Interaction Notes

Note 401

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ELECTROSTATIC FIELD PENETRATION OF A CONDUCTING SPHERICAL SHELL THROUGH A MESH-LOADED CIRCULAR APERTURE*

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Abstract

The effectiveness of wire-mesh loading as a hardening technique for apertures in otherwise closed shield surfaces is assessed for electric field penetration by considering a canonical problem of potential theory in which the shield surface is spherical and the aperture is circular. Exact and approximate solutions to this canonical problem are obtained and are found to agree closely over a wide range of aperture opening angles.

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I. INTRODUCTION

In previous Notes [1-4], we have considered several problems of low-frequency or quasistatic electromagnetic field penetration of apertures. Electric and magnetic penetration of a loaded circular aperture in an infinitely extended ground plane was considered in [1], and of an open circular aperture in a perfectly conducting spherical shell in [2]. Quasistatic magnetic field penetration of a resistively loaded circular aperture in a perfectly conducting spherical shell was discussed in [3] and [4]. As the results in [3] and [4] can be readily adapted to mesh loading by replacing the sheet resistance R_s with Z_s, the sheet impedance of the mesh [1], the remaining problem to be considered is that of electrostatic penetration of a mesh-loaded circular aperture in a perfectly conducting spherical shell. We address this problem in this Note, considering the case where the external applied electric field is parallel to the symmetry axis of the structure. The penetrant field is maximized for this orientation of the external field [2].

As in [2-4], formulating this problem in terms of a scalar potential leads to a pair of dual series equations which are then reduced to a Fredholm integral equation. This integral equation is converted into a set of linear equations in the expansion coefficients for the potential; these are then readily solved on a computer. In this Note, we also introduce a "pseudo-variational" approach to the solution of the dual series equations and compare the results with the numerically computed exact results. We shall see that this approach yields a relatively simple closed-form expression which is also accurate over a wide range of parameters.

The formulation of the problem is carried out in the following section, and the solution of the dual series equations is presented in section III. Numerical results are discussed in section IV, and section V concludes the Note.

II. FORMULATION OF THE PROBLEM

The geometry of the problem is shown in Figure 1. A perfectly conducting open spherical shell is centered at the origin of the spherical coordinates (r,θ,ϕ). The shell has radius a and extends from $\theta = 0$ to $\theta = \alpha$ $(\alpha \leq \pi)$. The remainder of the surface $r = a$ (i.e., $\alpha < \theta \leq \pi$) is a bonded-junction wire mesh. The individual meshes are square and of size a_s; the wire radius is $r_w << a_s$. It is assumed that the mesh is electrically connected to the perfectly conducting shell at $\theta = \alpha$. The structure is immersed in a uniform electrostatic field $E_o\bar{a}_z$. The object of the analysis is to determine the electric field everywhere; in particular, we wish to find the electric field inside the spherical shell and to evaluate the electric polarizability of the structure.

We express the electric field $\bar{E}$ in terms of the scalar potential V as

$$\bar{E} = -\nabla V \tag{1}$$

in which

$$V(r,\theta) = -E_o r \cos\theta + V_1(r,\theta) \tag{2}$$

where

$$\nabla^2 V_1 = 0 \qquad (r \neq a) \tag{3}$$

$$\lim_{r\to\infty} rV_1 = Q_o/4\pi\varepsilon_o \tag{4}$$

$$V_1 = V_o + E_o a \cos\theta \qquad (r = a,\ 0 \leq \theta < \alpha) \tag{5}$$

$$\left.\frac{\partial V_1}{\partial r}\right|_{r=a+} - \left.\frac{\partial V_1}{\partial r}\right|_{r=a-} = \frac{3K}{a}\,[V(a,\theta) - V_o] \quad (\alpha < \theta \leq \pi) \tag{6}$$

The derivation of the boundary condition (6) is carried out in [1]. Q_o and V_o denote respectively the net charge and the potential on the structure and K is a parameter given by

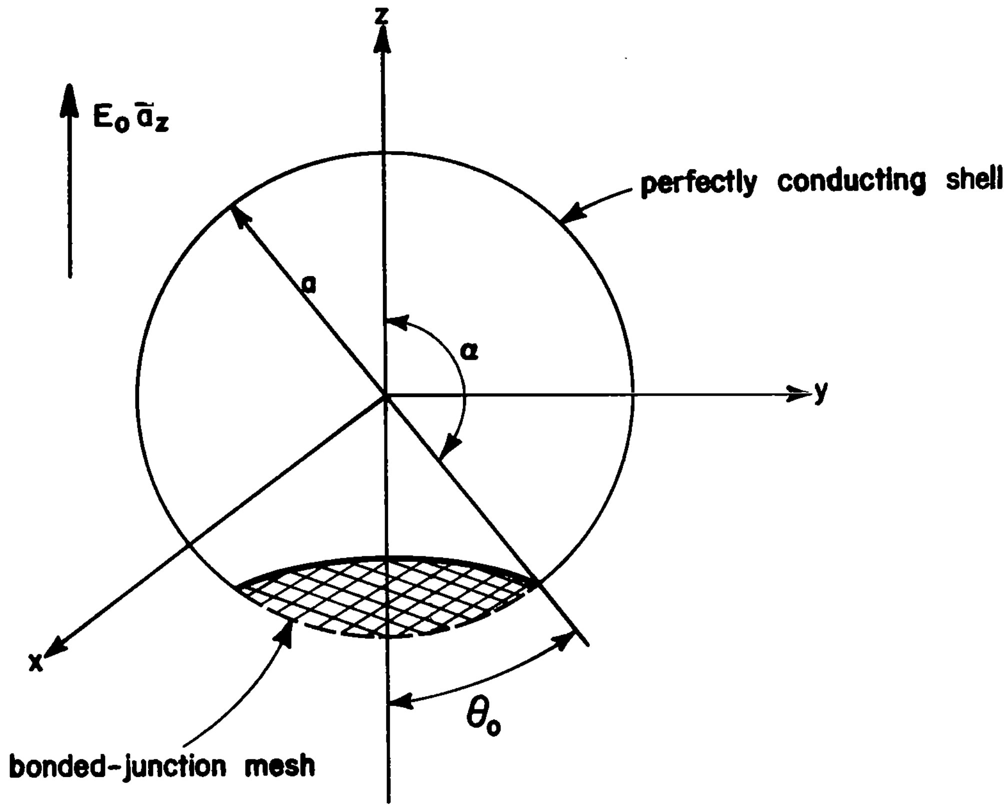

Figure 1. Geometry of the problem.

$$\frac{1}{K} = \frac{3a_s}{4\pi a} \ell n\left(1 - e^{-2\pi r_w/a_s}\right)^{-1} \tag{7}$$

Now $V_1(r,\theta)$ can be expressed as

$$r \leq a: \quad V_1(r,\theta) = -E_o a \sum_{n=0}^{\infty} a_n (\frac{r}{a})^n P_n(\cos\theta) \tag{8a}$$

$$r \geq a: \quad V_1(r,\theta) = -E_o a \sum_{n=0}^{\infty} a_n (\frac{r}{a})^{-n-1} P_n(\cos\theta) \tag{8b}$$

in which $P_n(\cdot)$ denotes the Legendre polynomial of degree n and where the coefficients a_n are to be determined. Applying the boundary conditions (5) and (6), we obtain the following set of dual series equations for the unknown coefficients:

$$\sum_{n=0}^{\infty} b_n P_n(\cos\theta) = 0 \qquad (0 \leq \theta < \alpha) \tag{9}$$

$$\sum_{n=0}^{\infty} (2n+1) b_n P_n(\cos\theta) + 3K \sum_{n=0}^{\infty} b_n P_n(\cos\theta) = \frac{V_o}{E_o a} + 3\cos\theta \qquad (\alpha < \theta \leq \pi) \tag{10}$$

in which the coefficients b_n are related to a_n by

$$b_o = a_o + \frac{V_o}{E_o a} \tag{11a}$$

$$b_1 = a_1 + 1 \tag{11b}$$

$$b_n = a_n \quad (n \geq 2) \tag{11c}$$

and, from (4),

$$b_o = \frac{V_o}{E_o a} - \frac{Q_o}{4\pi\varepsilon_o E_o a^2} \tag{12}$$

We shall be concerned with the case for which $Q_o = 0$, so that $a_o = 0$ and

$$b_o\Big|_{Q_o=0} = \frac{V_o}{E_o a} \tag{13}$$

III. SOLUTION OF THE DUAL SERIES EQUATIONS

We express the coefficients b_n in terms of an unknown function $h(u)$ by [5]

$$b_n = \int_0^{\pi} \cos(n+\tfrac{1}{2})u\, h(u)H(u-\alpha)\, du \tag{14}$$

in which $H(\cdot)$ denotes the unit step function. Then making use of the relations [6]

$$\sum_{n=0}^{\infty} \cos(n+\tfrac{1}{2})u\, P_n(\cos\theta) = \frac{1}{\sqrt{2}} \frac{H(\theta-u)}{\sqrt{\cos u - \cos\theta}} \tag{15}$$

$$\sum_{n=0}^{\infty} \sin(n+\tfrac{1}{2})u\, P_n(\cos\theta) = \frac{1}{\sqrt{2}} \frac{H(u-\theta)}{\sqrt{\cos\theta - \cos u}} \tag{16}$$

we find that

$$\sum_{n=0}^{\infty} b_n P_n(\cos\theta) = \frac{1}{\sqrt{2}} \int_0^{\theta} \frac{h(u)H(u-\alpha)du}{\sqrt{\cos u - \cos\theta}} \tag{17}$$

$$\begin{aligned}\sum_{n=0}^{\infty} (2n+1)b_n P_n(\cos\theta) &= \sqrt{2} \int_0^{\pi} h(u)H(u-\alpha) \frac{d}{du}\left[\frac{H(u-\theta)}{\sqrt{\cos\theta - \cos u}}\right] du \\ &= -\sqrt{2}\, \cos\theta\, \frac{d}{d\theta} \int_0^{\pi} \frac{h(u)H(u-\alpha)H(u-\theta)\, \sin u\, du}{\sqrt{\cos\theta - \cos u}}\end{aligned} \tag{18}$$

for $0 \leq \theta \leq \pi$.

It is evident that the first of the dual series equations (9) is satisfied by this representation for b_n; and the second of these equations will be satisfied if $h(u)$ is the solution of the integral equation

$$\begin{aligned}-\frac{1}{\pi}\frac{d}{d\theta} \int_{\theta}^{\pi} \frac{h(u)\, \sin u\, du}{\sqrt{\cos\theta - \cos u}} &= \frac{\sin\theta}{\pi\sqrt{2}} \left(\frac{V_o}{E_o a} + 3\cos\theta\right) \\ &- \frac{3K}{2\pi} \sin\theta \int_{\pi}^{\theta} \frac{h(u)du}{\sqrt{\cos u - \cos\theta}} \qquad (\alpha < \theta \leq \pi)\end{aligned} \tag{19}$$

Now making use of the integral equation/solution pair [7]

$$\int_x^b \frac{f(t)dt}{\sqrt{\cos x - \cos t}} = g(x) \qquad (0 \leq a < x < b \leq \pi) \tag{20a}$$

$$f(t) = -\frac{1}{\pi}\frac{d}{dt}\int_t^b \frac{g(u)\ \sin u\, du}{\sqrt{\cos t - \cos u}} \qquad (a < t < b) \tag{20b}$$

we convert the integral equation (19) into the inhomogeneous Fredholm integral equation of the second kind

$$h(x) + \frac{3K}{2\pi}\int_\alpha^\pi K(x,u)h(u)du = \frac{2}{\pi}\left(\frac{V_o}{E_o a}\cos\frac{x}{2} + \cos\frac{3x}{2}\right) \qquad (\alpha < \theta \leq \pi) \tag{21}$$

in which the kernel K(x,u) is given by

$$K(x,u) = \int_o^\pi \frac{H(t-x)H(t-u)\ \sin t\, dt}{\sqrt{\cos x - \cos t}\ \sqrt{\cos u - \cos t}} \tag{22}$$

for $\alpha < x \leq \pi$, $\alpha < u \leq \pi$. It will be useful in the following to extend the domain of definition of K(x,u) to $(0,\pi) \times (0,\pi)$. Thus we obtain

$$h(x)H(x-\alpha) + \frac{3K}{2\pi} H(x-\alpha)\int_o^\pi K(x,u)h(u)H(u-\alpha)du$$
$$= \frac{2}{\pi}\left(\frac{V_o}{E_o a}\cos\frac{x}{2} + \cos\frac{3x}{2}\right)H(x-\alpha) \qquad (0 \leq x \leq \pi) \tag{23}$$

We now convert the integral equation (23) into a system of linear equations in the unknown coefficients b_n. The relation (14) is equivalent to

$$h(u)H(u-\alpha) = \frac{2}{\pi}\sum_{n=0}^{\infty} b_n \cos(n+\tfrac{1}{2})\, u \tag{24}$$

Substituting (24) into (23) and making use of the easily demonstrated fact that

$$\int_o^\pi K(x,u)\ \cos(n+\tfrac{1}{2})\, u\, du = \frac{\pi}{n+\frac{1}{2}}\cos(n+\tfrac{1}{2})\, x \tag{25}$$

we obtain

$$\sum_{n=0}^{\infty} b_n \cos(n+\tfrac{1}{2})x + 3KH(x-\alpha) \sum_{n=0}^{\infty} \frac{b_n}{2n+1} \cos(n+\tfrac{1}{2})x$$

$$= H(x-\alpha)\left(\frac{V_o}{E_o a}\cos\frac{x}{2} + \cos\frac{3x}{2}\right) \qquad (0 \le x \le \pi) \qquad (26)$$

Now multiplying equation (26) by $\frac{2}{\pi}\cos(m+\frac{1}{2})x$ and integrating with respect to x ($0 \le x \le \pi$), we obtain a system of linear equations in the coefficients b_n, viz.

$$b_m + 3K \sum_{n=0}^{\infty} \frac{b_n}{2n+1} I_{mn} = \frac{V_o}{E_o a} I_{mo} + I_{m1} \qquad (m \ge 0) \qquad (27)$$

where

$$I_{mn} = \frac{2}{\pi}\int_{\alpha}^{\pi} \cos(m+\tfrac{1}{2})x \, \cos(n+\tfrac{1}{2})x \, dx$$

$$= \frac{1}{\pi}\left[\pi - \alpha - \frac{\sin(2n+1)\alpha}{2n+1}\right] \qquad (m = n)$$

$$= -\frac{1}{\pi}\left[\frac{\sin(m-n)\alpha}{m-n} + \frac{\sin(m+n+1)\alpha}{m+n+1}\right] \qquad (m \ne n) \qquad (28)$$

When the structure carries no net charge, $b_o = V_o/E_o a$; thus the system of equations (27) becomes

$$b_m + 3K \sum_{n=0}^{\infty} \frac{b_n I_{mn}}{2n+1}\left(1 - \frac{\delta_{no}}{3K}\right) = I_{m1} \qquad (m \ge 0) \qquad (29)$$

where $\delta_{no} = 1$ if $n = 0$ and $\delta_{no} = 0$ otherwise. The system of equations (29) can be readily solved for any of the unknowns b_m by numerical means. We shall presently discuss numerical results; before proceeding, however, we consider a "capacitive shielding" hypothesis to obtain an approximate analytical solution. This approximate solution will be compared with the "exact" solution in the following section.

We hypothesize that*

$$b_n \simeq b_n^o(1 + \beta K)^{-1} \quad (n \geq 0) \tag{30}$$

where β is a parameter to be determined and the coefficients b_n^o satisfy the dual series equations

$$\sum_{n=0}^{\infty} b_n^o P_n(\cos\theta) = 0 \quad (0 \leq \theta < \alpha) \tag{31}$$

$$\sum_{n=0}^{\infty} (2n + 1)b_n^o P_n(\cos\theta) = \frac{V_o^o}{E_o a} + 3 \cos\theta \quad (\alpha < \theta \leq \pi) \tag{32}$$

Substitution of (30) into (9) and (10) and making use of (31) and (32) results in

$$\sum_{n=0}^{\infty} b_n^o P_n(\cos\theta) = \beta\cos\theta \; H(\theta-\alpha) \quad (0 \leq \theta \leq \pi) \tag{33}$$

This relation is not exact; we define the error $e(\theta)$ as

$$e(\theta) = \sum_{n=0}^{\infty} b_n^o P_n(\cos\theta) - \beta\cos\theta \; H(\theta-\alpha) \tag{34}$$

for $0 \leq \theta \leq \pi$ and minimize the mean-square error

$$\overline{e^2} = \frac{1}{2}\int_0^{\pi} e^2(\theta) \sin\theta \; d\theta \tag{35}$$

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*We term this hypothesis a "pseudovariational solution." Variational methods will yield a solution of the form

$$b_n \simeq \frac{b_n^o}{1 + \alpha_n K}$$

where the expression for α_n can be rather cumbersome [3,4] and, as a consequence, not very useful. Our "pseudovariational" approach will, as will be seen, yield a relatively simple and accurate result.

with respect to the parameter β. The resulting relation for β is

$$\beta = \frac{\int_\alpha^\pi \cos\theta \sin\theta \sum_{n=0}^{\infty} b_n^o P_n(\cos\theta)\, d\theta}{\int_\alpha^\pi \cos^2\theta \sin\theta \, d\theta} \tag{36}$$

Now the integral in the denominator is easily evaluated; we have

$$\int_\alpha^\pi \cos^2\theta \sin\theta \, d\theta = \frac{1}{3}(1 + \cos^3\alpha) \tag{37}$$

From (17) and (21), we obtain

$$\sum_{n=0}^{\infty} b_n^o P_n(\cos\theta) = \frac{\sqrt{2}}{\pi} \int_\alpha^\theta \left(\frac{V_o^o}{E_o a} \cos\frac{u}{2} + \cos\frac{3u}{2} \right) \frac{du}{\sqrt{\cos u - \cos\theta}} \tag{38}$$

from which it is easy to show that

$$\int_\alpha^\pi \cos\theta \sin\theta \sum_{n=0}^{\infty} b_n^o P_n(\cos\theta) d\theta = \frac{\sqrt{2}}{\pi} \int_\alpha^\pi \left(\frac{V_o^o}{E_o a} \cos\frac{u}{2} + \cos\frac{3u}{2} \right) du \cdot \int_u^\pi \frac{\cos\theta \sin\theta \, d\theta}{\sqrt{\cos u - \cos\theta}} \tag{39}$$

and since

$$\int_u^\pi \frac{\cos\theta \sin\theta \, d\theta}{\sqrt{\cos u - \cos\theta}} = \frac{2\sqrt{2}}{3} \cos\frac{3u}{2} \tag{40}$$

we find that

$$\int_\alpha^\pi \cos\theta \sin\theta \sum_{n=0}^{\infty} b_n^o P_n(\cos\theta) d\theta = \frac{2}{3} \left(\frac{V_o^o}{E_o a} I_{01} + I_{11} \right) \tag{41}$$

Now for the case which we are considering where the sphere carries no net charge,

$$\frac{V_o^o}{E_o a} = b_o^o = \frac{I_{01}}{1 - I_{00}} \tag{42}$$

Thus

$$\beta = 2(1 + \cos^3\alpha)^{-1}\left[\frac{I_{01}^2}{1 - I_{oo}} + I_{11}\right]$$

$$= \frac{2}{\pi}(1 + \cos^3\alpha)^{-1}\left[\frac{(\sin\alpha + \frac{1}{2}\sin 2\alpha)^2}{\alpha + \sin\alpha} + \pi - \alpha - \frac{1}{3}\sin 3\alpha\right] \tag{43}$$

A plot of β as a function of the aperture opening angle $\theta_o = \pi - \alpha$ is shown in Figure 2. When $\theta_o \lesssim 30^\circ$, a simple approximate expression for β is

$$\beta \simeq \frac{2\theta_o}{\pi} \tag{44}$$

It is interesting to note that β has a maximum near $\theta_o = 90^\circ$, where the structure is a perfectly conducting hemisphere joined to a wire-mesh hemisphere.

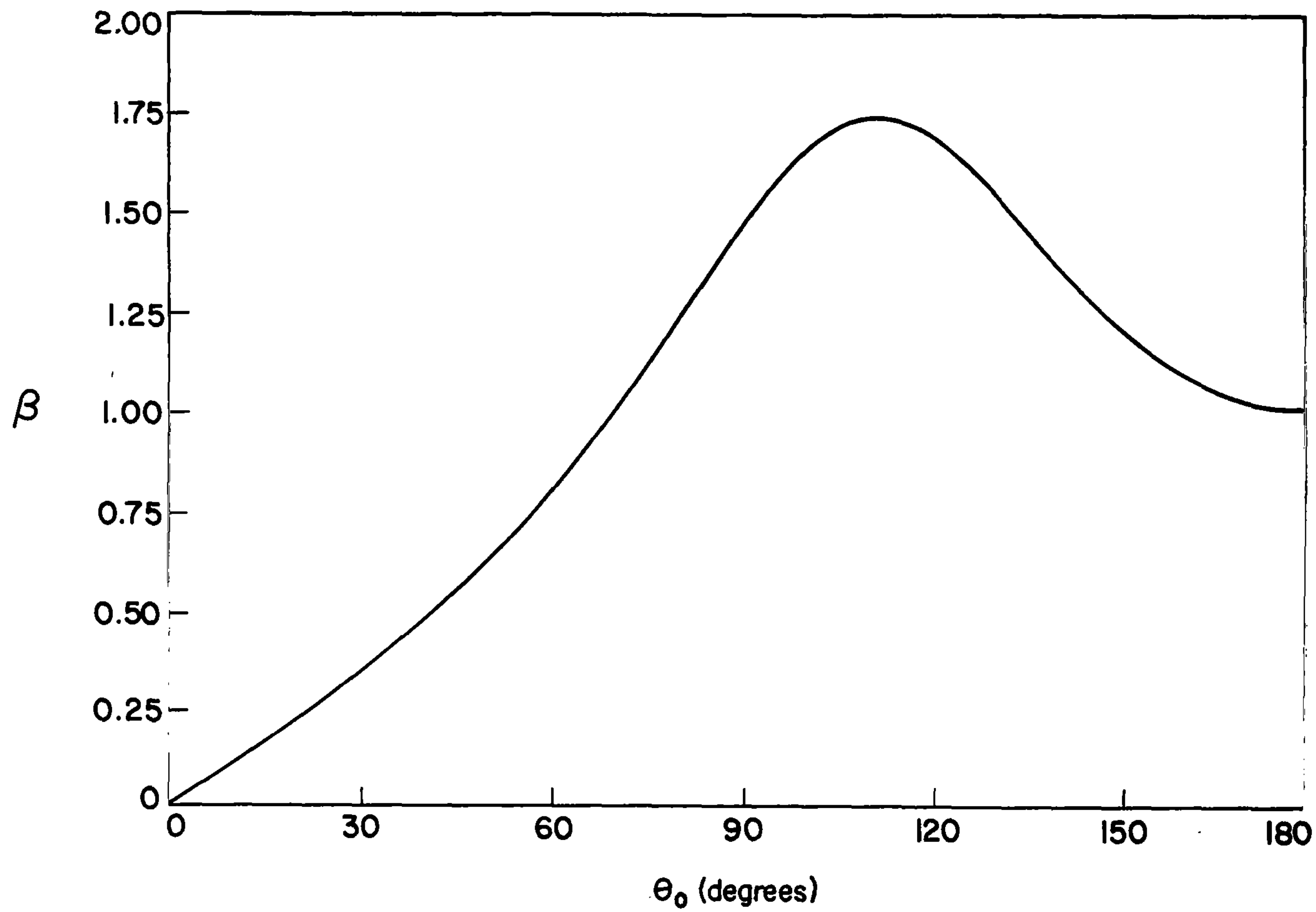

Figure 2. β as a function of aperture angle θ_o.

IV. NUMERICAL RESULTS

The lowest-order coefficients b_0^0, b_1^0, b_2^0, and b_3^0 are plotted as functions of θ_0 in Figure 3. The coefficient b_1 is the ratio of the electric field E_z at the center of the spherical shell to the applied field E_0:

$$b_1 = \frac{E_z(0)}{E_0} \tag{45}$$

Also, the dipole moment of the shell is given by

$$p_z = 4\pi a^3 \varepsilon_0 E_0 (1 - b_1) \tag{46}$$

so that the coefficient b_1 is the one of the greatest interest to us. The dipole moment p_{zo} of a closed spherical shell of radius <u>a</u> is

$$p_{zo} = 4\pi a^3 \varepsilon_0 E_0 \tag{47}$$

so that

$$\frac{p_z}{p_{zo}} = 1 - b_1 \simeq 1 - \frac{b_1^0}{1 + \beta K} \tag{48}$$

in which b_1^0 and β depend upon the aperture opening angle θ_0 and K depends upon the loading.

In Figures 4 and 5 are shown curves of b_1/b_1^0 as a function of the parameter K for various values of the angle θ_0. Both the "exact" and the "pseudovariational" results are shown and the excellent agreement between the two sets of results should be noted. The important point to note insofar as the effectiveness of the mesh shielding of the aperture is concerned is that $\beta K \gtrsim 1$ in order for significant reduction in the penetrant field to occur. For $\theta_0 \lesssim 30^\circ$, $\beta K \gtrsim 1$ implies that

$$\frac{8a\theta_0}{3a_s} \ln^{-1}\left(1 - e^{-2\pi r_w/a_s}\right)^{-1} \gtrsim 1 \tag{50}$$

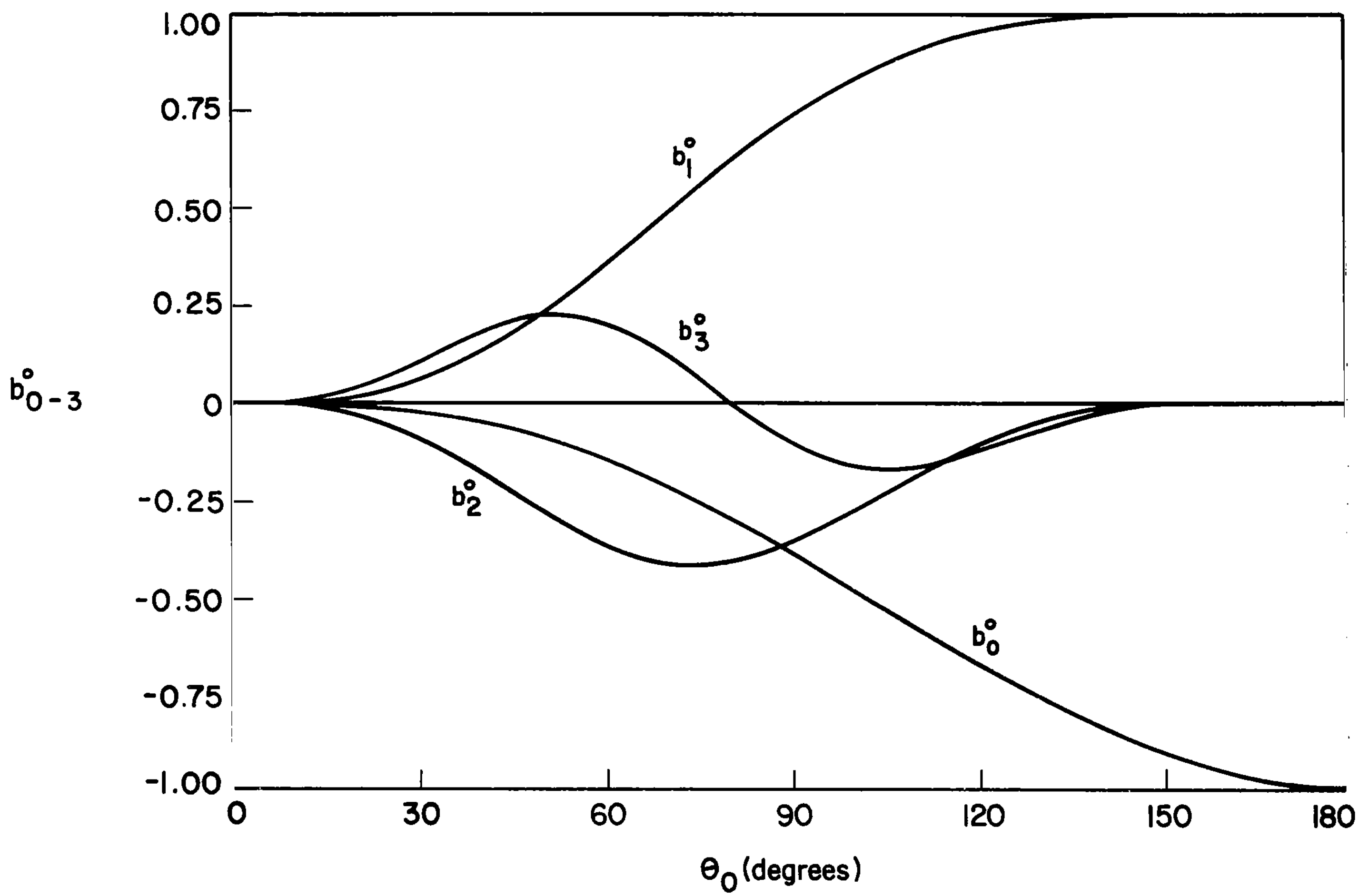

Figure 3. Coefficients b_0^0-b_3^0 as functions of aperture angle θ_0.

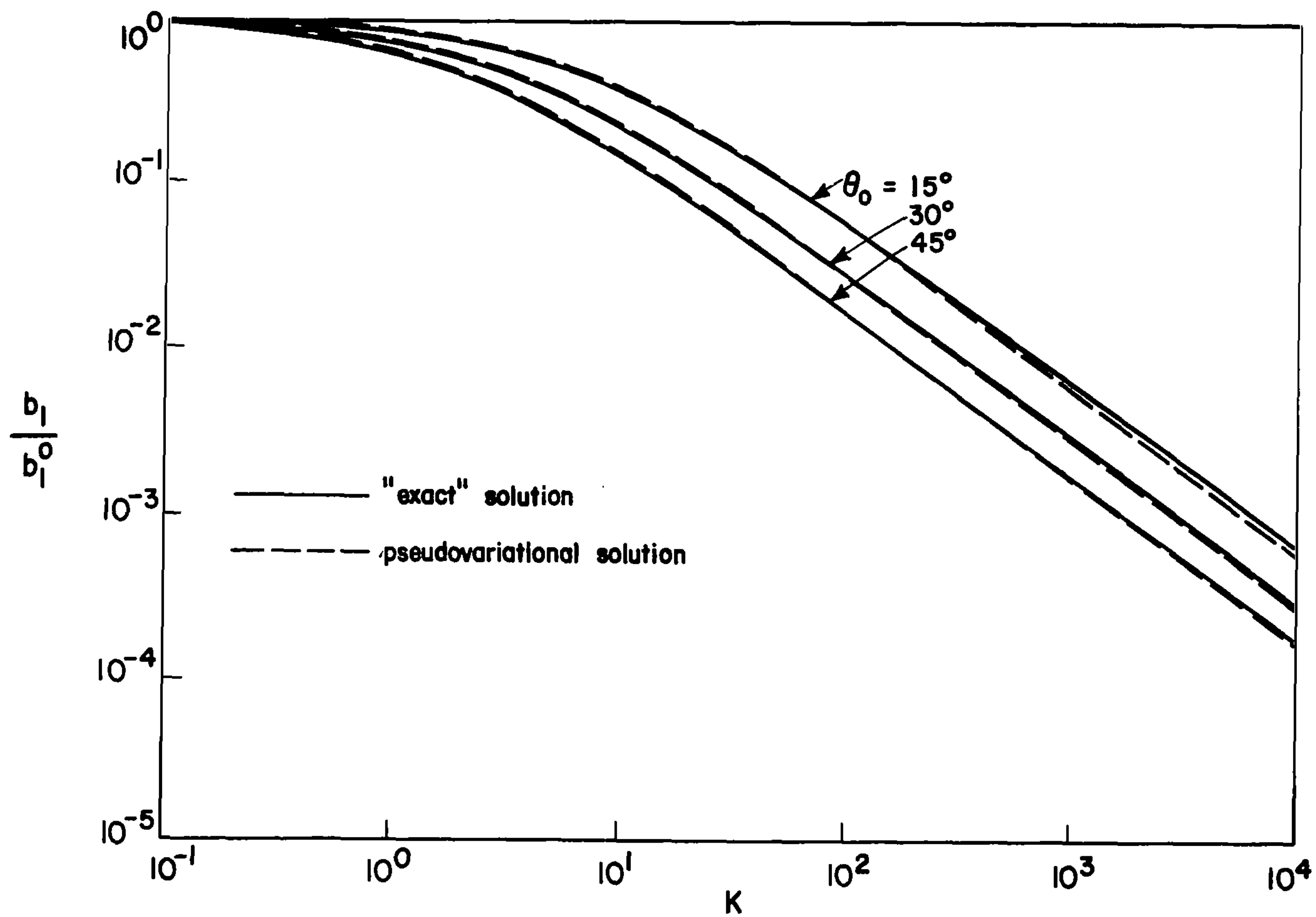

Figure 4. b_1/b_1^0 vs. K; θ_0 = 15°, 30°, 45°.

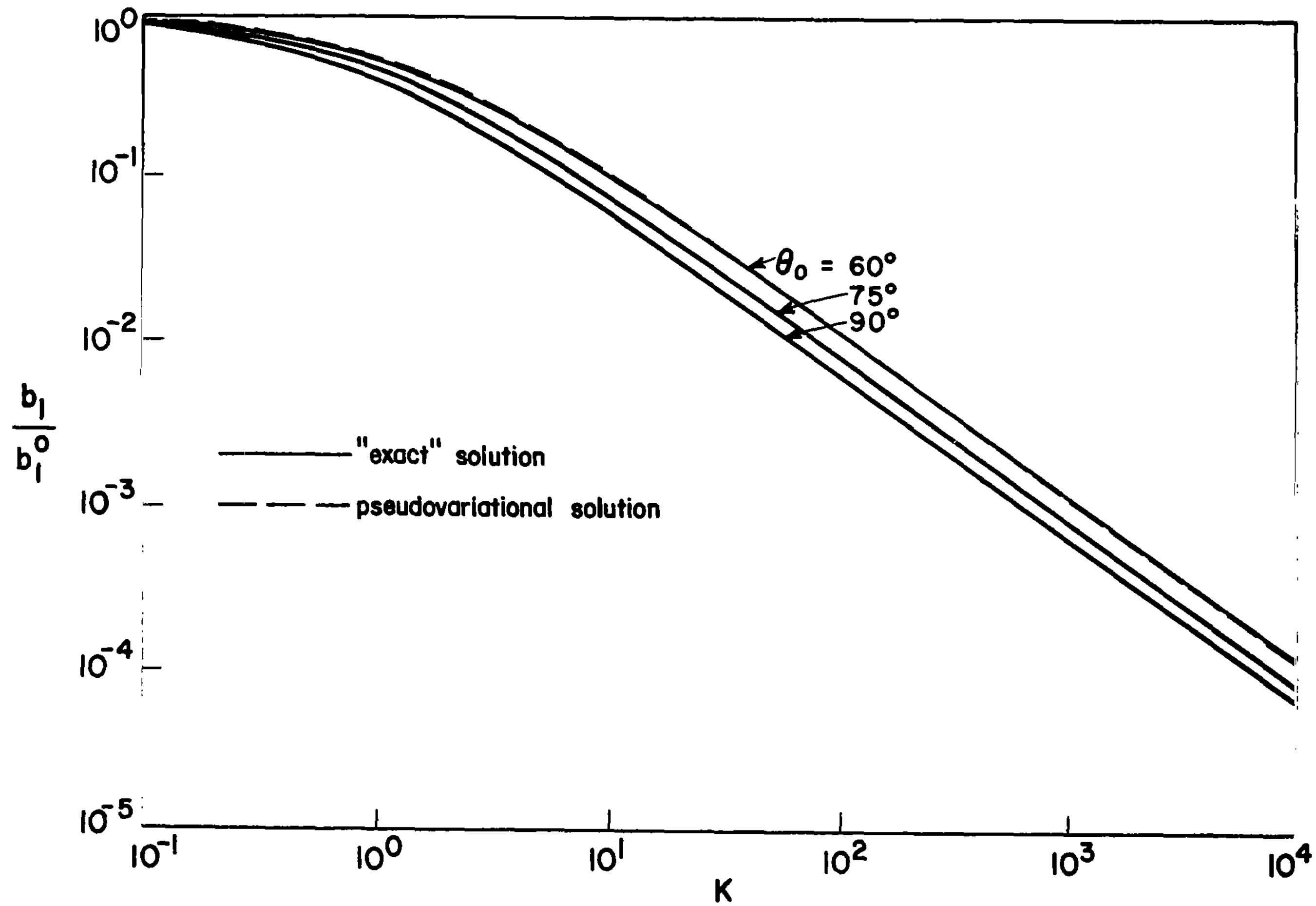

Figure 5. b_1/b_1^0 vs. K; $\theta_0 = 60^\circ, 75^\circ, 90^\circ$.

Since $a\theta_o/a_s >> 1$ in any practical case, we conclude that the mesh loading will significantly reduce the penetrant electric field in the enclosure.

V. CONCLUDING REMARKS

In this Note we have formulated and solved the last of a series of canonical boundary value problems concerning low-frequency electromagnetic penetration of loaded apertures. A contribution of the present Note, in addition to the solution of the problem itself, is the demonstration of the utility of the "pseudovariational" approach to the solution of the dual series equations which have arisen in connection with this problem. This approach seems to yield good results for the present problem and would appear to be worthy of further study in applications to related problems, including those considered in [1], [3], and [4].

ACKNOWLEDGMENT

It is a pleasure to acknowledge the contributions of Dr. K. C. Chen of AFWL to the pseudovariational method introduced in this Note.

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