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Interaction Notes

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Bounds on Norms of Scattering Matrices

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Abstract

Scattering matrices play an important role in transmission-line network theory and in electromagnetic topology. Norms of such matrices are used in bounding system response. Using power considerations, bounds for the 2 norm of such matrices can be found. Appropriate constraints on the normalizing admittance (impedance) matrix for the N-waves defining the scattering matrix are made. A renormalizing procedure is defined for the N-wave variables and scattering matrix based on a real-power condition. For use with EM topology a most significant result concerns the minimization of the norm of the scattering matrix leading to a choice of a real conductance times the identity as the normalizing admittance matrix.

I. Introduction

In the subject of electromagnetic topology, for the analysis and design of complex electromagnetic systems, scattering matrices are introduced to describe the transport of electromagnetic energy through various portions of the system described in topological terms as layers and shields, sublayers and subshields, and elementary volumes and elementary surfaces [2]. In transmission-line network theory scattering matrices also play an important role as can be seen by their explicit appearance in the BLT equation where scattering matrices are used to characterize the junctions [1].

In order to simplify the complex scattering equations one can obtain bounds on the system response through the use of norms [5] in conjunction with the scattering equations. In an EM topological context such norm bounds have been developed in [2,4]. In a transmission-line context such norm bounds have been developed in [3]. In these contexts the question has arisen concerning the 2 norm of scattering matrices connected with sublayers (or layers) in EM topology and with junctions in transmission-line networks as appropriate.

The basic question here concerns the choice of the admittance (or impedance) matrices with which to multiply appropriate current vectors (in complex-frequency domain) before addition to the appropriate voltage vectors in order to form the combined voltages or N-wave variables for use in defining the scattering matrices. Some previous results concern the renormalization of the wave variables for the case of a diagonal resistive admittance (or impedance) matrix with arbitrary choice of the diagonal conductance (resistance) values [7,10,12]. This note generalizes this procedure to arbitrary admittance (or impedance matrices.

II. Energy Considerations in Normalizing Admittance or Impedance

Keeping this discussion in a general form consider the real power either entering or leaving some volume in EM topology or junction in transmission-line network theory. This takes the general form

$$\begin{aligned}
&\frac{1}{2}\left[(\tilde{V}_n(j\omega))^* \cdot (\tilde{I}_n(j\omega)) + (\tilde{V}_n(j\omega)) \cdot (\tilde{I}_n(j\omega))^*\right] \geq 0 \\
&(\tilde{V}_n(s)) \equiv \text{voltage vector}, \quad (\tilde{I}_n(s)) \equiv \text{current vector} \\
&s \equiv \text{Laplace transform variable} \\
&\quad = j\omega \ (\omega \text{ real}) \text{ for present analysis}
\end{aligned} \tag{2.1}$$

Noting that this applies to the normalizing admittance as in fig. 2.1 for the wave variables as

$$(\tilde{I}_n(j\omega)) = (\tilde{Y}_{n,m}(j\omega)) \cdot \tilde{V}_n(j\omega) \tag{2.2}$$

(representing an N-wave propagating to the right in fig. 2.1). Then (2.1) takes the form

$$\begin{aligned}
&\frac{1}{2}\left[(\tilde{V}_n(j\omega))^* \cdot (\tilde{Y}_{n,m}(j\omega)) \cdot (\tilde{V}_n(j\omega))\right. \\
&\quad \left. + (\tilde{V}_n(j\omega)) \cdot (\tilde{Y}_{n,m}(j\omega))^* \cdot (\tilde{V}_n(j\omega))^*\right] \\
&\quad \geq 0
\end{aligned} \tag{2.3}$$

One way to visualize this is to temporarily take $(\tilde{Y}_{n,m})_T$ as $(\tilde{Y}_{n,m})$, for which case there is only an N-wave to the right.

Noting that only real ω are considered in this note suppress this variable for present purposes. Then (2.3) can be rearranged as

$$\begin{aligned}
&(\tilde{V}_n)^* \cdot (\tilde{G}_{n,m}) \cdot (\tilde{V}_n) \geq 0 \\
&\tilde{G}_{n,m} = \frac{1}{2}\left[(\tilde{Y}_{n,m}) + (\tilde{Y}_{n,m})^\dagger\right] \equiv \text{Hermitian part of } (\tilde{Y}_{n,m}) \\
&\dagger \equiv T^* \equiv \text{transpose conjugate} \equiv \text{adjoint}
\end{aligned} \tag{2.4}$$

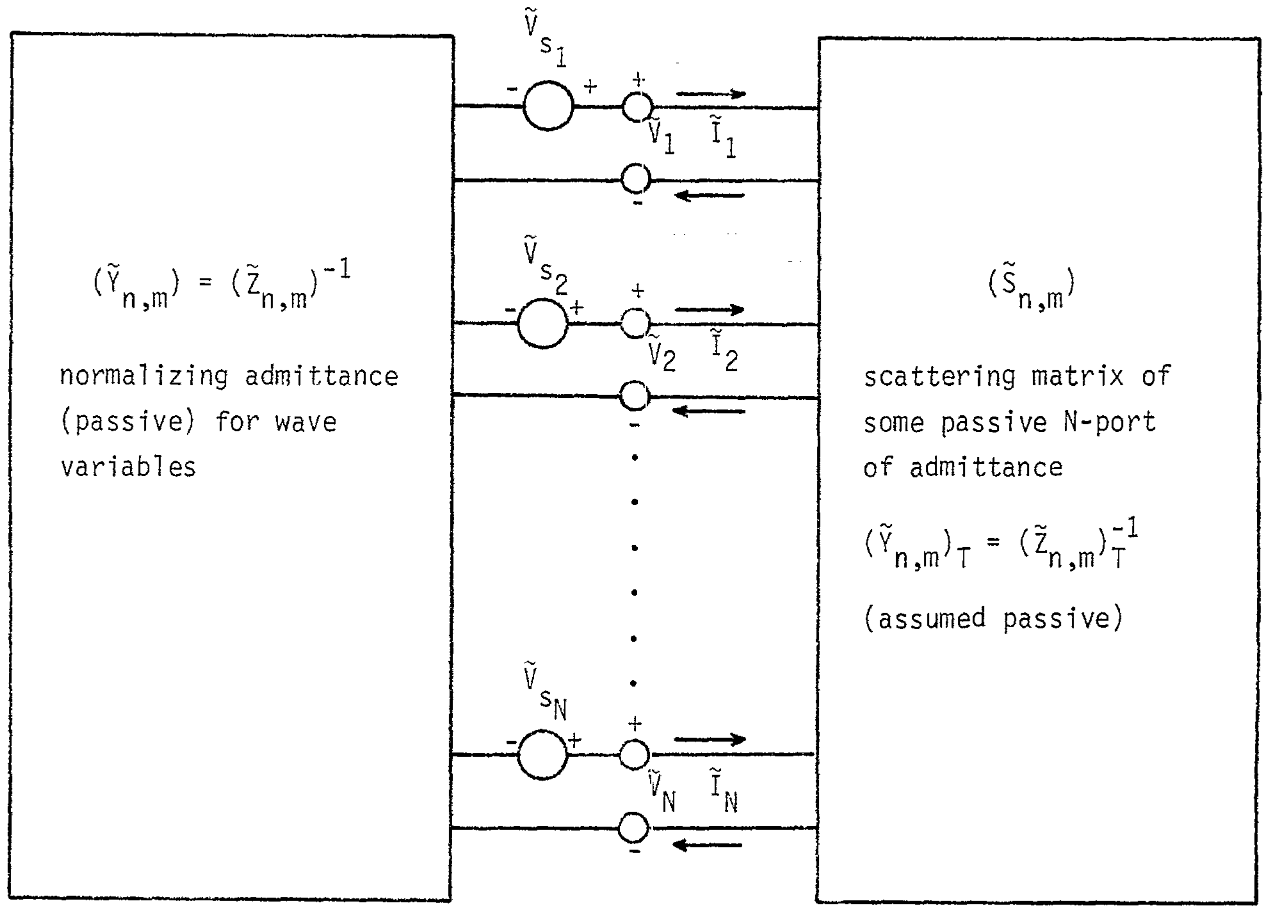

Fig. 2.1. Equivalent Circuit for Scattering at N-Port

The vectors indicated by a single subscript, and the matrices indicated by two subscripts, can be of any size, nominally for $n,m = 1,2,\ldots,N$. In some cases the vectors and matrices can be supervectors and supermatrices (as in the context of EM topology and the BLT equation for transmission-line network theory).

Now by our construction in (2.4)

$$(\tilde{G}_{n,m}) = (G_{n,m})^{\dagger} = \text{a Hermitian matrix} \tag{2.5}$$

Expand this in terms of eigenvalues and eignevectors as

$$\begin{aligned}
&(\tilde{G}_{n,m}) = \sum_{n=1}^{N} \tilde{\lambda}_\beta (\tilde{x}_n)_\beta (\tilde{x}_n)_\beta^* \\
&(\tilde{G}_{n,m}) \cdot (\tilde{x}_n) = \tilde{\lambda}_\beta (\tilde{x}_n)_\beta \qquad \text{(right eigenvectors)} \\
&(\tilde{x}_n)_\beta^* \cdot (\tilde{G}_{n,m}) = \tilde{\lambda}_\beta (\tilde{x}_n)_\beta^* \qquad \text{(left eigenvectors)} \\
&(\tilde{x}_n)_\beta \cdot (\tilde{x}_n)_{\beta'}^* = \begin{cases} 0 \text{ for } \beta \neq \beta' & \text{(orthogonality)} \\ 1 \text{ for } \beta = \beta' & \text{(orthonormalization)} \end{cases}
\end{aligned} \tag{2.6}$$

The eigenvalues may be all distinct, or if any are degenerate the above results still apply to Hermitian matrices for which a set of N orthonormal eigenvectors can always be constructed [8,9,11].

Stated another way, every Hermitian matrix is similar to a diagonal matrix as [8,9,11]

$$(\tilde{X}_{n,m})^{\dagger} \cdot (\tilde{G}_{n,m}) \cdot (\tilde{X}_{n,m}) = (\tilde{\Lambda}_{n,m})$$

$$(\tilde{\Lambda}_{n,m}) = \begin{pmatrix} \tilde{\lambda}_1 & & & \bigcirc \\ & \tilde{\lambda}_2 & & \\ & & \ddots & \\ \bigcirc & & & \tilde{\lambda}_N \end{pmatrix} = (\tilde{\lambda}_n 1_{n,m})$$

$$(\tilde{X}_{n,m}) = ((\tilde{x}_n)_1, (\tilde{x}_n)_2, \ldots, (\tilde{x}_n)_N) \tag{2.7}$$

$((\tilde{x}_n)_\beta$ inserted as columns ($\beta \rightarrow$ second index)

$$(\tilde{X}_{n,m})^\dagger = \begin{pmatrix} (\tilde{x}_n)_1^* \\ (\tilde{x}_n)_2^* \\ \cdot \\ \cdot \\ \cdot \\ (\tilde{x}_n)_N^* \end{pmatrix}$$

$((\tilde{x}_n)_\beta^*$ inserted as rows ($\beta \rightarrow$ first index)

$$(\tilde{X}_{n,m})^\dagger = (X_{n,m})^{-1}$$

Additionally it is known that all the eigenvalues of a Hermitian matrix are real. Furthermore, selecting

$$(\tilde{V}_n) = (x_n)_\beta \quad \text{for } \beta = 1,2,\ldots,N \tag{2.8}$$

implies from (2.4)

$$\lambda_\beta \geq 0 \quad \text{for } \beta = 1,2,\ldots,N \tag{2.9}$$

i.e., all the eigenvalues are nonnegative. This makes $(\tilde{G}_{n,m})$ a positive semi-definite matrix. This is another way of saying that we have restricted $(\tilde{Y}_{n,m})$ to be passive. As used in (2.2) the admittance matrix $(\tilde{Y}_{n,m})$ used to relate $(\tilde{V}_n)$ and $(\tilde{I}_n)$ might be the characteristic admittance matrix of a multiconductor transmission line (tube) or that used to combine voltage and current variables into wave variables.

These results for $(\tilde{Y}_{n,m}(s))$ and for $(\tilde{Z}_{n,m}(s))$ with

$$(\tilde{Z}_{n,m}(s)) = (\tilde{Y}_{n,m}(s))^{-1} \tag{2.10}$$

are more generally referred to as the p.r. (positive real) properties of such admittance and impedance matrices [6].

III. Renormalization of Voltage Variables

Now write

$$(\tilde{G}_{n,m}) = (\tilde{g}_{n,m})^2$$

$$(\tilde{g}_{n,m}) = \sum_{\beta=1}^{N} \tilde{\lambda}_{\beta}^{\frac{1}{2}} (\tilde{x}_n)_{\beta} (\tilde{x}_n)_{\beta}, \tag{3.1}$$

$$\lambda_{\beta}^{\frac{1}{2}} \geq 0 \quad \text{for } \beta = 1,2,\ldots,N$$

so that we can think of $(\tilde{g}_{n,m})$ as the principal square root of $(\tilde{G}_{n,m})$ which is then always possible for a non-negative definite Hermitian matrix. Since $(\tilde{g}_{n,m})$ is also Hermitian

$$\begin{aligned} (\tilde{g}_{n,m}) &= (\tilde{g}_{n,m})^{\dagger} \\ (\tilde{G}_{n,m}) &= (\tilde{g}_{n,m})^{\dagger} \cdot (\tilde{g}_{n,m}) \end{aligned} \tag{3.2}$$

Our expression of the passive nature of $(\tilde{G}_{n,m})$ in (2.4) then takes the form

$$\begin{aligned} &(\tilde{V}_n)^{*} \cdot (\tilde{g}_{n,m})^{\dagger} \cdot (\tilde{g}_{n,m}) \cdot (\tilde{V}_n) \geq 0 \\ &[(\tilde{V}_n) \cdot (\tilde{g}_{n,m})^{T}]^{*} \cdot (\tilde{g}_{n,m}) \cdot (\tilde{V}_n) \geq 0 \\ &[(\tilde{g}_{n,m}) \cdot (\tilde{V}_n)]^{*} \cdot [(\tilde{g}_{n,m}) \cdot (\tilde{V}_n)] \geq 0 \end{aligned} \tag{3.3}$$

From which we can readily define a renormalized voltage as

$$(\tilde{v}_n) \equiv (\tilde{g}_{n,m}) \cdot (\tilde{V}_n) \tag{3.4}$$

giving

$$(\tilde{v}_n)^{*} \cdot (\tilde{v}_n) \geq 0$$

In this form $(\tilde{G}_{n,m})$ has been effectively removed or "factored out" by considering these new "effective" voltage vectors. Of course, the inner product in (3.5) must be real and, more generally, non-negative. However, getting to this point relies on the properties of $(\tilde{G}_{n,m})$ which is consistent with this inner-product result.

IV. The Scattering Matrix

Now let us construct the scattering matrix of our N-port from appropriate admittance and impedance matrices. This N-port as in fig. 2.1 can be characterized by an admittance $(\tilde{Y}_{n,m}(s))_T$ or an impedance $(\tilde{Z}_{n,m}(s))_T$ so that

$$
\begin{aligned}
(\tilde{V}_n(s)) &= (\tilde{Z}_{n,m}(s))_T \cdot (\tilde{I}_n(s)) \\
(\tilde{I}_n(s)) &= (\tilde{Y}_{n,m}(s))_T \cdot (\tilde{V}_n(s)) \\
(\tilde{Z}_{n,m}(s))_T &= (\tilde{Y}_{n,m}(s))_T^{-1}
\end{aligned}
\tag{4.1}
$$

This N-port of interest can be a black box, a junction or collection of junctions in transmission-line network theory, or some volume or collection of volumes in EM topology.

As discussed in [1] the voltage and current N-vectors can be combined to form independent wave variables (N-vectors) of the form

$$
\begin{aligned}
(\tilde{V}_n(s))_{in} &\equiv (\tilde{V}_n(s)) + (\tilde{Z}_{n,m}(s)) \cdot (\tilde{I}_n(s)) \\
(\tilde{V}_n(s))_{out} &\equiv (\tilde{V}_n(s)) - (\tilde{Z}_{n,m}(s)) \cdot (\tilde{I}_n(s))
\end{aligned}
\tag{4.2}
$$

In the context of transmission-line network theory these N-waves can be thought of as propagating on an N-wire (plus reference) transmission line with $(\tilde{V}(s))_{in}$ representing an incident N-wave and $(\tilde{V}(s))_{out}$ representing a scattered (or reflected) N-wave. In an EMP topological context these N-wave variables have the same role except that the reference transmission line(s) can be shrunk to zero length while the reference admittance matrix $(\tilde{Y}_{n,m}(s))$ can be chosen as a convenient normalization matrix with, of course, the limitation that it should not be singular and should have finite elements (at least for frequencies of interest), as well as have the usual restriction of passivity.

The scattering matrix of interest is then defined by

$$
(\tilde{V}_n(s))_{out} \equiv (\tilde{S}_{n,m}(s)) \cdot (\tilde{V}_n(s))_{in} \tag{4.3}
$$

Let us now explore some of the properties of this scattering matrix.

The scattering matrix can be computed from the various impedance and/or admittance matrices. Following the derivation in [1] we have

$$(\tilde{S}_{n,m}(s)) = [(\tilde{Z}_{n,m}(s))_T \cdot (\tilde{Y}_{n,m}(s)) + (1_{n,m})]^{-1}$$

$$\cdot [(\tilde{Z}_{n,m}(s))_T \cdot (\tilde{Y}_{n,m}(s)) - (1_{n,m})]$$

$$= [(1_{n,m}) + (\tilde{Z}_{n,m}(s)) \cdot (\tilde{Y}_{n,m}(s))_T]^{-1}$$

$$\cdot [(\tilde{1}_{n,m}) - (\tilde{Z}_{n,m}(s)) \cdot (\tilde{Y}_{n,m}(s))_T] \tag{4.4}$$

There are other forms which can be obtained. Noting that

$$[(\tilde{b}_{n,m}) + (1_{n,m})] \cdot [(\tilde{b}_{n,m}) - (1_{n,m})]$$

$$= [(\tilde{b}_{n,m})^2 - (1_{n,m})]$$

$$= [(\tilde{b}_{n,m}) - (1_{n,m})] \cdot [(\tilde{b}_{n,m}) + (1_{n,m})] \tag{4.5}$$

dot multiplying on both left and right by $[(\tilde{b}_{n,m}) + (1_{n,m})]^{-1}$ gives

$$[(\tilde{b}_{n,m}) - (1_{n,m})] \cdot [(\tilde{b}_{n,m}) + (1_{n,m})]^{-1}$$

$$= [(\tilde{b}_{n,m}) + (1_{n,m})]^{-1} \cdot [(\tilde{b}_{n,m}) - (1_{n,m})] \tag{4.6}$$

showing an important property of this bilinear form, namely commutativity. This results because one of the matrices is the identity matrix (or it would also work for a constant times the identity matrix).

Another general property of this bilinear form is found by placing it in a form using $(\tilde{b}_{n,m})^{-1}$ as

$$[(\tilde{b}_{n,m}) + (1_{n,m})]^{-1} \cdot [(\tilde{b}_{n,m}) - (1_{n,m})]$$

$$= \left\{(\tilde{b}_{n,m}) \cdot [(1_{n,m}) + (\tilde{b}_{n,m})^{-1}]\right\}^{-1} \cdot (\tilde{b}_{n,m}) \cdot [(1_{n,m}) - (\tilde{b}_{n,m})^{-1}]$$

$$= [(1_{n,m}) + (\tilde{b}_{n,m})^{-1}]^{-1} \cdot (\tilde{b}_{n,m})^{-1} \cdot (\tilde{b}_{n,m}) \cdot [(1_{n,m}) - (\tilde{b}_{n,m})^{-1}]$$

$$= [(1_{n,m}) + (\tilde{b}_{n,m})^{-1}]^{-1} \cdot [(1_{n,m}) - (\tilde{b}_{n,m})^{-1}] \tag{4.7}$$

Note that by (4.6) this last result can also be reversed in order (i.e., the terms commute). So (4.4) can be augmented by the following two additional forms

$$(\tilde{S}_{n,m}(s)) = [(\tilde{Z}_{n,m}(s))_T \cdot (\tilde{Y}_{n,m}(s)) - (1_{n,m})]$$

$$\cdot [(\tilde{Z}_{n,m}(s))_T \cdot (\tilde{Y}_{n,m}(s)) + (1_{n,m})]^{-1}$$

$$= [(1_{n,m}) - (\tilde{Z}_{n,m}(s)) \cdot (\tilde{Y}_{n,m}(s))_T]$$

$$\cdot [(1_{n,m}) + (\tilde{Z}_{n,m}(s)) \cdot (\tilde{Y}_{n,m}(s))_T]^{-1} \tag{4.8}$$

There are three matrices in these equations: $(\tilde{S}_{n,m}(s))$, $(\tilde{Z}_{n,m}(s))_T$, and $(\tilde{Y}_{n,m}(s))$ (which can also be considered in terms of their inverses). Note however that the latter two appear as the product $(\tilde{Z}_{n,m}(s))_T \cdot (\tilde{Y}_{n,m}(s))$ or as the inverse of this product, i.e., $(\tilde{Z}_{n,m}(s)) \cdot (\tilde{Y}_{n,m}(s))_T$. This product can also be determined in terms of $(\tilde{S}_{n,m}(s))$. Taking the general form

$$(\tilde{S}_{n,m}) = [(\tilde{b}_{n,m}) + (1_{n,m})]^{-1} \cdot [(\tilde{b}_{n,m}) - (1_{n,m})]$$

$$[(\tilde{b}_{n,m}) + (1_{n,m})] \cdot (\tilde{S}_{n,m}) = [(b_{n,m}) - (1_{n,m})]$$

$$(\tilde{b}_{n,m}) \cdot [(\tilde{S}_{n,m}) - (1_{n,m})] = -[(\tilde{S}_{n,m}) + (1_{n,m})]$$

$$(\tilde{b}_{n,m}) = [(1_{n,m}) + (\tilde{S}_{n,m})] \cdot [(1_{n,m}) - (\tilde{S}_{n,m})]^{-1} \tag{4.9}$$

Then replacing $(\tilde{b}_{n,m})$ by $(\tilde{Z}_{n,m})_T \cdot (\tilde{Y}_{n,m})$, and noting the general result for bilinear forms involving the identity matrix in (4.6) gives

$$(\tilde{Z}_{n,m}(s))_T \cdot (\tilde{Y}_{n,m}) = [(1_{n,m}) + (\tilde{S}_{n,m}(s))] \cdot [(1_{n,m}) - (\tilde{S}_{n,m}(s))]^{-1}$$

$$= [(1_{n,m}) - (\tilde{S}_{n,m}(s))]^{-1} \cdot [(1_{n,m}) + (\tilde{S}_{n,m}(s))] \tag{4.10}$$

Various other forms are also possible, including the use of $(\tilde{S}_{n,m})^{-1}$. An interesting form results from taking the inverse of both sides of (4.10) giving

$$(\tilde{Z}_{n,m}(s)) \cdot (\tilde{Y}_{n,m}(s))_T = [(1_{n,m}) - (\tilde{S}_{n,m}(s))] \cdot [(1_{n,m}) + (\tilde{S}_{n,m}(s))]^{-1}$$

$$= [(1_{n,m}) + (\tilde{S}_{n,m}(s))]^{-1} \cdot [(1_{n,m}) - (\tilde{S}_{n,m}(s))] \tag{4.11}$$

V. Constraint on the Scattering Matrix

Linearity and passivity of our general N-port require that the real power into it be non-negative as expressed by

$$\frac{1}{2}\left[(\tilde{V}_n(j\omega)\cdot(\tilde{I}_n(j\omega)^* + (\tilde{V}_n(j\omega)^*\cdot(\tilde{I}_n(j\omega))\right] \geq 0 \tag{5.1}$$

Again noting that only real ω are considered, suppress this variable for present purposes. Substituting from (4.2) as

$$\begin{aligned}(\tilde{V}_n) &= \frac{1}{2}\left[(\tilde{V}_n)_{in} + (\tilde{V}_n)_{out}\right]\\ (\tilde{I}_n) &= \frac{1}{2}(\tilde{Y}_{n,m})\cdot\left[(\tilde{V}_n)_{in} - (\tilde{V}_n)_{out}\right]\end{aligned} \tag{5.2}$$

Then (5.1) becomes

$$\begin{aligned}\frac{1}{4}\{&[(\tilde{V}_n)_{in} + (\tilde{V}_n)_{out}]\cdot(\tilde{Y}_{n,m})^*\cdot[(\tilde{V}_n)^*_{in} - (\tilde{V}_n)^*_{out}]\\ &+ [(\tilde{V}_n)^*_{in} + (\tilde{V}_n)^*_{out}]\cdot(\tilde{Y}_{n,m})\cdot[(\tilde{V}_n)_{in} - (\tilde{V}_n)_{out}]\} \geq 0\end{aligned} \tag{5.3}$$

This can be manipulated into the form

$$\begin{aligned}\frac{1}{4}\{&(\tilde{V}_n)^*_{in}\cdot[(\tilde{Y}_{n,m}) + (\tilde{Y}_{n,m})^\dagger]\cdot(\tilde{V}_n)_{in} - (\tilde{V}_n)^*_{out}\cdot[(\tilde{Y}_{n,m}) + (\tilde{Y}_{n,m})^\dagger]\cdot(\tilde{V}_n)_{out}\\ &- (\tilde{V}_n)^*_{in}\cdot[(\tilde{Y}_{n,m}) - (\tilde{Y}_{n,m})^\dagger]\cdot(\tilde{V}_n)_{out} + (\tilde{V}_n)^*_{out}\cdot[(\tilde{Y}_{n,m}) - (\tilde{Y}_{n,m})^\dagger]\cdot(\tilde{V}_n)_{in}\}\\ &\equiv p + q \geq 0\end{aligned} \tag{5.4}$$

These terms are

$$p \equiv \frac{1}{2}\{(\tilde{V}_n)^*_{in}\cdot(\tilde{G}_{n,m})\cdot(\tilde{V}_n)_{in} - (\tilde{V}_n)^*_{out}\cdot(\tilde{G}_{n,m})\cdot(\tilde{V}_n)_{out}\}$$

$$q \equiv \frac{1}{2}\{-(\tilde{V}_n)^*_{in}\cdot(\tilde{G}'_{n,m})\cdot(\tilde{V}_n)_{out} + (\tilde{V}_n)^*_{out}\cdot(\tilde{G}'_{n,m})\cdot(\tilde{V}_n)_{in}\}$$

$$(\tilde{G}_{n,m}) \equiv \frac{1}{2}[(\tilde{Y}_{n,m}) + (\tilde{Y}_{n,m})^{\dagger}] \qquad (5.5)$$

$$\equiv \text{Hermitian part of } (\tilde{Y}_{n,m})$$

$$(\tilde{G}'_{n,m}) \equiv \frac{1}{2}[(\tilde{Y}_{n,m}) - (\tilde{Y}_{n,m})^{\dagger}]$$

$$\equiv \text{anti-Hermitian part of } (\tilde{Y}_{n,m}) \text{ (or skew Hermitian)}$$

$$(\tilde{Y}_{n,m}) = (\tilde{G}_{n,m}) + (\tilde{G}'_{n,m})$$

$$(\tilde{G}_{n,m}) = (\tilde{G}_{n,m})^{\dagger} \quad , \quad (\tilde{G}'_{n,m}) = -(\tilde{G}'_{n,m})^{\dagger}$$

Since $(\tilde{G}_{n,m})$ is Hermitian then both terms comprising p are Hermitian forms and must be real [9]. Hence

$$p = \text{real}$$
$$q = \text{real} \qquad (5.6)$$

Which implies

$$q = q^{*}$$

$$-(\tilde{V}_n)^{*}_{in} \cdot (\tilde{G}'_{n,m}) \cdot (\tilde{V}_n)_{out} + (\tilde{V}_n)^{*}_{out} \cdot (\tilde{G}'_{n,m}) \cdot (\tilde{V}_n)_{in}$$

$$= (\tilde{V}_n)^{*}_{out} \cdot (\tilde{G}'_{n,m}) \cdot (\tilde{V}_n)_{in} - (\tilde{V}_n)^{*}_{in} \cdot (\tilde{G}'_{n,m}) \cdot (\tilde{V}_n)_{out} \qquad (5.7)$$

which is consistent.

Let us now require

$$q = 0 = -\frac{1}{2}(\tilde{V}_n)^{*}_{in} \cdot (\tilde{G}'_{n,m}) \cdot (\tilde{V}_n)_{out} + \frac{1}{2}(\tilde{V}_n)^{*}_{out} \cdot (\tilde{G}'_{n,m}) \cdot (\tilde{V}_n)_{in} \qquad (5.8)$$

so that

$$p = \frac{1}{2}(\tilde{V}_n)^{*}_{in} \cdot (\tilde{G}_{n,m}) \cdot (\tilde{V}_n)_{in} - \frac{1}{2}(\tilde{V}_n)^{*}_{out} \cdot (\tilde{G}_{n,m}) \cdot (\tilde{V}_n)_{out}$$
$$\geq 0 \qquad (5.9)$$

represents a real power relationship for the wave variables $(\tilde{V}_n)_{in}$ and $(\tilde{V}_n)_{out}$ with incoming power no less than outgoing power (with equality for a lossless scattering network). Note also that

$$\begin{aligned} &\frac{1}{2}(\tilde{V}_n)^*_{in} \cdot (\tilde{G}_{n,m}) \cdot (\tilde{V}_n)_{in} \geq 0 \\ &\frac{1}{2}(\tilde{V}_n)^*_{out} \cdot (\tilde{G}_{n,m}) \cdot (\tilde{V}_n)_{out} \geq 0 \end{aligned} \tag{5.10}$$

since these are separately Hermitian forms and $(\tilde{G}_{n,m})$ is positive semi-definite.

Returning to q write this as

$$\begin{aligned} q = 0 &= -\frac{1}{2}(\tilde{V}_n)^*_{in} \cdot (\tilde{G}'_{n,m}) \cdot (\tilde{V}_n)_{out} - \frac{1}{2}[(\tilde{V}_n)^*_{in} \cdot (\tilde{G}'_{n,m}) \cdot (\tilde{V}_n)_{out}]^* \\ &= -\frac{1}{2}(\tilde{V}_n)^*_{in} \cdot (\tilde{G}'_{n,m}) \cdot (\tilde{V}_n)_{out} - \frac{1}{2}(\tilde{V}_n)_{in} \cdot (\tilde{G}'_{n,m})^* \cdot (\tilde{V}_n)^*_{out} \end{aligned} \tag{5.11}$$

showing that q is of the form of a term plus its conjugate. Substituting from (4.3) we have

$$\begin{aligned} 0 &= (\tilde{V}_n)^*_{in} \cdot (\tilde{G}'_{n,m}) \cdot (\tilde{S}_{n,m}) \cdot (\tilde{V}_n)_{in} + (\tilde{V}_n)_{in} \cdot (\tilde{G}'_{n,m})^* \cdot (\tilde{S}_{n,m})^* \cdot (\tilde{V}_n)^*_{in} \\ &= (\tilde{V}_n)^*_{in} \cdot (\tilde{G}'_{n,m}) \cdot (\tilde{S}_{n,m}) \cdot (\tilde{V}_n)_{in} + (\tilde{V}_n)^*_{in} \cdot (\tilde{S}_{n,m})^\dagger \cdot (\tilde{G}'_{n,m})^\dagger \cdot (\tilde{V}_n)_{in} \\ &= (\tilde{V}_n)^*_{in} \cdot (\tilde{a}_{n,m}) \cdot (\tilde{V}_n)_{in} \\ (\tilde{a}_{n,m}) &\equiv (\tilde{G}'_{n,m}) \cdot (\tilde{S}_{n,m}) + (\tilde{S}_{n,m})^\dagger \cdot (\tilde{G}'_{n,m})^\dagger \\ &= (\tilde{a}_{n,m})^\dagger \quad \text{(Hermitian)} \end{aligned} \tag{5.12}$$

The first of (5.12) is then a Hermitian form for $(\tilde{a}_{n,m})$. Choosing $(\tilde{V}_n)_{in}$ to be an arbitrary eigenvector of $(\tilde{a}_{n,m})$, then the eigenvalues of $(\tilde{a}_{n,m})$ are zero. Since $(\tilde{a}_{n,m})$ is similar to a diagonal matrix with the eigenvalues on the diagonal [8,9,11] then $(\tilde{a}_{n,m})$ is similar to a zero matrix and hence is the zero matrix, i.e.,

$$(0_{n,m}) = (\tilde{a}_{n,m}) = (\tilde{G}'_{n,m}) \cdot (\tilde{S}_{n,m}) + (\tilde{S}_{n,m})^{\dagger} \cdot (\tilde{G}'_{n,m})^{\dagger} \tag{5.13}$$

Now consider some of the various possible choices allowed for $(\tilde{S}_{n,m})$. Writing from (4.4)

$$(\tilde{S}_{n,m}) = [(\tilde{Z}_{n,m})_T \cdot (\tilde{Y}_{n,m}) + (1_{n,m})]^{-1} \cdot [(\tilde{Z}_{n,m})_T \cdot (\tilde{Y}_{n,m}) - (1_{n,m})] \tag{5.14}$$

choose $(\tilde{Z}_{n,m})_T$ as

$$(\tilde{Z}_{n,m}(s))_T \equiv (\tilde{Z}_{n,m}(s)) + \tilde{\delta}(s)(1_{n,m}) \tag{5.15}$$

This is evidently possible since $(\tilde{Z}_{n,m})$ is passive by hypothesis and is in "series" with another term $\tilde{\delta}(1_{n,m})$ which can be chosen to be passive. Then (5.14) becomes

$$\begin{aligned}(\tilde{S}_{n,m}(s)) &= [(2(1_{n,m}) + \tilde{\delta}(s)(\tilde{Y}_{n,m}(s))]^{-1} \cdot [\tilde{\delta}(s)(\tilde{Y}_{n,m}(s))] \\ &= \frac{1}{2}\,\tilde{\delta}(s)\tilde{Y}_{n,m}(s) + O(\tilde{\delta}^2) \quad \text{as } \tilde{\delta} \to 0\end{aligned} \tag{5.16}$$

This term $\tilde{\delta}$ is taken small and represents some small impedance in series with each of the N ports with the N-port taken as characterized by $(\tilde{Z}_{n,m})$.

Turning to (5.13) insert these results giving

$$\begin{aligned}(0_{n,m}) &= \tilde{\delta}(\tilde{G}'_{n,m}) \cdot (\tilde{Y}_{n,m}) + \tilde{\delta}^{*}(\tilde{Y}_{n,m})^{\dagger} \cdot (\tilde{G}'_{n,m})^{\dagger} + O(\tilde{\delta}^2) \\ &\text{as } \tilde{\delta} \to 0\end{aligned} \tag{5.17}$$

First, choose $\tilde{\delta}$ small and real (say a small series resistance) giving

$$(0_{n,m}) = (\tilde{G}'_{n,m}) \cdot (\tilde{Y}_{n,m}) + (\tilde{Y}_{n,m})^{\dagger} \cdot (\tilde{G}'_{n,m})^{\dagger} \tag{5.18}$$

Second, choose $\tilde{\delta}$ small and imaginary (say a small series inductance as $j\omega L_{\delta}$) giving

$$(0_{n,m}) = (\tilde{G}'_{n,m}) \cdot (\tilde{Y}_{n,m}) - (\tilde{Y}_{n,m})^{\dagger} \cdot (\tilde{G}'_{n,m})^{\dagger} \tag{5.19}$$

Combining (adding) these two last equations gives

$$(\tilde{G}'_{n,m}) \cdot (\tilde{Y}_{n,m}) = (0_{n,m}) \tag{5.20}$$

Assuming $(\tilde{Y}_{n,m})$ is in general non-singular, we must have

$$(\tilde{G}'_{n,m}) = (0_{n,m}) \tag{5.21}$$

demonstrating that the anti-Hermitian part of $(\tilde{Y}_{n,m})$ must be $(0_{n,m})$ and hence that

$$(\tilde{Y}_{n,m}(j\omega)) = (\tilde{Y}_{n,m}(j\omega))^{\dagger} \qquad \text{(Hermitian)} \tag{5.22}$$

for the real power relationship of (5.8) to hold for the N-wave variables.

With this interesting result we can introduce a common assumption. Suppose the normalizing admittance $(\tilde{Y}_{n,m})$ for the wave variables is reciprocal, i.e., suppose

$$(\tilde{Y}_{n,m}(s)) = (\tilde{Y}_{n,m}(s))^{T} \tag{5.23}$$

This is appropriate to common N-wire transmission-line characteristic admittances (or equivalently, impedances) since multiconductor cables are normally constructed from reciprocal media. Similarly for N-port passive electrical networks these are typically reciprocal being LRC in nature, including transformers. For this typical case of reciprocity then (5.22) and (5.23) combine to give

$$(\tilde{Y}_{n,m}(j\omega)) = (\tilde{Y}_{n,m}(j\omega))^{T*} = (\tilde{Y}_{n,m}(j\omega))^{*}$$

$$\mathrm{Im}[(\tilde{Y}_{n,m}(j\omega))] = (0_{n,m}) \quad \text{or} \quad (\tilde{Y}_{n,m}(j\omega)) \text{ real} \tag{5.24}$$

This remarkable result significantly limits the acceptable $(\tilde{Y}_{n,m}(j\omega))$. An acceptable example is the characteristic admittance (or impedance) of a lossless N-wire transmission line. A purely resistive N-port network also meets this requirement. Other examples, however, might include lossless (but dispersive) waveguides operating in a single TE or TM (non-TEM) propagating mode.

VI. Bounds on Scattering-Matrix 2 Norm

From (5.9) and (5.10) we have the real-power relationships for the N-wave variables (for $s = j\omega$)

$$\frac{1}{2}(\tilde{V}_n)^*_{in} \cdot (\tilde{G}_{n,m}) \cdot (\tilde{V}_n)_{in} \geq \frac{1}{2} (\tilde{V}_n)^*_{out} \cdot (\tilde{G}_{n,m}) \cdot (\tilde{V}_n)_{out} \geq 0 \tag{6.1}$$

and from (4.3) we have the scattering matrix in

$$(\tilde{V}_n)_{out} = (\tilde{S}_{n,m}) \cdot (\tilde{V}_n)_{in} \tag{6.2}$$

Referring to section 3 we have a renormalization of the voltage variables using

$$\begin{gathered}(\tilde{G}_{n,m}) = (\tilde{g}_{n,m})^{\dagger} \cdot (\tilde{g}_{n,m}) \quad , \quad (\tilde{g}_{n,m}) = (\tilde{g}_{n,m})^{\dagger} \\ (\tilde{v}_n) \equiv (\tilde{g}_{n,m}) \cdot (\tilde{V}_n)\end{gathered} \tag{6.3}$$

where $(\tilde{g}_{n,m})$ is Hermitian (or even real symmetric if reciprocity is assumed) and is in general bounded and non-singular. Applying this normalization to the wave variables and scattering matrix define

$$\begin{gathered}(\tilde{v}_n)_{in} \equiv (\tilde{g}_{n,m}) \cdot (\tilde{V}_n)_{in} \\ (\tilde{v}_n)_{out} \equiv (\tilde{g}_{n,m}) \cdot (\tilde{V}_n)_{out} \\ (\tilde{S}'_{n,m}) \equiv (\tilde{g}_{n,m}) \cdot (\tilde{S}_{n,m}) \cdot (\tilde{g}_{n,m})^{-1}\end{gathered} \tag{6.4}$$

Now (6.1) can be changed as

$$\frac{1}{2} (\tilde{V}_n)^*_{in} \cdot (\tilde{g}_{n,m})^{\dagger} \cdot (\tilde{g}_{n,m}) \cdot (\tilde{V}_n)_{in}$$

$$\geq (\tilde{V}_n)^*_{out} \cdot (\tilde{g}_{n,m})^{\dagger} \cdot (\tilde{g}_{n,m}) \cdot (\tilde{V}_n)_{out} \geq 0$$

$$\frac{1}{2}[(\tilde{g}_{n,m})\cdot(\tilde{V}_n)_{in}]^*\cdot(\tilde{g}_{n,m})\cdot(\tilde{V}_n)_{in}$$

$$\geq \frac{1}{2}[(\tilde{g}_{n,m})\cdot(\tilde{V}_n)_{out}]^*\cdot(\tilde{g}_{n,m})\cdot(\tilde{V}_n)_{out}\geq 0 \tag{6.5}$$

$$\frac{1}{2}(\tilde{v}_n)^*_{in}\cdot(\tilde{v}_n)_{in}\geq\frac{1}{2}(\tilde{v}_n)^*_{out}\cdot(\tilde{v}_n)_{out}\geq 0$$

and (6.2) becomes

$$(\tilde{g}_{n,m})\cdot(\tilde{V}_n)_{out}=(\tilde{g}_{n,m})\cdot(\tilde{S}_{n,m})\cdot(\tilde{g}_{n,m})^{-1}\cdot(\tilde{g}_{n,m})\cdot(\tilde{V}_n)_{in}$$

$$(\tilde{v}_n)_{out}=(\tilde{S}'_{n,m})\cdot(\tilde{v}_n)_{in} \tag{6.6}$$

In this renormalized form both the real-power and scattering equations assume simple forms.

Substituting (6.6) into (6.5) gives

$$\frac{1}{2}(\tilde{v}_n)^*_{in}\cdot(\tilde{v}_n)_{in}\geq\frac{1}{2}(\tilde{v}_n)^*_{in}\cdot(\tilde{S}'_{n,m})^\dagger\cdot(\tilde{S}_{n,m})\cdot(\tilde{v}_n)_{in}\geq 0 \tag{6.7}$$

For any eigenvector $(\tilde{E}_n(s))_\beta$ and eigenvalue $\tilde{\chi}_\beta$ of the Hermitian, positive, semi-definite matrix $(\tilde{S}'_{n,m})^\dagger\cdot(\tilde{S}'_{n,m})$ we have

$$(\tilde{E}_n)^*_\beta\cdot(\tilde{S}'_{n,m})^\dagger\cdot(\tilde{S}'_{n,m})\cdot(\tilde{E}_n)_\beta=\tilde{\chi}_\beta(\tilde{E}_n)^*_\beta\cdot(\tilde{E}_n)_\beta \tag{6.8}$$

However, from (6.7) with $(\tilde{v}_n)_{in}$ chosen as $(\tilde{E}_n)_\beta$

$$(\tilde{E}_n)^*_\beta\cdot(\tilde{E}_n)_\beta\geq(\tilde{E}_n)^*_\beta\cdot(\tilde{S}'_{n,m})^\dagger\cdot(\tilde{S}'_{n,m})\cdot(\tilde{E}_n)_\beta \tag{6.9}$$

which combines with (6.8) and the positive semi-definiteness to give

$$0\leq\tilde{\chi}_\beta\leq 1 \qquad \text{for all } \beta=1,2,\ldots,N \tag{6.10}$$

Now since

$$\|(\tilde{S}'_{n,m})\|_2=\max_\beta \chi_\beta^{1/2} \tag{6.11}$$

then

$$0 \leq \|(\tilde{S}'_{n,m})\|_2 \leq 1 \tag{6.12}$$

Thus it is the 2 norm of $(\tilde{S}'_{n,m})$ that is bounded by 1, which makes this renormalized scattering matrix a bounded real matrix [6].

Next consider $(\tilde{S}_{n,m})$. From (6.4) we have

$$\begin{aligned} \|(\tilde{S}'_{n,m})\|_2 &\leq \|(\tilde{g}_{n,m})\|_2 \; \|(\tilde{S}_{n,m})\|_2 \; \|(\tilde{g}_{n,m})^{-1}\|_2 \\ &= \frac{\tilde{\lambda}^{1/2}_{\beta_{max}}}{\tilde{\lambda}^{1/2}_{\beta_{min}}} \|(\tilde{S}_{n,m})\|_2 \end{aligned} \tag{6.13}$$

where the $\tilde{\lambda}_\beta$ are the real non-negative eigenvalues of $(\tilde{G}_{n,m})$, the Hermitian part of $(\tilde{Y}_{n,m})$, introduced in section 2; the $\tilde{\lambda}^{1/2}_\beta$ (positive square root) are the eigenvalues of $(\tilde{g}_{n,m})$. Solving for $(\tilde{S}_{n,m})$ we have

$$(\tilde{S}_{n,m}) = (\tilde{g}_{n,m})^{-1} \cdot (\tilde{S}'_{n,m}) \cdot (\tilde{g}_{n,m}) \tag{6.14}$$

from which we obtain another bound

$$\begin{aligned} \|(\tilde{S}_{n,m})\|_2 &\leq \|(\tilde{g}_{n,m})\|_2^{-1} \|(\tilde{S}'_{n,m})\|_2 \; \|(\tilde{g}_{n,m})\|_2 \\ &= \frac{\tilde{\lambda}^{1/2}_{\beta_{max}}}{\tilde{\lambda}^{1/2}_{\beta_{min}}} \|(\tilde{S}'_{n,m})\|_2 \end{aligned} \tag{6.15}$$

The condition number of a matrix (square) is given by [3]

$$K((\tilde{b}_{n,m})) = \|(\tilde{b}_{n,m})\| \, \|(\tilde{b}_{n,m})^{-1}\| \geq 1 \tag{6.16}$$

Using a subscript 2 to indicate the use of the 2 norm we have

$$K_2((\tilde{G}_{n,m})) = \frac{\tilde{\lambda}_{\beta_{max}}}{\tilde{\lambda}_{\beta_{min}}}$$

$$K_2((\tilde{g}_{n,m})) = \frac{\tilde{\lambda}^{\frac{1}{2}}_{\beta_{max}}}{\tilde{\lambda}^{\frac{1}{2}}_{\beta_{min}}} = [K_2((\tilde{G}_{n,m}))]^{\frac{1}{2}} \tag{6.17}$$

These bounds can then be summarized as

$$0 \le [K_2((\tilde{G}_{n,m}))]^{-\frac{1}{2}} \|(\tilde{S}'_{n,m})\|_2 \le \|(\tilde{S}_{n,m})\|_2$$
$$\le [K_2((\tilde{G}_{n,m}))]^{\frac{1}{2}} \|(\tilde{S}'_{n,m})\|_2 \le [K_2((\tilde{G}_{n,m}))]^{\frac{1}{2}} \tag{6.18}$$

and

$$0 \le [K_2((\tilde{G}_{n,m}))]^{-\frac{1}{2}} \|(\tilde{S}_{n,m})\|_2 \le \|(\tilde{S}'_{n,m})\|_2$$
$$\le \text{lesser of } \{[K_2((\tilde{G}_{n,m}))]^{\frac{1}{2}} \|(\tilde{S}_{n,m})\|_2 \ , \ 1\} \tag{6.19}$$

VII. Lossless Scattering Matrices

In some cases the N-port network of interest may consist of lossless elements such as inductors and capacitors, or even open or short circuits. In such cases no real power is absorbed by the network. Then (6.1) is replaced by

$$\frac{1}{2}\,(\tilde{V}_n)^*_{in} \cdot (\tilde{G}_{n,m}) \cdot (\tilde{V}_n)_{in} = \frac{1}{2}\,(\tilde{V}_n)^*_{out} \cdot (\tilde{G}_{n,m}) \cdot (\tilde{V}_n)_{out} \geq 0 \qquad (7.1)$$

Using the same renormalization procedure as in section 6, the difference occurs at (6.7) and (6.9), the latter becoming

$$(\tilde{E}_n)^*_\beta \cdot (\tilde{E}_n)_\beta = (\tilde{E}_n)^*_\beta \cdot (\tilde{S}'_{n,m})^\dagger \cdot (\tilde{S}'_{n,m}) \cdot (\tilde{E}_n)_\beta \qquad (7.2)$$

which combines with (6.8) to give

$$\tilde{\chi}_\beta = 1 \quad \text{for all } \beta = 1,2,\ldots,N \qquad (7.3)$$

Since $(\tilde{S}'_{n,m})^\dagger \cdot (\tilde{S}'_{n,m})$ is Hermitian and every Hermitian matrix is similar to a diagonal matrix (as in (2.7)) with the diagonal elements as the eigenvalues (now all 1), then $(\tilde{S}'_{n,m})^\dagger \cdot (\tilde{S}'_{n,m})$ is similar to the identity matrix. However a similarity transformation of the identity matrix gives the identity matrix, and hence

$$(\tilde{S}'_{n,m})^\dagger \cdot (\tilde{S}'_{n,m}) = (1_{n,m}) \qquad (7.4)$$

This is the property of a unitary matrix [11]. Stated in another form this is

$$(\tilde{S}'_{n,m})^{-1} = (\tilde{S}'_{n,m})^\dagger \qquad (7.5)$$

The lossless renormalized scattering matrix then has the 2 norm

$$\|(\tilde{S}'_{n,m})\|_2 = 1 \qquad (7.6)$$

For the unrenormalized scattering matrix $(\tilde{S}_{n,m})$ the bound in (6.18) becomes for the lossless case

$$0 \leq [K_2((\tilde{G}_{n,m}))]^{-\frac{1}{2}} \leq \|(\tilde{S}_{n,m})\|_2 \leq [K_2((\tilde{G}_{n,m}))]^{\frac{1}{2}} \tag{7.7}$$

Comparing this result to (7.6) shows a clear advantage obtained for the bounding process by renormalization.

VIII. Choice of Admittance Matrix for N-Waves

Now we are entering the home stretch. The previous results can now be used to tell us something about the optimal choice of the normalizing admittance $(\tilde{Y}_{n,m}(s))$ for the definition of our wave variables. In transmission-line network theory $(\tilde{Y}_{n,m})$ would seem to be appropriately chosen to be the characteristic admittance of the N-wire transmission line(s) involved. However, in EM topology the choice is somewhat more flexible. In the good-shielding approximation [2] the various terms in the product of matrices correspond to shields (or subshields) and layers (or sublayers). The volume terms (layers or sublayers) have terms of the form [4]

$$(\tilde{I}_{n,m})^{-1} = [(1_{n,m}) - (\tilde{S}_{n,m}(s))]^{-1} \tag{8.1}$$

This form has considerable interest in the context of matrix norms. We have the results (applied to 2 norms) [3]

$$[(1_{n,m}) - (\tilde{S}_{n,m})]^{-1} \leq \begin{cases} \dfrac{1}{1 - \|(\tilde{S}_{n,m})\|_2} & \text{if } \|(\tilde{S}_{n,m})\|_2 < 1 \\[2ex] \dfrac{\|(\tilde{S}_{n,m})^{-1}\|_2}{1 - \|(\tilde{S}_{n,m})^{-1}\|_2} & \text{if } \|(\tilde{S}_{n,m})^{-1}\|_2 < 1 \end{cases} \tag{8.2}$$

Noting that

$$\begin{aligned} \|(\tilde{S}_{n,m})\|_2 &= [\text{maximum eigenvalue of } (\tilde{S}_{n,m})^{\dagger} \cdot (\tilde{S}_{n,m})]^{\frac{1}{2}} \\ \|(\tilde{S}_{n,m})^{-1}\|_2 &= [\text{minimum eigenvalue of } (\tilde{S}_{n,m})^{\dagger} \cdot (\tilde{S}_{n,m})]^{-\frac{1}{2}} \end{aligned} \tag{8.3}$$

Then if

$$\|(\tilde{S}_{n,m})\|_2 > 1 \quad \text{and/or} \quad \|(\tilde{S}_{n,m})^{-1}\|_2 > 1 \tag{8.4}$$

which can occur consistent with (6.18), the bounds in (8.2) do not apply.

In the context of EM topology then we would like to be able to constrain

$$0 \leq \|(\tilde{S}_{n,m})\|_2 \leq 1 \tag{8.5}$$

similar to the result for $(\tilde{S}'_{n,m})$ in (6.12). Considering the bounds in (6.18) this can be achieved for arbitrary $(\tilde{S}'_{n,m})$ (consistent with (6.12)) by noting that

$$0 \leq \|(\tilde{S}_{n,m})\|_2 \leq [K_2((\tilde{G}_{n,m}))]^{\frac{1}{2}} \tag{8.6}$$

and then constraining

$$K_2((\tilde{G}_{n,m})) = 1 \tag{8.7}$$

From (6.17) this implies

$$\tilde{\lambda}_{\beta_{max}} = \tilde{\lambda}_{\beta_{min}} \tag{8.8}$$

and hence

$$\tilde{\lambda}_\beta = \tilde{\lambda} \quad \text{independent of } \beta = 1,2,\ldots,N \tag{8.9}$$

Now these eigenvalues apply to the matrix $(\tilde{G}_{n,m})$ as displayed in (2.6). What kind of Hermitian matrix such as this has all eigenvalues the same? Considering (2.7) we have the result that this matrix is similar to a diagonal matrix as

$$\begin{aligned}(\tilde{X}_{n,m})^{-1} \cdot (\tilde{G}_{n,m}) \cdot (\tilde{X}_{n,m}) &= (\tilde{\Lambda}_{n,m}) \\ &= \tilde{\lambda}(1_{n,m})\end{aligned} \tag{8.10}$$

However this in turn implies

$$(\tilde{G}_{n,m}(j\omega)) = \tilde{\lambda}(j\omega)(1_{n,m}) \tag{8.11}$$

so that the Hermitian part of $(\tilde{Y}_{n,m}(j\omega))$ is now constrained to be a "constant" times the identity matrix. This constant is the N eigenvalues of $(\tilde{G}_{n,m}(j\omega))$ which is Hermitian and should not be the zero matrix in general (since (2.4) would become trivial). Thus we constrain

$$\tilde{\lambda}(j\omega) > 0 \tag{8.12}$$

at least for frequencies of interest.

In (5.21) we have the result that the anti-Hermitian part of $(\tilde{Y}_{n,m})$ is the zero matrix. Hence

$$(\tilde{Y}_{n,m}(j\omega)) = \tilde{\lambda}(j\omega)(1_{n,m}) \tag{8.13}$$

Since this is symmetric then the constraint in (8.7) makes $(\tilde{Y}_{n,m})$ satisfy a condition of reciprocity, i.e.,

$$(\tilde{Y}_{n,m}(j\omega)) = (\tilde{Y}_{n,m}(j\omega))^T \tag{8.14}$$

without having to invoke this as a separate assumption as in (5.23).

There are various possible choices of $\tilde{\lambda}(j\omega)$ in (8.13). A very simple choice is

$$\tilde{\lambda}(j\omega) \equiv g > 0 \tag{8.15}$$

which is a frequency independent conductance.

IX. Conclusion

After this exercise in matrix algebra we have some interesting results. First there is a general renormalization procedure which is generally applicable based on the Hermitian part of $(\tilde{Y}_{n,m}(j\omega))$, the normalizing admittance, as in (3.4) and (6.4). Second, from (5.21) we have that $(\tilde{Y}_{n,m}(j\omega))$ should be Hermitian if the real-power relation for the N-waves in (5.8) is to apply. Third, an assumption of reciprocity for $(\tilde{Y}_{n,m}(j\omega))$ gives the result that it should also be real. Fourth, a constraint that $\|(\tilde{S}_{n,m}(j\omega)\|_2 \leq 1$ for use in EM topology (in the good-shielding approximation) for a general N-port network (passive) makes $(\tilde{Y}_{n,m}(j\omega))$ a constant times the identity; this constant is conveniently chosen as a simple frequency-independent conductance which could be say $(50\ \Omega)^{-1}$ or $(100\ \Omega)^{-1}$ or whatever value seems most appropriate to the system in question.

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