Interaction Notes

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Statistical Relationship Between Testing and Predictions of EMP Interaction

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Abstract

In our investigation of the general relationship between electromagnetic pulse (EMP) testings and predictions, we first show that such a relationship must be statistical in nature due to the many uncontrollable and uncertain elements and the shear complexity of the problem. Then, we devise a theoretical framework that decomposes the overall problem into different stages. For each stage, we identify the input information required, delineate the output information produced, and outline the nature of the effort needed. In particular the methodology used is explicitly specified and the difficulties encountered, of both physical and mathematical origins, are pointed out. While we have suggested ideas and commented on the extent of efforts needed to resolve some of these difficulties, in this report we can merely indicate others as being open questions. In short, for the general EMP testing-prediction problem we have spelled out the ingredients needed to arrive at a final assessment and have outlined the ideas

and procedures for making use of these ingredients.

To the part of the statistical relationship between EMP coupling test data and prediction results, we have devoted a detailed effort. Based on a linear model that links the subsystem level black box points to their dominant points of entries (POE), we have developed a statistical formalism that enables one to calibrate the uncertainties in the theoretical (analytical or computational) prediction capability by the uncertainties in the test data under simulated EMP environments. The thus calibrated prediction capability then will be used to predict, with quantified estimates of confidence levels and intervals, the statistical behavior of the system responses at that subsystem level to a threat EMP stress. Following this formalism, we use a simplified example to illustrate how the procedures are applied and implemented. In the example, test data and prediction results are statistically compared and aggregated to yield an EMP coupling assessment that has quantified confidence measures. Such a coupling assessment, combined with the other subsequent ingredients outlined in the general theoretical framework, could be used either forwardly to facilitate the overall assessment or backwardly to specify the amount of hardening required for improving the unsatisfactory coupling parts.

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SECTION I. INTRODUCTION AND SUMMARY

1.1 INTRODUCTION

In this report, we investigate the relationship between, and the strategy for making improved use of, experimental test data and theoretical prediction results for nuclear electromagnetic pulse (EMP) interaction with a complicated system, e.g., an E-4A Air Borne National Command Post.

The urgency and importance of such an investigation are obvious. On the one hand, we have a number of methods, including simplified analytical formulas and big computer codes, to compute and predict EMP interaction results. On the other hand, we have collected much EMP test data from experimental simulations, and are likely to collect more. Considerable efforts have been devoted to both areas separately. Not nearly enough has been done to bring these separate results together toward realizing the desired final technology for predicting EMP interaction, a technology that is validated and quantified in terms of system performances and can thus provide overall assessment and fix requirements. To attain such a prediction technology, it appears that at present we have more need to examine the relationship between tests and predictions than to generate further results in either category separately. In examining this relationship, we find many outstanding problems addressing such questions as what are the tests for; what are the implications and utility of test data in regard to the capability of a prediction technology; what is an appropriate formalism to quantify these answers; and how can the quantification formalism lead to strategic allocation of limited resources?

As an exploratory investigation toward closing the gap between EMP test data and prediction results, we must first recognize that the former is collected primarily for the purpose of calibrating and improving the latter. Based on this premise, we shall focus our theme in the subsequent text.

Furthermore, in view of the complexity and uncertainties in the EMP system interaction, we recognize that both the calibration mechanism of the prediction capability itself and the eventual prediction results made by that capability are statistical in nature. Therefore, we shall examine the relationship between EMP tests and predictions from a statistical perspective. Finally, it should also be stated that with this brief investigation what we are trying to achieve is very limited. We are not at all so ambitious as to preempt the field by answering all relevant questions; rather, we shall identify clearly the major areas of concern, suggest a theoretical framework to quantify this complicated problem, and illustrate the methodology with a simplified example.

In the following text, Section 1.2 briefly summarizes the findings of this report; Section 2 presents the general scheme of the statistical relationship between EMP tests and predictions; and Section 3 illustrates those ideas with an example.

1.2 SUMMARY

The contents of this report are briefly summarized as follows. Regarding the general relationship between EMP testings and predictions, we first show that such a relationship must be statistical in nature due to the many uncontrollable and uncertain elements and the shear complexity of the problem. Second, we devise a theoretical framework that decomposes the overall problem into a number of different stages. For each of such stages, we identify the input information required, spell out the output information produced, and outline the nature of effort to achieve each. In particular, the methodology of the mathematical manipulation used at each stage is explicitly specified, and the difficulties stemming from both physical and mathematical origins are pointed out. For some, but not all, of these difficulties, ideas to resolve them are suggested and the extent of effort needed is commented on. Other difficulties are indicated to be open questions. In general, the framework not only provides a clear picture of the problem of the assessment/hardening of EMP-system interaction as to what efforts are needed in which areas, but also provides a systematic basis for making decisions as to why and to what extent each of those areas needs attention to achieve a balanced effort in an overall program. In short, for the general EMP testing prediction problem we have spelled out the ingredients needed for arriving at a final assessment, and have outlined the ideas and procedures to make use of these ingredients.

To the statistical relationship between test data and prediction results of EMP couplings, we have devoted a detailed effort. Based on a linear model that links the subsystem level black-box points to their dominant points of entries (POE) (see Section II-2.1 and eq. (1) for detailed description), we have developed a statistical theory that enables one to calibrate the uncertainties in the theoretical (analytical or computational) prediction capability by the uncertainties in the test data under simulated EMP environments. With those test-data calibrated uncertainties, the prediction capability can be used to predict, with quantified estimates of confidence levels and intervals, the statistical behavior of the system responses at that subsystem level to a threat EMP stress. Following the general formalism, we illustrate the procedures for its application and implementation by using a simplified example. In the example, test data and prediction results are statistically compared and aggregated to yield an EMP coupling assessment with quantified confidence measures. Such a coupling assessment, combined with those other subsequent ingredients needed for the overall assessment of the system performance as outlined in the general relationship, could be used either forwardly to facilitate that overall assessment or backwardly to specify the amount of hardening required for the unsatisfactory coupling parts. Finally, we emphasize that although the example represents merely a gross simplification of reality, a closer look at the statistical methodology used does show that it has a wide applicability and serves at least as a prototype of a practically feasible method.

SECTION II. STATISTICAL RELATIONSHIP BETWEEN TESTING DATA AND PREDICTION RESULTS OF THE EMP INTERACTION

2.1 GENERAL REMARKS

In both the historical past and conceivable future, the purposes and uses of experimental testings are

a) to obtain the phenomenological facts for the cases tested;

b) to calibrate and to revise/establish a theoretical capability such as to enable one to predict phenomenological results of "similar" but untested cases and to "understand" the outcomes of all interesting cases; and

c) to iterate the state of the art in the above two categories toward their mutual enhancement and convergence.

Here, of the theoretical capability in b), the prediction aspect is to satisfy real operational needs and the understanding aspect is to satisfy merely our subjective mental imagination. But these two aspects are so interwoven in our mode of thinking that in reality they cannot be clearly separated from each other. Therefore, it is such a combined theoretical capability the experimental testings are to calibrate and/or revise quantitatively [1].

Furthermore, the above process from a) to b) is never deterministic nor even unique -- be it at a basic level of revising/establishing a theoretical capability or at an applied level of calibrating that capability using a "well-established" physical theory. Such features of non-determinancy and non-uniqueness are caused by at least four factors:

a) the inherent uncertainties in the test data;

b) the difficulties in separating and isolating causes and their consequent effects;

c) the inferences from the tested cases to the untested "similar" ones (including, of course, the ones in reality for which the test cases were intended in the first place);

d) the subjectivity, resulting unavoidably from our imaginary thinking process, in the choices of a theoretical framework mentally created and used to interpolate and extrapolate the test data.

Consequently, as in many other areas, the problem of relating tests and predictions in EMP interactions with a complicated electronic system contains uncertainties which are statistical in nature [2]. Such a statistical nature exists in both the very inherent mechanism of calibrating/revising the prediction capability, via comparisons of test data and prediction results, and the eventual hardness assessments using the prediction capability.

To be specific, even assuming a full and accurate knowledge of the EMP environment, probability and statistics still enter the EMP interaction through the following mechanisms:

1) the random deviations of the system characteristics within the macroscopic design and specification tolerances--these deviations may be tolerable under normal system-operating constraints because of design safety factors, but could yield relatively disparate responses when pushed near or beyond such safety limits by EMP;

2) our lack of complete knowledge of these deviations;

3) our making guesses about the effects of the above via imperfect and partial test data and prediction results.

Therefore, our basic task is to statistically quantify the implications of these factors in an overall EMP technology. Here, the deviations in 1) are measures of how widely spread the reality is, and will be referred to as variations. These could stand for variations from each other among similar but distinct systems or among similar electric subcomponents used in one system. On the contrary, the lack of knowledge in 2) is a measure of the accuracy and limitations in our capability to describe the reality, and will be referred to as uncertainties. We emphasize the distinction between variations in the reality and uncertainties in knowing the reality by noticing that uncertainties could very well exist even if there were only one component that exhibited no variation.

2.2 STATISTICAL RELATIONSHIP BETWEEN TEST AND PREDICTIONS

2.2.1 From Testings and Predictions to Calibrated Prediction Capability of EMP Coupling

For clarity, we present our ideas concerning the statistical relationship between EMP-coupling testings and predictions by using a block diagram, as shown in Figure 1, and examine these ideas accordingly as follows.

Suppose, for the present, we are interested in the electric responses of the electronic system at some subsystem level, say the responses at a large number of pins or at some critical black box circuit points. We label these subsystem points by x_i, $i=1, 2 . . . . I$, and label the collection of these points by $\{x_i\}_{i=1}^{I}$.*

To obtain the electric response to an EMP at a point x_i, we can make use of some theoretical model and predict the response by solving the model either analytically and/or numerically. Typically, we predict the response by identifying and modeling the dominant coupling paths to the reponse point x_i from its

*For the notations used, see Glossary at the end of the report.

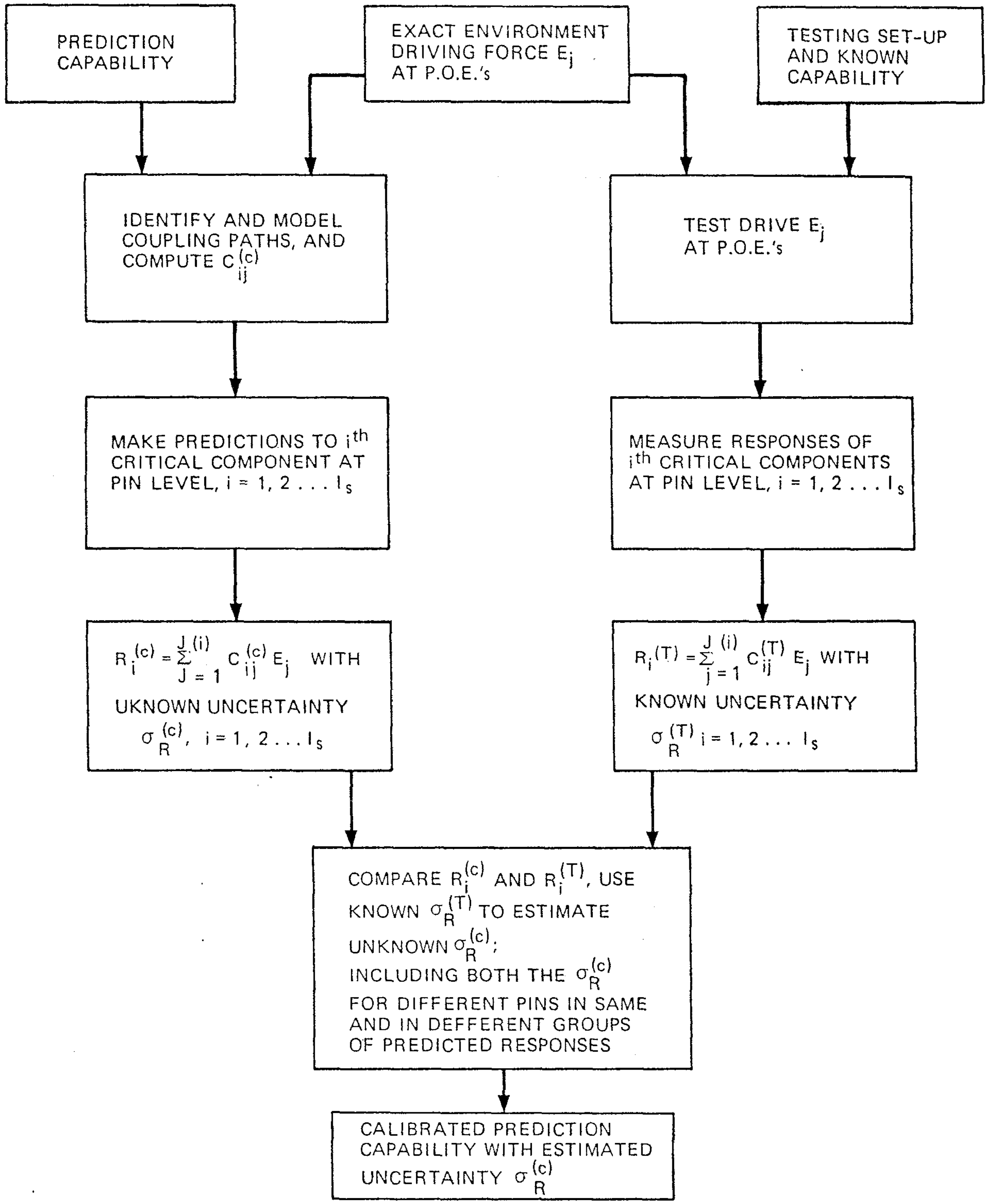


Figure 1. Block Diagram of EMP Interaction Testing and Prediction Relationship

points of entry (POE's) Y_j, $j=1,2,\ldots J(i)$, computing the coupling matrix $C_{ij}^{(c)}$, and obtaining [3]

$$R_i^{(c)} = \sum_{j=1}^{J(i)} C_{ij}^{(c)} E_j \ . \qquad (1)$$

Here, the superscript (c) stands for computed results. Also, we have assumed that the EMP environment E_j at the $j^{\underline{th}}$ POE is known accurately. We will come back to this assumption later and remark on how to relax it. At this stage, we do not know how accurate the prediction $R_i^{(c)}$ is, and we represent <u>conceptually</u> its uncertain error, which is to be found, simply by $\sigma_R^{(c)}$.

To quantify the $\sigma_R^{(c)}$ for the predicted $R_i^{(c)}$, and to further improve that prediction, we need to resort to experimental tests. First, to be able to make use of the testing results at all, we must have a knowledge (most likely itself statistical in nature) of the accuracies of the testing set-up. Second, the testing is usually conducted in a less-than-full-threat environment which actually simulates those E_j's at POEs, a number but not all at each time, according to the dominant coupling paths identified in the prediction process. Thus, assuming again an accurate control of the simulated driven environment E_j at the POEs, from the tests we can measure the responses at x_i (see Figure 1):

$$R_i^{(T)} = \sum_{j=i}^{J(i)} C_{ij}^{(T)} E_j \ . \qquad (2)$$

Here the superscript (T) stands for test results. Now the known degree of accuracy of the test set-up must be translated into the degree of accuracy of the test responses $R_i^{(T)}$, when the latter is subjected to the simulated driving environment E_j at the predicted dominant POEs. The $R_i^{(T)}$ then, with its known degree of accuracy, or uncertainty, which is symbolically represented by $\sigma_R^{(T)}$, contains all the information we have gained through the experiment and can be exploited to calibrate our prediction capability.

The actual quantification of the unknown $\sigma_R^{(c)}$ against the known $\sigma_R^{(T)}$ obviously should consist of comparing the $R_i^{(c)}$ and $R_i^{(T)}$ at those subsystem points (such as pins) selected for predictions and testings. Then statistical inferences could be made regarding the prediction capability as a whole in its uncertainty limits. A thus quantified prediction capability will then be used to make further predictions for responses at untested subsystem points and for responses subject to different environments. In doing so, there are two difficulties that we shall point out and indicate how, at least conceptually and partially, to resolve.

First, out of the large number of all subsystem points, usually we can only select a certain fraction of them to make predictions for and to run tests against. In executing that selection we perhaps have grouped similar subsystem points according to their EMP responses, and have selected randomly some members from within each such group in the hope that those selected ones could somewhat represent their respective groups. A grouping and selection of this type has already employed part of the prediction capability. To learn and calibrate this part of the prediction capability optimally, via comparisons with testing data, it is not clear whether the subsystem points selected for testing within each such group should be identical to those selected for predictions, or should be according to some other algorithm, e.g. independent of those selected for predictions. At present, it seems likely to the author that an optimal choice may substantially depend upon the nature of the true variation of those similar responses within the group -- i.e., depend upon the level of detail we are willing to expend and the nature of uncertainties in the very prediction capability itself used to make such groupings in the first place. This area does present a realistic problem, and we do not pretend to know the answer. However, one statistical principle is lucidly clear: it is always more advantageous to employ known information in selecting response points for testings and predictions than not to, even if that information is of an uncertain nature. In other words, unless we can afford testing fully all response points (and, incidentally, thereby do away with the need for prediction), it is always more optimal, in terms of obtaining most information with the least number of tests, therefore least cost, to group similar response points as much as we can and to distribute our resources accordingly by randomly selecting points from within such groups than to randomly and indiscriminately select among all the response points without grouping them all.

Second, to measure the response $R_i^{(T)}$ excited by driving the environments only at those POEs predicted to have major coupling links to a response point and to make inferences based on a comparison of such a measured $R_i^{(T)}$ to the predicted $R_i^{(c)}$ could lead to erroneous but self-deceiving conclusions about the prediction capability. Such erroneous conclusions will lead to an erroneous vulnerability assessment of the system to real EMP threat. This is because that in the above procedure we may have inadvertently ignored significant coupling paths to the response point from unrecognized POEs that contribute substantially to the response at the response point. It is also because the same procedure can never by itself reveal such a defect so long as the $C_{ij}^{(c)}$ and the $C_{ij}^{(T)}$ in (1) and (2) are linking the response point only to the same set of recognized POEs. It seems very difficult to get out of this loop of fallacy if we do not have tests at full threat conditions--both in EMP amplitude level, time shape, and spatial extent. In fact, the problem of unraveling the inadvertent effects of partial-threat simulation to recover and calibrate the prediction capability in predicting responses to a full-EMP-threat is among the most outstanding EMP problems [4]. Short of full-threat testings, nevertheless, there is one potentially fruitful idea that could shine light onto this dilemma and break the loop of self-deception. That idea is to drive the environment at POEs selected randomly out of those predicted not to give rise to any significant responses at a given subsystem response point, and to measure and check the response at that given point. A comparison and inference based on testings and predictions of such a complementary nature to a number of randomly selected response points could quantify and calibrate the uncertainty errors in the predictions due to ignorant exclusions of significant coupling paths.

2.2.2 From Calibrated Prediction Capability of EMP Coupling to System Performance Assessment

The flow of ideas for using the test-calibrated prediction capability in EMP coupling and other inputs to arrive at an overall assessment to system performance under EMP stress is depicted by Figure 2.

The prediction capability is first used to predict subsystem responses to an EMP threat at a number of sampled response points, often larger in number than those selected for testing. From these sampled prediction responses (see remarks in Section 3.4 for some associated difficulties), we can employ the calibrated uncertainties in the prediction mechanism and make statistical inferences about the distributions of the responses at all the critical subsystem points, with estimated confidence intervals and levels, for each of the various similar groups of points. Combining these distributions and those of the subsystem component malfunction thresholds T_i, as further known inputs, we could obtain prediction-inferred distributions for the safety margins [5]

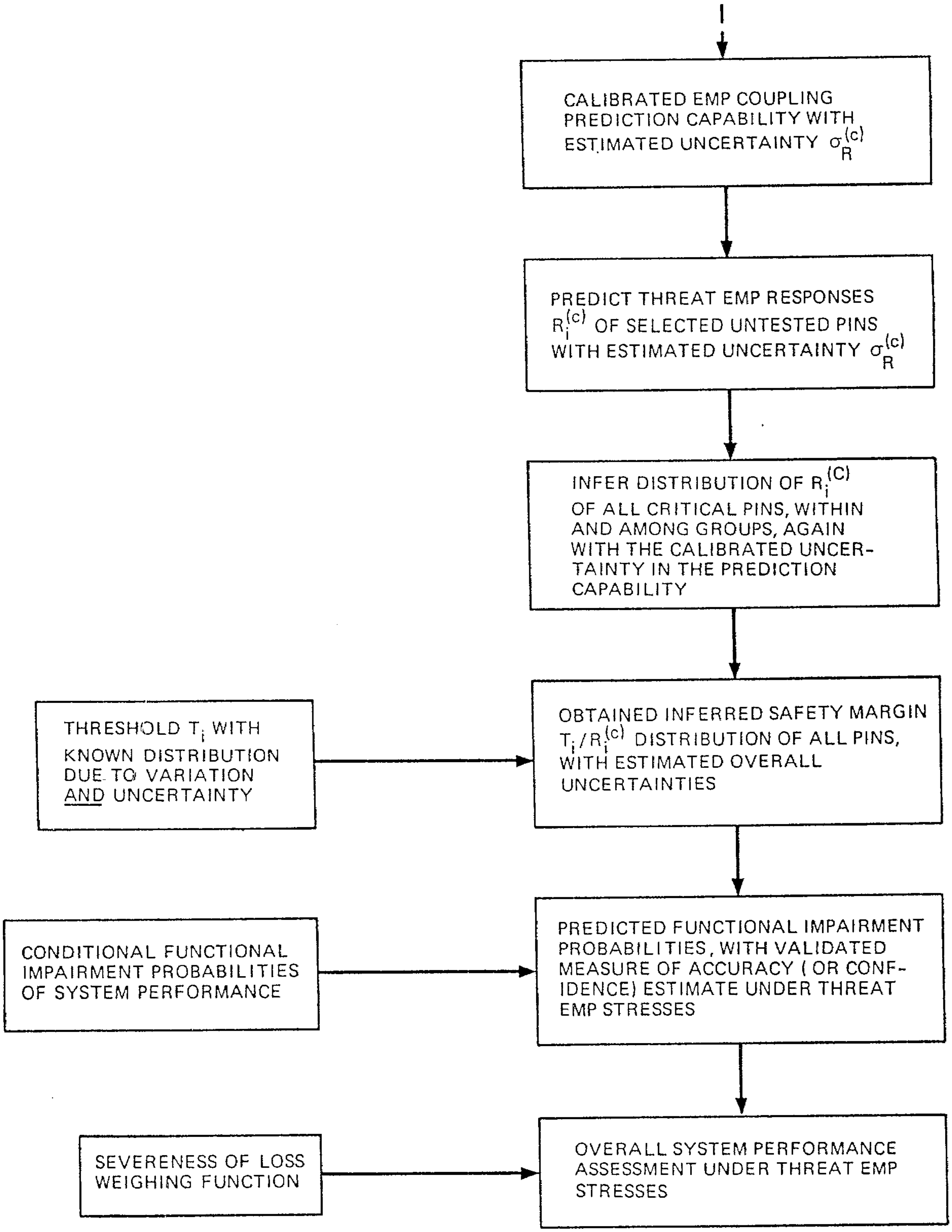


Figure 2. Block Diagram of EMP Interaction Testing and Prediction Relationship (Continuation of Figure 1)

$$S_i^{(c)} \equiv \frac{T_i}{R_i^{(c)}} \tag{3}$$

for the subsystem components due to the EMP threat. Here, the distributions are to be inferred according to the groupings of similar subsystem response points.

Up to this stage, we have been concerned only with the electric coupling from the EMP threat environments to the system at some subsystem black-box component level. The effects of these electric responses at the component level on the functional performances of the system are, of course, the ones of eventual concern. The information needed to link the electric malfunction safety margins of the subsystem components to their impact on system functional performance is the conditional functional impairment probabilities. These are the probabilities of impairments to a certain type of system functional performances (e.g., the dial tone of a telephone, the routing of a switch center, the correct process of a coded signal at a terminal) caused by a specified and given failure status of the electric subcomponents. Often, such conditional probabilities of functional performance impairments are not easy to obtain, even for the system designers, because an electronic system is usually designed and implemented on a sufficient basis, i.e., it works satisfactorily as long as the electric signals, their amplitudes and bandwidths, are within the designed norm. What exactly the functional impairments are and how they come about, if the set of subsystem components operate outside the signal norm and fail in specified ways, are usually not well delineated and need substantial effort to unravel. However, interested here mainly in the EMP coupling mechanism, we assume the availability of such a linking information, and thus, in principle, can arrive at the system's functional performance predictions under EMP stress, with estimated confidence measures obtained from and validated by the testing-prediction calibration procedure.

A final input to the overall assessment is a weighting function that judges and weighs the severeness of the loss due to the above impairments of system functional performances. Again, the determination of this loss function (or utility function) is not in the hands of EMP analysts and testers. It encompasses the nature of the mission of the particular system as a unit and the strategic deployment of that unit in an overall communication-command-and-control network. We would like to point out, however, the decision concerning the adequacy of the system performances as evaluated and the amount of hardening needed do depend critically on that loss function.

2.2.3 Improving the System Vulnerability Assessment

If, as a result of the overall assessment effort, the system in part or as a whole is found not to meet some required performance specifications under EMP stress, we must improve the system. This could proceed in two ways, separately or combined, as depicted by Figure 3. If some performance impairments are judged too severe, we may have to trace back according to the conditional probabilities of functional impairments as outlined in Section 2.2.2, find the weak subcomponents, harden them, and then repeat the testing and prediction process again. If the assessment confidence is too low or the accuracy is too vague, we may have to refine the prediction by adjusting our prediction model as indicated by the test data and/or by taking more test data and making more predictions, so as to adequately narrow down the uncertainties in our prediction capability. Of course, these two undertakings could be combined and cross-iterated in repeating the whole test-prediction process until a satisfactory performance assessment is reached.

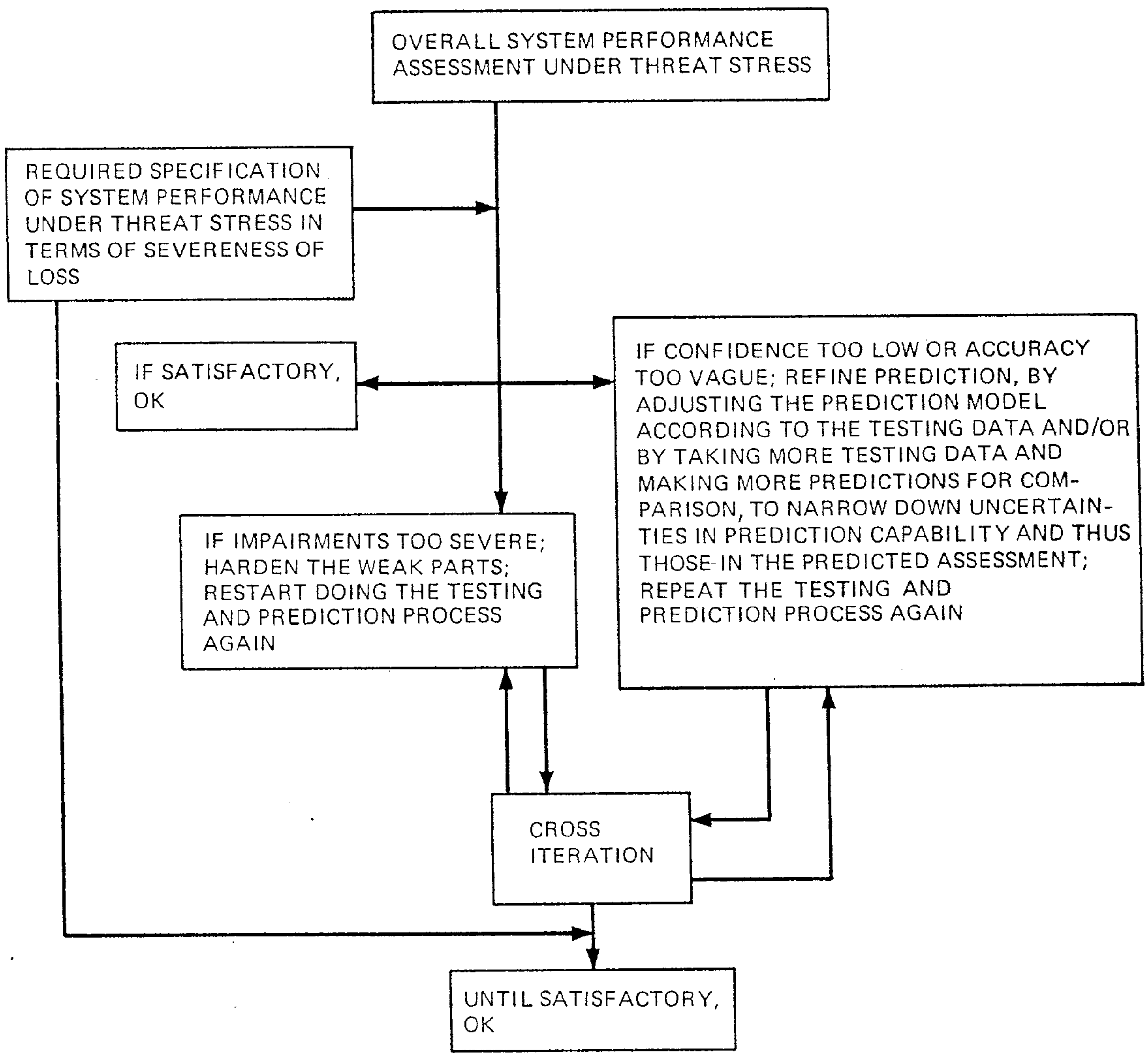


Figure 3. Block Diagram of EMP Interaction Testing and Prediction Relationship (Continuation of Figure 2)

2.2.4 Conclusion

In Sections 2.1 to 2.2.3, we have presented and analyzed the ideas regarding the statistical relationship between calculated prediction results and experimental test data of EMP coupling to electronic systems. Within the framework of our methodology, we have pointed out some difficult areas which need substantial attention. Furthermore, there are several necessary pieces of information (see Figures 1,2,3) that we simply assumed as known quantities and used as inputs to deduce assessment conclusions. In fact, attaining these pieces of assumed information generates complicated problem areas which in themselves deserve substantial investigation. In short, we have outlined the ideas and the procedures to solve the testing-prediction problem for EMP coupling, and have spelled out the ingredients needed to arrive at a final assessment, rather than solved in detail any particular problem for an explicitly specified system. In the next section, we shall illustrate the detailed pursuit and execution of our ideas by presenting a relatively simple example.

SECTION III. AN EXAMPLE

3.1 A SIMPLIFIED MODEL OF THE TESTING-PREDICTION PROBLEM IN EMP COUPLING

To illustrate the application and the implementation of our heretofore advocated ideas, we shall consider the following simplified model. Suppose we have an electronic system and wish to determine its EMP-induced responses at a large number n subsystem-level black-box points x_i, $i=1,2...,n$, say the input pins to the instrumentation panel. Further, the largeness of n and the testing constraints in reality forbid us from performing a full-scale EMP test of all the pins. Therefore, we have to resort to prediction.

Without compromising any essential concepts, we can further simplify the model by assuming that all the n pins we are interested in can be classified as a single group in that the variation distribution of their true responses R_i^{oE} to a given EMP environment is normal with mean μ_g^{oE} and standard deviation (s.d.) σ_g^{oE}:

$$R_i^{oE} \in N\left(\mu_g^{oE},\ \sigma_g^{oE}\right),\ i = 1, \cdots n \tag{4}$$

Here, for the superscripts, the o denotes the true value and the E denotes the given EMP environment; and for the subscripts, the

i denotes the ith pin and the g denotes the group. Our objective is to quantitatively estimate [6] the μ_g^{OE} and the σ_g^{OE} via a statistical comparison of prediction results and test data of EMP coupling.

Now, we choose randomly n_s of the pins (usually $n_s \ll n$); identify for each of them their dominant coupling paths leading from that pin to its POEs and compute the coupling matrices C_{ij}^C, $i=1, 2 \ldots n_s$, $j=1, 2 \ldots J(i)$; predict at these sampled pins the EMP responses R_i^C due to a simulated environment consisting of driving sources E_j at the j^{th} POE according to

$$R_i^C = \sum_{j=1}^{J(i)} C_{ij}^C E_j , \quad i = 1,2 \cdots n_s \quad ; \tag{5}$$

and measure the corresponding testing responses

$$R_i^T = \sum_{j=1}^{J(i)} C_{ij}^T E_j , \quad i = 1,2 \cdots n_s \tag{6}$$

Again, as in (1) and (2), we assumed an accurate control and knowledge of the environment E_j. A final simplifying assumption we shall make is the unbiased normality of both R_i^C and R_i^T, i.e.,

$$R_i^C \in N \left(R_i^O, \sigma_i^C\right), \; i = 1,2 \cdots n_s \tag{7}$$

$$R_i^T \in N \left(R_i^O, \sigma_i^T\right), \; i = 1,2 \cdots n_s \tag{8}$$

with unknown standard deviation (s.d.)

$$\sigma_i^C = \sigma^C, \ i=1,2\ldots n_s \tag{9}$$

and known s.d.

$$\sigma_i^T = \sigma^T, \ i=1,2\ldots n_s \tag{10}$$

Here, the R_i^O is the true response at x_i to the simulated testing environment. Notice that the s.d.'s assumed in (9) and (10) are independent of the true responses rather than proportional to them. This represents a high degree of simplification of the reality and chosen to simplify the mathematical manipulations. We shall come back to this later.

From the above description of the simplified model, our tasks are to estimate the values of and to quantify the uncertainties in the σ^C, the R_i^O for $i=1,2\ldots n_s$; the R_i^{OE} for $i=1,2\ldots n_E$; and finally the μ_g^{OE} and σ_g^{OE}. Here, the n_E is the number of sampled points $\{x_i\}_{i=1}^{n_E}$ whose responses to a threat EMP environment will be predicted, via the testing-calibrated prediction technology, and used in the system vulnerability assessment to that threat EMP. Often $n_s \lesssim n_E \ll n$ and the sets of points $\{x_i\}_{i=1}^{n_E}$ and $\{x_i\}_{i=1}^{n_s}$ may overlap to a considerable extent but not necessarily be the same. The following subsections carry out these tasks one at a time.

3.2 ESTIMATING THE σ^C

Following the assumed forms of R_i^C and R_i^T, (7) to (10), their differences are normal:

$$D_i \equiv R_i^C - R_i^T \in N\left\{0, \sqrt{(\sigma^C)^2+(\sigma^T)^2}\right\}, \quad i=1,2\cdot\cdot n_s \quad (11)$$

Thus, we can make use of the D_i to estimate the $(\sigma^C)^2 + (\sigma^T)^2$ and thus the $(\sigma^C)^2$. The actual procedure of doing so is slightly complicated by that of incorporating the known knowledge of σ^T into our statistical estimate of $(\sigma^C)^2 + (\sigma^T)^2$, and thereby into $(\sigma^C)^2$. In fact, the only way to do so which we have been able to discover is via a Bayesian Fiducial estimation method, a method not entirely rigorous and satisfactory [7]. However, for lack of anything better we have to use it, and the result is (see Appendix A):

$$P_f\left\{(\hat\sigma_-^C)^2 < (\sigma^C)^2 < (\hat\sigma_+^C)\right\} \geq \frac{\gamma\left[a,\ \min\left(b, \frac{(n_s-1)\ \hat\sigma_s^2}{(\sigma^T)^2}\right);\ n_s\right]}{\gamma\left[0,\ \frac{(n_s-1)\hat\sigma_s^2}{(\sigma^T)^2};\ n_s\right]} \quad (12)$$

Here, the notations are defined

$$(\hat\sigma_-^C)^2 \equiv \max\left(0, \frac{(n_s-1)\hat\sigma_s^2}{b} - (\sigma^T)^2\right) \quad (13)$$

$$(\hat\sigma_+^C)^2 \equiv \frac{(n_s-1)\hat\sigma_s^2}{a} - (\sigma^T)^2 \quad (14)$$

$\gamma(a, b; n_s) \equiv$ probability that a $\chi^2(n_s)$-distributed random variable lies in the interval (a, b), where the $o<a<b$ and the $\chi^2(n_s)$ is a chi-square distribution of n_s degree of freedom (15)

$$a < \frac{(n_s-1)\hat{\sigma}_s^2}{(\sigma^T)^2} \tag{16}$$

$$\hat{\sigma}_s^2 \equiv \frac{\sum_{i=1}^{n_s} (D_i - 0)^2}{n_s-1} \Rightarrow \frac{\hat{\sigma}_s^2}{\frac{(\sigma^T)^2+(\sigma^C)^2}{n_s-1}} \varepsilon \; \chi^2(n_s) \tag{17}$$

The inequality (12) states that, as a result of calibrating the prediction uncertainty by using experimental testing data, we have at least the confidence described by the right-hand side of (12) that the unknown uncertainty spread, the s.d. σ^C, in the prediction technology is indeed between $\hat{\sigma}_-^C$ and $\hat{\sigma}_+^C$.

3.3 ESTIMATING THE SAMPLED R_i^O

The best estimates of the values of the true responses R_i^O, i=1, 2, · · n_s, at the sampled pins, while being driven by the known simulated environment E_j's, are to be made solely by using the test data (6), in conjunction with the assumed known testing accuracy described by (8) and (10). No extra advantage can be gained by employing the predicted values R_i^C, i=1, 2, ··n_s, because the very uncertainties in these predictions are themselves not known before being calibrated by the test data. Thus, the estimates of R_i^O are simply

$$P\left\{R_i^T - \sigma^T d < R_i^O < R_i^T + \sigma^T d\right\} = \lambda(-d,d), \; i=1,2,\cdot\cdot n_s \tag{18}$$

where $d>0$ and $\lambda(-d,d) \equiv P\{|X|>d \mid X \varepsilon N(0,1)\}$.

However, these estimates (18) of R_i^O are of little direct interest, because they are responses at those pins sampled for testing under merely a controlled (see (5)) and simulated EMP environment rather than under a real EMP threat environment.

3.4 ESTIMATING THE R_i^{OE}

Having estimated the uncertainty error σ^C in the prediction technology, we can use the thus-calibrated prediction technology to predict the EMP responses at the pins under a real EMP threat environment. We must make two remarks before we proceed.

First, we have assumed that the same prediction capability prevails in making coupling predictions for both the EMP simulation responses and the EMP threat responses, whereas only the former capability has been calibrated against by statistically comparing its results with the test data. This assumption is very important. However, usually it is not mentioned but is assumed and employed implicitly in the rationale for obtaining and using the testing data. The assumption is approximately and reasonably true in reality when we genuinely apply the same level of technical effort in making the coupling prediction for both the simulated testing and the real threat EMP environments. In fact, this must be the case unless one makes grossly careless coupling predictions for the real threat case after having carefully learned a lot from the simulated prediction-testing efforts and then ignored that learning. Although this circumstance is possible, we exclude it from our technical investigation-- a circumstance under which one can make as much inaccuracy as he wishes, independent of any previously accumulated knowledge, in predicting coupling responses to real EMP threats.

Second, we have assumed that it is unfeasible to perform the real threat EMP coupling experiments due to difficulties on at least two counts: the creation of the various full threat environments and measurements of the responses at as many pins as we could theoretically predict. Otherwise, in the theme of calibrating and establishing the prediction capability by using known experimental capabilities, there would be no operational need for any prediction capability at all.

Now, in accord with the above remarks, we have

$$R_i^{cE} \in N\left(R_i^{oE}, \sigma^c\right), \quad i = 1, 2, \cdots n_E \tag{19}$$

for the sampled and predicted responses to an EMP threat -

$$R_i^{cE} \equiv \sum_{j=1}^{J(i)} C_{ij}^{c} E_j^{E}, \quad i = 1, 2, \cdots n_E \tag{20}$$

Here, the C^C_{ij} is the predicted coupling matrix from the j^{th} dominant POE to the i^{th} pin. Again, we have assumed an accurate knowledge of the EMP environment E^E_j. Using the R^{CE}_i as the estimator for the R^{OE}_i and incorporating the statistical knowledge of σ^C by using its fiducial probability density, we immediately obtain estimates for R^{OE}_i (see Appendix B):

$$P_f\left\{-b+R^{CE}_i < R^{OE}_i < b+R^{CE}_i\right\} = \int_0^\infty p_{(\sigma^C)^2}(\eta)\,\lambda\left(\frac{-b}{\sqrt{\eta}},\frac{b}{\sqrt{\eta}}\right)d\eta \qquad (21)$$

$$i = 1, 2, \cdots n_s,\ b>0,$$

where $p_{(\sigma^C)^2}(\eta)$ is the fiducial probability density of the estimated parameter $(\sigma^C)^2$ obtained from (12) (see Appendix B)

$$p_{(\sigma^C)^2}(\eta) = \begin{cases} \dfrac{\dfrac{(n_s-1)\hat{\sigma}^2_s}{[\eta+(\sigma^T)^2]^2}\cdot p_{\chi^2(n_s)}\left(\dfrac{(n_s-1)\hat{\sigma}^2_s}{\eta+(\sigma^T)^2}\right)}{\gamma\left(0,\dfrac{(n_s-1)\hat{\sigma}^2_s}{(\sigma^T)^2};\ n_s\right)}, & \text{if } \eta>0 \\ = 0, & \text{if } \eta<0 \end{cases} \qquad (22)$$

and the $p_{\chi^2(n_s)}(\xi)$ is the probability density of a chi-square distribution of n_s degrees of freedom evaluated at ξ. The one simple special case of (21) is for a large sample size and small $(\sigma^T)^2$ such that the $(\sigma^C)^2$ is narrowly distributed around the $\hat{\sigma}^2_s$ and (21) gives approximately

$$P\left|-b + R^{CE}_i < R^{OE}_i < +b + R^{CE}_i\right| \approx \lambda\left(\frac{-b}{\hat{\sigma}_s},\ \frac{b}{\hat{\sigma}_s}\right) \qquad (23)$$

This result for R^{OE}_i is similar to the result (18) for R^O_i, but with the corresponding uncertainties in (11), the difference between the prediction and the testing in the prediction-testing calibration process, virtually all attributed to the prediction part. That last statement is easily seen by rewriting $b/\hat{\sigma}_s \equiv d$ in (23). In general, the confidence estimate (21) can be obtained easily by a numerical calculation.

3.5 ESTIMATING THE μ_g^{OE} and σ_g^{OE}

The procedure for estimating the μ_g^{OE} and the σ_g^{OE}, under the normality assumption (4), is similar to that for estimating R_i^{OE}. Namely, we first estimate conditionally assuming σ^c is known, then we use the fiducial probability distribution of σ^c obtained previously to unravel that condition. The results are (see Appendix C)

$$P_f\left\{-b + \hat{\mu}_g^{cE} < \mu_g^{OE} < b + \hat{\mu}_g^{cE}\right\}$$

$$= \int_0^\infty d\eta \; P_{(\sigma^c)^2}(\eta) \int_0^\infty d\xi \; P_{\sigma^2|(\sigma^c)^2}(\xi|\eta) \; . \qquad (24)$$

$$\lambda\left(\frac{-b\sqrt{n_s}}{\xi}, \frac{\sqrt{n_s}}{\xi}\right) \qquad b > 0$$

where

$$\hat{\mu}_g^{cE} \equiv \frac{\sum_{i=1}^{n_E} R_i^{cE}}{n_E} \qquad (25)$$

$$P_{\sigma^2|(\sigma^c)^2}(\xi|\eta) = 0 \;, \text{ if } \xi < \eta$$

$$= \frac{\dfrac{(n_s-1)\left(\hat{\sigma}_g^{cE}\right)^2}{\xi} \; P_{\chi^2(n_E-1)}\left(\dfrac{(n_E-1)\left(\hat{\sigma}_g^{cE}\right)^2}{\xi}\right)}{\lambda\left(0, \dfrac{(n_E-1)\left(\hat{\sigma}_g^{cE}\right)^2}{\eta}; \; n_E-1\right)} \;, \text{ if } \xi > \eta \qquad (26)$$

$$\left(\hat{\sigma}_g^{cE}\right)^2 \equiv \frac{\sum_{i=1}^{n_E}\left(R_i - \hat{\mu}_g^{cE}\right)^2}{(n_E-1)} \qquad (27)$$

and

$$P_f\left\{a_1 < \left(\sigma_g^{OE}\right)^2 < a_2\right\} = \int_0^\infty d\eta \; p_{\left(\sigma^C\right)^2}(\eta)\int_{a_1+\eta}^{a_2+\eta} p_{\sigma^2|\left(\sigma^C\right)^2}(\xi|\eta)\, d\xi \qquad (28)$$

Notice that the (28) recovers precisely the familiar expression

$$P\left\{\max\left(0, \frac{\left(n_E-1\right)\left(\hat{\sigma}_g^{cE}\right)^2}{b} - \left(\sigma^{cl}\right)^2\right) < \left(\sigma_g^{OE}\right)^2 < \frac{\left(n_E-1\right)\left(\hat{\sigma}_g^{cE}\right)^2}{a} - \left(\sigma^{cl}\right)^2\right\}$$

$$\geqslant \frac{\gamma\left(a, \min\left[b, \frac{\left(n_E-1\right)\left(\hat{\sigma}_g^{cE}\right)^2}{\left(\sigma^{cl}\right)^2}\right]; \left(n_E-1\right)\right)}{\gamma\left(0, \frac{\left(n_E-1\right)\left(\hat{\sigma}_g^{cE}\right)^2}{\left(\sigma^{cl}\right)^2}; n_E-1\right)} \qquad (29)$$

in the special case when the $(\sigma^C)^2$ is a known constant (or nearly known with high confidence in a narrow range about that constant) $(\sigma^{cl})^2$ and the parameters a_1 and a_2 are

$$a_1 \equiv \frac{\left(n_E-1\right)\left(\hat{\sigma}_g^{cE}\right)^2}{b} - \left(\sigma^{cl}\right)^2$$

$$a_2 \equiv \frac{\left(n_E-1\right)\left(\hat{\sigma}_g^{cE}\right)^2}{a} - \left(\sigma^{cl}\right)^2 \qquad (30)$$

Inequality (29) is the same as (12), except for the obvious replacement of the corresponding quantities in the two respective cases. This is, of course, as it should be, since the estimation of σ_g^{OE} with the σ^C as a known constant σ^{cl} is the same as the estimation of σ^C with the σ^T as a known constant (see Appendices A and C).

3.6 FROM CALIBRATED COUPLING PREDICTION TO SYSTEM PERFORMANCE ASSESSMENT AND SOME REMARKS

Now that we have estimated the statistical uncertainties in the sampled and predicted coupling responses to an EMP threat environment and estimated the variation distribution of all responses, we can proceed to attain the system performance assessment as outlined in Section II by making use of those other pieces of information as further needed inputs. We shall not elaborate on this, because our main interest here is to obtain a testing-prediction relationship for EMP coupling. For the above simplified example, we have obtained that relationship in explicit detail. However, several remarks concerning relaxing some of the simplifications made in the example are in order.

First, in reality there may be many groups of pins and each group has its members' response to EMP as we illustrated in the example. Thus, we would apply our illustrated method to each such group. This multiplies our efforts by as many times as there is the number of distinct groups.

Second, the various probability distributions assumed may not be normal, or even closely so. This, in principle, poses no difficulties, because we could assume other appropriate forms of distributions as the situation should dictate and estimate accordingly, though with possibly more complicated mathematics, or we could directly deal with percentile points by using distribution free-methods.

Third, even for the assumed normal distributions, the uncertainties (9) and (10) assumed and used in the example are oversimplifications in that σ_i^T and σ_i^C are independent of the true response R_i^O. A more realistic assumption would be errors proportional to responses given by

$$\sigma_i^C = K^C |R_i^O| \tag{31}$$

$$\sigma_i^T = K^T |R_i^O| \tag{32}$$

instead of (9) and (10). Then the first task would be to estimate the K^C, starting with a known K^T. Similarly, the σ^C in (19) should also be replaced by $K^C R_i^{OE}$. Such an assumption leads to slightly more complicated estimation calculations than those illustrated, but causes no unusual difficulties (Appendix D).

Fourth, when the environments at the POEs are themselves not accurately known, we need to unravel those uncertainties by one further step of aggregating our conditional probabilities, while making use of, as an additional known input, the statistically quantified uncertainties in those environments.

Fifth, the <u>direct</u> use of the prediction uncertainties obtained from the partially simulated testing-prediction situations in real EMP threat predictions is not correct even if we do apply a consistent level of prediction effort to both cases as assumed at the beginning of Section 3.4. The probability of the self-deceiving fallacy, to which such a direct extrapolation of prediction uncertainties is susceptible as pointed out at the end of Section 2.2.1, in inadvertently ignoring possible major coupling paths must be accounted for. This may slightly or considerably reduce the prediction accuracy as yielded directly by the testing-prediction calibration mechanism illustrated. Although we have suggested a plausible way of complementary testing (see Section 2.2.1) to account for this effect, we have not yet performed any in-depth investigation of this.

REFERENCES AND FOOTNOTES

1. This is always the situation except when one has test data for all possible cases of interest. Under that exceptional circumstance, those test data by themselves alone constitute the full knowledge. As such, the theoretical capability is not operationally needed at all and, thus, any fantasy in our imaginary thinking process to gain understanding is immaterial. Obviously, however, the data from limited, incomplete, and imperfect EMP interaction testings fall far short of that exceptional circumstance.

2. See, e.g., W. R. Graham and C. T. C. Mo, "Probability Distribution of CW Induced Currents on Randomly Oriented Sub-Resonant Loops and Wires," AFWL Interaction Note 321, May 1976.

   M. A. Morgan and F. M. Tesche, "Statistical Analysis of Critical Load Excitations Induced on A Random Cable System by An Incident Driving Field: Basic Concepts and Methodology," AFWL Interaction Note 249, July 1975.

3. Assume the interactions are linear. If not, the subsequent analysis can at most give an indication of the extent to which linear model starts breaking down.

4. See, e.g., C. E. Baum, "Extrapolation Techniques for Interpreting the Results of Tests in EMP Simulators in Terms of EMP Criteria," AFWL EMP Sensor and Simulation Note 222 (March 1977).

5. Notice that we could define the safety margin by a monotonic function of the ratio of the T_i and the $R_i^{(c)}$, instead of the T_i and the $R_i^{(c)}$ themselves, if it is more convenient for computational purposes to do so. For example we could define

5. (cont.)

$$S_i^{(c)} \equiv \ln \frac{T_i}{R_i^{(c)}}$$

if the logarithm of the responses allows simple mathematical manipulation statistically. The safety margins so defined serve as normalized subsystem malfunction threshold indicators.

6. For the statistical methods used in this report, see any typical text book on mathematical statistics, e.g., M. G. Kendall and A. Stuart, " The Advanced Theory of Statistics, Vol. 2, Inference and Relationship," (Second Edition, 1967, Hafner Publishing Co.).

7. We must point out that this is a controversial area in statistical inference. In fact, there is <u>no</u> way of incorporating the a priori knowledge rigorously into the statistical estimate theory. The Bayesian renormalization procedure seems to be an intuitively plausible method, and yields intuitively sound results, although it rests on somewhat undefined and shaky mathematical foundations. Furthermore, it seems to be the <u>only</u> method in existence to do this incorporation. In view of such, we have no choice but use it, keeping in mind its being less than totally satisfactory.

APPENDIX A

DERIVATION OF CONDITIONAL CONFIDENCE ESTIMATES WITH PRIOR PARTIAL KNOWLEDGE

A rigorous theory of statistical estimation that incorporates prior knowledge of the parameters being estimated is known to not exist, and is even considered by many to be impossible to find. The only existing formalism for incorporating such prior knowledge in a statistical estimation is via a Bayesian formalism or a fiducial estimation method---which are the same if we use as our best guess a uniform fiducial probability density in the region of the parameter unrestricted by the prior knowledge. Although yielding intuitively plausible results, that formalism is somewhat mathematically undefined and unclarified at its very foundation. In view of the lack of anything better, we have to use it. However, in this appendix, we will carry the rigorous probabilistic formalism as far as possible, and only at the final stage clearly and exactly point out the approximations involved to reach a practically useful statistical estimation formula. In particular, we shall identify the approximation that yields the Bayesian estimate. The following is pertinent to estimating the standard deviation of a normal population when its mean is known.

Suppose we draw random samples D_i, $i = 1, 2 \cdots n_s$, from a normal population $N(0,\sigma)$, where the σ is unknown, except that

$\sigma \geq \sigma_0$ with σ_0 being a known positive number, and is to be estimated from the sampled values D_i. To estimate σ, with a known prior knowledge $\sigma \geq \sigma_0$, we use the Bayesian fiducial method. First, the unbiased estimator of the population variance, using the known population mean being 0, given by

$$\hat{\sigma}^2 = \frac{\sum_i^{n_s} D_i^2}{(n_s - 1)} \tag{A-1}$$

is a random variable with its distribution given by

$$\frac{\hat{\sigma}^2}{\left(\frac{\sigma^2}{n_s - 1}\right)} \varepsilon \chi^2(n_s), \tag{A-2}$$

where $\chi^2(n_s)$ is a chi-square distribution of n_s degree of freedom with a probability density $p_{\chi^2(n_s)}(\xi)$. Using (A-1) and (A-2), we obtain (see Figure A-1)

$$P\left\{a < (n_s - 1) \cdot \frac{\hat{\sigma}^2}{\sigma^2} < b\right\} = \gamma(a, b; n_s) \tag{A-3}$$

where $\gamma(a,b;n_s)$ is the area between a and b, where $o<a<b$, beneath a $\chi^2(n_s)$-ly distributed probability density function. Merely rewriting (A-3) as

$$P\left\{\frac{(n_s - 1)\hat{\sigma}^2}{b} < \sigma^2 < \frac{(n_s - 1)\hat{\sigma}^2}{a}\right\} = \gamma(a, b; n_s) \tag{A-4}$$

we have the probability $\gamma(a,b;n_s)$ that the random interval $I \equiv \left((n_s-1)\hat{\sigma}^2/b,\ (n_s-1)\hat{\sigma}^2/a\right)$ indeed encompasses the true parameter value σ^2. Using as a fiducial probability for the distribution of σ^2, a probability intuitively associated to a particular interval obtained from a particular sample (size n_s) as the value assumed by the random interval I, we re-interpret (A-4) as the <u>fiducial</u> probability that the σ^2 lies in that interval I. Notice that this is a <u>completely</u> <u>new</u> definition and is a measure of the fuzziness of our knowledge about the true σ^2 which is a constant, although unknown, <u>number</u>. Customarily, we just say that we have a confidence $\gamma(a,b;\, n_s)$ that the σ^2 lies in I, with all those fine details just mentioned as to what it really means kept in the back of our mind. Now to incorporate the prior knowledge of $\sigma \geq \sigma_o$ into our estimation, we make use of the Bayesian conditional probability expression for the fiducial probability for σ^2

$$P_f \left\{ \frac{(n_s-1)\hat{\sigma}^2}{b} < \sigma^2 < \frac{(n_s-1)\hat{\sigma}^2}{a} \,\middle|\, \sigma^2 > \sigma_o^2 \right\}$$

$$= \frac{P_f \left\{ \sigma^2 > \sigma_o^2 \text{ and } \frac{(n_s-1)\hat{\sigma}^2}{b} < \sigma^2 < \frac{(n_s-1)\hat{\sigma}^2}{a} \right\}}{P_f \left\{ \sigma^2 > \sigma_o^2 \right\}} \qquad \text{(A-5)}$$

and rewrite it into

$$P_f \left\{ \frac{(n_s-1)\hat{\sigma}^2}{b} < \sigma^2 < \frac{(n_s-1)\hat{\sigma}^2}{a} \,\middle|\, \sigma^2 > \sigma_o^2 \right\}$$

$$= \frac{\gamma \left(a,\ \min \left[b,\ \frac{(n_s-1)\hat{\sigma}^2}{\sigma_o^2} \right];\ n_a \right)}{\gamma \left(0,\ \frac{(n_s-1)\hat{\sigma}^2}{\sigma_o^2}\ ;\ n_s \right)} \qquad \text{(A-6)}$$

by using (A-4) in the fiducial sense. Here, the parameter a must be so chosen that

$$o < a < \frac{(n_s-1)\hat{\sigma}^2}{\sigma_o^2} \qquad \text{(A-7)}$$

for the particular sample value $\hat{\sigma}^2$ used. The meaning of (A-6) is the following (see Figure A-1): given the prior knowledge that $\sigma^2 \geq \sigma_o^2$, then for a sampled value of $\hat{\sigma}^2$ of size n_s and for some arbitrarily chosen constants a, b such that $b > a$ and a satisfies (A-7), we have the

$$\text{confidence} = \frac{\gamma\ (a,\ b;\ n_s)}{\left(0,\ \frac{(n_s-1)\,\hat{\sigma}^2}{\sigma^2};\ n_s\right)} > \gamma\ (a,\ b;\ n_s) \qquad \text{(A-8)}$$

that the true σ^2 lies within I if the known σ_o^2 and the sampled value of $\hat{\sigma}^2$ happen to satisfy $\sigma_o^2 < \frac{(n_s-1)\,\hat{\sigma}^2}{b}$; and we have the

$$\text{confidence} = \frac{\gamma\left(a,\ \frac{(n_s-1)\,\hat{\sigma}^2}{\sigma_o^2};\ n_s\right)}{\gamma\left(0,\ \frac{(n_s-1)\,\hat{\sigma}^2}{\sigma_o^2};\ n_s\right)} < \frac{\gamma\ \left(a,\ b;\ n_s\right)}{\gamma\left(0,\ \frac{(n_s-1)\,\hat{\sigma}^2}{\sigma_o^2};\ n_s\right)} \qquad \text{(A-9)}$$

that the true σ^2 lies within I if the known σ_o^2 and the sampled value of $\hat{\sigma}^2$ happen to satisfy $\sigma_o^2 > \frac{(n_s-1)\,\hat{\sigma}^2}{b}$.

Notice that (A-6) implies

$$P_f\left\{\frac{(n_s-1)\,\hat{\sigma}^2}{b} < \sigma^2 < \frac{(n_s-1)\,\hat{\sigma}^2}{a}\ \middle|\ \sigma^2 > \sigma_o^2\right\} \qquad \text{(A-10)}$$

$$\xrightarrow[\sigma_o \to o]{} \quad \gamma\ (a,\ b;\ n_s)\ ,$$

as it should when there is no prior knowledge about σ^2. Further, notice that the <u>confidence</u> (A-8) is enhanced as compared to the no-prior knowledge case, but the estimation interval

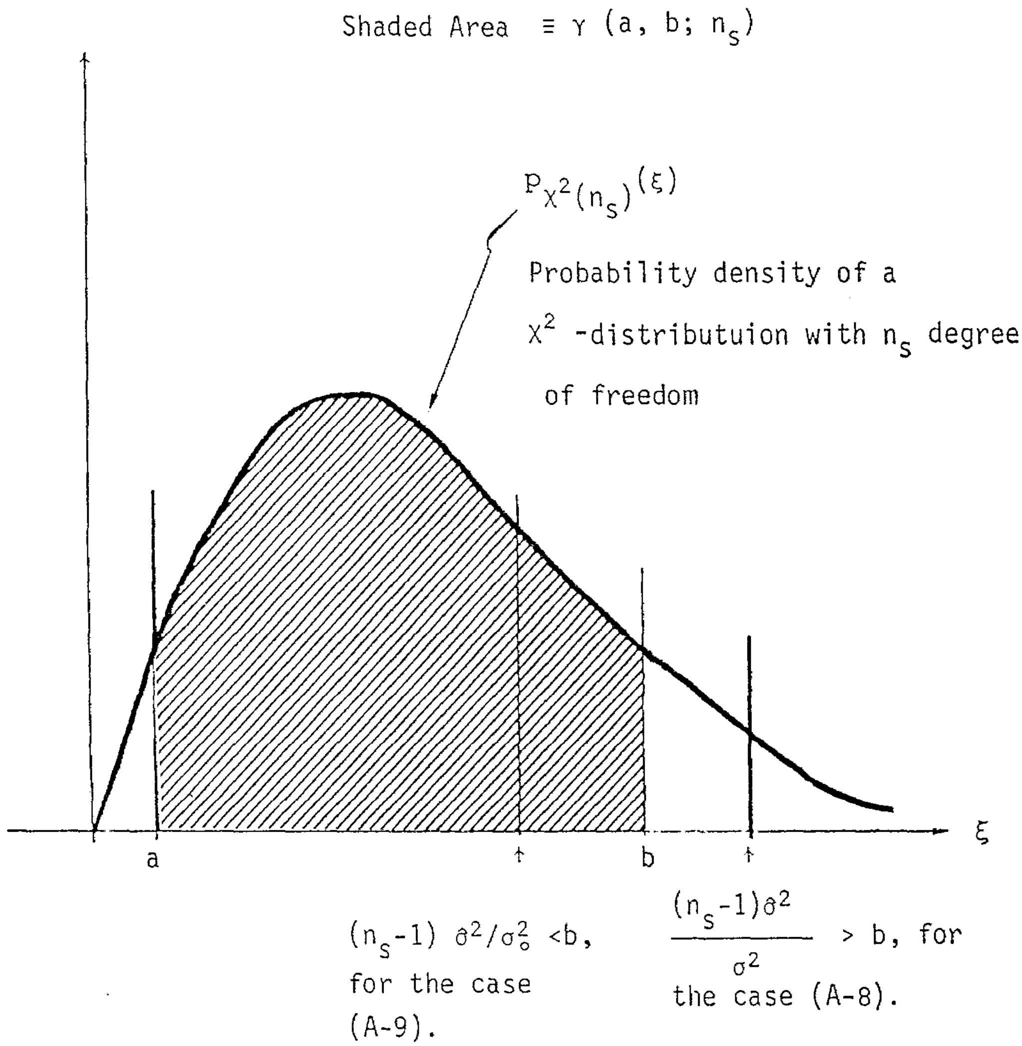


Figure A-1. The Chi-Square Distribution and Confidence Intervals is the same as the original one of the no-prior knowledge case if the estimation interval does not conflict with the prior knowledge, i.e., the lower boundary $(n_s-1)\hat{\sigma}^2/b$ of the estimation interval is greater than the σ_o^2. However, if the estimation interval does conflict with the prior knowledge, i.e., $\sigma_o^2 > (n_s-1)\hat{\sigma}^2/b$, we can only infer a confidence reduced from its fully enhanced value, but for a shortened (tighter) estimation interval with the portion of the original interval conflicting the prior knowledge cut-off. In the particular case when the original estimation interval for the sample taken and for the constants a, b chosen lies completely outside the region permitted by the prior knowledge, i.e., $\sigma_o^2 > (n_s-1)\hat{\sigma}^2/a$, we have, in view of the prior knowledge about σ^2, gained no further knowledge about σ^2 from that original estimation interval and can conclude nothing in the prior-knowledge permitted region at that original confidence level corresponding to those originally chosen a and b.

Finally, putting (A-6) and (A-7) together, we have the shorthand expression for the $\sigma^2 \geq \sigma_o^2$ incorporated estimate for σ^2:

$$p_f \left\{ \max \left[\sigma_o^2, \frac{(n_s-1)\hat{\sigma}^2}{b} \right] < \sigma^2 < \frac{(n_s-1)\hat{\sigma}^2}{a} \;\middle|\; \sigma^2 > \sigma_o^2 \right\}$$

$$= \frac{\gamma\left(a, \max\left[a, \min\left(b, \frac{(n_s-1)\hat{\sigma}^2}{\sigma_o^2}\right)\right]; n_s\right)}{\gamma\left(0, \frac{(n_s-1)\hat{\sigma}^2}{\sigma_o^2}; n_s\right)} \qquad \text{(A-11)}$$

Now we would like to carry a formally clearer estimation theory and see the link that its final result has to the one just obtained above from using the mathematically unclarified and not-rigorously defined Bayesian fiducial procedures. With the value of the σ^2 being some constant, although unknown, number, rigorously we have the same probability statements (A-3) for the true random variable $\hat{\sigma}^2$ normalized by a constant factor $\sigma^2/(n_s-1)$. Again, (A-4) remains a valid statement for the <u>random interval</u> I regardless of the prior knowledge about σ, $\sigma^2 \geq \sigma_o^2$, with σ_o^2 another constant but of known value. However, in making a probabilistic statement of which random interval encompasses the constant number σ^2, we can use the prior knowledge $\sigma^2 \geq \sigma_o^2$ and pose the probabilistic statement in a better way than without using that knowledge. We ask for the probability that the higher one of the two true random intervals, $\left(\sigma_o^2,\ (n_s-1)\ \hat{\sigma}^2/a\right)$ and $\left((n_s-1)\ \hat{\sigma}^2/b,\ (n_s-1)\ \hat{\sigma}^2/a\right)$, indeed encompasses σ^2, given that $\sigma^2 \geq \sigma_o^2$, by excluding those impossible cases that have the upper bound of the interval beneath the σ_o^2:

$$P\left\{\max\left(\sigma_o^2, \frac{(n_s-1)\hat{\sigma}^2}{b}\right) < \sigma^2 < \frac{(n_s-1)\hat{\sigma}^2}{a} \,\middle|\, \sigma^2 > \sigma_o^2 \text{ and } \hat{\sigma}^2 > \frac{a\sigma_o^2}{(n_s-1)}\right\} = ? \tag{A-12}$$

Now we shall clearly deviate from the rigorous definition of the conditional probability, which dictates away the condition $\sigma^2 \geq \sigma_o^2$ since it is a given known truth, by <u>formally</u> writing

$$P\left\{\max\left(\sigma_o^2, \frac{(n_s-1)\hat{\sigma}^2}{b}\right) < \sigma^2 < \frac{(n_s-1)\hat{\sigma}^2}{a} \,\middle|\, \sigma^2 > \sigma_o^2 \text{ and } \hat{\sigma}^2 > \frac{a\sigma_o^2}{(n_s-1)}\right\}$$

$$= P\left\{a < \frac{(n_s-1)\hat{\sigma}^2}{\sigma^2} < \min\left(b, \frac{(n_s-1)\hat{\sigma}^2}{\sigma_o^2}\right) \,\middle|\, \frac{a\sigma_o^2}{\sigma^2} < \frac{(n_s-1)\hat{\sigma}^2}{\sigma^2} < \frac{(n_s-1)\hat{\sigma}^2}{\sigma_o^2}\right\} \tag{A-13}$$

and then manipulating with the property of a conditional probability of the right-hand side of (A-13) to get

$$P\left\{\max\left(\sigma_o^2, \frac{(n_s-1)\hat{\sigma}^2}{b}\right) < \sigma^2 < \frac{(n_s-1)\hat{\sigma}^2}{a} \,\middle|\, \sigma^2 > \sigma_o^2 \text{ and } \hat{\sigma}^2 > \frac{a\sigma_o^2}{(n_s-1)}\right\}$$

$$= \frac{\gamma\left(a, \min\left(b, \frac{(n_s-1)\hat{\sigma}^2}{\sigma_o^2}\right); n_s\right)}{\gamma\left(\frac{a\sigma_o^2}{\sigma^2}, \frac{(n_s-1)\hat{\sigma}^2}{\sigma_o^2}; n_s\right)} \tag{A-14}$$

If we substitute for the unknown σ^2 at the right-hand side of (A-14) by its maximum possible value, ∞, we get a smaller and more conservative measure for our confidence. This more conservative confidence is identical to the Bayesian fiducial confidence (A-6), or (A-11) since the (A-14) itself implies (A-7), obtained previously. Thus, the undefined treatment of σ^2 as a fiducial random variable is equivalent to the formal substitutions just described in (A-13) and (A-14). This gives the inequality relations in (12) and (29).

For the lack of anything better and because the (A-11) is intuitively plausible, as we have already shown in checking its various limits, we use (A-11) in making our statistical estimate with the given prior knowledge $\sigma^2 \geq \sigma_o^2$. We conclude by reemphasizing that (A-11) or (A-14), unsatisfactory as they are, is the <u>only</u> type of formalism available at all in making such statistical estimates with prior knowledge.

Applying (A-11), using $\hat{\sigma}^2 = \hat{\sigma}_S^2$, $\sigma^2 = (\sigma^T)^2 + (\sigma^C)^2$, and $\sigma_o \equiv \sigma^T$, we obtain (12) to (17) for the estimation of σ^C in the text.

Finally, we write down the fiducial probability density for σ^2 corresponding to (A-6):

$$p_{\sigma^2}(\eta) = \frac{\dfrac{(n_S-1)\hat{\sigma}^2}{\eta^2} \; p_{\chi^2(n_S)}\left(\dfrac{(n_S-1)\hat{\sigma}^2}{\eta}\right)}{\gamma\left(0, \dfrac{(n_S-1)\hat{\sigma}^2}{\sigma_o^2} ; n_S\right)} \quad , \text{ if } \eta > \sigma_o^2 \qquad \text{(A-15)}$$

$$= 0 \quad , \text{ if } \eta < \sigma_o^2$$

APPENDIX B

THE DERIVATION OF THE MARGINAL ESTIMATES OF R_i^{cE} USED IN (21)

From (19), we have

$$p_{\left(R_i^{cE}-R_i^{oE}\right)\mid\left(\sigma^c\right)^2}(\xi\mid\eta) = \frac{e^{\frac{-\xi^2}{2\eta}}}{\sqrt{2\pi\eta}}\ , \quad -\infty<\xi<\infty \tag{B-1}$$

Now, using the fiducial probability density of σ^c, we get

$$p_{\left(R_i^{cE}-R_i^{oE}\right)}(\xi) = \int_0^\infty d\eta\ p_{\left(\sigma^c\right)^2}(\eta)\ p_{\left(R_i^{cE}-R_i^{oE}\right)\mid\left(\sigma^c\right)^2}(\xi\mid\eta) \tag{B-2}$$

where the $p_{\left(\sigma^c\right)^2}(\eta)$ is obtained from (A-15) or (12):

$$p_{\left(\sigma^c\right)^2}(\eta) = \frac{\dfrac{\left(n_s-1\right)\hat{\sigma}_s^2}{[\eta+(\sigma^T)^2]^2}\, p_{\chi^2(n_s)}\left(\dfrac{\left(n_s-1\right)\hat{\sigma}_s^2}{\eta+(\sigma^T)^2}\right)}{\gamma\left(0,\ \dfrac{(n_s-1)\hat{\sigma}_s^2}{(\sigma^T)^2}\ ;\ n_s\right)}\ , \quad \text{if } \eta>0,$$

$$= 0 \qquad\qquad , \quad \text{if } \eta<0 \tag{B-3}$$

Thus we obtain

$$P_f \left\{ -b + R_i^{cE} < R_i^{oE} < b + R_i^{cE} \right\}$$

$$= P \left\{ -b < R_i^{cE} - R_i^{oE} < b \right\}$$

$$= \int_{-b}^{b} p_{(R_i^{cE} - R_i^{oE})}(\xi) \; d\,\xi$$

$$= \int_{o}^{\infty} d\eta \quad p_{(\sigma^c)^2}(\eta) \quad \int_{-b}^{b} p_{(R_i^{cE} - R_i^{oE}) \mid (\sigma^c)^2}(\xi \mid \eta) \; d\,\xi$$

$$= \int_{o}^{\infty} d\eta \quad p_{(\sigma^c)^2}(\eta) \quad \int_{-\frac{b}{\eta}}^{\frac{b}{\eta}} \cdot \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \; dx$$

$$= \int_{o}^{\infty} d\eta \quad p_{(\sigma^c)^2}(\eta) \quad \cdot \; \lambda \left(\frac{-b}{\sqrt{\eta}} \cdot \frac{b}{\sqrt{\eta}} \right) \qquad \text{(B-4)}$$

APPENDIX C

DERIVATION OF THE ESTIMATES FOR μ_g^{OE} and AND σ_g^{OE} of (24) and (25)

From (4) and (19), we have

$$R_i^{cE} \equiv R_i^{cE} - R_i^{OE} + R_i^{OE} \in N\left(\mu_g^{OE}, \sqrt{(\sigma^C)^2 + (\sigma_g^{OE})^2}\right) \quad \text{(C-1)}$$

Thus, we have the estimates

$$\hat{\mu}_g^{cE} \equiv \frac{\sum_{i=1}^{n_E} R_i^{cE}}{n_E} \in N\left(\mu_g^{OE}, \sqrt{\frac{(\sigma^C)^2 + (\ \ _g^{OE})^2}{n_E}}\right) \equiv N\left(\mu_g, \frac{\sigma}{\sqrt{n_E}}\right) \quad \text{(C-2)}$$

$$\left(\hat{\sigma}_g^{cE}\right)^2 \equiv \frac{\left(\sum_{i=1}^{n_E} R_i^{cE} - \hat{\mu}_g^{cE}\right)^2}{n_E-1}$$

$$\Rightarrow \frac{\left(\hat{\sigma}_g^{cE}\right)^2}{\frac{\sigma^2}{(n_E-1)}} \in \chi^2 \ (n_E-1) \quad \text{(C-3)}$$

and the independence of these estimators. We can find the estimate region for μ_g^{OE} and σ_g^{OE} simultaneously as follows.

First, ignoring the $\sigma^2 \equiv (\sigma^C)^2 + (\sigma_g^{OE})^2 \geq (\sigma^C)^2$ temporarily and estimating for σ and μ_g^{OE} unconditioned by <u>any</u> prior knowledge of σ, we get the fiducial probability

$$P_f \left\{ -\sigma d + \hat{\mu}_g^{CE} < \mu_g^{OE} < \hat{\mu}_g^{CE} + \sigma d \text{ and } \frac{(n_E-1)\left(\hat{\sigma}_g^{CE}\right)^2}{b} \right.$$

$$\left. < \sigma^2 < \frac{\left(n_E-1\right)\ \hat{\sigma}_g^{CE\ 2}}{a} \right\} = P_f \left\{ -\sigma d + \hat{\mu}_g^{CE} < \mu_g^{OE} < \hat{\mu}_g^{CE} + \sigma d \right\}$$

$$\cdot\ P \left\{ \frac{(n_E-1)\left(\hat{\sigma}_g^{CE}\right)^2}{b} < \sigma^2 < \frac{(n_E-1)\left(\hat{\sigma}_g^{CE}\right)^2}{a} \right\}.$$

$$= \lambda\ (-d,d)\ \gamma\ \left(a,\ b;\ n_E\ -1\right) \qquad \text{(C-4)}$$

where $o<a<b$, $o\ d$, and the $\lambda(-d,\ d)$ and $\gamma\ (a,\ b;\ n_E-1)$ are as definded in Appendix A.

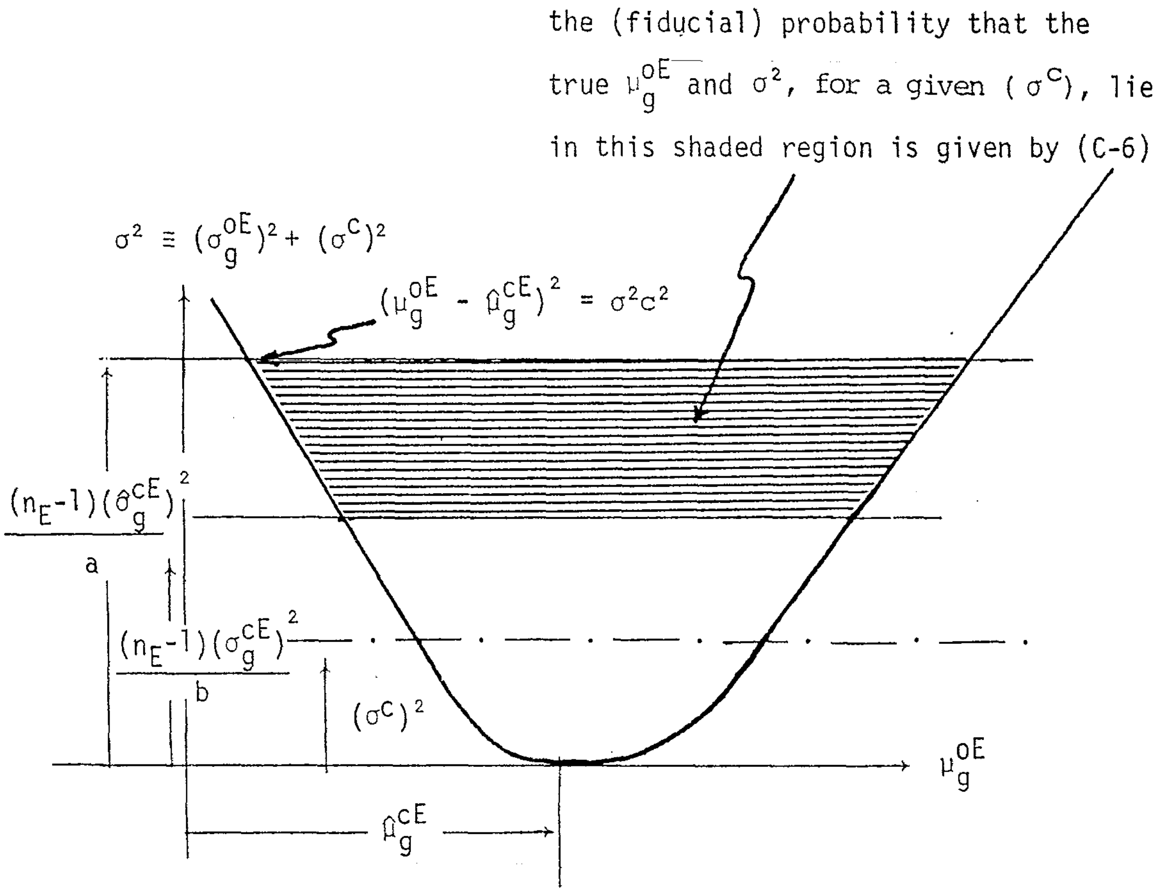
Second, we incorporate the knowledge of $\sigma^2 \equiv (\sigma^C)^2 + (\sigma_g^{OE})^2 \geq (\sigma^C)^2$, treating $(\sigma^C)^2$ as if it were a given constant, according to (A-11) but with the following substitutions

$$\sigma_o \longrightarrow \sigma^C$$

$n_S \longrightarrow n_E$, except in the order of χ^2 where $n_S \to n_E-1$

$$\hat{\sigma}^2 \longrightarrow \hat{\sigma}_g^{CE\ 2} \qquad \text{(C-5)}$$

and obtain (see Figure C-1)



Shaded Region: bounded by the parabola, the <u>uppermost</u> horizontal line, and the higher of the two lower horizontal lines.

Figure C-1. Estimating Conditionally the σ_g^{oE} and μ_g^{oE}, Where $o<a< \min\left(b, \frac{(n_s-1)(\hat{\sigma}_g^{cE})^2}{(\sigma^c)^2}\right)$, for a Given Fixed σ^c

$$P_f\left\{\left(-\sigma c + \hat{\mu}_g^{cE} < \mu_g^{oE} < \sigma d + \hat{\mu}_g^{cE}\right) \text{ and } \left(\max\left[(\sigma^c)^2,\ \frac{(n_E-1)\left(\hat{\sigma}_g^{cE}\right)^2}{b}\right]\right.\right.$$

$$\left.\left. < \sigma^2 < \frac{(n_E-1)\left(\hat{\sigma}_g^{cE}\right)^2}{a} \ \middle|\ \sigma^2 > \left[\sigma^c\right]^2\right)\right\}$$

$$= \lambda\ (-c,\ d)\ \frac{\gamma\left\{a,\ \max\left(a,\ \min\left[b,\ \frac{(n_E-1)\left(\hat{\sigma}_g^{cE}\right)^2}{(\sigma^c)^2}\right]\right);\ n_E-1\right\}}{\gamma\left\{0,\ \frac{(n_E-1)\left(\hat{\sigma}_g^{cE}\right)^2}{(\sigma^c)^2};\ n_E-1\right\}} \tag{C-6}$$

Finally, we accommodate the statistical nature in our grasp of the true prediction uncertainty spread σ^c itself by employing the fiducial probability density $p_{(\sigma^c)^2}(\xi)$ of $(\sigma^c)^2$ to obtain an overall expected or average confidence estimate subject to that statistical spread of σ^c. The procedure is similar to that used in Appendix B. Namely, using the (C-6) and the $p_{(\sigma^c)^2}(\xi)$ from (B-3) with substitute (C-5), we find after some calculation that

$$P_f\left\{\hat{\mu}_g^{cE} + \delta_1 < \mu_g^{oE} < \mu_g^{cE} + \delta_2 \text{ and } \alpha < (\sigma_g^{oE})^2 < \beta\right\}$$

$$= \int_0^\infty d\xi\ p_{(\sigma^c)^2}(\xi) \int_{\alpha+\xi}^{\beta+\xi} d\eta \int_{\hat{\mu}_g^{cE}+\delta_1}^{\hat{\mu}_g^{cE}+\delta_2} d\rho\ P_{\sigma^2,\ \mu_g^{oE}\ |\ (\sigma^c)^2}(\eta,\rho|\xi) \tag{C-7}$$

where

$$p_{\sigma^2,\ \mu_g^{OE} \mid (\sigma^C)^2}(\eta,\ \rho \mid \xi) \equiv \frac{\partial^2}{\partial\eta\ \partial\rho}\ P_{\sigma^2,\ \mu_g^{OE} \mid (\sigma^C)^2}\left\{ \sigma^2 < \eta,\ \mu_g^{OE} < \rho \mid (\sigma^C)^2 = \xi \right\} \tag{C-8}$$

and the last probability in (C-8) can be obtained by transforming random variables.

The above procedure may be too lengthy to carry out. A much simpler alternative is to estimate σ_g^{OE} and μ_g^{OE} separately but less tightly. Making use of (C-3) and (22), we instantly have

$$p_{(\sigma_g^{OE})^2 \mid (\sigma^C)^2}(\eta \mid \xi) = \frac{\dfrac{(n_E-1)(\hat{\sigma}_g^{CE})^2}{(\eta+(\sigma^C)^2)^2}\ p_{\chi^2(n_E-1)}\left(\dfrac{(n_E-1)(\hat{\sigma}_g^{CE})^2}{\eta+(\sigma^C)^2}\right)}{\gamma\left(0,\ \dfrac{(n_E-1)(\hat{\sigma}_g^{CE})^2}{(\sigma^C)^2}\ ;\ n_E-1\right)} \quad \text{if } \eta > 0$$

$$= 0 \quad \text{if } \eta < 0 \tag{C-9}$$

which immediately gives the unconditional fiducial probability density for $(\sigma_g^{OE})^2$:

$$p_{(\sigma_g^{OE})^2}(\eta) = \int_0^\infty p_{(\sigma_g^{OE})^2 \mid (\sigma^C)^2}(\eta \mid \xi)\ p_{(\sigma^C)^2}(\xi)\ d\ \xi \tag{C-10}$$

where the $p_{(\sigma^C)^2}(\xi)$ is given by (22). Equations (C-9) and (C-10) enable us to evaluate the confidence estimate for $(\sigma_g^{OE})^2$ via

$$P\left\{a_1 < \left(\sigma_g^{OE}\right)^2 < a_2\right\} = \int_{a_1}^{a_2} p_{\left(\sigma_g^{OE}\right)^2}(\eta)\, d\eta \tag{C-11}$$

The separate estimate for μ_g^{OE} is easy once we have estimated σ^2. From (C-27) and using Appendix B, we obtain

$$P_f\left\{-b + \hat{\mu}_g^{cE} < \mu_g^{OE} < +b + \hat{\mu}_g^{cE}\right\}$$

$$= \int_0^{\infty} \xi\, p_{(\sigma c)^2}(\xi) \int_0^{\infty} d\eta\; p_{\sigma^2|(\sigma^C)^2}(\eta|\xi)\; \lambda\left(-b\left(\frac{n_E}{\eta}\right)^{\frac{1}{2}}, b\left(\frac{n_E}{\eta}\right)^{\frac{1}{2}}\right) \tag{C-12}$$

where $p_{\sigma^2|(\sigma^C)^2}(\eta|\xi)$ is given by (A-15) with substitutions (C-5) and ξ is the dummy integration variable assumed by $(\sigma^C)^2$.

APPENDIX D

GENERALIZED METHOD FOR DISTRIBUTION OF (31),(32)

If we have for $i=1,2...n_s$

$$R_i^T \; \varepsilon \; N\left(R_i^O, \; |R_i^O| K^T\right) \tag{D-1}$$

$$R_i^C \; \varepsilon \; N\left(R_i^O, \; |R_i^O| K^C\right) \tag{D-2}$$

where $K^T \geq o$ is known and $K^C \geq O$ is unknown, we can easily estimate R_i^O by using

$$\frac{R_i^T - R_i^O}{|R_i^O|} \; \varepsilon \; N\left(0, \; K^T\right) \tag{D-3}$$

The results, for $R_i^T \neq 0$, are

$$P\left\{ \min\left(\hat{R}_{i-}^O, \; \hat{R}_{i+}^O\right) \leqslant R_i^O \leqslant \max\left(\hat{R}_{i-}^O, \; \hat{R}_{i+}^O\right) \right\} = \gamma(-b, \; b), \quad \text{if } K^T b \leqslant 1 \tag{D-4}$$

$$P\left\{ R_i^O \geqslant \max\left(\hat{R}_{i-}^O, \; \hat{R}_{i+}^O\right) \text{ or } R_i^O \leqslant \min\left(\hat{R}_{i-}^O, \; \hat{R}_{i+}^O\right) \right\} = \gamma(-b, \; b) \quad \text{if } K^T b > 1 \tag{D-5}$$

where

$$\hat{R}^O_{i-} \equiv \frac{-R^T_i}{K^T b-1} \qquad \text{(D-6)}$$

$$\hat{R}^O_{i+} \equiv \frac{R^T_i}{K^T b+1} \qquad \text{(D-7)}$$

Notice that for $K^T b=1$, both (D-4) and (D-5) give the same limiting results

$$P\left\{ R^O_i \geqslant \frac{R^T_i}{2} \right\} = \gamma\left(\frac{-1}{K^T}, \frac{1}{K^T}\right), \text{ if } R^T > 0 \qquad \text{(D-8)}$$

$$P\left\{ R^O_i \leqslant \frac{R^T_i}{2} \right\} = \gamma\left(\frac{-1}{K^T}, \frac{1}{K^T}\right), \text{ if } R^T_i < 0 \qquad \text{(D-9)}$$

The result when $R^T_i=0$ is

$$P\left[-\infty < R^O_i < \infty\right] = \gamma(-b, b), \text{ for } b > \frac{1}{K^T} \qquad \text{(D-10)}$$

and cannot give any conclusion in a tighter interval for a parameter b of lower confidence. All above can most easily be seen by plotting the parabolas $\left(K^T b\right)^2$ $\left(R^O_i\right)^2$ and $\left(R^O_i - R^T_i\right)^2$ as functions of R^O_i .

To estimate the K^C, we can form the random variable

$$z_i \equiv \frac{R_i^C}{R_i^T} \equiv \frac{R_i^C/R_i^O}{R_i^T/R_i^O} \tag{D-11}$$

which has the probability density

$$p_{z_i}(z) = \int_{-\infty}^{\infty} dy \; \frac{e^{\frac{-(y-1)^2}{2\left(K^T\right)^2}}}{\sqrt{2\pi\left(K^T\right)^2}} \cdot \frac{y \; e^{\frac{-(yz-1)^2}{2\left(K^C\right)^2}}}{\sqrt{2\pi\left(K^C\right)^2}} \tag{D-12}$$

independent of i. Thus, the probability density of

$$z \equiv \frac{\sum_{i=1}^{n} z_i}{n} \tag{D-13}$$

is

$$p_z(z) = n \int_{-\infty}^{\infty} \left(\phi_{z_i}(t)\right)^n e^{-inzt} \; \frac{dt}{2\pi} \tag{D-14}$$

where

$$\phi_{z_i}(t) \equiv \int_{-\infty}^{\infty} p_{z_i}(z) e^{izt} \; dz \tag{D-15}$$

With this $p_z(z)$, which is parameterized by K^T and K^C, we can straightforwardly estimate the distribution of K^C. Then we can estimate the R_i^{OE} from the R_i^{CE} of an EMP by using

$$R_i^{CE} \;\varepsilon\; N\left(R_i^{OE},\ |R_i^{OE}|K^C\right)\ ,\ i = 1,2,\ldots n_E \qquad \text{(D-16)}$$

first conditioned on a given K^C $\left(\text{similar to the estimation of } R_i^O \text{ from } R_i^T\right)$, and then unfolding that condition by using the fiducial probability of K^C itself just estimated.

To estimate the μ_g^{OE} of the

$$R_i^{OE} \;\varepsilon\; N\left(\mu_g^{OE},\ \sigma_g^{OE}\right)\ ,\ i = 1,2,\ldots n_E \qquad \text{(D-17)}$$

we can use the student-t estimator to the sample mean $\sum_{i=1}^{n_E} R_i^{CE}/n_E$, since

$$R_i^{CE} \equiv \left(R_i^{CE} - R_i^{OE}\right) + R_i^{OE} \;\varepsilon\; N\left(\mu_g^{OE}\ ,\ \sqrt{\left(\sigma_g^{OE}\right)^2 + \left(K^C\ R_i^{OE}\right)^2}\right) \qquad \text{(D-18)}$$

Finally, to estimate the σ_g^{OE}, we do it first by using the estimator $\sum_i \left(R_i^{OE} - \sum_j R_j^{OE}/n\right)^2 / \left(\sigma_g^{OE}\right)^2$ being a $\chi^2(n-1)$ distribution with assumed values of the R_i^{OE}, and then unfolding those values of the R_i^{OE} by their fiducial distribution about the calculated R_i^{CE} just obtained.

Of course, if the logrithms of the responses have simple mathematical behaviors, we can always estimate the mean, the spread, etc. for those logrithms and thus transform it back for the quantity itself. In doing so, the multiplicative scaling factors in (A-1) and (A-2) become merely additive terms corresponding to shifts of the mean value, and subsequent mathematics becomes much simplified.

LIST OF KEY NOTATIONS

Notation	Meaning
x_i	The position of the $i^{\underline{th}}$ subsystem points at which EMP response is of interest.
$\{x_i\}_{i=1}^{I}$	The set of the collection of all subsystem points where I is the number of all such points.
Y_j	The position of the $j^{\underline{th}}$ point of entry (POE).
$R_i^{(c)}$ or R_i^{c}, $R_i^{(T)}$ or R_i^{T}	The calculated and the tested EMP response at x_i.
$C_{ij}^{(c)}$ or C_{ij}^{c}, $C_{ij}^{(T)}$ or C_{ij}^{T}	The calculated and tested coupling matrix linking the response at x_i to the environmental driving strength at Y_j.
$J(i)$	The number of POE's linked to x_i.
E_j	The environmental driving strength at Y_j.
$\sigma_R^{(c)}$, $\sigma_R^{(T)}$	The uncertainty errors, also used as the standard deviations, in $R^{(c)}$ and $R^{(T)}$.
T_i	The malfunction threshold response at x_i.
$S_i = \frac{T_i}{R_i}$	The safety margin at x_i.

LIST OF KEY NOTATIONS (Con't)

Notation	Meaning
$p_X(x)$	The probability density function of the random variable X evaluated at the value x assumed by X.
$p_{X\|Y}(x\|y)$	The conditional probability density of the random variable X evaluated at x given that the random variable Y assumes value y.
$P\left\{a \leq X \leq b\right\}$	The probability that the random variable lies between a and b.
$\hat{\sigma}$	The statistical estimator of the parameter σ.
$\chi^2(n)$	The chi-square distribution with n-degree of freedom.
$\gamma(a, b; n)$	The probability that the random variable $\chi^2(n)$ lies between a and b.
$\lambda(\alpha, \beta)$	The probability that a standard normal random variable, with zero mean and unit standard deviation, lies between α and β.