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EEM AND SEM PARAMETER VARIATION DUE TO SURFACE PERTURBATIONS OF A METALLIC SPHERE

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ABSTRACT

In this report we utilize first-order degenerate perturbation theory to determine the changes in the eigenfunctions, eigenvalues and the corresponding singularity expansion method parameters due to surface perturbations of a metallic sphere. The main motivational factor for this study is the desire to provide a basis for determining errors in the SEM parameters due to uncertainties in the geometric configuration of a sphere-like metallic body and for judiciously extrapolating the results to more complicated structures.

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SECTION I

INTRODUCTION

The objective for this study is to utilize the only known analytical solution for electromagnetic scattering from a finite metallic body, i.e., the sphere, obtained via the Eigenmode Expansion Method (EEM) or equivalently the Singularity Expansion Method (SEM), to arrive at a solution for a metallic body slightly deviating from the sphere. The main motivational factor is the desire to provide a basis for estimating the errors in the SEM parameters of a sphere-like metallic body for a given degree of uncertainty in its geometrical configuration and for judiciously extrapolating the results to more complicated structures.

The objective is realized by tackling the problem via a perturbation method and, in particular, first-order degenerate perturbation theory, due to the degeneracy of the sphere eigenfunctions of the Magnetic Field Integral Equation (MFIE) operator. The first-order corrections to the eigenvalues and eigenfunctions are calculated in terms of the known unperturbed parameters of the sphere problem. These corrections involve integrals that, in general, cannot be evaluated analytically except perhaps for specific geometrical configurations. Application of perturbation theory is not as straightforward as it may appear because of the following two reasons: 1) The unperturbed eigenfunctions (ref. 1, 2) form a complete set for smooth vector fields that are tangential to the sphere, whereas the perturbed eigenfunctions are tangential to the perturbed sphere but not to the unperturbed sphere.

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1. Baum, C. E., On the Singularity Expansion Method for the Solution of Electromagnetic Interaction Problems, Interaction Note 88, Air Force Weapons Laboratory, December 1971.

2. Marin, L., Natural-Mode Representation of Transient Scattering from Rotationally Symmetric, Perfectly Conducting Bodies and Numerical Results for a Prolate Spheroid, Interaction Note 119, Air Force Weapons Laboratory, September 1972.

We overcame this difficulty by showing that a perturbed eigenfunction can be obtained, to any desired order, by applying a suitable "inverse" projection operator to the tangential-to-the-sphere component of this eigenfunction which, in turn, can be calculated via perturbation theory. (2) The Magnetic Field Integral operator is not self-adjoint and, consequently, one must solve for both its regular and adjoint eigenfunctions. Instead of evaluating the adjoint to the sphere and perturbation operators, we appealed to the results obtained in reference 3 which show that the adjoint operator need not be evaluated because the adjoint eigenfunctions can be explicitly expressed in terms of the regular eigenfunctions.

Once the perturbed eigenvalues and eigenfunctions are known, one can evaluate the shift in the pole locations (or natural frequencies), the natural modes and the coupling coefficients to obtain the SEM solution. We assumed that the appropriate coupling coefficients are class 1, i.e., the question of whether an additional entire function is present is not dealt with in this report. However, our formalism allows us to evaluate the perturbed eigenfunctions and eigenvalues, and, consequently, one may apply the procedure outlined in Section VII of reference 3 to probe the question of the presence or absence of the entire function when the coupling coefficients are class 1. (See Section V for more details.) A future effort may be directed toward this matter.

The final goal of the perturbation study is to estimate how uncertainties in the geometric configuration of a metallic structure affect its SEM parameters. For a complex structure such a task is formidable if not impossible. The present "perturbed sphere" problem offers an understanding of how the

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3. Sancer, M. I., A. D. Varvatsis and S. Siegel, Pseudosymmetric Eigenmode Expansion for the Magnetic Field Integral Equation and SEM Consequences, Interaction Note 355, Air Force Weapons Laboratory, October 1978.

various natural modes and frequencies are affected by specific deviations from the sphere geometry and also provides a basis for a statistical correlation between the deviations in the geometric configuration and in the SEM parameters. Such correlations may be applicable to more complex structures. As an initial effort, this report investigated only the perturbational effects necessary to the statistical correlation characterization. This characterization needs to be pursued in the next phase of the statistical EMP investigation.

Section II presents the derivation of the perturbation operator and explains how the set of the unperturbed sphere eigenfunctions can still be used to evaluate the perturbed eigenfunctions that are not tangential to the sphere. Section III applies first-order degenerate perturbation theory to calculate the shift in the eigenvalues and eigenfunctions and explicitly shows that the set of the perturbed eigenfunctions suffices to determine the response of the perturbed sphere to an incident field, i.e., no adjoint eigenfunctions are necessary. Section IV is concerned with the determination of the appropriate SEM parameters and discusses the meaning of an SEM perturbation solution in the time domain. Section V discusses the findings in this report and offers suggestions for future research. Finally two appendices give computational details to supplement the derivation of the main results. Appendix A provides the algebraic steps that led to the formula for the perturbation operator. Appendix B serves a two-fold purpose. It shows how one can reshuffle singular integrals to insure that the perturbation operator is nonsingular and also provides sufficient details to show how integrands with removable singularities can be transformed to become suitable for numerical integration.

SECTION II

DERIVATION OF THE PERTURBATION OPERATOR

We start with the Magnetic Field Integral Equation

$$L\underline{J} \equiv \frac{1}{2}\,\underline{J}(\underline{r}) - \int_S \hat{n}(\underline{r})\times\left[\nabla G\times\underline{J}(\underline{r}')\right]dS' = \underline{J}^{inc}(\underline{r}) \qquad (1)$$

where $\underline{J}(\underline{r})$ is the surface current density induced on a perfectly conducting body enclosed by the surface S, $\hat{n}$ is the unit outward normal to S, $\underline{J}^{inc} = \hat{n}\times\underline{H}^{inc}$, $\underline{H}^{inc}$ is the incident magnetic field and $G = \exp(-\gamma R)/4\pi R$ is the free space Green's function. The rest of the quantities are γ and R defined as follows:

$$\gamma = \frac{s}{c}$$

s: complex frequency, c: speed of light

$R = |\underline{R}|$, $\underline{R} = \underline{r}-\underline{r}'$

$\underline{r},\underline{r}'$: radii vectors to points on S.

We will assume that the surface S can be described by the parametric equation

$$\underline{r} = \underline{r}_0(\theta,\phi)\left[1+\varepsilon f(\theta,\phi)\right] \qquad (2)$$

where $\underline{r}_0(\theta,\phi)$ is the radius vector for the sphere S_0 inscribed in S, θ and ϕ are the usual angular spherical coordinates, f is a smooth function and ε a perturbation parameter. Representation (2) guarantees that for each $\Omega = (\theta,\phi)$ there is only <u>one</u> value for $\underline{r}$.

We are interested in solving Eq. 1 by employing the Singularity Expansion Method, and, consequently, we must determine the natural modes, natural frequencies or pole locations and coupling coefficients for the problem expressed by Eq. 1 and 2. This will be accomplished by first solving the eigenvalue problem for the MFIE operator L, i.e.,

$$L\underline{J}_p = \lambda_p \underline{J}_p \tag{3}$$

with the aid of first-order perturbation theory. The unperturbed operator is the MFIE operator L_0 for the problem of the inscribed sphere S_0. The unperturbed eigenfunctions and eigenvalues are well known (ref. 2), and we will present their explicit expressions in Section III. In applying perturbation theory the perturbed eigenfunctions are expanded in terms of the unperturbed ones that form a complete set. In the present case this procedure cannot be employed in a straightforward manner because the sphere eigenfunctions form a complete set for the class of functions that are <u>tangential</u> to the sphere, whereas the perturbed eigenfunctions are tangential to S but in general have a nonvanishing normal component with respect to the sphere. If we write

$$\underline{J}_p(\underline{r}) = \underline{J}_{pt}(\underline{r}) + \underline{J}_{pn}(\underline{r}) \tag{4}$$

where

$$\underline{J}_{pt}(\underline{r}) \equiv - \hat{n}_0(r) \times \hat{n}_0(\underline{r}) \times \underline{J}_p(\underline{r}) = (\underline{\underline{I}} - \hat{n}_0\hat{n}_0) \cdot \underline{J}_p(\underline{r})$$

$$\underline{J}_{pn}(\underline{r}) \equiv \hat{n}_0(\underline{r})\hat{n}_0(\underline{r}) \cdot \underline{J}_p(\underline{r})$$

and $\hat{n}_0$ is the outward unit normal to the <u>sphere</u>, then $\underline{J}_{pt}(\underline{r})$ is, by construction, tangential to S_0 and can be expanded in terms of the unperturbed eigenfunctions. The normal component $\underline{J}_{pn}$ is simply related to $\underline{J}_{pt}$; i.e., $\underline{J}_p(\underline{r})$ can be determined once $\underline{J}_{pt}$ is known. To see how this comes about, we dot Eq. 4 with $\hat{n}$, the unit outward normal to S.

$$0 = \hat{n}\cdot\underline{J}_p = \hat{n}\cdot\underline{J}_{pt} + \hat{n}\cdot\underline{J}_{pn} \tag{5}$$

Next we write

$$\left.\begin{aligned} \hat{n} &= \hat{n}_0 + \epsilon\underline{n}^{(1)} + \epsilon^2\underline{n}^{(2)} + \cdots \\ \underline{J}_{pt} &= \underline{J}_{pt}^{(0)} + \epsilon\underline{J}_{pt}^{(1)} + \epsilon^2\underline{J}_{pt}^{(2)} + \cdots \\ \underline{J}_{pn} &= \underline{J}_{pn}^{(0)} + \epsilon\underline{J}_{pn}^{(1)} + \epsilon^2\underline{J}_{pn}^{(2)} + \cdots \end{aligned}\right\} \tag{6}$$

and noting that $\hat{n}_0\cdot\underline{J}_{pt} = 0$, Eqs. 5 and 6 can be combined to yield the following relationships:

$$\begin{aligned} \underline{J}_{pn}^{(0)} &= 0 \\ \hat{n}_0\cdot\underline{J}_{pn}^{(1)} &= -\underline{n}^{(1)}\cdot\underline{J}_{pt}^{(0)} \\ \underline{n}^{(1)}\cdot\underline{J}_{pn}^{(1)} + \hat{n}_0\cdot J_{pn}^{(2)} &= -\underline{n}^{(1)}\cdot\underline{J}_{pt}^{(0)} \end{aligned} \tag{7}$$

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Eq. 7 shows that once $\underline{J}_{pt}$ has been determined to any given order in ϵ the normal component $\underline{J}_{pn}$ can also be evaluated up to the next higher order. In what follows we will restrict our attention to first-order perturbation and consequently we can write

$$\underline{J}_{pn} = -\varepsilon\hat{n}_0\underline{n}^{(1)}\cdot\underline{J}_{pt} \equiv \varepsilon A\underline{J}_{pt}$$

$$\underline{J}_p = \underline{J}_{pt} - \varepsilon\hat{n}_0\underline{n}^{(1)}\cdot\underline{J}_{pt} \equiv M\underline{J}_{pt} \qquad (8)$$

Notice that Eq. 8 is correct in first order in ε only.

Before we go on to consider the perturbation solution, we note that since the MFIE operator L is not self-adjoint, the adjoint eigenfunctions $\underline{J}_p^\dagger$ are needed for the calculation of the coupling coefficients in the SEM solution or the expansion coefficients in the EEM solution of Eq. 1. (To be precise the coupling coefficients involve the coupling vectors that are obtained by evaluating the adjoint eigenfunctions at the pole locations.) It was shown, however, in reference 3 that the adjoint eigenfunctions $\underline{J}_p^\dagger$ are related to the eigenfunctions of L as follows:

$$\underline{J}_p^{\dagger *} = \tilde{\underline{J}}_p \times \hat{n}$$

where $\tilde{\underline{J}}_p$ is an eigenfunction of L with eigenvalue $1-\lambda_p$; i.e.,

$$L\tilde{\underline{J}}_p = (1-\lambda_p)\tilde{\underline{J}}_p$$

Naturally, the eigenvalue λ_p corresponds to $\underline{J}_p$ satisfying

$$L\underline{J}_p = \lambda_p\underline{J}_p$$

Thus we can dispense with the adjoint eigenfunctions by introducing a pseudosymmetric inner product

$$\left\{\underline{\Phi},\underline{\Psi}\right\} \equiv \int_S \underline{\Phi}\cdot(\hat{n}\times\underline{\Psi})\,dS \qquad (9)$$

and employing the following orthogonality relationships

$$\left\{\underline{\tilde{J}}_p, J_n\right\} = \left(\underline{J}_p^{\dagger}, \underline{J}_n\right) = N_n \delta_{pn}$$

where (10a)

$$\left(\underline{J}_p^{\dagger}, \underline{J}_n\right) \equiv \int_S \underline{J}_p^{\dagger *} \cdot \underline{J}_n dS$$

Moreover if we group our eigenfunctions into two categories, $\underline{J}_p$ and $\underline{\tilde{J}}_p$, we also have that

$$\left\{\underline{J}_p, \underline{J}_n\right\} = 0 \quad \left\{\underline{\tilde{J}}_p, \underline{\tilde{J}}_n\right\} = 0 \tag{10b}$$

for any p and n.

If we now write

$$\underline{\tilde{J}}_p = \underline{\tilde{J}}_{pt} + \underline{\tilde{J}}_{pn} = M\underline{\tilde{J}}_{pt}$$

then $\underline{\tilde{J}}_{pt}$ is an eigenfunction of the same operator as $\underline{J}_{pt}$ since

$$LM\underline{J}_{pt} = \lambda_p M\underline{J}_{pt}$$

$$LM\underline{\tilde{J}}_{pt} = (1-\lambda_p)M\underline{\tilde{J}}_{pt}$$

or

$$M^{-1}LM\underline{J}_{pt} = \lambda_p \underline{J}_{pt} \tag{11a}$$

$$M^{-1}LM\underline{\tilde{J}}_{pt} = (1-\lambda_p)\underline{\tilde{J}}_{pt} \tag{11b}$$

and $M^{-1}\underline{f} = (\underline{\underline{I}} - \hat{n}_0\hat{n}_0)\cdot\underline{f}$. Notice that $M^{-1}M\underline{J}_{pt} = \underline{J}_{pt} + 0(\varepsilon^2)$.

In what follows we will employ Eq. 11 which is correct in first order in ε and cast $M^{-1}LM$ in the form

$$M^{-1}LM = L_0 + \varepsilon P$$

where L_0 is the MFIE sphere operator. In the process the following relationships will be employed (see appendix A)

$$dS = dS_0 + 2\varepsilon f dS_0$$

$$\hat{n} = \hat{n}_0 + \varepsilon \underline{n}^{(1)}$$

$$\underline{n}^{(1)} = -\left(\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{\sin\theta}\frac{\partial f}{\partial \phi}\hat{e}_\phi\right)$$

$$\nabla G = \nabla G_0 + \varepsilon \underline{\Delta}$$

$$\underline{\Delta} = \left[\left(\gamma + \frac{1}{R_0}\right)\left(3\hat{R}_0\hat{R}_0 - \underline{\underline{I}}\right) + \gamma^2 \frac{\hat{R}_0\hat{R}_0}{R_0}\right]\cdot\frac{\Delta\underline{R}}{R_0} G_0$$

$$G_0 = \exp(-\gamma R_0)/4\pi R_0$$

$$R_0 = |\hat{n}_0 - \hat{n}_0'|, \quad \hat{R}_0 = (\hat{n}_0 - \hat{n}_0')/R_0$$

$$\Delta\underline{R} = f(\theta,\phi)\hat{n}_0(\theta,\phi) - f(\theta',\phi')\hat{n}_0(\theta',\phi')$$

$$\underline{J}_{pn} = -\varepsilon\hat{n}_0\underline{n}^{(1)}\cdot\underline{J}_{pt} \equiv \varepsilon A\underline{J}_{pt} \qquad (12)$$

Notice that the radius of the sphere has been set equal to unity. If Eq. 12 is employed, Eq. 11 can be rewritten as

$$(L_0+\varepsilon P)\underline{J}_{pt} = \lambda_p J_{pt} \tag{13a}$$

$$(L_0+\varepsilon P)\tilde{\underline{J}}_{pt} = (1-\lambda_p)\tilde{J}_{pt} \tag{13b}$$

where

$$L_0\underline{J}_{pt} = \frac{1}{2}\,\underline{J}_{pt}(\Omega) - \int_{S_0} \hat{n}_0(\Omega) \times \left[\nabla G_0 \times \underline{J}_{pt}(\Omega')\right] dS_0$$

$$P\underline{J}_{pt} = -2\int_{S_0} \hat{n}_0(\Omega) \times \left[\nabla G_0 \times \underline{J}_{pt}(\Omega')\right] f(\Omega')\, dS_0$$

$$+ \int_{S_0} \hat{n}_0(\Omega) \times \left[\nabla G_0 \times A J_{pt}(\Omega')\right] dS_0$$

$$- \int_{S_0} \hat{n}_0(\Omega) \times \left[\underline{\Delta}(\Omega,\Omega') \times \underline{J}_{pt}(\Omega')\right] dS_0$$

$$- \left[\underline{\underline{I}} - \hat{n}_0(\Omega)\hat{n}_0(\Omega)\right] \cdot \int_{S_0} \underline{n}^{(1)}(\Omega) \times \left[\nabla G_0 \times \underline{J}_{pt}(\Omega')\right] dS_0 \tag{14}$$

The reason the integration is over S_0 rather than S is that all the quantities in the integrands are evaluated at points on S_0. Notice that $dS_0 = r_0 d\Omega = d\Omega = \sin\theta d\theta d\phi$.

Before we proceed with the perturbation solution of Eq. 13, we observe that the integrands in the last three of Eq. 14 have nonintegrable singularities. It is shown, however, in Appendix B that these integrands can be expanded, rearranged and grouped so that the resulting integrands have integrable singularities. This should indeed be the case because both L and L_0 involve integrals with integrable singularities, and consequently P should not be singular.

SECTION III

THE PERTURBATION SOLUTION

In this section we determine the perturbed eigenfunction and eigenvalues in terms of the unperturbed ones. The unperturbed eigenfunctions are the vector spherical harmonics $\underline{R}_{nm}$, $\underline{Q}_{nm}$ satisfying equations

$$L_0 \underline{R}_{nm} = \lambda_n^{(0)R} \underline{R}_{nm}$$
$$L_0 \underline{Q}_{nm} = \lambda_n^{(0)Q} \underline{Q}_{nm} \tag{15}$$

where (ref. 2)

$$\left.\begin{aligned} \underline{R}_{nm} &= -\frac{\partial Y_{nm}}{\partial\theta}\hat{e}_\phi + \frac{1}{\sin\theta}\frac{\partial Y_{nm}}{\partial\phi}\hat{e}_\theta \\ \underline{Q}_{nm} &= \frac{1}{\sin\theta}\frac{\partial Y_{nm}}{\partial\phi}\hat{e}_\phi + \frac{\partial Y_{nm}}{\partial\theta}\hat{e}_\theta \end{aligned}\right\} \begin{aligned} n\geq 1 \\ |m|\leq n \end{aligned} \tag{16}$$

$$\lambda_n^{(0)R} = \left[\gamma a i_n(\gamma a)\right]' \left[\gamma a k_n(\gamma a)\right]$$
$$\lambda_n^{(0)Q} = -\left[\gamma a i_n(\gamma a)\right]\left[\gamma a k_n(\gamma a)\right]' \tag{17}$$

and a is the radius of the sphere (a=1 in our case). Thus the sphere eigenfunctions are degenerate; i.e., for each n there are 2n+1 R-eigenfunctions with eigenvalue $\lambda_n^{(0)R}$ and 2n+1 Q-eigenfunctions with eigenvalue $\lambda_n^{(0)Q}$. It has been shown in reference 3 that

$$\tilde{\underline{R}}_{nm} = \underline{Q}_{nm}$$
$$\tilde{\underline{Q}}_{nm} = \underline{R}_{nm} \tag{18}$$

and consequently we should have $\lambda_n^{(0)R} + \lambda_n^{(0)Q} = 1$ which can be verified by appealing to Eq. 17 and employing the appropriate Wronskian. Then orthogonality relationships, Eq. 10, can be translated into the following

$$\left|\tilde{\underline{R}}_{nm}, \underline{R}_{pl}\right| = \left|\underline{Q}_{nm}, \underline{R}_{pl}\right| = N^R_{nm}\delta_{np}\delta_{ml}$$

$$\left|\tilde{\underline{Q}}_{nm}, \underline{Q}_{pl}\right| = \left|\underline{R}_{nm}, \underline{Q}_{pl}\right| = N^Q_{nm}\delta_{np}\delta_{ml} \tag{19}$$

$$\left|\underline{R}_{nm}, \underline{R}_{pl}\right| = 0, \ \left|\underline{Q}_{nm}, Q_{pl}\right| = 0$$

From Eq. 16 we see that

$$\underline{Q}_{nm} = \hat{n}_0 \times \underline{R}_{nm}$$

$$\underline{R}_{nm} = -\hat{n}_0 \times \underline{Q}_{nm}$$

and consequently

$$N^R_{nm} = \int \underline{Q}_{nm} \cdot \left(\hat{n}_0 \times \underline{R}_{nm}\right) d\Omega = \int \underline{Q}_{nm} \cdot \underline{Q}_{nm} d\Omega = \int \underline{R}_{nm} \cdot \underline{R}_{nm} d\Omega$$

$$N^Q_{nm} = \int \underline{R}_{nm} \cdot \left(\hat{n}_0 \times Q_{nm}\right) d\Omega = -\int \underline{Q}_{nm} \cdot \underline{Q}_{nm} d\Omega$$

i.e.,

$$N^R_{nm} = -N^Q_{nm} \equiv N^0_{nm} \tag{20}$$

The presence of the perturbation will usually resolve the degeneracy. As $\varepsilon \rightarrow 0$ we should have $\underline{J}_{pt} \rightarrow$ unperturbed eigenfunction and the question arises as to which (among the ones corresponding to index p) this is. To answer this question, we write

$$\underline{J}^{R}_{nmt} = \underline{\check{R}}_{nm} + \varepsilon \sum_{p} \sum_{\ell} \left[a^{RR}_{nmp\ell} \underline{R}_{p\ell} + a^{RQ}_{nmp\ell} \underline{Q}_{p\ell} \right] \tag{21a}$$

$$\underline{J}^{Q}_{nmt} = \underline{\check{Q}}_{nm} + \varepsilon \sum_{p} \sum_{\ell} \left[a^{QR}_{nmp\ell} \underline{R}_{p\ell} + a^{QQ}_{nmp\ell} \underline{Q}_{p\ell} \right] \tag{21b}$$

$$\underline{\tilde{J}}^{R}_{nmt} = \underline{\tilde{\check{R}}}_{nm} + \varepsilon \sum_{p} \sum_{\ell} \left[\tilde{a}^{RR}_{nmp\ell} \underline{R}_{p\ell} + \tilde{a}^{RQ}_{nmp\ell} \underline{Q}_{p\ell} \right] \tag{21c}$$

$$\underline{\tilde{J}}^{Q}_{nmt} = \underline{\tilde{\check{Q}}}_{nm} + \varepsilon \sum_{p} \sum_{\ell} \left[\tilde{a}^{QR}_{nmp\ell} \underline{R}_{p\ell} + \tilde{a}^{QQ}_{nmp\ell} \underline{Q}_{p\ell} \right] \tag{21d}$$

where the eigenfunctions with the inverted hat correspond to those special eigenfunctions and can be written as

$$\underline{\check{R}}_{nm} = \sum_{k} b^{R}_{nmk} \underline{R}_{nk} \tag{22a}$$

$$\underline{\check{Q}}_{nm} = \sum_{k} b^{Q}_{nmk} \underline{Q}_{nk} \tag{22b}$$

$$\underline{\check{\tilde{R}}}_{nm} = \sum_k \tilde{b}^R_{nmk} \underline{Q}_{nk} \tag{22c}$$

$$\underline{\check{\tilde{Q}}}_{nm} = \sum_k \tilde{b}^Q_{nmk} \underline{R}_{nk} \tag{22d}$$

These expansion coefficients must now be determined. Let's begin with $\underline{J}^R_{nmt}$, satisfying

$$(L_0 + \varepsilon P) \underline{J}^R_{nmt} = \lambda^R_{nm} \underline{J}^R_{nmt} \tag{23}$$

Inserting $\underline{J}^R_{nmt}$ in Eq. 21 into Eq. 23 and writing

$$\lambda^R_{nm} = \lambda^{(0)R}_n + \varepsilon \lambda^{(1)R}_{nm} \tag{24}$$

we obtain

$$L_0 R_{nm} = \lambda^{(0)R}_n \underline{R}_{nm} \tag{25}$$

$$P\underline{\check{R}}_{nm} + \sum_p \sum_\ell \left[a^{RR}_{nmp\ell} \lambda^{(0)R}_p \underline{R}_{p\ell} + a^{RQ}_{nmp\ell} \lambda^{(0)Q}_p \underline{Q}_{p\ell} \right]$$

$$= \lambda^{(1)R}_{nm} \underline{\check{R}}_{nm} + \lambda^{(0)R}_n \sum_p \sum_\ell \left[a^{RR}_{nmp\ell} \underline{R}_{p\ell} + a^{RQ}_{nmp\ell} Q_{p\ell} \right] \tag{26}$$

Let us now form the pseudosymmetric inner product (defined by Eq. 9) of $\underline{Q}_{ns}$ with Eq. 26. If we take Eq. 10 and Eq. 22a into account, we obtain

$$\left|\underline{Q}_{ns}, P\check{\underline{R}}_{nm}\right| = \lambda_{nm}^{(1)R} \, b_{nms}^{R} N_{ns}^{0} \tag{27}$$

If we combine Eq. 27 and 22a, we arrive at

$$\sum_{k} \left\{\underline{Q}_{ns}, P\underline{R}_{nk}\right\} b_{nmk}^{R} = \lambda_{nm}^{(1)R} \, b_{nms}^{R} N_{ns}^{0}$$

or

$$\sum_{k} \left(P_{nsk}^{R}/N_{nk}^{0}\right)\left(b_{nmk}^{R} N_{nk}^{0}\right) = \lambda_{nm}^{(1)R}\left(b_{nms}^{R} N_{ns}^{0}\right) \tag{28}$$

where

$$P_{nsk}^{R} \equiv \left|\underline{Q}_{ns}, P\underline{R}_{nk}\right| \tag{29}$$

Thus to determine $\lambda_{nm}^{(1)}$ and the expansion coefficients b_{nmk}^{R}, we must solve the eigenvalue problem, Eq. 28. If there is no degeneracy, we will obtain 2n+1 eigenvalues $\lambda_{n1}^{(1)R}$, $\lambda_{n2}^{(1)R}$, ..., $\lambda_{2n+1}^{(1)R}$ and 2n+1 eigenvectors b_{n1}^{R}, b_{n2}^{R}, ..., $b_{n,2n+1}^{R}$, each having 2n+1 components. We now return to Eq. 26 and form the pseudosymmetric inner products first of $\underline{Q}_{rs}$ $(r \neq n)$ and then of $\underline{R}_{rs}$ with this equation. We obtain

$$a_{nmrs}^{RR} = \frac{\left|\underline{Q}_{rs}, P\check{\underline{R}}_{nm}\right|}{\left(\lambda_{n}^{(0)R} - \lambda_{r}^{(0)R}\right) N_{rs}^{0}} \qquad (n \neq r) \tag{30a}$$

$$a^{RQ}_{nmrs} = - \frac{\left\{\underline{R}_{rs}, P\check{\underline{R}}_{nm}\right\}}{\left(\lambda^{(0)R}_{n} - \lambda^{(0)Q}_{r}\right) N^{0}_{rs}} \tag{30b}$$

If we now consider $\underline{J}^{Q}_{nmt}$ given by Eq. 21b, we can follow the same procedure as for $\underline{J}^{R}_{nmt}$ to obtain

$$\left.\begin{aligned} \sum_{k} \left(P^{Q}_{nsk}/N^{0}_{nk}\right)\left(b^{Q}_{nmk}N^{0}_{nk}\right) &= -\lambda^{(1)Q}_{nm}\left(b^{Q}_{nms}N^{0}_{ns}\right) \\ P^{Q}_{nsk} &\equiv \left\{\underline{R}_{ns}, PQ_{nk}\right\} \end{aligned}\right\} \tag{31}$$

$$\left.\begin{aligned} a^{QR}_{nmrs} &= \frac{\left\{\underline{Q}_{rs}, P\check{\underline{Q}}_{nm}\right\}}{\left(\lambda^{(0)Q}_{n} - \lambda^{(0)R}_{r}\right) N^{0}_{rs}} \\ a^{QQ}_{nmrs} &= - \frac{\left\{\underline{R}_{rs}, P\check{\underline{Q}}_{nm}\right\}}{\left(\lambda^{(0)Q}_{n} - \lambda^{(0)Q}_{r}\right) N^{0}_{rs}} \qquad (r \neq n) \end{aligned}\right. \tag{32}$$

Thus Eq. 31 determines the eigenvalue corrections $\lambda^{(1)Q}_{nm}$ and the expansion coefficients b^{Q}_{nmk} whereas Eq. 32 gives the expansion coefficients in Eq. 21b. Finally, the same procedure can be employed for $\tilde{J}^{R}_{nmt}$ and $\tilde{J}^{Q}_{nmt}$, satisfying Eq. 13b, to obtain

$$\left.\begin{aligned} \sum_{k} \left(\tilde{P}^{R}_{nsk}/N^{0}_{nk}\right)\left(\tilde{b}^{R}_{nmk}N^{0}_{nk}\right) &= \lambda^{(1)R}_{nm}\left(\tilde{b}^{R}_{nms}N^{0}_{ns}\right) \\ \tilde{P}^{R}_{nsk} &\equiv \left\{R_{ns}, PQ_{nk}\right\} \end{aligned}\right\} \tag{34}$$

$$\tilde{a}^{RR}_{nmrs} = \frac{\left\{Q_{rs}, P\tilde{\tilde{R}}_{nm}\right\}}{\left(\lambda_n^{(0)Q} - \lambda_r^{(0)R}\right)N^0_{rs}} \quad (35a)$$

$$\tilde{a}^{RQ}_{nmrs} = - \frac{\left\{R_{rs}, P\tilde{\tilde{R}}_{nm}\right\}}{\left(\lambda_n^{(0)Q} - \lambda_r^{(0)Q}\right)N^0_{rs}} \quad (r \neq n) \quad (35b)$$

$$\left.\begin{aligned} \sum_k \left(\tilde{P}^Q_{nsk}/N^0_{nk}\right)\left(\tilde{b}^Q_{nmk}N^0_{nk}\right) &= -\lambda^{(1)}_{nm}\tilde{b}^Q_{nms}N^0_{ns} \\ \tilde{P}^Q_{nsk} &\equiv \left\{Q_{ns}, PR_{nk}\right\} \end{aligned}\right\} \quad (36)$$

$$\tilde{a}^{QR}_{nmrs} = \frac{\left\{\underline{Q}_{rs}, P\tilde{\tilde{Q}}_{nm}\right\}}{\left(\lambda_n^{(0)R} - \lambda_r^{(0)R}\right)N_{rs}} \quad (r \neq n) \quad (37a)$$

$$\tilde{a}^{QQ}_{nmrs} = - \frac{\left\{\underline{R}_{rs}, P\tilde{\tilde{Q}}_{nm}\right\}}{\left(\lambda_n^{(0)R} - \lambda_r^{(0)Q}\right)N^0_{rs}} \quad (37b)$$

Comparing Eqs. 29 and 36, we see that $P^R_{nsk} = \tilde{P}^Q_{nsk}$ and consequently the corresponding eigenvalue problems are identical. Thus

$$\lambda^{(1)Q}_{nm} = -\lambda^{(1)R}_{nm} \quad (38a)$$

and if there is no degeneracy

$$\tilde{b}^Q_{nmk} = cb^R_{nmk} \quad (38b)$$

where c is a constant that can be set equal to unity without loss of generality; consequently

$$\check{R}_{nm} = \tilde{\check{Q}}_{nm}$$

Then it is a simple matter to show that

$$\tilde{a}^{QR}_{nmrs} = a^{RR}_{nmrs} \qquad (39a)$$

$$\tilde{a}^{QQ}_{nmrs} = a^{RQ}_{nmrs} \qquad (39b)$$

Since it is shown in perturbation theory that the expansion coefficients a^{RR}_{nmns}, a^{QQ}_{nmns} are not needed, we conclude that

$$\tilde{J}^{Q}_{nmt} = J^{R}_{nmt} \qquad (40)$$

Similarly we obtain

$$\tilde{\check{R}}_{nm} = \check{Q}_{nm}$$

$$\tilde{J}^{R}_{nmt} = J^{Q}_{nmt} \qquad (41)$$

Notice that the result given by Eq. 38a in conjunction with the assumption of nondegeneracy is sufficient to show the validity of Eqs. 40 and 41. Thus we start with

$$L\tilde{J}^{Q}_{nm} = \left(1-\lambda^{Q}_{nm}\right)\tilde{J}^{Q}_{nm} = \left(1-\lambda^{(0)Q}_{n}-\varepsilon\lambda^{(1)Q}_{nm}\right)\tilde{J}^{Q}_{nm}$$

and then consider

$$LJ^R_{nm} = \lambda^R_{nm} J^R_{nm} = \left(\lambda^{(0)R}_{n} + \varepsilon\lambda^{(1)R}_{nm}\right) J^R_{nm}$$

$$= \left(1-\lambda^{(0)Q}_{n} + \varepsilon\lambda^{(1)R}_{nm}\right) J^R_{nm} = \left(1-\lambda^{(0)Q}_{n} - \varepsilon\lambda^{(1)Q}_{nm}\right) J^R_{nm}$$

Thus, if there is no degeneracy, $\tilde{J}^Q_{nm} = J^R_{nm}$ and a similar proof holds for $\tilde{J}^R_{nm} = J^Q_{nm}$. (Again we have set the multiplicative constant equal to unity.)

We are now in a position to determine the perturbed eigenfunctions of the original eigenvalue problems given by Eq. 3 by invoking the second of Eq. 8 and Eq. 21; i.e.,

$$\underline{J}^R_{nm} = M\underline{J}^R_{nmt} = \check{\underline{R}}_{nm} - \varepsilon\left(\hat{n}_0\underline{n}^{(1)}\cdot\check{\underline{R}}_{nm}\right)$$

$$+ \varepsilon\sum_{p\neq n}\sum_{\ell} a^{RR}_{nmp\ell}\,\underline{R}_{p\ell} + \varepsilon\sum_{p}\sum_{\ell} a^{RQ}_{nmp\ell}\,\underline{Q}_{p\ell} \qquad (42a)$$

$$\underline{J}^Q_{nm} = M\underline{J}^Q_{nmt} = \check{\underline{Q}}_{nm} - \varepsilon\left(\hat{n}_0\underline{n}^{(1)}\cdot\check{\underline{Q}}_{nm}\right)$$

$$+ \varepsilon\sum_{p}\sum_{\ell} a^{QR}_{nmp\ell}\,\underline{R}_{p\ell} + \varepsilon\sum_{p\neq n}\sum_{\ell} a^{QQ}_{nmp\ell}\,\underline{Q}_{p\ell} \qquad (42b)$$

$$\tilde{\underline{J}}^R_{nm} = \underline{J}^Q_{nm},\quad \tilde{\underline{J}}^Q_{nm} = \underline{J}^R_{nm} \qquad (42c)$$

where $\check{\underline{R}}_{nm}$ is given by Eq. 22a and the expansion coefficients a^{RR}_{nmpl}, a^{RQ}_{nmpl} are determined by Eq. 30. Similarly $\check{\underline{Q}}_{nm}$ is defined by Eq. 22b and the corresponding expansion coefficients

$a^{QR}_{nmp\ell}$, $a^{QQ}_{nmp\ell}$ are given by Eq. 32. The perturbed eigenvalues are determined by

$$\lambda^R_{nm} = \lambda^{(0)R}_{nm} + \varepsilon\lambda^{(1)R}_{nm}$$

$$\lambda^Q_{nm} = \lambda^{(0)Q}_{nm} + \varepsilon\lambda^{(1)Q}_{nm} = 1-\lambda^{(0)R}_{nm} - \varepsilon\lambda^{(1)R}_{nm}$$

$$= 1-\lambda^R_{nm} \qquad (43)$$

and $\lambda^{(1)R}_{nm} = -\lambda^{(1)Q}_{nm}$ are determined by solving the eigenvalue problem Eq. 28.

SECTION IV

SEM RESULTS

The quantities of importance for the Singularity Expansion Method are the pole locations, natural modes and coupling coefficients. The question of the additional entire function in the SEM expansion is not dealt with here.

1. POLE LOCATIONS

The pole locations $\gamma_{nmn'}$ are the zeros of the following equation

$$\lambda_{nm}(\gamma_{nmn'}) = 0 \tag{44}$$

where λ_{nm} are the eigenvalues. (We have dropped the superscripts R and Q for simplicity.) In first order

$$\lambda_{nm} = \lambda_n^{(0)} + \varepsilon\lambda_{nm}^{(1)} \tag{45}$$

$$\gamma_{nmn'} = \gamma_{nn'}^{(0)} + \varepsilon\gamma_{nmn'}^{(1)} \tag{46}$$

where $\gamma_{nn'}^{(0)}$ are the zeros of

$$\lambda_n^{(0)}\left(\gamma_{nn'}^{(0)}\right) = 0$$

Combining Eqs. 44, 45 and 46 we obtain

$$\lambda_n^{(0)}\left(\gamma_{nn'}^{(0)} + \varepsilon\gamma_{nmn'}^{(1)}\right) + \varepsilon\lambda_{nm}^{(1)}\left(\gamma_{nn'}^{(0)} + \varepsilon\gamma_{nmn'}^{(1)}\right) = 0$$

Expanding and retaining only first order terms, we arrive at the following equation

$$\lambda_n^{(0)}\left(\gamma_{nn'}^{(0)}\right) + \varepsilon\gamma_{nmn'}^{(1)}\left[\frac{d\lambda_n^{(0)}}{d\gamma}\right]_{\gamma=\gamma_{nn'}^{(0)}} + \varepsilon\lambda_{nm}^{(1)}\left(\gamma_{nn'}^{(0)}\right) = 0$$

or

$$\gamma_{nmn'}^{(1)R} = -\left[\frac{\lambda_{nm}^{(1)R}(\gamma)}{d\lambda_n^{(0)R}/d\gamma}\right]_{\gamma=\gamma_{nn'}^{(0)R}} \tag{47a}$$

$$\gamma_{nmn'}^{(1)Q} = -\left[\frac{\lambda_{nm}^{(1)Q}(\gamma)}{d\lambda_n^{(0)Q}/d\gamma}\right]_{\gamma=\gamma_{nn'}^{(0)R}} \tag{47b}$$

$$\gamma_{nmn'}^{R} = \gamma_{nn'}^{(0)R} + \varepsilon\gamma_{nmn'}^{(1)R} \tag{47c}$$

$$\gamma_{nmn'}^{Q} = \gamma_{nn'}^{(0)Q} + \varepsilon\gamma_{nmn'}^{(1)Q} \tag{47d}$$

Notice that $\lambda_{nm}^{(1)R}(\gamma) = -\lambda_{nm}^{(1)Q}(\gamma)$ and $\left(d\lambda_n^{(0)R}/d\gamma\right) = -\left(d\lambda_n^{(0)Q}/d\gamma\right)$; i.e., $\lambda_{nm}^{(1)R}(\gamma)/\left(d\lambda_n^{(0)R}/d\gamma\right) = \lambda_{nm}^{(1)Q}(\gamma)/\left(d\lambda_n^{(0)Q}/d\gamma\right)$, but $\gamma_{nmn'}^{(1)R} \neq \gamma_{nmn'}^{(1)Q}$ since $\gamma_{nn'}^{(0)R} \neq \gamma_{nn'}^{(0)Q}$.

2. NATURAL MODES

The natural modes are the perturbed eigenfunctions (determined in Section III) evaluated at the appropriate pole locations; i.e.,

$$\underline{\nu}_{nmn'}(\underline{r}) = \underline{J}_{nm}(\underline{r}, \gamma_{nmn'})$$

Notice that for the sphere the eigenfunctions are independent of the complex variable γ and consequently they coincide with the natural modes. Recalling that $\gamma_{nmn'} = \gamma_{nn'}^{(0)} + \varepsilon\gamma_{nmn'}^{(1)}$, and $\underline{J}_{nm}(\gamma,\underline{r}) = \underline{\check{J}}_{nm}^{(0)}(\Omega)+\varepsilon J_{nm}^{(1)}(\gamma,\Omega)$ we understand that the natural modes are evaluated as follows:

$$\underline{\nu}_{nmn'} = \underline{\check{J}}_{nm}^{(0)}(\Omega) + \varepsilon\underline{J}_{nm}^{(1)}\left(\gamma_{nn'}^{(0)},\Omega\right) + 0(\varepsilon^2) \qquad (48)$$

$$\underline{\nu}_{nmn'}^{R}(\underline{r}) = \underline{J}_{nm}^{R}\left(\underline{r},\gamma_{nn'}^{(0)R}\right) \quad , \quad \underline{\nu}_{nmn'}^{Q}(\underline{r}) = \underline{J}_{nm}^{Q}\left(\underline{r},\gamma_{nn'}^{(0)Q}\right) \qquad (49)$$

$\underline{J}_{nm}^{R}$ and $\underline{J}_{nm}^{Q}$ are given by Eqs. 42a and 42b respectively where the expansion coefficients given by Eqs. 30 and 32 have been evaluated at $\gamma_{nn'}^{(0)R}$ and $\gamma_{nn'}^{(0)Q}$ respectively.

3. COUPLING COEFFICIENTS

We will examine coupling coefficients of class 1 since these are the ones appropriate for the sphere problem. By definition (ref. 2)

$$\eta_{nmn'p}(\gamma) = e^{(\gamma_{nmn'}-\gamma)ct_0} \left\{ \frac{\left|\underline{\tilde{J}}_{nm},\underline{J}_p^{inc}\right|}{N_{nm}\left[d\lambda_{nm}/d\gamma\right]} \right\}_{\gamma=\gamma_{nmn'}} \qquad (50)$$

where again we have dropped the superscripts R and Q for simplicity. The index p signifies polarization direction for the incident field, t_0 is the time at which the incident wave front first hits the body and

$$N_{nm} = \left|\underline{\tilde{J}}_{nm},\underline{J}_{nm}\right|$$

Notice that

$$\left\{ \tilde{\underline{J}}_{nm}, \underline{J}_p^{inc} \right\} = \int_S \tilde{\underline{J}}_{nm} \cdot \hat{n} \times \underline{J}_p^{inc} \, ds$$

$$\left\{ \tilde{\underline{J}}_{nm}, \underline{J}_{nm} \right\} = \int_S \tilde{\underline{J}}_{nm} \cdot \hat{n} \times \underline{J}_{nm} \, dS$$

i.e., the pseudosymmetric inner products have as their domain of integration the original surface S. Since $\underline{J}_p^{inc}$ is tangential to S, we can write

$$\underline{J}_p^{inc} = M\underline{J}_{pt}^{inc}$$

and it is easy to show that

$$\left\{ \tilde{\underline{J}}_{nm}, \underline{J}_p^{inc} \right\} = \int_S \tilde{\underline{J}}_{nmt} \cdot \hat{n} \times \underline{J}_{pt}^{inc} \, ds + O(\varepsilon^2)$$

$$\left\{ \tilde{\underline{J}}_{nm}, \underline{J}_{nm} \right\} = \int_S \tilde{\underline{J}}_{nmt} \cdot \hat{n} \times \underline{J}_{nmt} \, dS + O(\varepsilon^2)$$

If we express $\underline{J}_{nmt}$, $\tilde{J}_{nmt}$, $\gamma_{nmn'}$, $d\lambda_{nm}/d\gamma$, $\hat{n}$ and dS in terms of the appropriate unperturbed quantities, we can write the coupling coefficients as

$$\eta_{nmn'p}(\gamma) = \eta^{(0)}_{nmn'p}(\gamma) + \varepsilon\eta^{(1)}_{nmn'p}(\gamma) \qquad (51)$$

where the correction term can be obtained after some rather laborious algebraic manipulations. Notice that in doing so $\underline{J}_{nmt}$ and $\tilde{\underline{J}}_{nmt}$ are evaluated at $\gamma_{nn}^{(0)}$, rather than $\gamma_{nmn'}$, and

$$\left.\frac{d\lambda_{nm}}{d\gamma}\right|_{\gamma=\gamma_{nmn'}} = \left.\frac{d}{d\gamma}\left[\lambda_n^{(0)} + \varepsilon\lambda_{nm}^{(1)}\right]\right|_{\gamma=\gamma_{nmn'}}$$

$$= \left[\frac{d\lambda_n^{(0)}}{d\gamma} + \varepsilon\frac{d\lambda_{nm}^{(1)}}{d\gamma} + \varepsilon\frac{d^2\lambda_n^{(0)}}{d\gamma^2}\,\gamma_{nmn'}^{(1)}\right]_{\gamma=\gamma_{nn'}^{(0)}}$$

$$= \left[\frac{d\lambda_n^{(0)}}{d\gamma} + \varepsilon\frac{d\lambda_{nm}^{(1)}}{d\gamma} - \varepsilon\frac{d^2\lambda_n^{(0)}}{d\gamma^2}\,\frac{\lambda_{nm}^{(1)}}{\left[d\lambda_n^{(0)}/d\gamma\right]}\right]_{\gamma=\gamma_{nn'}^{(0)}} \tag{52}$$

Notice that both terms on the right-hand side of Eq. 51 contain $\exp(-\gamma ct_0)$ as a multiplicative factor to ensure that the excitation begins as soon as the wavefront hits the perturbed sphere. Thus $\eta_{nmn'p}^{(0)}(\gamma)$ differs from the coupling coefficient for the sphere problem in that it contains $\exp(-\gamma ct_0)$ rather than $\exp(-\gamma ct_0^{(0)})$, where $t_0^{(0)}$ is the time at which the wavefront hits the unperturbed sphere. Also recall that $\exp(-\gamma ct_0)$ contains the only γ dependence of the coupling coefficient.

4. SEM SOLUTION

Neglecting the possible additional entire function, the exact SEM solution to a delta function excitation can be written, in a simplified notation, as

$$\underline{J}(\underline{r},\gamma) = \sum_{\alpha} \eta_{\alpha}(\gamma)\frac{1}{\gamma-\gamma_{\alpha}}\underline{\nu}_{\alpha}(\underline{r}) \tag{53}$$

If the coupling coefficient is class 1, then

$$\eta_{\alpha}(\gamma) = e^{(\gamma_{\alpha}-\gamma)ct_0}\ \eta_{\alpha}(\gamma_{\alpha})$$

and the Laplace inversion of Eq. 53 gives

$$\underline{J}(\underline{r},t) = u(t-t_0)\sum_{\alpha}\eta_{\alpha}(\gamma_{\alpha})\underline{\nu}_{\alpha}(\underline{r})e^{\gamma_{\alpha}t} \tag{54}$$

If the MFIE operator is a function of a parameter ε (as in the perturbed sphere problem), we will assume that the following perturbation expansions are true:

$$\gamma_{\alpha} = \gamma_{\alpha}^{(0)} + \varepsilon\gamma_{\alpha}^{(1)} + \varepsilon^2\gamma_{\alpha}^{(2)} + \ldots \tag{55a}$$

$$\eta_{\alpha} = \eta_{\alpha}^{(0)} + \varepsilon\eta_{\alpha}^{(1)} + \varepsilon^2\eta_{\alpha}^{(2)} + \ldots \tag{55b}$$

$$\underline{\nu}_{\alpha} = \underline{\nu}_{\alpha}^{(0)} + \varepsilon\underline{\nu}_{\alpha}^{(1)} + \varepsilon^2\underline{\nu}_{\alpha}^{(2)} + \ldots \tag{55c}$$

Using Eq. 55a and 55b, Eq. 54 can be rewritten as

$$\underline{J}(\underline{r},t) = u(t-t_o)\Big\{ \sum_\alpha \eta_\alpha^{(0)} \underline{\nu}_\alpha^{(0)}(\underline{r}) e^{\gamma_\alpha t} + \varepsilon \sum_\alpha \eta_\alpha^{(1)} \underline{\nu}_\alpha^{(0)}(\underline{r}) e^{\gamma_\alpha t}$$

$$+ \varepsilon \sum_\alpha \eta_\alpha^{(0)} \underline{\nu}_\alpha^{(1)}(\underline{r}) e^{\gamma_\alpha t} + 0(\varepsilon^2)\Big\} \qquad (56)$$

Eq. (56) is just a rewriting of Eq. 54 under the assumption of Eq. 55. We can now choose ε such that all the terms of order ε^2 and higher in Eq. 55b, 55c, and 56 can be neglected and Eq. 56 gives

$$\underline{J}(\underline{r},t) = u(t-t_o)\Big\{ \sum_\alpha \eta_\alpha^{(0)} \underline{\nu}_\alpha^{(0)}(\underline{r}) + \varepsilon \sum_\alpha \eta_\alpha^{(0)} \underline{\nu}_\alpha^{(1)}(\underline{r})$$

$$+ \varepsilon \sum_\alpha \eta_\alpha^{(1)} \underline{\nu}_\alpha^{(0)}(\underline{r}) \Big\} e^{\gamma_\alpha t} \qquad (57)$$

Finally, we neglect all terms higher than order ε in Eq. 55a by imposing (if we have to) an additional restriction on ε. Thus the final <u>first order</u> SEM solution is

$$\underline{J}(\underline{r},t) = u(t-t_0)\Big\{ \sum_\alpha \eta_\alpha^{(0)} \underline{\nu}^{(0)}(\underline{r}) + \varepsilon \sum_\alpha \eta_\alpha^{(0)} \underline{\nu}_\alpha^{(1)}(\underline{r})$$

$$+ \varepsilon \sum_\alpha \eta_\alpha^{(1)} \underline{\nu}_\alpha^{(0)}(\underline{r}) \Big\} e^{(\gamma_\alpha^{(0)} + \varepsilon \gamma_\alpha^{(1)})t} \qquad (58)$$

Notice that this is not the only first-order solution since we can always expand $\exp(\varepsilon\gamma_{\alpha}^{(1)}t)$ and neglect terms higher than order ε. This way, however, we must impose a restriction on the maximum time t for which such an approximation is valid. Naturally, we do not have to impose such a restriction if our first-order solution is the one given by Eq. 58.

SECTION V

SUMMARY AND DISCUSSION

The main results of this report are as follows: (1)Perturbation theory can be applied even though the unperturbed sphere eigenfunctions form a complete set for functions tangential to the sphere, and the perturbed eigenfunctions are not tangential to the sphere. The saving feature is the relationship between normal and tangential components with respect to the surface of sphere of the perturbed eigenfunctions expressed by Eq. 7 and 8. (2) By employing first-order perturbation theory, we have shown that the perturbed eigenfunctions can be categorized in the same manner as the unperturbed sphere eigenfunctions $\underline{R}_{nm}$, $\underline{Q}_{nm}$; i.e., Eq. 42c and 43. (3) The perturbation operator consists of four integrals, three of which are singular. It is shown, however, in Appendix B, that their integrands can be regrouped into three new integrands that have removable singularities; i.e., the singularities cancel out and the perturbation operator is nonsingular as it should be. (4) The corrections to the eigenvalues and the eigenfunctions involve integrals whose integrands contain removable singularities, and Appendix B shows how these singularities can be treated to render the integrals suitable for numerical integration. (5) Once the corrections to the eigenvalues and eigenfunctions are calculated, the perturbed pole locations and natural modes can simply be evaluated with the aid of Eq. 47 and 49. Notice that, in first order, the corresponding expressions are evaluated at the unperturbed pole locations. (6) The coupling coefficients are assumed to be class 1 and are given by Eq. 50. Guidelines are presented for casting them into the form given by Eq. 51, but the final formula for the correction is not given because the question of the absence or presence of the entire function in the SEM expansion is not examined here.

Several aspects of the problem solved here have not been addressed. (1) Convergence of the series representing the correction to the unperturbed eigenfunctions (Eq. 21) has not been considered. (2) Second order corrections have not been evaluated, and no error bounds are given for our perturbation treatment. (3) The resulting perturbed eigenfunctions have not been tested for biorthogonality relationships, i.e., whether they satisfy Eq. 10 to second order. (4) No attempt has been made to answer the question of the existence of the entire function in the SEM expansion if class 1 coupling coefficients are employed. (5) No specific example has been worked out. (6) The second objective of this work and a motivating factor for the perturbation solution, i.e., providing a basis for estimating errors in the SEM parameters of a metallic sphere-like object due to uncertainties in its geometrical configuration, has not been explored.

Finally, we present ideas for future research based on the present treatment. (1) Write down an EEM expansion for the current density induced on a metallic sphere-like body illuminated by a plane wave based on the knowledge of the perturbed eigenfunctions and eigenvalues. The next step is to attempt to transform the EEM expansion into an equivalent SEM expansion following the guidelines given in Section VII of reference 3 and thus test if class 1 coupling coefficients are correct; i.e., if no additional entire function is required. (2) Work out specific examples to determine how various perturbations to the surface of the sphere affect the first few natural modes and frequencies. This can provide intuition that may be applicable to more complex structures. (3) Assuming that the perturbation to the surface of the sphere is a random variable with its first few moments known, explore the possibility of expressing the average values and standard deviations of the natural modes and frequencies in terms of these moments. The relationships to be employed are Eq. 47

for the natural frequencies and Eq. 49 for the natural modes. Notice that these expressions are evaluated at the unperturbed pole locations, and, consequently, the randomized variables are the corrections to the eigenvalues and expansion coefficients. These depend on the perturbation operator which in turn depends on the deviation from the surface of the sphere.

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APPENDIX A

In this appendix we derive Eq. 12.

a. By definition the differential solid angle $d\Omega$ is given by

$$d\Omega = \frac{\hat{n}_0 \cdot \underline{dS}}{r^2} = \frac{\hat{n}_0 \cdot \hat{n} dS}{r^2} \quad \text{and} \quad d\Omega = \frac{dS_0}{r_0^2} = dS_0$$

i.e.

$$\begin{aligned} dS &= r^2 d\Omega \left(\hat{n}_0 \cdot \hat{n}\right) \\ &= \left(1+\varepsilon f\right)^2 \, \hat{n}_0 \cdot \left(\hat{n}_0 + \varepsilon \underline{n}^{(1)}\right) d\Omega \\ &= \left(1+2\varepsilon f\right) \left(1+\varepsilon \hat{n}_0 \cdot \underline{n}^{(1)}\right) dS_0 \\ &= \left(1+2\varepsilon f\right) dS_0 \end{aligned}$$

since as we will show $\hat{n}_0 \cdot \hat{\underline{n}}^{(1)} = 0(\varepsilon^2)$

b. The unit outward normal is defined as

$$\hat{n} \equiv \left(\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi}\right) \Big/ \left|\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi}\right| \qquad \text{(A-1)}$$

Define

$$\underline{n} \equiv \frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} \equiv \underline{n}_0 + \varepsilon \Delta \underline{n} \qquad \text{(A-2)}$$

where from $\underline{r} = \hat{n}_0(1+\varepsilon f)$

$$\underline{n}_0 = \frac{\partial \hat{n}_0}{\partial \theta} \times \frac{\partial \hat{n}_0}{\partial \phi} \ , \ n_0 \equiv |\underline{n}_0|$$

$$\Delta\underline{n} = \frac{\partial \hat{n}_0}{\partial \theta} \times \frac{\partial\left(\hat{n}_0 f\right)}{\partial \phi} + \frac{\partial\left(\hat{n}_0 f\right)}{\partial \theta} \times \frac{\partial \hat{n}_0}{\partial \phi}$$

From Eqs. (A-1) and (A-2) we see that

$$\hat{n} = \frac{\underline{n}}{|\underline{n}|} = \frac{\underline{n}_0+\varepsilon\Delta\underline{n}}{|\underline{n}_0+\varepsilon\Delta\underline{n}|} = \frac{\underline{n}_0+\varepsilon\Delta\underline{n}}{n_0\left(1+\varepsilon\frac{\Delta\underline{n}\cdot\underline{n}_0}{n_0^2}\right)}$$

$$= \frac{\underline{n}_0}{n_0} + \varepsilon\left[\frac{\Delta\underline{n}}{n_0} - \frac{\underline{n}_0\cdot\Delta\underline{n}}{n_0^3}\,\underline{n}_0\right]$$

$$= \hat{n}_0 + \varepsilon\left(\underline{\underline{I}}-\hat{n}_0\hat{n}_0\right)\cdot\frac{\Delta\underline{n}}{n_0} + 0(\varepsilon^2) \qquad \text{(A-3)}$$

We have

$$\Delta\underline{n} = \frac{\partial \hat{n}_0}{\partial \theta} \times \frac{\partial \hat{n}_0}{\partial \phi} f + \left(\frac{\partial \hat{n}_0}{\partial \theta} \times \hat{n}_0\right) \frac{\partial f}{\partial \phi} + \frac{\partial \hat{n}_0}{\partial \theta} \times \frac{\partial \hat{n}_0}{\partial \phi} f$$

$$+\left(\hat{n}_0 \times \frac{\partial \hat{n}_0}{\partial \phi}\right) \frac{\partial f}{\partial \theta}$$

It is easy to show that

$$\frac{\partial \hat{n}_0}{\partial \theta} = \hat{e}_\theta, \ \frac{\partial \hat{n}_0}{\partial \phi} = \sin\theta\hat{e}_\phi$$

Similarly,

$$\frac{\underline{R}}{R^3} = \frac{1}{R_0^3}\left[\underline{R}_0+\varepsilon\left(\Delta\underline{R}-3\;\frac{\underline{R}_0}{R_0}\;\Delta R\right)\right]$$

and

$$e^{-\gamma R} = e^{-\gamma R_0}(1-\varepsilon\gamma\Delta R)$$

Combining the previous equation with Eq. (A-5), we obtain

$$4\pi\nabla G = -\left(\gamma+\frac{1}{R}\right)\frac{\underline{R}}{R^2}\;e^{-\gamma R}$$

$$= -\;\frac{\gamma}{R_0^2}\left[\underline{R}_0+\varepsilon\left(\Delta\underline{R}-2\;\frac{\underline{R}_0}{R_0}\;\Delta R\right)\right]\left[1-\varepsilon\gamma\Delta R\right]\;e^{-\gamma R_0}$$

$$-\;\frac{1}{R_0^3}\left[\underline{R}_0+\varepsilon\left(\Delta\underline{R}-3\;\frac{\underline{R}_0}{R_0}\;\Delta R\right)\right]\left[1-\varepsilon\gamma\Delta R\right]\;e^{-\gamma R_0}$$

or

$$\nabla G = \nabla G_0 + \varepsilon\left[\left(\gamma+\frac{1}{R_0}\right)\left(3\hat{R}_0\hat{R}_0-\underline{\underline{I}}\right)+\gamma^2\;\frac{\hat{R}_0\hat{R}_0}{R_0}\right]\cdot\frac{\Delta\underline{R}}{R_0}\;G_0$$

where $\hat{R}_0 = \underline{R}_0/R_0 = \left(\hat{n}_0-\hat{n}_0'\right)/\left|\hat{n}_0-\hat{n}_0'\right|$.

i.e.,

$$\left.\begin{aligned} n_0 &= |-\sin\theta\hat{n}_0| = \sin\theta \\ \Delta\underline{n} &= 2\sin\theta\hat{n}_0 f - \sin\theta\frac{\partial f}{\partial\theta}\,\hat{e}_\theta - \frac{\partial f}{\partial\phi}\,\hat{e}_\phi \end{aligned}\right\} \quad (A-4)$$

Combining Eqs. (A-3) and (A-4), we obtain

$$\hat{n} = \hat{n}_0 - \varepsilon\left(\frac{\partial f}{\partial\theta}\,\hat{e}_\theta + \frac{1}{\sin\theta}\,\frac{\partial f}{\partial\phi}\,\hat{e}_\phi\right) \equiv \hat{n}_0 + \varepsilon\underline{n}^{(1)}$$

Notice that $\hat{n}\cdot\hat{n} = 1 + 0(\varepsilon^2)$ since $\hat{n}_0\cdot\underline{n}^{(1)} = 0(\varepsilon^2)$

c. We have

$$4\pi\nabla G = \nabla\frac{e^{-\gamma R}}{R} = -\left(\gamma+\frac{1}{R}\right)\frac{\underline{R}}{R^2}\,e^{-\gamma R} \quad (A-5)$$

and

$$\begin{aligned} R = \underline{r}-\underline{r}' &= \hat{n}_0 - \hat{n}_0' + \varepsilon\left[\hat{n}_0 f(\theta,\phi) - \hat{n}_0' f(\theta',\phi')\right] \\ &\equiv \underline{R}_0 + \varepsilon\Delta\underline{R} \end{aligned}$$

$$\begin{aligned} R \equiv |\underline{R}| &= |\underline{R}_0 + \varepsilon\Delta\underline{R}| = R_0 + \varepsilon\underline{R}_0\cdot\Delta\underline{R}/R_0 \\ &\equiv R_0 + \varepsilon\Delta R \end{aligned}$$

$$\begin{aligned} \frac{\underline{R}}{R^2} &= \frac{\underline{R}_0 + \varepsilon\Delta\underline{R}}{\left(R_0+\varepsilon\Delta R\right)^2} = \left(\underline{R}_0+\varepsilon\Delta\underline{R}\right)\left(1-2\varepsilon\frac{\Delta R}{R_0^2}\right)\Big/R_0^2 \\ &= \frac{1}{R_0^2}\left[\underline{R}_0+\varepsilon\left(\Delta\underline{R}-2\,\frac{\underline{R}_0}{R_0}\,\Delta R\right)\right] \end{aligned}$$

APPENDIX B

In this appendix we show that the perturbation term given by Eq. 14 is nonsingular. If we examine the four integrals comprising $P\underline{J}_{pt}$ we observe that:

a. The first integral apart from $f(\Omega')$, which is a smooth function, has the form of the MFIE operator and consequently its integrand has an integrable singularity.

b. The integrand of the second integral can be written as

$$\underline{B} = -\left[\hat{n}_0(\Omega)\cdot\hat{n}_0(\Omega')\underline{n}^{(1)}(\Omega')\cdot\underline{J}_{pt}(\Omega')\right]\nabla G_0$$

$$+ (\hat{n}_0(\Omega)\cdot\nabla G_0)\hat{n}_0(\Omega')\underline{n}^{(1)}(\Omega')\cdot\underline{J}_{pt}(\Omega')$$

The second term contains the smooth function $\hat{n}_0(\Omega')\underline{n}^{(1)}(\Omega')\cdot\underline{J}_{pt}(\Omega')$ and $\hat{n}_0(\Omega)\cdot\nabla G_0$ which has an integrable singularity. However, the first term of B contains ∇G_0 which has a non-integrable singularity at $\Omega' = \Omega$ and the smooth term $\hat{n}_0(\Omega) \cdot \hat{n}_0(\Omega')\underline{n}^{(1)}(\Omega')\cdot J_{pt})\Omega')$ which in general does not vanish as $\Omega'\to\Omega$. Thus B is non-integrable.

c. If we expand the integrand of the third integral, we obtain

$$\underline{C} = -\hat{n}_0(\Omega)\cdot\underline{J}_{pt}(\Omega')\underline{\Delta}(\Omega,\Omega') + \hat{n}_0(\Omega)\cdot\underline{\Delta}(\Omega,\Omega')\underline{J}_{pt}(\Omega')$$

where

$$\underline{\Delta} = \left[\left(\gamma+\frac{1}{R_0}\right)\left(3\hat{R}_0\hat{R}_0-\underline{\underline{I}}\right) + \gamma^2\,\frac{\hat{R}_0\hat{R}_0}{R_0}\right]\cdot\frac{\Delta\underline{R}}{4\pi R_0^2}\,e^{-\gamma R_0}\;.$$

The highest singularity in $\underline{\Delta}$ is $\Delta\underline{R}/R_0^3$, which is non-integrable, but as $\Omega'\to\Omega$ the term $n_0(\Omega)\cdot\underline{J}_p(\Omega')$ goes to zero fast enough to make the first term in C integrable. However, the second term is non-integrable.

d. The integrand of the fourth integral has the form

$$\underline{D} = -\left[\underline{\underline{I}}-\hat{n}_0(\Omega)\hat{n}_0(\Omega)\right] \cdot \left[\underline{n}^{(1)}(\Omega)\cdot\underline{J}_{pt}(\Omega')\nabla G_0 - \underline{n}^{(1)}(\Omega)\cdot\nabla G_0 \underline{J}_{pt}(\Omega')\right]$$

Both terms are non-integrable as $\Omega \to \Omega'$.

Let us now add the singular integrands and regroup them as follows:

$$\begin{aligned}\underline{B} + \underline{C} + \underline{D} = &\left[-\hat{n}_0(\Omega)\cdot\hat{n}_0(\Omega')\left[\underline{n}^{(1)}(\Omega')\cdot\underline{J}_{pt}(\Omega')\right]\nabla G_0 \right.\\ &\left. +\left[\underline{n}^{(1)}(\Omega)\cdot\underline{J}_{pt}(\Omega')\right]\left[\underline{\underline{I}}-\hat{n}_0(\Omega)\hat{n}_0(\Omega)\right]\cdot\nabla G_0\right]\\ &+\left[\left[\hat{n}_0(\Omega)\cdot\nabla G_0\right]\hat{n}_0(\Omega')\underline{n}^{(1)}(\Omega')\cdot J_{pt}(\Omega')\right]\\ &+\left[\left[\hat{n}_0(\Omega)\cdot\underline{\Delta}(\Omega,\Omega') - \underline{n}^{(1)}(\Omega)\cdot\nabla G_0\right]\underline{J}_{pt}(\Omega')\right] \qquad \text{(B-1)}\end{aligned}$$

As $\Omega' \to \Omega$ $\left[I-n_0(\Omega)n_0(\Omega)\right]\cdot\nabla G_0 \to \nabla G_0$ and $n_0(\Omega)\cdot n_0(\Omega')\left[\underline{n}^{(1)}(\Omega')\cdot\underline{J}_{pt}(\Omega')\right] - \underline{n}^{(1)}(\Omega)\cdot\underline{J}_{pt}(\Omega')$ goes to zero fast enough to offset the non-integrable singularity of ∇G_0. Thus the first term has an integrable singularity. The second term is integrable as we mentioned earlier. The third term also has an integrable singularity, but the reason is not so obvious. The important terms are

$$\hat{n}_0(\Omega)\cdot\frac{\Delta\underline{R}}{4\pi R_0^3}\, e^{-\gamma R_0} - \underline{n}^{(1)}(\Omega)\cdot\frac{\underline{R}_0}{4\pi R_0^3}\, e^{-\gamma R_0}$$

$$= \left[\hat{n}_0(\Omega)\cdot\Delta\underline{R} - \underline{n}^{(1)}(\Omega)\cdot\underline{R}_0\right]\frac{e^{-\gamma R_0}}{4\pi R_0^3}$$

Let's expand $f(\theta',\phi')\hat{n}_0(\theta',\phi')$ and $\hat{n}_0(\theta',\phi')$ about θ ,ϕ :

$$\Delta\underline{R} = f(\theta,\phi)\hat{n}_0(\theta,\phi) - \left[f(\theta,\phi)\hat{n}_0(\theta,\phi) + \frac{\partial}{\partial\phi}(f\hat{n}_0)\delta\phi + \frac{\partial}{\partial\theta}(f\hat{n}_0)\delta\theta + 0(\delta^2)\right] = -\left[\left(\frac{\partial\hat{n}_0}{\partial\phi} f + \hat{n}_0 \frac{\partial f}{\partial\phi}\right)\delta\phi + \left(\frac{\partial\hat{n}_0}{\partial\phi} f + \hat{n}_0 \frac{\partial f}{\partial\phi}\right)\delta\theta + 0(\delta^2)\right]$$

where $\delta\theta = \theta'-\theta$, $\delta\phi = \phi'-\phi$.

$$\underline{R}_0 = \hat{n}_0(\theta,\phi) - \hat{n}_0(\theta',\phi') = \hat{n}_0(\theta,\phi) - \left[\hat{n}_0(\theta,\phi) + \frac{\partial\hat{n}_0}{\partial\phi}\delta\phi + \frac{\partial\hat{n}_0}{\partial\theta}\delta\phi + 0(\delta^2)\right]$$

$$= -\frac{\partial\hat{n}_0}{\partial\phi}\delta\phi + \frac{\partial\hat{n}_0}{\partial\theta}\delta\theta + 0(\delta^2)$$

Recalling that

$$\frac{\partial\hat{n}_0}{\partial\phi} = \sin\theta\hat{e}_\phi , \quad \frac{\partial\hat{n}_0}{\partial\theta} = \hat{e}_\theta$$

we obtain

$$\hat{n}_0(\Omega)\cdot\Delta\underline{R} = \frac{\partial f}{\partial\phi}\delta\phi + \frac{\partial f}{\partial\theta}\delta\theta + 0(\delta^2)$$

$$\underline{n}^{(1)}(\Omega)\cdot\underline{R}_0 = -\left(\frac{\partial f}{\partial\theta}\hat{e}_\theta + \frac{1}{\sin\theta}\frac{\partial f}{\partial\phi}\hat{e}_\phi\right)\cdot\left(\sin\theta\hat{e}_\phi\delta\phi + \hat{e}_\theta\delta\theta + 0(\delta^2)\right)$$

$$= -\frac{\partial f}{\partial\phi}\delta\phi - \frac{\partial f}{\partial\theta}\delta\theta + 0(\delta^2)$$

i.e.,

$$\hat{n}_0(\Omega)\cdot\Delta\underline{R} - \underline{n}^{(1)}(\Omega)\cdot\underline{R}_0 = 0(\delta^2)$$

and the third term in Eq. (B-1) is integrable.

From the point of view of numerical evaluation of the matrix element $\left|J_r^{(0)}, PJ_p^{(0)}\right|$, the regrouped integrands as they stand in Eq. (B-1) are useless because they diverge as $\Omega'\to\Omega$. We will illustrate how they can be cast into a numerically suitable form by supplying the details for the first term in Eq. (B-1). Thus the term of interest is

$$\underline{I}(\Omega) = \int_{S_0} \left\{ -\hat{n}_0(\Omega)\cdot\hat{n}_0(\Omega')\left[\underline{n}^{(1)}(\Omega')\cdot\underline{J}_p^{(0)}(\Omega')\right]\nabla G_0 + \left[\underline{n}^{(1)}(\Omega)\cdot\underline{J}_p^{(0)}(\Omega')\right]\left[\underline{\underline{I}}-\hat{n}_0(\Omega)\hat{n}_0(\Omega)\right]\cdot\nabla G_0\right\} d\Omega' \qquad \text{(B-2)}$$

and the matrix elements involve integrals of the form $\int_{S_0} J_n^{(0)}\cdot\hat{n}_0\times\underline{I}(\Omega)d\Omega$. Consider the integral

$$\underline{I}_1(\Omega) = -\int_{S_0} \left[-\hat{n}_0(\Omega)\cdot\hat{n}_0(\Omega')\left[\underline{n}^{(1)}(\Omega')\cdot\underline{J}_p^{(0)}(\Omega')\right] + \left[\underline{n}^{(1)}(\Omega)\cdot\underline{J}_p^{(0)}(\Omega')\right]\right] \frac{\underline{R}_0}{4\pi R_0^3}\, d\Omega'$$

i.e., retain the highest singularity in $\nabla G_0 = -\left(\gamma+\frac{1}{R_0}\right)\frac{\underline{R}_0}{4\pi R_0^2}\, e^{-\gamma R_0}$ only.

Again we expand in a Taylor series about Ω

$$B_1 \equiv \left[-\hat{n}_0(\Omega)\cdot\hat{n}_0(\Omega')\left[\underline{n}^{(1)}(\Omega')\cdot\underline{J}_p^{(0)}(\Omega')\right] + \left[\underline{n}^{(1)}(\Omega)\cdot\underline{J}_p^{(0)}(\Omega')\right]\right]\underline{R}_0$$

$$= \left[\left[-\hat{n}_0\cdot\left(\hat{n}_0 + \frac{\partial\hat{n}_0}{\partial\phi}\,\delta\phi + \frac{\partial\hat{n}_0}{\partial\theta}\,\delta\theta + 0(\delta^2)\right)\right.\right.$$

$$\left(\underline{n}^{(1)} + \frac{\partial\underline{n}^{(1)}}{\partial\theta}\,\delta\theta + \frac{\partial\underline{n}^{(1)}}{\partial\phi}\,\delta\phi + 0(\delta^2)\right)$$

$$\left.\cdot\left(\underline{J}_p^{(0)} + \frac{\partial\underline{J}_p^{(0)}}{\partial\phi}\,\delta\phi + \frac{\partial\underline{J}_p^{(0)}}{\partial\theta}\,\partial\theta + 0(\delta^2)\right)\right]$$

$$\left. + \underline{n}^{(1)}\cdot\left(\underline{J}_p^{(0)} + \frac{\partial\underline{J}_p^{(0)}}{\partial\phi}\delta\phi + \frac{\partial\underline{J}_p^{(0)}}{\partial\theta}\delta\theta + 0(\delta^2)\right)\right]$$

$$\left[\hat{n}_0 - \left(\hat{n}_0 + \frac{\partial\hat{n}_0}{\partial\phi}\,\delta\phi + \frac{\partial\hat{n}_0}{\partial\theta}\,\delta\theta + 0(\delta^2)\right)\right]$$

$$= \left[-\underline{J}_p^{(0)}\left(\frac{\partial\underline{n}^{(1)}}{\partial\phi}\,\delta\phi + \frac{\partial\underline{n}^{(1)}}{\partial\theta}\,\delta\theta\right) + 0(\delta^2)\right]$$

$$\left[-\left(\frac{\partial\hat{n}_0}{\partial\phi}\delta\phi + \frac{\partial\hat{n}_0}{\partial\theta}\,\delta\theta\right) + 0(\delta^2)\right]$$

$$= \left(\underline{J}_p^{(0)}\cdot\frac{\partial\underline{n}^{(1)}}{\partial\phi}\right)\frac{\partial\hat{n}_0}{\partial\phi}(\delta\phi)^2 + \left[\left(\underline{J}_p^{(0)}\cdot\frac{\partial\underline{n}^{(1)}}{\partial\theta}\right)\frac{\partial\hat{n}_0}{\partial\phi}\right.$$

$$\left. + \left(\underline{J}_p^{0}\cdot\frac{\partial\underline{n}^{(1)}}{\partial\phi}\right)\frac{\partial\hat{n}_0}{\partial\theta}\right](\delta\phi\delta\theta) + \left(\underline{J}_p^{(0)}\cdot\frac{\partial\underline{n}^{(1)}}{\partial\theta}\right)\frac{\partial n_0}{\partial\theta}\,(\delta\theta)^2 + 0(\delta^3)$$

Recalling that

$$\underline{n}^{(1)} = -\left(\frac{\partial f}{\partial\theta}\,\hat{e}_\theta + \frac{1}{\sin\theta}\,\frac{\partial f}{\partial\phi}\,\hat{e}_\phi\right)$$

we find

$$\frac{\partial \underline{n}^{(1)}}{\partial \phi} = -\left(\frac{\partial f}{\partial\theta\partial\phi}\,\hat{e}_\theta + \frac{\partial f}{\partial\theta}\,\frac{\partial \hat{e}_\theta}{\partial\phi} + \frac{1}{\sin\theta}\,\frac{\partial^2 f}{\partial\phi^2}\,\hat{e}_\phi + \frac{1}{\sin\theta}\,\frac{\partial f}{\partial\phi}\,\frac{\partial\hat{e}_\phi}{\partial\phi}\right)$$

$$= -\left[\frac{\partial f}{\partial\theta\partial\phi}\,\hat{e}_\theta + \frac{\partial f}{\partial\theta}\cos\theta\,\hat{e}_\phi + \frac{1}{\sin\theta}\,\frac{\partial^2 f}{\partial\phi^2}\,\hat{e}_\phi + \frac{1}{\sin\theta}\,\frac{\partial f}{\partial\phi}\left(-\sin\theta\,\hat{n}_0 + \cos\theta\,\hat{e}_\theta\right)\right]$$

$$\frac{\partial \underline{n}^{(1)}}{\partial \theta} = -\left[\frac{\partial^2 f}{\partial\theta^2}\,\hat{e}_\theta + \frac{\partial f}{\partial\theta}\,\frac{\partial\hat{e}_\theta}{\partial\theta} - \frac{\cos\theta}{\sin^2\theta}\,\frac{\partial f}{\partial\phi}\,\hat{e}_\phi + \frac{1}{\sin\theta}\,\frac{\partial f}{\partial\theta\partial\phi}\,\hat{e}_\phi + \frac{1}{\sin\theta}\,\frac{\partial f}{\partial\phi}\,\frac{\partial\hat{e}_\phi}{\partial\theta}\right]$$

Thus

$$\begin{aligned}\underline{B}_1 = &-\left[J^{(0)}_{p\theta}\left(\sin\theta\,\frac{\partial f}{\partial\theta\partial\phi} + \cos\theta\,\frac{\partial f}{\partial\phi}\right) + J^{(0)}_{p\phi}\left(\sin\theta\cos\theta\,\frac{\partial f}{\partial\theta} + \frac{\partial^2 f}{\partial\phi^2}\right)\right]\hat{e}_\phi\,(\delta\phi)^2 \\ &- \left[J^{(0)}_{p\theta}\,\frac{\partial^2 f}{\partial\theta^2} + J^{(0)}_{p\phi}\left(-\frac{\cos\theta}{\sin^2\theta}\,\frac{\partial f}{\partial\phi} + \frac{1}{\sin\theta}\,\frac{\partial^2 f}{\partial\theta\partial\phi}\right)\right]\hat{e}_\theta\,(\delta\theta)^2 \\ &- \left\{\left[J^{(0)}_{p\theta}\,\sin\theta\,\frac{\partial^2 f}{\partial\theta^2} + J^{(0)}_{p\phi}\left(-\frac{\cos\theta}{\sin\theta}\,\frac{\partial f}{\partial\phi} + \frac{\partial^2 f}{\partial\theta\partial\phi}\right)\right]\hat{e}_\phi \right. \\ &+ \left[J^{(0)}_{p}\left(\frac{\partial f}{\partial\theta\partial\phi} + \frac{\cos\theta}{\sin\theta}\,\frac{\partial f}{\partial\phi}\right) \right. \\ &\left.\left. + J^{(0)}_{p\phi}\left(\cos\theta\,\frac{\partial f}{\partial\theta} + \frac{1}{\sin\theta}\,\frac{\partial^2 f}{\partial\phi^2}\right)\right]\hat{e}_\theta\right\}(\delta\theta\delta\phi) + 0(\delta^3)\end{aligned}$$

We can also obtain

$$R_0 = 2(1-\sin\theta\sin\theta'\cos(\phi-\phi')-\cos\theta\cos\theta')^{1/2}$$

$$= \left[(\delta\theta)^2+\sin^2\theta(\delta\phi)^2+0(\delta^3)\right]^{1/2}$$

Ignoring the $0(\delta^3)$ terms in the numerator and denominator and denoting the remaining integral by I_1', we obtain

$$\underline{I}_1'(\Omega) = \int_0^{2\pi}\int_0^{\pi} \frac{\underline{F}_1(\Omega)(\delta\phi)^2+\underline{F}_2(\Omega)(\delta\theta)^2+\underline{F}_3(\Omega)(\delta\theta\delta\phi)}{4\pi\left[(\delta\theta)^2+\sin^2\theta(\delta\phi)^2\right]^{3/2}} \sin\theta d\theta' d\phi'$$

where the numerator is identical to $\underline{B}_1$ and $\underline{F}_1$, $\underline{F}_2$, and $\underline{F}_3$ are defined in an obvious manner. Notice that $d\Omega' = \sin\theta' d\theta d\phi'$; i.e., we have replaced $\sin\theta'$ by $\sin\theta$ because we are ignoring $0(\delta^3)$ terms. Writing $\delta\phi=u$, $\delta\theta=v$ for simplicity, we find

$$\underline{I}_1'(\Omega) = \int_{-\phi}^{2\pi-\phi}\int_{-\theta}^{\pi-\theta} \frac{\underline{F}_1(\Omega)u^2+\underline{F}_2(\Omega)v^2+\underline{F}_3(\Omega)uv}{4\pi\left[v^2+\sin^2\theta\, u^2\right]^{3/2}} \sin\theta dv du$$

All three integrals with respect to u, v are elementary and can be performed explicitly. We will not give the answer here, but we should note that $\underline{I}'(\Omega)$ is singular at $\theta=0$ and so is $\underline{I}(\Omega)$ that has the same behavior as $I_1'(\Omega)$. However, when we calculate the matrix elements that have the form $\int_{S_0} \underline{J}_r^0\cdot\left(\hat{n}_0 \times \underline{I}(\Omega)\right) d\Omega$, we obtain an extra $\sin\theta$ from $d\Omega$

and one can show that the integrand consists of terms that are either zero or finite as $\theta \to 0$. Thus $\theta = 0$ can be omitted as an integration point when calculating $\int_{S_0} \underline{J}^0 r \cdot \hat{n}_0 \times \underline{I}(\Omega)\, d\Omega$.

The behavior of the rest of the integrals having as integrands the first two terms in Eq. (B-1) is similar. We are finally in a position to rewrite Eq. (B-2) in a form suitable for numerical integration:

$$\underline{I}(\Omega) = \int_{S_0} \Bigg\{ \left[-\hat{n}_0(\Omega)\cdot\hat{n}_0(\Omega') \left[\underline{n}^{(1)}(\Omega')\cdot\underline{J}_{pt}(\Omega')\right] \nabla G_0 \right.$$

$$+ \left[\underline{n}^{(1)}(\Omega)\cdot\underline{J}_{pt}(\Omega')\right]\left[I - \hat{n}_0(\Omega)\hat{n}_0(\Omega)\right]\cdot\nabla G_0 \Bigg]$$

$$- \frac{\underline{F}_1(\Omega)(\phi'-\phi)^2 + \underline{F}_2(\Omega)(\theta'-\theta)^2 + \underline{F}_3(\Omega)(\phi'-\phi)(\theta'-\theta)}{4\pi\left[(\theta'-\theta)^2 + \sin^2\theta\,(\phi'-\phi)^2\right]^{3/2}} \frac{\sin\theta}{\sin\theta'} \Bigg\}\, d\Omega'$$

$$+ \underline{I}_1'(\Omega)$$

As $\Omega \to \Omega$ the integrand in the first integral goes as $O(\delta^0)$, and consequently we can ignore the point $\Omega' = \Omega$ in the numerical integration.