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Electromagnetic Penetration Through a Circular Aperture in a Plane Screen Separating a Conducting Medium and a Non-Conducting Medium

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Abstract

A low frequency approximation is made to the Fourier Transform of EMP in the presence of an infinite conducting screen. A circular aperture is introduced in the screen with air on the shadow side. A particular solution of Maxwell's equations is added to a general solution involving oblate spheriodal coordinates to match the boundary conditions. Analytical expressions are given for the fields on the screen, along the axis, in the aperture and for large distances from the aperture.

The electromagnetic diffraction by a circular aperture in a plane screen between different media was given by D. R. Thomas[1]. He assumed a low frequency incoming wave and no conductivity in either medium and gave an approximate result. C. M. Butler[2] generalized the problem to an arbitrarily shaped aperture and described a numerical procedure by the method of moments but only gave results for an infinite strip. D. R. Marston[6] and others found the currents induced in underground cables by EMP.

In this paper we shall assume a thin, infinite, perfectly conducting screen illuminated by a low frequency plane wave. We seek an exact analytical solution with a conductivity which is not assumed to be zero in the incident medium.

The physical problem which motivates the study is that of an electromagnetic pulse (EMP) coming through the ground and entering an aperture in a silo or a window on a submerged vessel or underground communication center (see Figure 1).

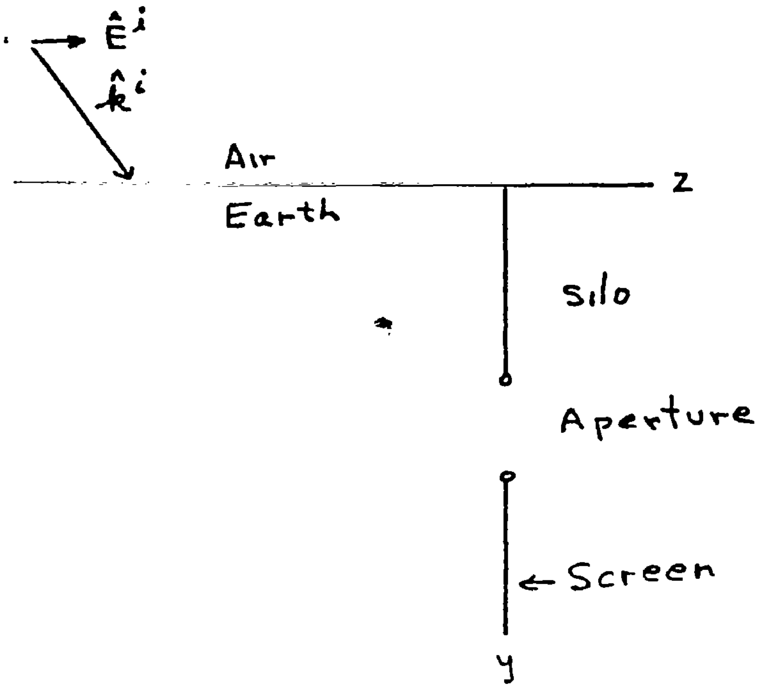

Figure 1A EMP Penetration of a Silo

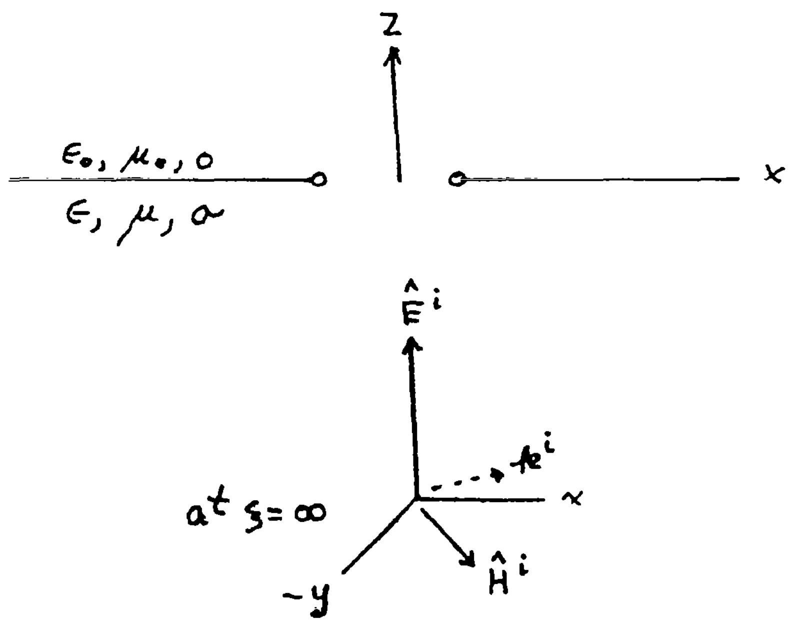

Figure 1B Standardized Coordinate System

The Fourier transform of the electric field in the earth of a plane EMP whose $\vec{E}$ is parallel to the surface of the ground is given as

$$\tilde{\vec{E}} = \hat{z} E_o(\omega) e^{\frac{-j\omega}{c(\omega)} \hat{k} \cdot \vec{r}}$$

where

$$E_o(\omega) = E_1 \left(\frac{1}{\alpha + j\omega} - \frac{1}{\beta + j\omega}\right) \frac{2 \cos \theta_o}{\cos \theta_o + \frac{\mu_o}{\mu} \sqrt{\frac{\varepsilon\mu}{\varepsilon_o \mu_o} - \frac{j\sigma\omega}{\varepsilon_o \mu_o \omega} - \sin^2 \theta_o}}$$

$$c(\omega) = \frac{1}{\sqrt{\varepsilon\mu - \frac{j\mu\sigma}{\omega}}}$$

β = 1/rise time of EMP

α = 1/fall time of EMP

$$\hat{k} = -k_3 \hat{x} + k_1 \hat{y}$$

$$k_3 = c(\omega) \frac{\sin \theta_o}{c_o}$$

$$c_o = \frac{1}{\sqrt{\varepsilon_o \mu_o}}$$

$$k_1 = \sqrt{1 - k_3^2}$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

The magnetic field is

$$\tilde{\vec{H}} = \frac{\hat{k} \times \tilde{\vec{E}}}{Z(\omega)}$$

where

$$Z(\omega) = \mu c(\omega)$$

If $\omega << \frac{\sigma}{\varepsilon}$, then $\tilde{\vec{E}}$ and $\tilde{\vec{H}}$ can be expanded in powers of $\omega^{\frac{1}{2}}$. Retaining only the first two terms,

$$\tilde{\vec{E}} = E_z^o \hat{z} + O(\omega)$$

$$\tilde{\vec{H}} = (H_x^o - \sigma E_z^o y)\hat{x} + H_y^o \hat{y} + O(\omega)$$

where

$$E_z^o = 2E_1 (\frac{1}{\alpha} - \frac{1}{\beta}) \sqrt{\frac{j\omega\varepsilon_o\mu}{\mu_o\sigma}} \cos\theta_o$$

$$H_x^o = 2E_1 (\frac{1}{\alpha} - \frac{1}{\beta}) \sqrt{\frac{\varepsilon_o}{\mu_o}} (1 - \sqrt{\frac{j\omega\varepsilon_o\mu}{\mu_o\sigma}} \cos\theta_o) \cos\theta_o$$

$$H_y^o = 2E_1 (\frac{1}{\alpha} - \frac{1}{\beta}) \varepsilon_o \sqrt{\frac{j\omega}{\mu\sigma}} \sin\theta_o \cos\theta_o$$

For very small ω, E_o is zero. Let us consider terms of order $\omega^{\frac{1}{2}}$ but neglect terms of order ω. Maxwell's equations on the conducting side of an aperture are then

$$\nabla \times \vec{\tilde{H}} = \sigma \vec{\tilde{E}}, \quad \nabla \times \vec{\tilde{E}} = 0, \quad \nabla \cdot \vec{\tilde{E}} = 0, \quad \nabla \cdot \vec{\tilde{H}} = 0$$

This approximation is good if $|\omega| << \sigma/\varepsilon$

Price[5] gives values $\sigma = 3 \times 10^{-3}$ mho/m

$\varepsilon = 10 \times 8.854 \times 10^{-12}$ f/m

For a frequency of 1 MHz, $\frac{\omega\varepsilon}{\sigma} = .185$.

On the shadow side of the screen $(z>0)$ $\sigma=0$ and Maxwell's equations are

$$\nabla \times \vec{\tilde{E}} = 0 \qquad \nabla \times \vec{\tilde{H}} = 0$$

$$\nabla \cdot \vec{\tilde{E}} = 0 \qquad \nabla \cdot \vec{\tilde{H}} = 0$$

The tildas will be suppressed from this point on.

The natural coordinates to use are oblate spheroidal coordinates ξ, η, ϕ, given in Fig. 2.

$$\rho = \sqrt{x^2+y^2} = \frac{1}{a}\sqrt{(\xi+a^2)\ (\eta+a^2)} \qquad 0<\xi<\infty$$

$$z = \frac{\text{sign } z}{a}\sqrt{-\xi\eta} \qquad -a^2<\eta<o$$

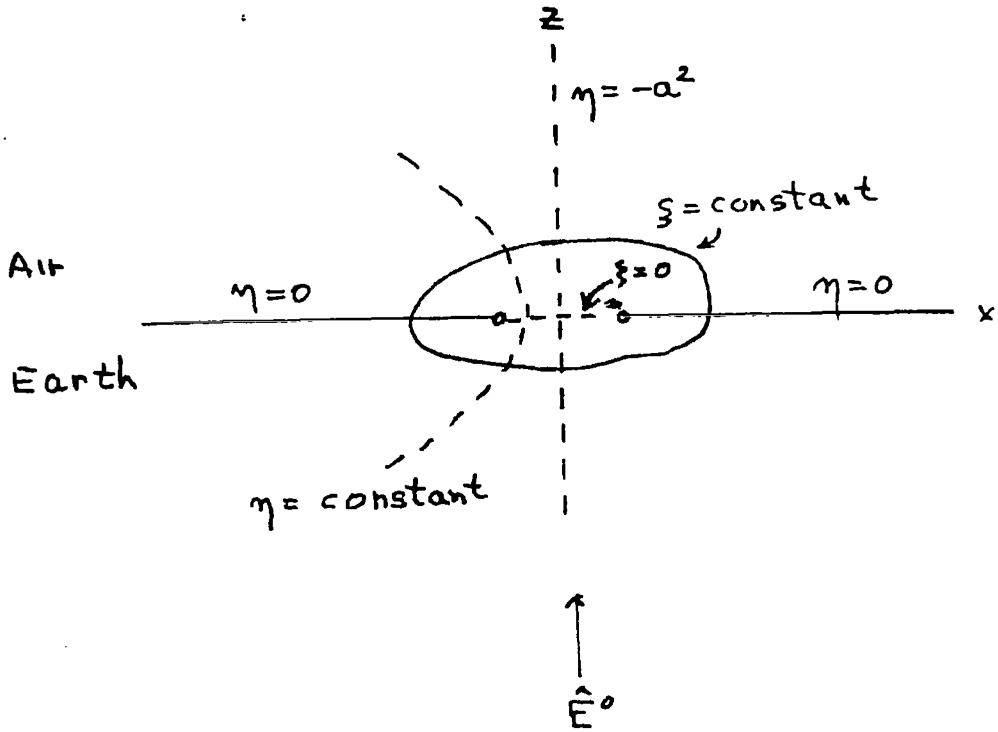

Figure 2 Oblate Spheroidal Coordinates

ϕ is the regular polar coordinate.

The boundary conditions are such that the tangential electric field and the normal magnetic field are zero on the screen. Both fields have continuous tangential components at the aperture. The normal component of magnetic induction $\vec{B}$ is continuous at the aperture. Both fields are zero at $z = +\infty$ and

$$\vec{E} = E_z^0\hat{z}; \ \vec{H} = (H_x^0 - \sigma E_z^0 y) + H_y^0\hat{y} \quad \text{at } z = -\infty.$$

The boundary condition at the aperture is that the current flowing into the aperture is the increase of surface charge at the aperture:

$$J_z^-(0,\eta,\phi) = \frac{\partial \rho_s}{\partial t} = j\omega\rho_s = j\omega[\varepsilon_o E_z^+(0,\eta,\phi) - \varepsilon E_z^-(0,\eta,\phi)]$$

$$= \sigma E_z^-(0,\eta,\phi)$$

Since we have assumed $\frac{\omega\varepsilon}{\sigma} << 1$ this is equivalent to $E_z^-(0,\eta,\phi) = 0$. It should be noted that the boundary condition reduces to $E_z^+(0,\eta,\phi) = E_z^-(0,\eta,\phi)$ if $\varepsilon = \varepsilon_o$ and $\sigma = 0$. However, if ω is allowed to go to zero first, then the boundary condition $E_z^-(0) = 0$ does not reduce to $E_z^+(0,\eta,\phi) = E_z^-(0,\eta,\phi)$ and hence the solution of the problem with $\sigma > 0$ does not reduce to the case of an aperture with both sides in air.

The complete mathematical problem is given as follows:

$$\nabla \times \vec{E}^- = 0 \qquad \nabla \times \tilde{\vec{H}}^- = \sigma E^-$$

$$\nabla \times \vec{E}^+ = 0 \qquad \nabla \times \vec{H}^+ = 0$$

$$\nabla \cdot \vec{E}^- = 0 \qquad \nabla \cdot \vec{H}^- = 0$$

$$\nabla \cdot \vec{E}^+ = 0 \qquad \nabla \cdot \vec{H}^+ = 0$$

At $\xi = 0$,

$$E_x^- = E_x^+$$

$$E_y^- = E_y^+$$

$$E_z^- = 0$$

$$H_x^- = H_x^+$$

$$H_y^- = H_y^+$$

$$\mu H_z^- = \mu_o H_z^+$$

At $\eta = 0$,

$$E_x^- = 0$$

$$E_x^+ = 0$$

$$E_y^- = 0$$

$$E_y^+ = 0$$

$$H_z^- = 0$$

$$H_z^+ = 0$$

At $\xi = \infty$, $z < 0$,

$$E_x^- = 0$$

$$E_y^- = 0$$

$$E_z^- = E_z^o$$

$$H_x^- = H_x^o - E_z^o \sigma y$$

$$H_y^- = H_y^o$$

$$H_z^- = 0$$

At $\xi = \infty$, $z > 0$

$$E_x^+ = 0$$

$$E_y^+ = 0$$

$$E_z^+ = 0$$

$$H_x^+ = 0$$

$$H_y^+ = 0$$

$$H_z^+ = 0$$

This is a total of 16 equations with 24 boundary conditions. They are sufficient to determine a unique solution. The minus subscript indicates the earth with constants ε, μ, σ. The plus subscript indicates the shadow side of the screen with constants ε_0, μ_0, 0:

The procedure to calculate a solution is outlined as follows.

1. We first formulate the Laplace Equation in oblate spheroidal coordinates. Since $\nabla \times \vec{E} = 0$, then $\vec{E} = -\nabla\Phi$ and $\nabla^2\Phi = 0$. We label

$$\Phi = \Phi^+ \quad \text{if } z \geq 0$$

$$\Phi = \Phi^- \quad \text{if } z \leq 0$$

2. We separate variables and find separable solutions.

3. We apply the boundary conditions and determine $\vec{E}$. This can be done without using the $\vec{H}$ equations.

4. We split $\vec{H}$ into three parts, $\vec{H} = \vec{H}_1 + \vec{H}_2 + \vec{H}_3$, such that $\vec{H}_1$ takes care of H_x^o and H_y^o at $z = -\infty$, $\vec{H}_2$ is the complementary solution and $\vec{H}_3$ is the particular solution satisfying $\nabla \times \vec{H}_3^- = \sigma \vec{E}^-$. Then $\vec{H}_1 = -\nabla\Phi_1^*$ and $\vec{H}_2 = -\nabla\Phi_2^*$, where $\nabla^2\Phi_1^* = \nabla^2\Phi_2^* = 0$. $H_{3x}^-(\xi = 0) = -\sigma E_z^o y$, and $\vec{H}_3^+ \equiv 0$.

5. We solve for $\vec{H}_1$ using the boundary condition that $\mu H_z^- = \mu_o H_z^+$ at the aperture.

6. We solve for $\vec{H}_3$ without specifying continuity at the aperture.

7. We solve for $\vec{H}_2$ without specifying continuity at the aperture.

8. We apply the continuity boundary conditions to the sum of $\vec{H}_2$ and $\vec{H}_3$ and evaluate $\vec{H}$.

The Laplace equation in oblate spheroidal coordinates is given by

$$\nabla^2\Phi = \frac{4}{\xi-\eta}\left\{\xi^{\frac{1}{2}}\frac{\partial}{\partial\xi}\left[\xi^{\frac{1}{2}}(\xi + a^2)\frac{\partial\Phi}{\partial\xi}\right]\right.$$

$$\left. + \sqrt{-\eta}\,\frac{\partial}{\partial\eta}\left[\sqrt{-\eta}\,(\eta + a^2)\frac{\partial\Phi}{\partial\eta}\right]\right\} + \frac{a^2}{(\xi+a^2)\ (\eta+a^2)}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

In order that $E_z^-(\xi = \infty) = E_z^o$ we shall pick a particular solution for Φ given by

$$\Phi_p^- = -E_z^o z = +\frac{E_z^o}{a}\sqrt{-\xi\eta}$$

Let us look for a separable complementary solution of the form $\Phi = f(\xi)g(\eta)h(\phi)$. Then it follows that

$$\xi^{\frac{1}{2}}\frac{d}{d\xi}\left[\xi^{\frac{1}{2}}(\xi+a^2)\frac{df}{d\xi}\right] + \left[\frac{a^2m^2}{\xi+a^2} - n\ (n+1)\right]\frac{f}{4} = 0$$

$$\sqrt{-\eta}\,\frac{d}{d\eta}\left[\sqrt{-\eta}\,(\eta+a^2)\frac{dg}{d\eta}\right] - \left[\frac{a^2m^2}{\eta+a^2} - n(n+1)\right]\frac{g}{4} = 0$$

$$\frac{d^2h}{d\phi^2} + m^2h = 0$$

The solution to the first two equations can be found by letting $\frac{i\xi^{\frac{1}{2}}}{a}$ and $\frac{\sqrt{-\eta}}{a}$ be new variables and recognizing that these are the associated Legendre equations:

$$f(\xi) = AP_n^m\left(\frac{i\xi^{1/2}}{a}\right) + BQ_n^m\left(\frac{i\xi^{1/2}}{a}\right)$$

$$g(\eta) = CP_n^m\left(\frac{\sqrt{-\eta}}{a}\right) + DQ_n^m\left(\frac{\sqrt{-\eta}}{a}\right)$$

$$h(\phi) = E\cos m\phi + F\sin m\phi$$

We are looking for solutions which have period 2π in ϕ so that m is chosen to be an integer. We want solutions which are well behaved at $\eta=0$ and $\eta=-a^2$ so that we will choose D=0 and n is an integer. If Φ is to be zero at ∞, then A=0. Some of the remaining functions are tabulated below:

$$Q_0^0\left(\frac{i\xi^{1/2}}{a}\right) = -i\alpha$$

$$Q_1^0\left(\frac{i\xi^{1/2}}{a}\right) = \frac{\xi^{1/2}}{a}\alpha - 1$$

$$Q_2^0\left(\frac{i\xi^{1/2}}{a}\right) = \frac{i}{2}\left[\left(\frac{3\xi}{a^2} + 1\right)\alpha - \frac{3\xi^{1/2}}{a}\right]$$

$$Q_1^1\left(\frac{i\xi^{1/2}}{a}\right) = \left(1+\frac{\xi}{a^2}\right)^{1/2}\left(\alpha - \frac{a\xi^{1/2}}{\xi+a^2}\right)$$

$$Q_2^1\left(\frac{i\xi^{1/2}}{a}\right) = i\left(1+\frac{\xi}{a^2}\right)^{1/2}\left(\frac{3\xi^{1/2}}{a}\alpha - 3 + \frac{a^2}{\xi+a^2}\right)$$

$$Q_2^2\left(\frac{i\xi^{1/2}}{a}\right) = i\left[3\left(1+\frac{\xi}{a^2}\right)\alpha - \frac{3\xi^{1/2}}{a} - \frac{2a\xi^{1/2}}{\xi+a^2}\right]$$

where $\alpha = \cot^{-1}\frac{\xi^{1/2}}{a}$

$$P_0^0\left(\frac{\sqrt{-\eta}}{a}\right) = 1$$

$$P_1^0\left(\frac{\sqrt{-\eta}}{a}\right) = \frac{\sqrt{-\eta}}{a}$$

$$P_2^0\left(\frac{\sqrt{-\eta}}{a}\right) = -1/2\left(\frac{3\eta}{a} + 1\right)$$

$$P_1^1\left(\frac{\sqrt{-\eta}}{a}\right) = -\left(1 + \frac{\eta}{a^2}\right)^{\frac{1}{2}}$$

$$P_2^1\left(\frac{\sqrt{-\eta}}{a}\right) = -\frac{3\sqrt{-\eta}}{a}\left(1 + \frac{\eta}{a^2}\right)^{\frac{1}{2}}$$

$$P_2^2\left(\frac{\sqrt{-\eta}}{a}\right) = 3\left(1 + \frac{\eta}{a^2}\right)$$

The complementary solution where $z<0$ is of the form

$$\phi_c^- = \Sigma\, Q_n^m\left(\frac{i\xi^{\frac{1}{2}}}{a}\right) P_n^m\left(\frac{\sqrt{-\eta}}{a}\right)(a_{mn}\cos m\phi + b_{mn}\sin m\phi)$$

The x component of the electric field is given by $E_x^- = -\frac{\partial\phi}{\partial x}$

$$= -\Sigma\left[\frac{dQ_n^m}{d\xi}\left(\frac{i\xi^{\frac{1}{2}}}{a}\right)\frac{\partial\xi}{\partial x} P_n^m\left(\frac{\sqrt{-\eta}}{a}\right) h_{mm}(\phi)\right.$$

$$+ Q_n^m\left(\frac{i\xi^{\frac{1}{2}}}{a}\right)\frac{dP_n^m}{d\eta}\left(\frac{\sqrt{-\eta}}{a}\right)\frac{\partial\eta}{\partial x} h_{mm}(\phi)$$

$$\left. + Q_n^m\left(\frac{i\xi^{\frac{1}{2}}}{a}\right) P_n^m\left(\frac{\sqrt{-\eta}}{a}\right)\frac{dh_{mn}(\phi)}{d\phi}\frac{\partial\phi}{\partial x}\right]$$

where $\frac{\partial\xi}{\partial x} = \frac{2x\xi}{\xi-\eta}$, $\frac{\partial\eta}{\partial x} = -\frac{2x\eta}{\xi-\eta}$, $\frac{\partial\phi}{\partial x} = -\frac{\sin\phi}{\rho}$

$$h_{mn}(\phi) = a_{mn}\cos m\phi + b_{mn}\sin m\phi$$

In order that this be zero at $\eta = 0$

we take $P_n^m(0) = 0$ or $m + n$ = an odd integer.

The z component of the electric field is given by

$$E_z^- = E_z^o - \sum_{\substack{m,n=0\\ m+n=\text{odd}}}^{\infty} \Big[\frac{dQ_n^m}{d\xi}\left(\frac{i\xi^{1/2}}{a}\right) \frac{\partial\xi}{\partial z} P_n^m\left(\frac{\sqrt{-\eta}}{a}\right) h_{mn}(\phi)$$

$$+ Q_n^m\left(\frac{i\xi^{1/2}}{a}\right) \frac{dP_n^m}{d\eta}\left(\frac{\sqrt{-\eta}}{a}\right) \frac{\partial\eta}{\partial z} h_{mn}(\phi) \Big]$$

If this is to vanish at $\xi = 0$, then

$$E_z^o = \sum_{\substack{m,n=0\\ m+n=\text{odd}}}^{\infty} \left[\frac{dQ_n^m\left(\frac{i\xi^{1/2}}{a}\right)}{d\left(\frac{i\xi^{1/2}}{a}\right)} \frac{i}{2a\xi^{1/2}} \frac{\partial\xi}{\partial z}\right]_{\xi=0} P_n^m\left(\frac{\sqrt{-\eta}}{a}\right) h_{mn}(\phi)$$

Since the left side is independent of ϕ, then $m = 0$.

$$\frac{\partial\xi}{\partial z} = \frac{2z(\xi+a^2)}{\xi-\eta} = -\frac{2(\xi+a^2)\sqrt{-\xi\eta}}{a(\xi-\eta)}$$

$$E_z^o = \sum_{\substack{n\\ \text{odd}}} \left[Q_n'\left(\frac{i\xi^{1/2}}{a}\right)\left(-\frac{i}{a^2}\right)\frac{\sqrt{-\eta}\,(\xi+a^2)}{\xi-\eta}\right]_{\xi=0} P_n\left(\frac{\sqrt{-\eta}}{a}\right) a_{on}$$

$$= \sum_{\substack{n\\ \text{odd}}} Q_n'(0)\,\frac{i\sqrt{-\eta}}{\eta}\, P_n\left(\frac{\sqrt{-\eta}}{a}\right) a_{on}$$

Since the left side is independent of η, then $n = 1$.

Hence

$$\Phi_c^- = Q_1\left(\frac{i\xi^{1/2}}{a}\right) P_1\left(\frac{\sqrt{-\eta}}{a}\right) a_{o1}$$

$$= a_{o1}\left[\frac{\xi^{1/2}}{a}\alpha - 1\right]\left[\frac{\sqrt{-\eta}}{a}\right]$$

$$\Phi_c^- = -E_z^o z + C\left(\frac{\xi^{1/2}}{a}\alpha - 1\right)\sqrt{-\eta}$$

From this it follows that

$$E_x^- = -\frac{Ca^2 x\sqrt{-\eta}}{(\xi-\eta)(\xi+a^2)}$$

$$E_y^- = -\frac{Ca^2 y\sqrt{-\eta}}{(\xi-\eta)(\xi+a^2)}$$

$$E_z^- = E_z^o + C\left(\alpha - \frac{a\xi^{1/2}}{\xi-\eta}\right)$$

The condition that $E_z^- = 0$ at $\xi = 0$ implies that

$$C = -\frac{2}{\pi}E_z^o$$

In order that the tangential electric field be continuous at $\xi=0$, we take

$$\Phi^+(x,y,z) = \Phi_c^-(x,y,-z)$$

Then

$$E_x^+(x,y,z)=E_x^-(x,y,-z),\quad E_y^+(x,y,z)=E_y^-(x,y,-z)$$

$$E_z^+ = -C \left(\alpha - \frac{a\xi^{1/2}}{\xi-\eta}\right)$$

This is independent of σ, ε_o, and ε. It is interesting that as σ goes to zero and $\varepsilon \to \varepsilon_o$ this solution does not reduce to the case of medium 1 and 2 being the same (as in air).

The latter case has $E_z^- = E_z^+ = \frac{E_z^o}{2}$ and no surface charge exists at the aperture. In the case of a small σ, $E_z^- = 0$ and $E_z^+ = E_z^o$. This gives a free surface charge of $\rho_s = \varepsilon_o E_z^o$ at the aperture. Thus a small change in the parameter σ gives an instantaneous change in E_z and ρ_s. There is a singularity in this field at $\xi=\eta=0$, which is the edge of the aperture.

The case of $\vec{H} = \vec{H}^o = H_x^o\,\hat{x} + H_y^o\,\hat{y}$ at $z = -\infty$ has been treated by Fletcher and Harrison[3] and Chen[4] for the case when $\varepsilon = \varepsilon_o$, $\mu = \mu_o$ and $\sigma = 0$. A similar procedure leads to the following solution for $\vec{H}_1$:

$$H_{1x}^- = H_x^o + \frac{A_1 H_x^o}{a^3}\left(\alpha - \frac{a\xi^{1/2}}{\xi+a^2}\right) - \frac{2A_1 x \xi^{1/2} \vec{H}^o \cdot \vec{\rho}}{(\xi-\eta)\ (\xi+a^2)^2}$$

$$H_{1y}^- = H_y^o + \frac{A_1 H_y^o}{a^3}\left(\alpha - \frac{a\xi^{1/2}}{\xi+a^2}\right) - \frac{2A_1 y \xi^{1/2} \vec{H}^o \cdot \vec{\rho}}{(\xi-\eta)\ (\xi+a^2)^2}$$

$$H_{1z}^- = \frac{2A_1 \vec{H}^o \cdot \vec{\rho}\sqrt{-\eta}}{a(\xi+a^2)\ (\xi-\eta)}$$

$$H_{1x}^{+} = \frac{A_2 H_x^o}{a^3}\left(\alpha - \frac{a\xi^{\frac{1}{2}}}{\xi + a^2}\right) - \frac{2A_2 \vec{H}^o \cdot \vec{\rho}\, x\, \xi^{\frac{1}{2}}}{(\xi-\eta)\ (\xi+a^2)^2}$$

$$H_{1y}^{+} = \frac{A_2 H_y^o}{a^3}\left(\alpha - \frac{a\xi^{\frac{1}{2}}}{\xi + a^2}\right) - \frac{2A_2 \vec{H}^o \cdot \vec{\rho}\, y\, \xi^{\frac{1}{2}}}{(\xi-\eta)\ (\xi+a^2)^2}$$

$$H_{1z}^{+} = -\frac{2A_2 \vec{H}^o \cdot \vec{\rho}\sqrt{-\eta}}{a(\xi-\eta)(\xi+a^2)}$$

where $\vec{\rho}=x\hat{x}+y\hat{y}$. To satisfy the continuity conditions

$$A_1 = -\frac{2a^3\mu_o}{\pi(\mu+\mu_o)}, \quad A_2 = \frac{2a^3\mu}{\pi(\mu+\mu_o)}$$

If $\mu=\mu_o$, then $A_1=-A_2=-\frac{a^3}{\pi}$ which checks the result of Chen[4].

This result is independent of ε, σ.

We next solve for $\vec{H}_3$ which is a particular solution of $\nabla x\vec{H}_3^- = \sigma\vec{E}^-$, $\vec{H}_3^+ \equiv 0$. In this case $\vec{H}_3$ is not derivable from a potential.

For large ξ and negative z, $\nabla x\vec{H}^- = \sigma E_z^o\hat{z}$.
A particular solution is

$$H_{3x}^- = -E_z^o\ \sigma y$$

$$H_{3y}^- = 0$$

$$H_{3z}^- = 0$$

For small z, H_{3x}^{-} must satisfy Laplace's equation. A separable solution is given by

$$A_{mn}\, Q_n^m \left(\frac{i\xi^{\frac{1}{2}}}{a}\right) P_n^m \left(\frac{\sqrt{-\eta}}{a}\right) \begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix}$$

If m=1 and m=1 this is

$$A_{11}\, Q_1^1 \left(\frac{i\xi^{\frac{1}{2}}}{a}\right) P_1^1 \left(\frac{\sqrt{-\eta}}{a}\right) \begin{Bmatrix} \sin\phi \\ \cos\phi \end{Bmatrix}$$

$$= -\frac{A_{11}}{a}\begin{pmatrix} y \\ x \end{pmatrix}\left(\alpha - \frac{\xi^{\frac{1}{2}}a}{\xi+a^2}\right)$$

A_{11} is chosen to satisfy $\nabla x \vec{H}^{-} = \sigma\vec{E}^{-}$

$$H_{3x}^{-} = -E_z^o\sigma y - \frac{E_z^o\sigma y f(\xi)}{\pi}$$

$$H_{3y}^{-} = \frac{E_z^o\sigma x f(\xi)}{\pi}$$

$$H_{3z}^{-} = 0$$

$$\vec{H}_3^{+} = 0$$

where $f(\xi) = \frac{a\xi^{\frac{1}{2}}}{\xi+a^2} - \alpha$

The last part of $\vec{H}$ to calculate is $\vec{H}_2$, which gives the complementary solution necessary to satisfy the continuity conditions at the aperture. $\vec{H}_2$ is derivable from a potential since $\nabla x\vec{H}_2=0$. The potential can be found in a similar way to that in which the electric field was found. It is

$$\Phi_2^+ = \frac{ia^2A}{3} Q_2^2 \left(\frac{i\xi^{\frac{1}{2}}}{a}\right) P_2^2 \left(\sqrt{-\eta}/a\right) \sin 2\phi$$

$$\Phi_2^- = \frac{ia^2B}{3} Q_2^2 \left(\frac{i\xi^{\frac{1}{2}}}{a}\right) P_2^2 \left(\sqrt{-\eta}/a\right) \sin 2\phi$$

where the coefficients are chosen to simplify later expressions.

m and n were chosen equal to 2 so that at the aperture

$$\Phi_2^+ = -3\pi Axy$$

$$\Phi_2^- = -3\pi Bxy$$

making the x- and y- components of $\vec{H}_2$ linear in x and y at the aperture; thus the boundary condition of $\vec{H}_2$ and $\vec{H}_3$ can be matched at the aperture.

$$\Phi_2^+ = A\left[-3\left(1+\frac{\xi}{a^2}\right)\alpha + \frac{3\xi^{\frac{1}{2}}}{a} + \frac{2a\xi^{\frac{1}{2}}}{\xi+a^2}\right] (\eta+a^2) \sin 2\phi$$

$$\vec{H}_{2x}^+ = -A \left[\frac{2x\xi}{\xi-\eta} (\eta+a^2) \sin 2\phi \;\; h(\xi) - \frac{2\eta x}{\xi-\eta} \sin 2\phi \; g(\xi)\right.$$

$$\left. -2\left(\frac{\eta+a^2}{\rho}\right) \sin \phi \cos 2\phi \; g(\xi)\right]$$

$$H_{2y}^+ = -A \left[\frac{2y\xi}{\xi-\eta} (\eta+a^2) \sin 2\phi \; h(\xi) - \frac{2\eta y}{\xi-\eta} \sin 2\phi \; g(\xi)\right.$$

$$\left. +2\frac{(\eta+a^2)}{\rho} \cos \phi \cos 2\phi \; g(\xi)\right]$$

$$H_{2z}^+ = -\frac{2z}{\xi-\eta} \rho^2 a^2 A \sin 2\phi \; h(\xi) + \frac{2Az}{\xi-\eta}(\eta+a^2) \sin 2\phi \; g(\xi)$$

$$= -\frac{16A\sqrt{-\eta}\; xya^4}{(\xi-\eta)\;(\xi+a^2)^2}$$

where

$$h(\xi) = -\frac{3\alpha}{a^2} + \frac{3}{a\xi^{\frac{1}{2}}} + \frac{a(a^2-\xi)}{\xi^{\frac{1}{2}}(a^2+\xi)^2} = g'(\xi)$$

$$g(\xi) = -3(1+\frac{\xi}{a^2})\alpha + \frac{3\xi^{\frac{1}{2}}}{a} + \frac{2a\xi^{\frac{1}{2}}}{\xi+a^2}$$

H_{2x}^-, H_{2y}^-, H_{2z}^- are the same except that A is replaced by B and the sign of H_{2z} is changed. On the aperture it follows that

$$H_{2x}^+ (\xi=0) = 3\pi Ay \qquad H_{2x}^- (\xi=0) = 3\pi By$$

$$H_{2y}^+ (\xi=0) = 3\pi Ax \qquad H_{2y}^- (\xi=0) = 3\pi Bx$$

$$H_{2z}^+ (\xi=0) = \frac{-16Axy}{\sqrt{a^2-\rho^2}} \qquad H_{2z}^- (\xi=0) = \frac{16Bxy}{\sqrt{a^2-\rho^2}}$$

$\vec{H}_1$ has already been constrained to satisfy the continuity boundary conditions at the aperture. Applying these boundary conditions to the sum $\vec{H}_2 + \vec{H}_3$, one obtains

$$\Delta H_x = 3\pi Ay - 3\pi By + E_z^O\sigma y - \frac{E_z^O\sigma y}{2} = 0$$

$$\Delta H_y = 3\pi Ax - 3\pi Bx + E_z^O\sigma x = 0$$

$$\Delta B_z = -\frac{16Axy\mu_o}{\sqrt{a^2-\rho^2}} - \frac{16Bxy\mu}{\sqrt{a^2-\rho^2}} = 0$$

or

$$A = -\frac{E_z^o \sigma\mu}{6\pi(\mu_o+\mu)}$$

$$B = \frac{E_z^o \sigma\mu_o}{6\pi(\mu_o+\mu)}$$

This completes the evaluation of all constants. We have thus found the fields which satisfy all twenty-four boundary conditions. The final result and some special cases are tabulated below.

Table of Results

At a general point in space:

$$E_x^- = \frac{2a^2 E_z^o x\sqrt{-\eta}}{\pi(\xi+a^2)(\xi-\eta)} = E_x^+$$

$$E_y^- = \frac{2a^2 E_z^o y\sqrt{-\eta}}{\pi(\xi+a^2)(\xi-\eta)} = E_y^+$$

$$E_z^- = E_z^o\left[1 - \frac{2\alpha}{\pi} + \frac{2a\xi^{1/2}}{\pi(\xi-\eta)}\right]$$

$$E_z^+ = \frac{2E_z^o}{\pi}\left(\alpha - \frac{a\xi^{1/2}}{\xi-\eta}\right)$$

In the aperture ($z=0$, $\rho\leq a$, $\eta=\rho^2-a^2$):

$$E_x^+ = \frac{2xE_z^o}{\pi\sqrt{a^2-\rho^2}}$$

$$E_x^- = \frac{2xE_z^o}{\pi\sqrt{a^2-\rho^2}}$$

$$E_y^+ = \frac{2yE_z^o}{\pi\sqrt{a^2-\rho^2}}$$

$$E_y^- = \frac{2yE_z^o}{\pi\sqrt{a^2-\rho^2}}$$

$$E_z^+ = E_z^o$$

$$E_z^- = 0$$

$$\rho_s = \varepsilon_o E_z^o$$

in the aperture ($z=0$, $\rho\leq a$, $\eta=\rho^2-a^2$)

$$E_x^- = 0$$

$$E_y^- = 0$$

$$E_z^- = E_z^o \left[1 - \frac{2}{\pi}\left(\sin^{-1}\frac{a}{\rho} - \frac{a}{\sqrt{\rho^2-a^2}}\right) \right]$$

on the screen ($z=0^-$, $\rho>a$, $\eta=0$, $\xi=\rho^2-a^2$)

$$E_x^+ = 0$$

$$E_y^+ = 0$$

$$E_z^+ = \frac{2E_z^o}{\pi}\left(\sin^{-1}\frac{a}{\rho} - \frac{a}{\sqrt{\rho^2-a^2}}\right)$$

on the screen ($z=0^+$, $\rho>a$, $\eta=0$, $\xi=\rho^2-a^2$)

$$E_x^- = 0$$

$$E_y^- = 0$$

$$E_z^- = E_z^o \left[1- \frac{2}{\pi}\left(\cot^{-1}\frac{z}{a} - \frac{az}{Z^2+a^2}\right) \right]$$

along the Z axis ($x=y=0$, $z\leq 0$, $\eta=-a^2$, $\xi=z^2$)

$$E_x^+ = 0$$

$$E_y^+ = 0$$

$$E_z^+ = \frac{2E_z^O}{\pi}\left(\cot^{-1}\frac{z}{a} - \frac{az}{z^2+a^2}\right)$$

along the z axis ($x=y=0$, $z>0$, $\eta=-a^2$, $\xi=z^2$)

$$E_x^+ = \frac{2E_z^O x a^3 \cos\theta}{\pi r^4} = \frac{2E_z^O a^3 \cos^2\phi \cos\theta \sin\theta}{\pi r^3}$$

$$E_y^+ = \frac{2E_z^O y a^3 \cos\theta}{\pi r^4} = \frac{2E_z^O a^3 \sin\phi \cos\theta \sin\theta}{\pi r^3}$$

$$E_z^+ = -\frac{2E_z^O a^3}{3\pi r^3}(1-3\cos^2\theta),$$

$$\vec{E}^+ = \frac{3\hat{r}(\hat{r}\cdot\vec{P}_{\hat{E}}) - \vec{P}_{\hat{E}}}{4\pi\varepsilon_o r^3} \text{ where } \vec{P}_E = \frac{8}{3}\varepsilon_o a^3 E_z^O \hat{z}$$

in the far field ($z>>0$, $\frac{\xi}{r^2}\to 1$, $\eta\to -\frac{a^2z^2}{r^2} = -a^2\cos^2\theta$,)

$$E_x^- = \frac{2E_z^O a^3 \cos\phi\cos\theta\sin\theta}{\pi r^3}$$

$$E_y^- = \frac{2E_z^O a^3 \sin\phi\cos\theta\sin\theta}{\pi r^3}$$

$$E_z^- = E_z^O\left[1 + \frac{2a^3}{3\pi r^3}(1-3\cos^2\theta)\right]$$

for the far field ($z<<0$, $\frac{\xi}{r^2}\to 1$, $\eta = -\frac{a^2z^2}{r^2} = -a^2\cos^2\theta$, θ measured from positive real axis)

$$H_x^+ = -\frac{E_z^O\sigma\mu y}{\pi(\mu_o+\mu)}\left\{\alpha - \frac{a\xi^{1/2}}{3(\xi+a^2)^2}\left[3\xi+5a^2+\frac{8a^4x^2}{(\xi-\eta)\ (\xi+a^2)}\right]\right\}$$

$$+\frac{2\mu H_x^O}{\pi(\mu+\mu_o)}\left(\alpha - \frac{a\xi^{1/2}}{\xi+a^2}\right) - \frac{4a^3\mu x\xi^{1/2}\ \vec{H}^O\cdot\vec{\rho}}{\pi(\mu+\mu_o)(\xi-\eta)(\xi+a^2)^2}$$

$$H_y^+ = -\frac{E_z^O \sigma \mu x}{\pi(\mu_o+\mu)} \left\{ \alpha - \frac{a\xi^{\frac{1}{2}}}{3(\xi+a^2)^2} \left[3\xi+5a^2 + \frac{8a^4y^2}{(\xi-\eta)(\xi+a^2)} \right] \right\}$$

$$+ \frac{2\mu H_y^O}{\pi(\mu+\mu_o)} \left(\alpha - \frac{a\xi^{\frac{1}{2}}}{\xi+a^2} \right) - \frac{4a^3\mu y\xi^{\frac{1}{2}}\, \vec{H}^O \cdot \vec{\rho}}{\pi(\mu+\mu_o)(\xi-\eta)(\xi+a^2)^2}$$

$$H_z^+ = \frac{8E_z^O \sigma \mu a^4 xy\sqrt{-\eta}}{3\pi(\mu_o+\mu)(\xi-\eta)(\xi+a^2)^2} - \frac{4a^2\mu\, \vec{H}^O \cdot \vec{\rho}\sqrt{-\eta}}{\pi(\mu+\mu_o)(\xi-\eta)(\xi+a^2)}$$

$$H_x^- = -E_z^O \sigma y \left[1 - \frac{\alpha}{\pi} + \frac{a\xi^{\frac{1}{2}}}{\pi(\xi+a^2)} \right]$$

$$+ \frac{E_z^O \sigma \mu_o y}{\pi(\mu_o+\mu)} \left\{ \alpha - \frac{a\xi^{\frac{1}{2}}}{3(\xi+a^2)^2} \left[3\xi+5a^2 + \frac{8a^4x^2}{(\xi-\eta)(\xi+a^2)} \right] \right\}$$

$$+ H_x^O \left[1 - \frac{2\mu_o}{\pi(\mu+\mu_o)} \left(\alpha - \frac{a\xi^{\frac{1}{2}}}{\xi+a^2} \right) \right] + \frac{4a^3\mu_o x\xi^{\frac{1}{2}}\vec{H}^O \cdot \vec{\rho}}{\pi(\mu+\mu_o)(\xi-\eta)(\xi+a^2)^2}$$

$$H_y^- = \frac{E_z^O \sigma x}{\pi} \left(\frac{a\xi^{\frac{1}{2}}}{\xi+a^2} - \alpha \right)$$

$$+ \frac{E_z^O \sigma \mu_o x}{\pi(\mu_o+\mu)} \left\{ \alpha - \frac{a\xi^{\frac{1}{2}}}{3(\xi+a^2)^2} \left[3\xi+5a^2 + \frac{8a^4y^2}{(\xi-\eta)(\xi+a^2)} \right] \right\}$$

$$+ H_y^O - \frac{2\mu_o H_y^O}{\pi(\mu+\mu_o)} \left(\alpha - \frac{a\xi^{\frac{1}{2}}}{\xi+a^2} \right) + \frac{4a^3\mu_o y\xi^{\frac{1}{2}}\vec{H}^O \cdot \vec{\rho}}{\pi(\mu+\mu_o)(\xi-\eta)(\xi+a^2)^2}$$

$$H_z^- = +\frac{8E_z^O \sigma \mu_o a^4 xy\sqrt{-\eta}}{3\pi(\mu+\mu_o)(\xi-\eta)(\xi+a^2)^2} - \frac{4a^2\mu_o \vec{H}^O \cdot \vec{\rho}\sqrt{-\eta}}{\pi(\mu+\mu_o)(\xi+a^2)(\xi-\eta)}$$

$$H_x^- = -\frac{E_z^o \sigma \mu y}{2(\mu+\mu_o)} + \frac{\mu H_x^o}{\mu+\mu_o} = H_x^+$$

$$H_y^- = -\frac{E_z^o \sigma \mu x}{2(\mu+\mu_o)} + \frac{\mu H_y^o}{\mu+\mu_o} = H_y^+$$

$$H_z^+ = \frac{8E_z^o \sigma \mu xy}{3\pi(\mu+\mu_o)\sqrt{a^2-\rho^2}} - \frac{4\mu \vec{H}^o \cdot \vec{\rho}}{\pi(\mu+\mu_o)\sqrt{a^2-\rho^2}}$$

$$H_z^- = \frac{8E_z^o \sigma \mu_o xy}{3\pi(\mu+\mu_o)\sqrt{a^2-\rho^2}} - \frac{4\mu_o \vec{H}^o \cdot \vec{\rho}}{\pi(\mu+\mu_o)\sqrt{a^2-\rho^2}}$$

In the aperture ($z=0$, $\rho<a$, $\xi=0$, $\eta=\rho^2-a^2$)

$$H_x^+ = \frac{2\mu H_x^o}{\pi(\mu+\mu_o)} \left(\cot^{-1}\frac{z}{a} - \frac{az}{z^2+a^2}\right)$$

$$H_y^+ = \frac{2\mu H_y^o}{\pi(\mu+\mu_o)} \left(\cot^{-1}\frac{z}{a} - \frac{az}{z^2+a^2}\right)$$

$$H_z^+ = 0$$

$$H_x^- = H_x^o - \frac{2\mu_o H_x^o}{\pi(\mu+\mu_o)} \left(\cot^{-1}\frac{z}{a} - \frac{az}{z^2+a^2}\right)$$

$$H_y^- = H_y^o - \frac{2\mu_o H_y^o}{\pi(\mu+\mu_o)} \left(\cot^{-1}\frac{z}{a} - \frac{az}{z^2+a^2}\right)$$

$$H_z^- = 0$$

Along the axis ($x=y=0$, $\eta=-a^2$, $\xi=z^2$)

On the screen ($\eta=0$, $z=0$).

$$H_x^+ = -\frac{E_z^o \sigma \mu y}{\pi(\mu+\mu_o)} \left\{ \sin^{-1}\frac{a}{\rho} - \frac{a^2}{3\rho^4}\sqrt{\frac{\rho^2}{a^2}-1}\left[3\rho^2+2a^2 + \frac{8a^4x^2}{\rho^2(\rho^2-a^2)}\right]\right\}$$

$$+ \frac{2\mu H_x^o}{\pi(\mu+\mu_o)}\left(\sin^{-1}\frac{a}{\rho} - \frac{a^2}{\rho^2}\sqrt{\frac{\rho^2}{a^2}-1}\right) - \frac{4a^2\mu x\vec{H}^o\cdot\vec{\rho}}{\pi(\mu+\mu_o)\rho^4\sqrt{\frac{\rho^2}{a^2}-1}}$$

$$H_y^+ = -\frac{E_z^o \sigma \mu x}{\pi(\mu+\mu_o)} \left\{ \sin^{-1}\frac{a}{\rho} - \frac{a^2}{3\rho^4}\sqrt{\frac{\rho^2}{a^2}-1}\left[3\rho^2+2a^2 + \frac{8a^4y^2}{\rho^2(\rho^2-a^2)}\right]\right\}$$

$$+ \frac{2\mu H_y^o}{\pi(\mu+\mu_o)}\left(\sin^{-1}\frac{a}{\rho} - \frac{a^2}{\rho^2}\sqrt{\frac{\rho^2}{a^2}-1}\right) - \frac{4a^2\mu y\ \vec{H}^o\cdot\vec{\rho}}{\pi(\mu+\mu_o)\rho^4\sqrt{\frac{\rho^2}{a^2}-1}}$$

$$H_z^+ = 0$$

$$H_x^- = -E_z^o \sigma y \left(1 - \frac{1}{\pi}\sin^{-1}\frac{a}{\rho} + \frac{a^2}{\pi\rho^2}\sqrt{\frac{\rho^2}{a^2}-1}\right)$$

$$+ \frac{E_z^o\sigma\mu_o y}{\pi(\mu+\mu_o)}\left\{\sin^{-1}\frac{a}{\rho} - \frac{a^2}{3\rho^4}\sqrt{\frac{\rho^2}{a^2}-1}\left[3\rho^2+2a^2 + \frac{8a^4x^2}{\rho^2(\rho^2-a^3)}\right]\right\}$$

$$+ H_x^o\left\{1 - \frac{2\mu_o}{\pi(\mu+\mu_o)}\sin^{-1}\frac{a}{\rho} - \frac{a^2}{\rho^2}\sqrt{\frac{\rho^2}{a^2}-1}\right\}$$

$$+ \frac{4a^2\mu_o x\vec{H}^o\cdot\vec{\rho}}{\pi(\mu+\mu_o)\rho^4\sqrt{\frac{\rho^2}{a^2}-1}}$$

$$H_y^- = -\frac{E_z^O \sigma x}{\pi} \left(\sin^{-1}\frac{a}{\rho} - \frac{a^2}{\rho^2}\sqrt{\frac{\rho^2}{a^2}-1}\right) + \frac{E_z^O \sigma \mu_o x}{\pi(\mu+\mu_o)} \{\sin^{-1}\frac{a}{\rho}$$

$$- \frac{a^2}{3\rho^4}\sqrt{\frac{\rho^2}{a^2}-1}\ [3\rho^2+2a^2+\frac{8a^4y^2}{\rho^2(\rho^2-a^2)}]\} + H_y^O$$

$$- \frac{2\mu^O H_y^O}{\pi(\mu+\mu_o)} \left(\sin^{-1}\frac{a}{\rho} - \frac{a^2}{\rho^2}\sqrt{\frac{\rho^2}{a^2}-1}\right)$$

$$+ \frac{4a^2\mu_o y \vec{H}^O \cdot \vec{\rho}}{\pi(\mu+\mu_o)\rho^4\sqrt{\frac{\rho^2}{a^2}-1}}$$

$$H_z^- = 0$$

Far fields (up to terms of $\frac{1}{r^3}$)

$$H_x^+ = \frac{4a^3\mu}{3\pi(\mu+\mu_o)r^3}\left(H_x^O - 3H_x^O \sin^2\theta\cos^2\phi - 3H_y^O \sin^2\theta\sin\phi\cos\phi \right)$$

(θ measured from positive z-axis)

$$H_y^+ = \frac{4a^3\mu}{3\pi(\mu+\mu_o)r^3}\left(H_y^O - 3H_x^O \sin^2\theta\cos\phi\sin\phi - 3H_y^O \sin^2\theta\sin^2\phi \right)$$

$$H_z^+ = -\frac{4a^3\mu\cos\theta\sin\theta(H_x^O\cos\phi + H_y^O\sin\phi)}{\pi(\mu+\mu_o)r^3}$$

$$H_x^- = H_x^O - E_z^O\sigma y + \frac{2a^3E_z^O\sigma y}{3\pi r^3} - \frac{4a^3\mu_o}{3\pi(\mu+\mu_o)r^3}$$

$$(H_x^O - 3H_x^O\sin^2\theta\cos^2\phi - 3H_y^O\sin^2\theta\sin\phi\cos\phi)$$

$$H_y^- = H_y^o - \frac{2a^3E_z^o\sigma x}{3\pi r^3} - \frac{4a^3\mu_o}{3\pi(\mu+\mu_o)r^3}$$

$$(H_y^o - 3H_x^o\sin^2\theta\cos\phi\sin\phi - 3H_y^o\sin^2\theta\sin^2\phi)$$

$$H_z^- = -\frac{4a^3\mu_o\cos\theta\sin\theta}{\pi(\mu+\mu_o)r^3}\quad(H_z^o\cos\phi + H_y^o\sin\phi)$$

$$\vec{H}^+ = \frac{3\hat{r}(\hat{r}\cdot\vec{P}_M) - \vec{P}_M}{4\pi r^3}\qquad \vec{P}_M = -\frac{16a^3\mu}{3(\mu+\mu_o)}\,\vec{H}^o$$

The magnetic moment is independent of σ. If $\mu=\mu_o$, this is the same value as for the case of an aperture in air. For the incident side, the scattered magnetic far field is the same as a magnetic dipole of moment given by

$$\vec{P}_M = \frac{16a^3\mu_o\vec{H}^o}{3(\mu+\mu_o)} + \frac{8}{3}a^3\sigma\vec{E}^o x\vec{\rho}$$

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