered near the origin of the s plane. Define a set of short circuit natural frequencies by $A_{\alpha_{sc}}$, and similarly a set of open circuit natural frequencies by $A_{\alpha_{oc}}$. Then equation 14.1 becomes

$$\tilde{Z}_a(s) \simeq Ze^{\tilde{h}(s)} \frac{\prod_{\alpha_{sc} \epsilon A_{\alpha_{sc}}} (s - s_{\alpha_{sc}})}{\prod_{\alpha_{oc} \epsilon A_{\alpha_{oc}}} (s - s_{\alpha_{oc}})} \tag{14.2}$$

Here $A_{\alpha_{sc}}$ and $A_{\alpha_{oc}}$ are defined so as to include conjugate pairs of natural frequencies in the left half s plane plus natural frequencies on the negative Re[s] axis. To avoid exponential behavior in the right half s plane let us require $\tilde{h}(s) \to$ constant for $s \to \infty$ with $|\arg(s)| \leq \pi/2 - \delta$ with $\delta > 0$. Then to have asymptotic behavior proportional to s, a constant, or s^{-1} as $s \to \infty$ in the right half plane, let us constrain that the number of natural frequencies in $A_{\alpha_{sc}}$ and $A_{\alpha_{oc}}$ differ by at most one (1). Referring back to sections VIII and XI note that this type of rational function is already constrained; for consistency with these previous results let us set

$$\tilde{h}(s) \equiv 0 \tag{14.3}$$

Note that Z is now of necessity a real constant.

If we wish to make the impedance match the asymptotic form at low frequencies to contain the scaling constant, then it is convenient to write the impedance as

$$\tilde{Z}_a(s) \simeq \left\{ \begin{matrix} sL \\ R \\ \frac{1}{sC} \end{matrix} \right\} \frac{\prod\limits_{\alpha_{sc} \varepsilon A'_{\alpha_{sc}}} \left(1 - \frac{s}{s_{\alpha_{sc}}}\right)}{\prod\limits_{\alpha_{oc} \varepsilon A'_{\alpha_{oc}}} \left(1 - \frac{s}{s_{\alpha_{oc}}}\right)}$$

$$A'_{\alpha_{sc}} = A_{\alpha_{sc}} - (\text{any } s_{\alpha_{sc}} = 0, \text{ if such exists})$$

$$A'_{\alpha_{oc}} = A_{\alpha_{oc}} - (\text{any } s_{\alpha_{oc}} = 0, \text{ if such exists}) \tag{14.4}$$

Here three choices are listed.

a. inductive (short circuit natural frequency at $s = 0$)

$$\tilde{Z}_a(s) \to sL \quad \text{as } s \to 0 \tag{14.5}$$

b. resistive (no natural frequency at s = 0)

$$\tilde{Z}_a(s) \to R \quad \text{as } s \to 0 \tag{14.6}$$

c. capacitive (open circuit natural frequency at s = 0)

$$\tilde{Z}_a(s) \to \frac{1}{sC} \quad \text{as } s \to 0 \tag{14.7}$$

If in addition $A_{\alpha_{sc}}$ and $A_{\alpha_{oc}}$ are chosen to have the same number of natural frequencies then $\tilde{Z}_a(s)$ will tend to a real constant at high frequencies as desired. By observing that any real natural frequencies away from s = 0 are negative, and that all other natural frequencies come in conjugate pairs, and that L, R, and C in equations 14.4 are positive, we find that $\tilde{Z}_a(s)$ so chosen tends to a real and positive constant at high frequencies.

An alternate representation is based on high frequencies as

$$\tilde{Z}_a(s) \simeq A \frac{\prod\limits_{\alpha_{sc} \varepsilon A_{\alpha_{sc}}} \left(1 - \frac{s_{\alpha_{sc}}}{s}\right)}{\prod\limits_{\alpha_{oc} \varepsilon A_{\alpha_{oc}}} \left(1 - \frac{s_{\alpha_{oc}}}{s}\right)} \tag{14.8}$$

where again $A_{\alpha_{sc}}$ and $A_{\alpha_{oc}}$ are constrained to have the same number of natural frequencies as well as include natural frequencies in conjugate pairs when off the real axis of the s plane. Then as discussed in section VII we have

$$\tilde{Z}_a \to A > 0 \quad \text{for } s \to \infty \quad |\arg(s)| \le \frac{\pi}{2} - \delta \tag{14.9}$$

Hence if we know A (such as from the considerations in section VI), then with the short and open circuit natural frequencies an approximation to Z_a can be readily constructed. However this may have limited accuracy for low frequencies, or a large number of natural frequencies may be required.

XV. Summary

As one can see there is quite a lot of information to be obtained in considering the results of more than one boundary value problem together as a single problem related to a given antenna or scatterer viewed under different conditions of excitation. This note has considered a single object with external incident waves, with excitation at a gap (port), and under short circuit and open circuit conditions at the gap.

Owing to the similarities in the representation of the solution for incident waves and for gap excitation certain term-by-term correspondences can be made to give equivalent circuit representations, involving both sources and impedance elements, which represent the solution to both problems (at least approximately). Here we have treated the equivalent circuits on both short circuit and open circuit bases, and to some extent combined the two.

Other forms of equivalent circuits should be possible. Present considerations have centered around the poles from the SEM form of the solutions, with some assist from the EEM form. Other types of expansions might be able to produce other forms of infinite size networks, such as ladder forms, or others common in circuit theory. Along with these there is the problem of synthesizing realizable impedance elements with simple sources.

Another generalization of the present considerations is to N-port equivalent circuits for objects with N gaps (ports). The present techniques should provide a direct way of doing this but off-diagonal admittance and impedance elements are somewhat more complicated than self-admittance and self-impedance (driving point). Hopefully this note will have introduced some of the basic concepts of this type of synthesis to the reader.

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