Interaction Notes

Note 452

October 1985

TRANSIENT RESPONSE OF AN INFINITE CYLINDRICAL ANTENNA IN A DISSIPATIVE MEDIUM

Kenneth C. Chen
Electromagnetic Analysis Division
Sandia National Laboratories
Albuquerque, New Mexico 87185

ABSTRACT

The transient response of an infinite cylindrical antenna in a dissipative medium to an impulse excitation is calculated via an exact solution in the form of definite integrals and a simple, accurate asymptotic formula. The transmission line solution is shown to be a limiting case of the exact solution. By-products of this study include extension of a general asymptotic method to a more complicated integral and evaluation of a new definite integral.

PREFACE

In January 1982 the author invited Professor T. T. Wu of Harvard University to the Air Force Weapons Laboratories to review the source region EMP calculations, and in particular, the calculation of the EMP coupling to the long buried wire. The result of this review is a list of limitations and deficiencies of the existing calculations. This note addresses one of the most outstanding problems: the correct transmission line model for the buried wire and its limitations. The accompanying note deals with the numerical aspect and other theoretical considerations of the buried wire problem.

ACKNOWLEDGEMENTS

The author wishes to thank Professor T. T. Wu for his stimulating discussions. He would like to thank Mr. Terry L. Brown for performing the numerical integrations.

CONTENTS

SECTION 1 Introduction 6

SECTION 2 Antenna Versus Transmission Line................. 10

SECTION 3 Formulation of the Problem....................... 14

SECTION 4 Explicit Solution for $I(z,t)$.................... 21

SECTION 5 Simplifications of the Explicit Solution......... 31

SECTION 6 An Asymptotic Formula............................ 33

SECTION 7 Conclusions...................................... 35

APPENDIX A Derivation of the Governing Equations........... 37

APPENDIX B Derivation of an Alternative Formula for Numerical Results.............................. 39

APPENDIX C Evaluation of Definite Integrals................ 45

APPENDIX D Asymptotic Evaluation of I_1, I_2 and I........... 47

APPENDIX E Asymptotic Formula For Large τ................. 52

APPENDIX F Rigorous Transmission Line Theory............... 54

FIGURES

1 Transmission Line Model ($G/C = \frac{\sigma}{\varepsilon}$, $LC = \mu\varepsilon = c^{-2}$, $R = 0$) Used to Compare with the Infinite Antenna Response 13

2 Infinite Tubular Antenna with a Gap Generator Embedded in a Dissipative Medium.. 15

3 Branch Cuts in ω-Plane with Real k for Analytical Continuation of Equation 27.................................. 19

4 Branch Cuts in the ω'-Plane............................ 23

5 Integration Path in the ω' and ζ-Planes.................. 25

6 Transformation of the Path of Integration in the ω and ϕ-Planes.. 26

7 The Contour for the ϕ Integration in the ϕ-Plane......... 28

8 Contour Path of Integration ($C_{\zeta'}$) in the ζ-Plane......... 29

TABLES

1 Comparison in Milliamperes of Equations 3 and 6 for Varying Values of α. Entries Shown are $I_n(\tau) \equiv I(z,t)e^{\frac{\sigma t}{2\varepsilon}-\alpha\tau}$ 9

B1 Numerical comparison of I_2 Given by Equations B2 and B9 for $\alpha = 5x10^{-3}$; Values Given in Milliamperes. Entries Shown are $I_2\ e^{\frac{\sigma t}{2\varepsilon}-\alpha\tau}$ 44

D1 Numerical Comparison for I_1 and I_2 with the Approximate Formulas Versus τ for $\alpha = 10^{-2}$ 50

D2 Numerical Comparison for I_1 and I_2 with the Approximate Formulas Versus τ for $\alpha = 10^{-3}$ 50

D3 Numerical Comparison for I_1 and I_2 with the Approximate Formulas Versus τ for $\alpha = 10^{-4}$ 51

D4 Numerical Comparison for I_1 and I_2 with the Approximate Formulas Versus τ for $\alpha = 10^{-5}$ 51

SECTION 1
INTRODUCTION

The transient response of an infinite cylindrical antenna in free space has been a subject of considerable interest[1-5]. Recently it was shown that a simple formula provides with remarkable accuracy the description of the overall current waveform for a unit step voltage input at a delta gap[6].

The problem of the transient response of an infinite cylindrical antenna in a dissipative medium is the subject of the present investigation. The solution to this problem not only gives the early time transient behavior of a finite dipole antenna, it also provides an understanding of the transient response of a long conductor buried in earth or immersed in water. In the past[7], the treatment of electromagnetic coupling of a long buried conductor was via an ad hoc transmission line model with ad hoc line parameters. The rigorous solution obtained here provides a correct transmission line model and shows the limitations of such a model.

The current $\mathcal{I}(z,t)$ due to a unit impulse input voltage at a delta gap in an infinite cylindrical antenna embedded in a dissipative medium with conductivity σ and permittivity ε is shown to be

$$\mathcal{I}(z,t) = \left(\frac{\partial}{\partial t} + \frac{\sigma}{\varepsilon}\right) I(z,t) \qquad (1)$$

where

$$I(z,t) = \frac{4}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} \left[\int_0^{\alpha} I_0\left(\tau\sqrt{\alpha^2 - \eta^2}\right) \left\{ [J_0(\eta)]^2 + [Y_0(\eta)]^2 \right\}^{-1} \frac{d\eta}{\eta} \right.$$

$$\left. + \int_{\alpha}^{\infty} J_0\left(\tau\sqrt{\eta^2 - \alpha^2}\right) \left\{ [J_0(\eta)]^2 + [Y_0(\eta)]^2 \right\}^{-1} \frac{d\eta}{\eta} \right] . \tag{2}$$

Equation 2 is shown to be equivalent to the following numerically accurate integral:

$$I(z,t) = \frac{4}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} \left[\int_0^{\alpha} I_0\left(\tau\sqrt{\alpha^2 - \eta^2}\right) \left\{ [J_0(\eta)]^2 + [Y_0(\eta)]^2 \right\}^{-1} \frac{d\eta}{\eta} \right.$$

$$+ \frac{1}{2} \ln \left\{ \frac{\sqrt{\pi^2 + \left(\ln \frac{\alpha_0}{2} + \gamma\right)^2}}{-\left(\ln \frac{\alpha_0}{2} + \gamma\right)} \right\} K_0\left(\tau\sqrt{\alpha^2 - \alpha_0^2}\right)$$

$$\left. + \frac{\pi^2}{2} \int_{\alpha_0}^{\infty} \frac{I_0(\eta)}{K_0(\eta)} \frac{K_0\left(\tau\sqrt{\alpha^2 + \eta^2}\right)}{\left\{ [K_0(\eta)]^2 + \pi^2 [I_0(\eta)]^2 \right\}} \frac{d\eta}{\eta} \right] \tag{3}$$

where

$$\left. \begin{aligned} \zeta_0 &= \sqrt{\frac{\mu}{\varepsilon}} \\ \tau &= \sqrt{c^2 t^2 - z^2}/a \\ \alpha &= \frac{\sigma a}{2\varepsilon c} \end{aligned} \right\} \tag{4}$$

with a as the wire radius, c as the speed of light, γ is Euler's constant, and α_0 is taken such that

$$\alpha_o << \alpha \ . \tag{5}$$

J_o and Y_o are the Bessel functions of the first and second kind, and I_o and K_o are the modified Bessel functions of the first and second kind.

A simple formula based on the asymptotic evaluation of Equation 2 is derived in Appendix D and is given by

$$I(z,t) \sim \frac{2}{\zeta_o} e^{-\frac{\sigma t}{2\varepsilon}} I_o(\alpha\tau) \arctan\left[\frac{-\pi}{\ell n\left(\frac{\alpha}{\tau}\right) + \frac{K_o(\alpha\tau)}{I_o(\alpha\tau)} - \ell n\ 2 + \gamma}\right] \ . \tag{6}$$

The results obtained from numerically integrating Equation 3 and evaluating Equation 6 where we have shown $I_n(\tau) \equiv I(z,t)e^{\frac{\sigma t}{2\varepsilon}-\alpha t}$ are given in Table 1 for varying values of τ and with the parameter $\alpha = 10^{-5}$, 10^{-4}, 10^{-3} and 10^{-2}.

Briefly, in Section 2 the antenna response and the transmission line response are compared. Section 3 derives the governing integro-differential equation. A closed form solution for antenna current is obtained in Section 4. In Section 5, the solution is reduced to a numerically convenient formula. Asymptotic formulas are obtained in Section 6. Detailed derivations of all formulas are given in appendices. Results obtained here are used in Reference 8 to calculate the transient response of an infinite wire in a dissipative medium.

Table 1. COMPARISON IN MILLIAMPERES OF EQUATIONS 3 AND 6 FOR VARYING VALUES OF α.

ENTRIES SHOWN ARE $I_n(\tau) \equiv I(z,t)e^{\frac{\sigma t}{2\varepsilon} - \alpha\tau}$

τ \ α	10^{-5}		10^{-4}		10^{-3}		10^{-2}	
	Eq. 3	Eq. 6	Eq. 3	Eq. 6	Eq. 3	Eq. 6	Eq. 3	Eq. 6
1.00	8.99621	8.33901	8.99540	8.33826	8.98732	8.33076	8.70306	8.25628
1.25	7.85090	7.58985	7.85002	7.58900	7.84120	7.58047	7.55037	7.49602
1.50	7.06864	6.99791	7.06769	6.99697	7.05816	6.98753	6.76104	6.89430
1.75	6.49615	6.52207	6.49513	6.52104	6.48492	6.51079	6.18174	6.40964
2.00	6.05635	6.13277	6.05527	6.13167	6.04439	6.12065	5.73536	6.01220
2.25	5.70616	5.80895	5.70500	5.80778	5.69348	5.79604	5.37879	5.68072
2.50	5.41950	5.53552	5.41829	5.53427	5.40613	5.52184	5.08595	5.40001
2.75	5.17969	5.30150	5.17841	5.30019	5.16563	5.28710	4.84012	5.15904
3.00	4.97549	5.09884	4.97415	5.09747	4.96075	5.08374	4.63005	4.94966
3.50	4.64472	4.76483	4.64326	4.76333	4.62868	4.74837	4.28794	4.60283
4.00	4.38667	4.50017	4.38509	4.49855	4.36936	4.48241	4.01903	4.32599
5.00	4.00588	4.10484	4.00408	4.10299	3.98614	4.08460	3.61773	3.90777
7.50	3.44329	3.51509	3.44097	3.51272	3.41788	3.48914	3.00912	3.26677
10.00	3.12213	3.17793	3.11933	3.17507	3.09148	3.14671	2.64742	2.88421
12.00	2.94492	2.99230	2.94175	2.98907	2.91028	2.95709	2.44073	2.68536
15.00	2.75124	2.79000	2.74753	2.78624	2.71088	2.74906	2.20675	2.41733
20.00	2.53331	2.56330	2.52876	2.65869	2.48394	2.51333	1.92991	2.12311
30.00	2.27510	2.29619	2.26898	2.29001	2.20912	2.22957	1.57653	1.74508
50.00	2.01221	2.02603	2.00319	2.01695	1.91641	1.92953	1.18189	1.31668
75.00	1.84097	1.85105	1.82862	1.83862	1.71197	1.72126	0.91106	1.01649
100.00	1.73524	1.74337	1.71974	1.72780	1.57612	1.58339	0.74685	0.83095
150.00	1.60426	1.61035	1.58284	1.58885	1.39157	1.39667	0.56092	0.61681
250.00	1.46343	1.46776	1.43112	1.43535	1.16238	1.16548	0.39925	0.42741
500.00	1.30496	1.30778	1.24839	1.25107	0.85064	0.85174	0.26484	0.27258
1000.00	1.17335	1.17524	1.07524	1.07694	0.56179	0.56138	0.17847	0.17906

SECTION 2
ANTENNA VERSUS TRANSMISSION LINE

In a dissipative medium, the induced conduction current serves as the return current. When the conductivity is high, the wire current and the return current form a differential mode. Previous treatments[7] are based on ad hoc assumptions for line parameters, applied voltage, and frequency domain skin depths. Since a simple but accurate formula for the wire current has been obtained, it is possible to determine the correct parameters for such a transmission line model and its limitations. To compare antenna current to the transmission line current, the following conditions may be imposed:

$$\ln \frac{\tau}{\alpha} >> 1$$

and

$$\alpha\tau >> 1 \ .$$

It is easy to see that

$$\frac{K_0(\alpha\tau)}{I_0(\alpha\tau)} << 1$$

and

$$\frac{-\pi}{\ln \frac{\alpha}{\tau} + \frac{K_0(\alpha\tau)}{I_0(\alpha\tau)} - \ln 2 + \gamma} \sim \frac{\pi}{\ln \frac{\tau}{\alpha}} << 1 \ . \tag{7}$$

Thus

$$\arctan \frac{\pi}{\ln \frac{\tau}{\alpha}} \sim \frac{\pi}{\ln \frac{\tau}{\alpha}} \ . \tag{8}$$

In addition, using Equation 4, one can show

$$\ell n \frac{\tau}{\alpha} = 2 \ \ell n \frac{\delta_\tau}{a} \tag{9}$$

with

$$\delta_\tau = \left(\frac{2\sqrt{t^2 - z^2/c^2}}{\mu\sigma} \right)^{\frac{1}{2}} . \tag{10}$$

Therefore, Equation 6 is reduced to, for $t > z/c$,

$$I(z,t) \sim \frac{\pi}{\zeta_0 \ \ell n \frac{\delta_\tau}{a}} e^{-\frac{\sigma t}{2\varepsilon}} I_0\left(\frac{\sigma}{2\varepsilon}\sqrt{t^2 - z^2/c^2}\right) . \tag{11}$$

Consider now a transient response of a transmission line as shown in Figure 1. A delta function voltage V_0 is applied across the wire to the outer cylinder with radius b. The transient response of the line current is

$$i(z,t) = cV_0 C\left(\frac{\partial}{\partial t} + \frac{\sigma}{\varepsilon}\right) \left\{ e^{-\frac{\sigma}{2\varepsilon} t} I_0\left(\frac{\sigma}{2\varepsilon}\sqrt{t^2 - z^2/c^2}\right)\right\} \tag{12}$$

for $t > z/c$. Here the line capacitance is

$$C = \frac{2\pi\varepsilon}{\ell n \frac{b}{a}} . \tag{13}$$

Equation 13 is equivalent to Equations 1 and 11, if two conditions are met: (1) $b = \delta_\tau$, and (2) $V_0 = 1/2$. The first condition can be interpreted as an expanding outer radius exatly analogous to the antenna in free space (Reference 6). However, when $t >> z/c$, δ_τ does reduce to a time domain skin depth $\delta \sim (2t/\mu\sigma)^{1/2}$, which makes the transmission line approximation possible. The second condition is the source factor. The antenna voltage is defined from one end of the wire across the gap to the other end, while the transmission line voltage is from the wire to infinity in the radial direction of the wire. It is important to point out that the comparison has been made only for wires without termination. The termination problem for long conductors cannot be treated as a simple transmission line problem with static terminating impedance, but rather a scattering problem.

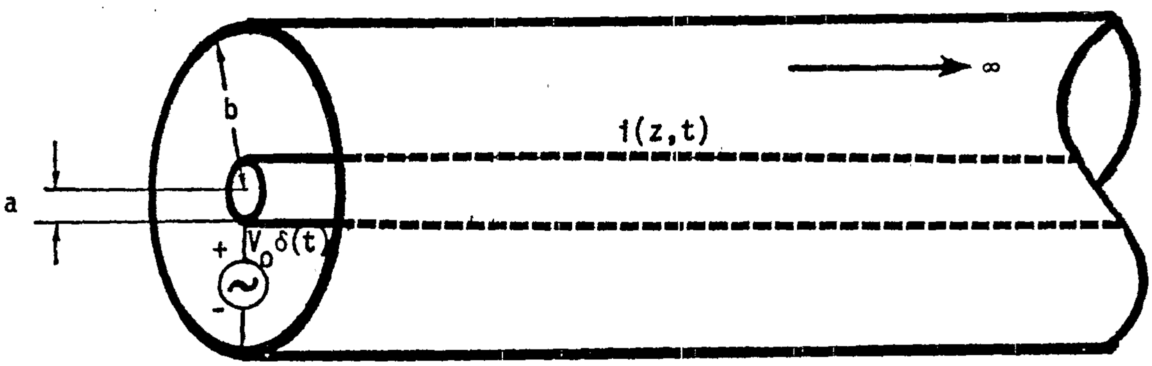


Figure 1. Transmission Line Model $\left(G/C = \frac{\sigma}{\varepsilon},\ LC = \mu\varepsilon = c^{-2},\ R = 0\right)$ Used to Compare with the Infinite Antenna Response.

SECTION 3
FORMULATION OF THE PROBLEM

Consider an infinite, circular, tubular antenna driven by a delta function voltage in time and a delta gap at the origin, as shown in Figure 2, where:

$$\mathscr{E}_z(a,\theta,z,t) = -V\delta(z)\delta(t) \quad . \tag{14}$$

The problem is rotationally symmetric with respect to θ, provided that

$$\mathscr{E}_\theta(a,\theta,z,t) = 0 \quad . \tag{15}$$

The governing equation for the vector potential on the antenna surface can be shown from Equations A7 and A13 (Appendix A) to be

$$\frac{\partial^2 \mathscr{A}_z}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \mathscr{A}_z}{\partial t^2} - \mu\sigma \frac{\partial \mathscr{A}_z}{\partial t} = \left(\frac{1}{c^2}\frac{\partial}{\partial t} + \mu\sigma\right)\mathscr{E}_z \quad . \tag{16}$$

The fact that $\mathscr{A}$ has only a z-component is used in deriving Equation 16. Use of Equations 14 and 16 gives, for $\rho = a$,

$$\frac{\partial^2 \mathscr{A}_z}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \mathscr{A}_z}{\partial t^2} - \mu\sigma \frac{\partial \mathscr{A}_z}{\partial t} = -\mu\left(\varepsilon\frac{\partial}{\partial t} + \sigma\right)[V\delta(z)\delta(t)] \quad . \tag{17}$$

On the other hand, let $\hat{z}\mathscr{I}(z,t)$ be the total current on the antenna. Then the vector potential $\mathscr{A}_z$ due to the current $\hat{z}\mathscr{I}(z,t)$ from Equation A11 with $\chi = 0$ is

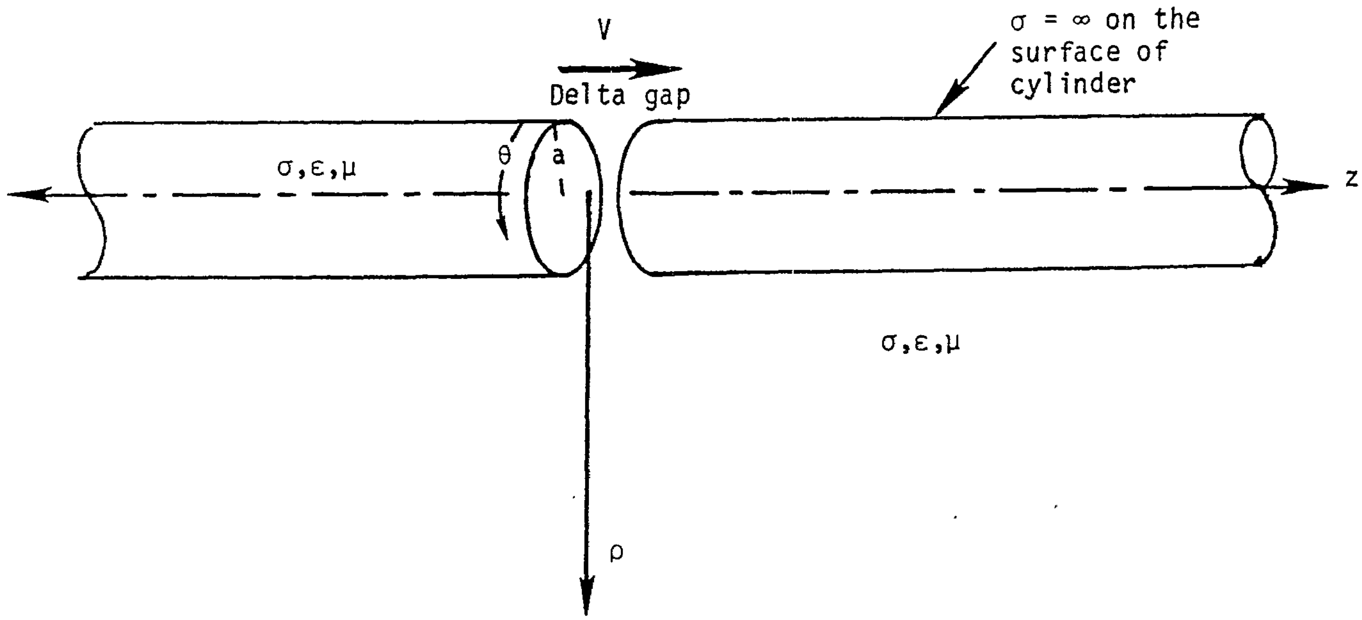


Figure 2. Infinite Tubular Antenna with a Gap Generator Embedded in a Dissipative Media.

$$\nabla^2 \mathscr{A}_z - \frac{1}{c^2}\frac{\partial^2 \mathscr{A}_z}{\partial t^2} - \mu\sigma \frac{\partial \mathscr{A}_z}{\partial t} = -\mu \mathscr{J} \; . \tag{18}$$

Let $\mathscr{G}(r,t)$ be the solution of

$$\nabla^2 \mathscr{G} - \frac{1}{c^2}\frac{\partial^2 \mathscr{G}}{\partial t^2} - \mu\sigma \frac{\partial \mathscr{G}}{\partial t} = -\delta(\vec{r})\delta(t) \tag{19}$$

subject to $\mathscr{G} = 0$, for $t < 0$.

Then,

$$\mathscr{A}_z(r,t) = \mu \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} \mathscr{J}(z',t')\, dt' \,\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathscr{G}(\vec{r}-\vec{r}',t-t')\, d\theta' \; . \tag{20}$$

The following discussion on $\mathscr{K}(z,t)$ is completely equivalent to those given in Reference 2. Define

$$\mathscr{K}(z,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathscr{G}(\vec{r}-\vec{r}',t)\, d\theta \; . \tag{21}$$

Before proceeding with the calculation for $\mathscr{K}(z,t)$, consider $\mathscr{G}(\vec{r},t)$. Applying the Fourier transform to Equation 19 gives

$$\mathscr{G}(r,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(r,\omega)\, e^{-i\omega t}\, d\omega \tag{22}$$

and

$$(\nabla^2 + k_1^2)\, G(r,\omega) = -\delta(r)$$

$$G(r,\omega) = (4\pi r)^{-1}\, e^{ik_1 r}$$

$$k_1 = \left(\frac{\omega^2}{c^2} + i\omega\mu\sigma\right)^{\frac{1}{2}} \tag{23}$$

$\mathscr{G}(r,t)$ in Equation 22 can be evaluated as

$$\mathscr{G}(r,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp\left[ir\sqrt{\frac{\omega^2}{c^2} + i\omega\mu\sigma} - i\omega t\right]}{4\pi r}\, d\omega$$

$$= \frac{1}{4\pi r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty} \frac{\exp\left[ir\sqrt{\frac{\omega^2}{c^2} + i\omega\mu\sigma} - i\omega t\right]}{2\pi i\sqrt{\frac{\omega^2}{c^2} + i\omega\mu\sigma}}\, d\omega$$

$$= \frac{-e^{-\frac{\sigma t}{2\varepsilon}}}{4\pi r}\frac{c\partial}{\partial r}\left[I_o\left(\frac{\sigma}{2\varepsilon}\sqrt{t^2 - r^2/c^2}\right) u(t - r/c)\right]$$

$$= \frac{e^{-\frac{\sigma t}{2\varepsilon}}}{4\pi r}\left\{\delta(t - r/c) - \frac{\frac{\sigma}{2\varepsilon}\frac{r}{c}}{\sqrt{t^2 - r^2/c^2}} I_1\left(\frac{\sigma}{2\varepsilon}\sqrt{t^2 - r^2/c^2}\right) u(t - r/c)\right\} . \tag{24}$$

To use Equations 23 and 24 in Equation 21, define r on the antenna as

$$r = [z^2 + (2a \sin \theta/2)^2]^{\frac{1}{2}} .$$

Since the time domain $\mathscr{K}(z,t)$ is cumbersome, its transform is given as

$$K(z,\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(\vec{r} - \vec{r}\,',\omega)\, d\theta$$

$$K(k,\omega) = \int_{-\infty}^{\infty} e^{-ikz} K(z,\omega)\, dz$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikz}\int_{-\pi}^{\pi} \frac{e^{ik_1 r}}{4\pi r}\, d\theta\, dz \tag{25}$$

$$K(k,\omega) = \frac{i}{4}\int_{-\pi}^{\pi} \frac{H_o^{(1)}\left(\sqrt{k_1^2 - k^2}\ 2a \sin \theta/2\right)}{2\pi} d\theta \tag{26}$$

$$= \frac{i}{4} J_o\left(a\sqrt{k_1^2 - k^2}\right) H_o^{(1)}\left(a\sqrt{k_1^2 - k^2}\right)$$

$$\text{for} \quad \omega > 0, \text{ any real } k\ . \tag{27}$$

The crucial step is from Equation 25 to 26 where $\omega > 0$ is imposed on Sommerfeld's outgoing condition in selecting $H_o^{(1)}$. The reduction from Equation 26 to 27 follows from an identity on page 441 of Watson[9].

To extend Equation 27 to other domains, note two very important identities

$$K(\pm k,-\omega) = [K(\mp k,\omega)]^*$$

$$K(\pm r,-\omega) = [K(\mp r,\omega)]^* \tag{28}$$

where * is the complex conjugate.

These identities follow from the fact that $\mathcal{K}(r,t)$ is real and they can be shown easily from the forward transform formula. Notice the sign of k makes no difference in Equation 28.

Use of Equation 28 gives

$$K(k,-\omega) = [K(-k,\omega)]^*$$

$$= \frac{-i}{4} J_o\left(a\sqrt{k_1^{*2} - k^2}\right) H_o^{(2)}\left(a\sqrt{k_1^{*2} - k^2}\right)$$

$$\text{for} \quad \omega < 0\ . \tag{29}$$

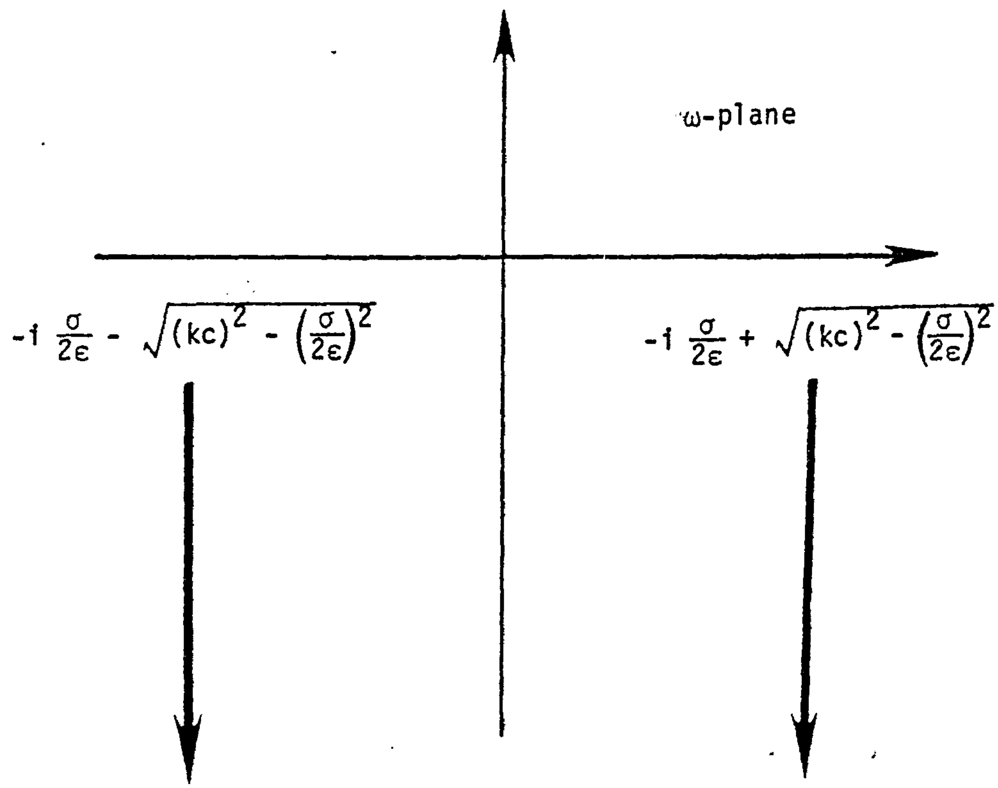


Figure 3. Branch Cuts in ω-Plane with Real k for Analytical Continuation of Equation 27.

Since there is no singularity in the upper half plane, it is possible to use Equation 27 as the definition of $K(k,\omega)$ for all values of ω and k with the branch cuts defined in Figure 3. This definition of $K(k,\omega)$ is consistent with Equation 29. A complete integral equation for current $\mathscr{I}(z,t)$ is obtained by combining Equation 17 and 20:

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \mu\sigma\frac{\partial}{\partial t}\right)\int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} \mathscr{I}(z',t')\mathscr{K}(z - z',t - t')\, dt'$$

$$= -\left(\varepsilon\frac{\partial}{\partial t} + \sigma\right) V\delta(z)\delta(t) \qquad (30)$$

SECTION 4
EXPLICIT SOLUTION FOR I(z,t)

Now solving for a Green's function denoted by I(z,t), we obtain

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \mu\sigma\frac{\partial}{\partial t}\right)\int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} dt' \; I(z',t')\,\mathcal{K}(z - z', t - t') = -\varepsilon\delta(t)\delta(z) \,. \tag{31}$$

Then the solution to Equation 30 is given by

$$\mathcal{I}(z,t) = V\left(\frac{\partial}{\partial t} + \frac{\sigma}{\varepsilon}\right) I(z,t) \tag{32}$$

$$I(z,t) = -(2\pi)^{-2}\,\varepsilon \int_{C_\omega} d\omega \int_{-\infty}^{\infty} dk[c^{-2}\omega^2 + i\sigma\mu\omega - k^2]^{-1}\left[K(k,\omega)\right]^{-1} \times \exp[i(kz - \omega t)] \tag{33}$$

where C_ω is the Bromwich contour, which is just above the real ω-axis, and $K(k,\omega)$ is defined in Equation 27. Introduce the transformation from the ω-variable to the ω'-variable with $\omega' = \omega + \frac{i\sigma}{\varepsilon}$ in Equation 33 as follows:

$$I(z,t) = -\left(\frac{c}{2\pi}\right)^2 \varepsilon\, e^{\left(-\frac{\sigma t}{2\varepsilon}\right)} \int_{C_{\omega'}} d\omega' \int_{-\infty}^{\infty} dk \left[\omega'^2 - k^2c^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2\right]^{-1}$$

$$\times\; [K'(k,\omega')]^{-1} \exp[i(kz - \omega' t)] \tag{34}$$

where

$$K'(k,\omega') = \frac{i}{4}\, J_0\left[a\sqrt{\omega'^2 - k^2c^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2}\Big/c\right] H_0^{(1)}\left[a\sqrt{\omega'^2 - k^2c^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2}\Big/c\right] \tag{35}$$

with branch cuts defined in Figure 4.

The next step in the simplification of Equation 34 is to transform the variables from ω', k to ζ, k to ζ, ϕ. Notice the transformation of variables is performed one variable at a time so that the theory of one complex variable can be used to deform the path of integration to be discussed later. The auxiliary transformations are from t and z/c to τ_1 and θ. All these transformations are as follows:

$$\zeta = \sqrt{\omega'^2 - (kc)^2} \qquad (36)$$
$$k = k$$

$$\zeta = \zeta$$
$$\phi = \operatorname{arcsinh} \frac{kc}{\zeta} \qquad (37)$$

and

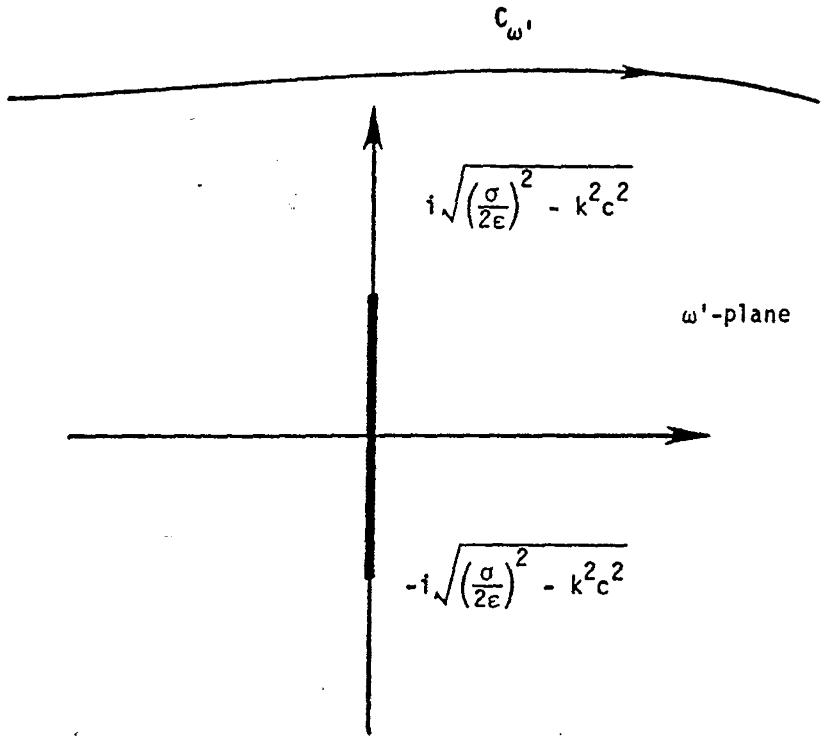
$$t = \tau_1 \cosh \theta$$
$$\text{or } \tau_1 = \sqrt{t^2 - z^2/c^2} \qquad (38)$$
$$z/c = \tau_1 \sinh \theta \ .$$

Equations 36 and 37 are equivalent to

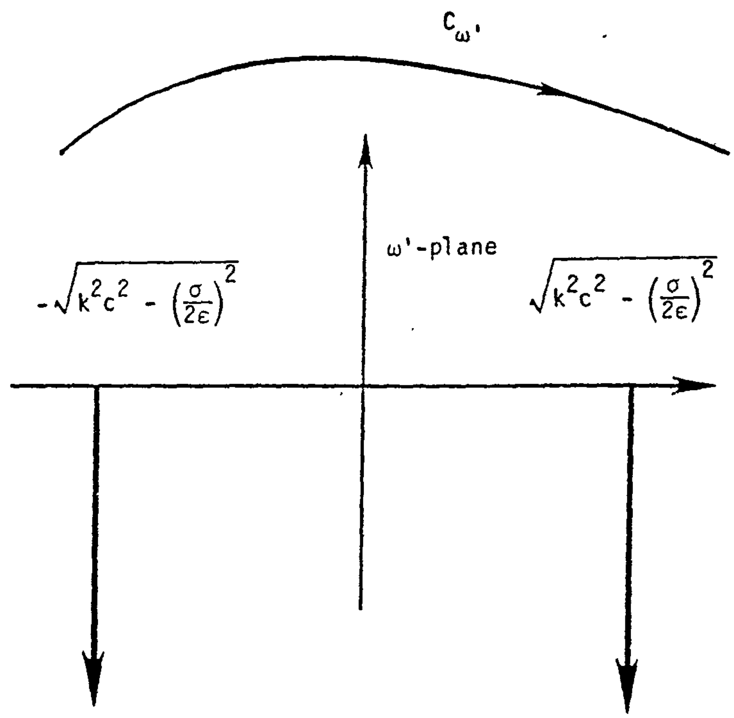
$$\zeta = \sqrt{\omega'^2 - (kc)^2} \qquad (39)$$

$$\tanh \phi = \frac{kc}{\omega'} \ . \qquad (40)$$

Notice the transformation as given in Equation 36 maps the ω',k space to ζ,k space. To evaluate the definite integral as given in Equation 34 in the ζ,k space, it is necessary to determine the path of integration C_ζ in the ζ,k space from the hypersurface formed by the contour C_{ω}' and $C_k(-\infty<k<\infty)$



(a) For $|kc| < \frac{\sigma}{2\varepsilon}$



(b) For $|kc| > \frac{\sigma}{2\varepsilon}$

Figure 4. Branch Cuts in the ω'-Plane.

as shown in Figure 4. Since C_ω' (Figure 4) is above all branch cuts and since the transformation from ω' to ζ (Figure 5) results in $\mathrm{Im}(\zeta) > \mathrm{Im}\,\omega'$, C_ζ is above all the branch cuts in the ζ-space.

Now hold ζ fixed and perform transformation 37. The main task here is to determine the path of integration C_ϕ in the ζ,ϕ space from the hyperspace formed by the contour C_ζ and $C_k(-\infty<k<\infty)$. Start with $\zeta = i\zeta_I + \zeta_R$ on C_ζ. As k varies from 0 to ∞, ϕ as given by Equation 37 traces a curve AA'. Again, as $\mathrm{Re}(\zeta)$ decreases, AA' sweeps across an area shown in the ϕ-plane. Now, $\zeta = i\zeta_I + \zeta_R$ in Figure 6 also corresponds to B at k=0 in the ζ-plane. As k varies from 0 to $-\infty$, ϕ traces a curve BB'. Furthermore, as $\mathrm{Re}(\zeta)$ decreases, BB' sweeps across a cross-hatched area shown in the ϕ-plane.

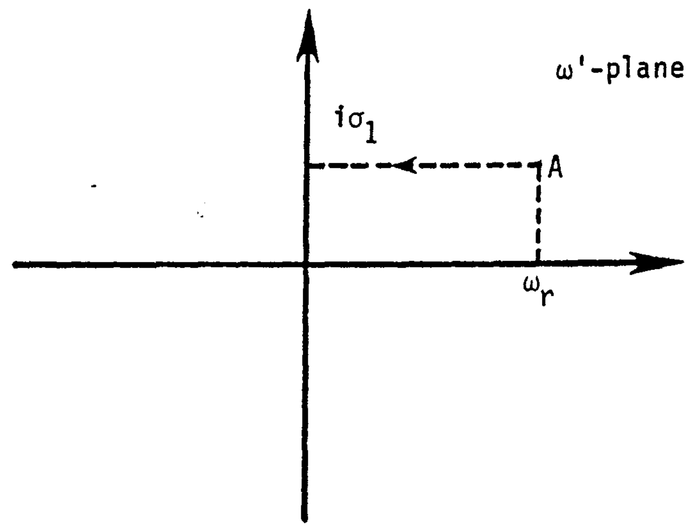
The path of integration in the ζ-plane and ϕ-plane can now be determined. Since the causality allows further deformation of contour above C_ω' for any k without changing the value of the integration, the path of integration in the ζ-plane for all k can be set to

$$C_\zeta(-\infty + i\infty \quad \text{to} \quad \infty + i\infty) \ .$$

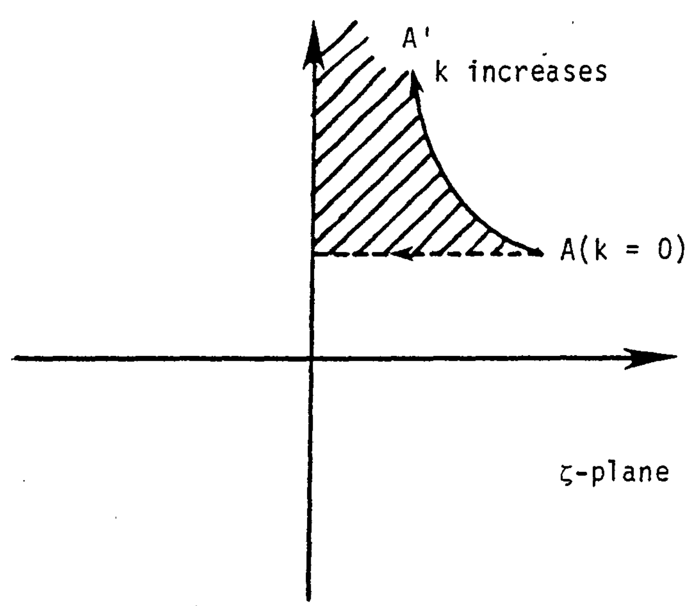
The path of integration in the ϕ-plane has been shown to be AB in Figure 6. Therefore, Equation 34 can be written as

$$I(z,t) = \frac{c\,\varepsilon i}{2\pi}\, e^{-\frac{\sigma t}{2\varepsilon}} \int_{C_\zeta} \zeta d\zeta \left[\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2\right]^{-1} \left|K(\zeta)\right|^{-1} T(\zeta\tau_1) \tag{41}$$

with

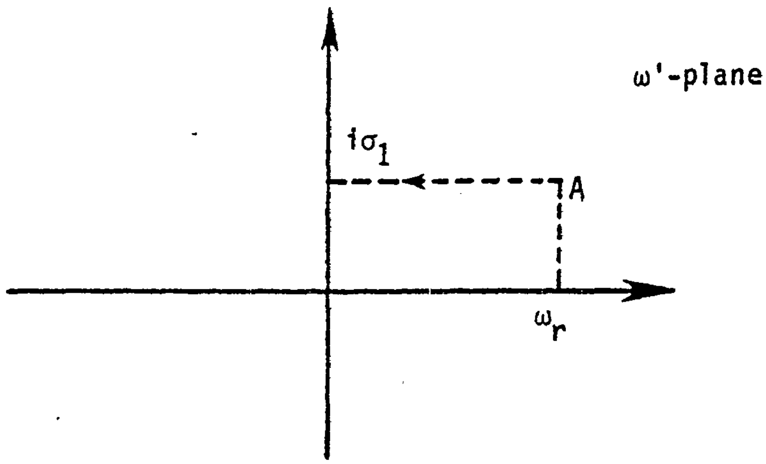


(a) Path in the ω'-Plane

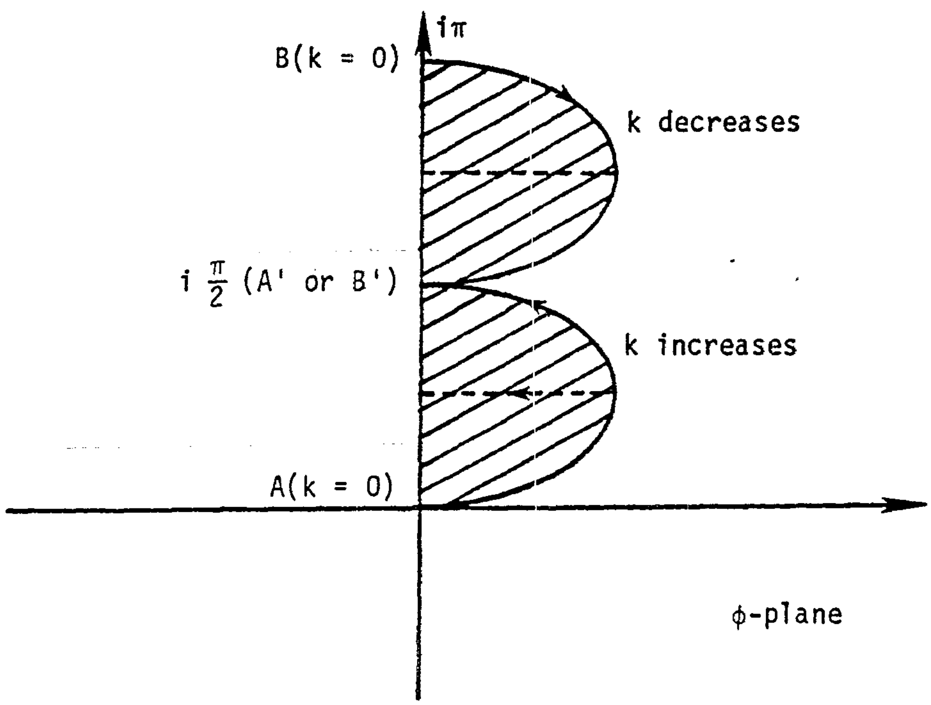


(b) ω'-Plane to ζ-Plane Mapping

Figure 5. Integration Path in the ω' and ζ-Planes.



(a) Path in the ω'-Plane



(b) Path in the ϕ-Plane

Figure 6. Transformation of the Path of Integration in the ω and ϕ-Planes.

$$T(\zeta\tau_1) = \frac{1}{2\pi i}\int_0^{i\pi} \exp[i\zeta\tau_1 \cosh(\phi - \theta)]\, d\phi \ . \tag{42}$$

Elementary simplifications of the above integral are in order. First, ϕ is replaced by $\phi_1 + i\pi$, which removes the negative sign in the exponent. Second, ϕ_1 is substituted by $-\phi$, which brings the limits of integration from 0 to $i\pi$. Third, the integration from 0 to ∞ and that from $i\pi$ to $i\pi + \infty$ are added to the integral, because these contributions cancel out each other. The resulting integral is as follows:

$$T(\zeta\tau_1) = \frac{1}{2\pi i}\int_{C_\phi} \exp[i\zeta\tau_1 \cosh(\phi - \theta)]\, d\phi \tag{43}$$

where C_ϕ is shown in Figure 19. Equation 43 can be identified as[10]

$$T(\zeta\tau_1) = J_o(\zeta\tau_1) \ . \tag{44}$$

Equations 41 and 44 give

$$I(z,t) = \frac{2}{\pi\zeta_o} e^{-\frac{\sigma t}{2\varepsilon}} \int_{C'_\zeta} \left[\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2\right]^{-1}$$

$$\left\{J_o\left[a\sqrt{\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2}/c\right] H_o^{(1)}\left[a\sqrt{\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2}/c\right]\right\}^{-1} \times J_o(\zeta\tau_1)\zeta d\zeta \ , \text{ for } \tau_1 > 0 \tag{45}$$

where ζ_o is defined in Equation 4.

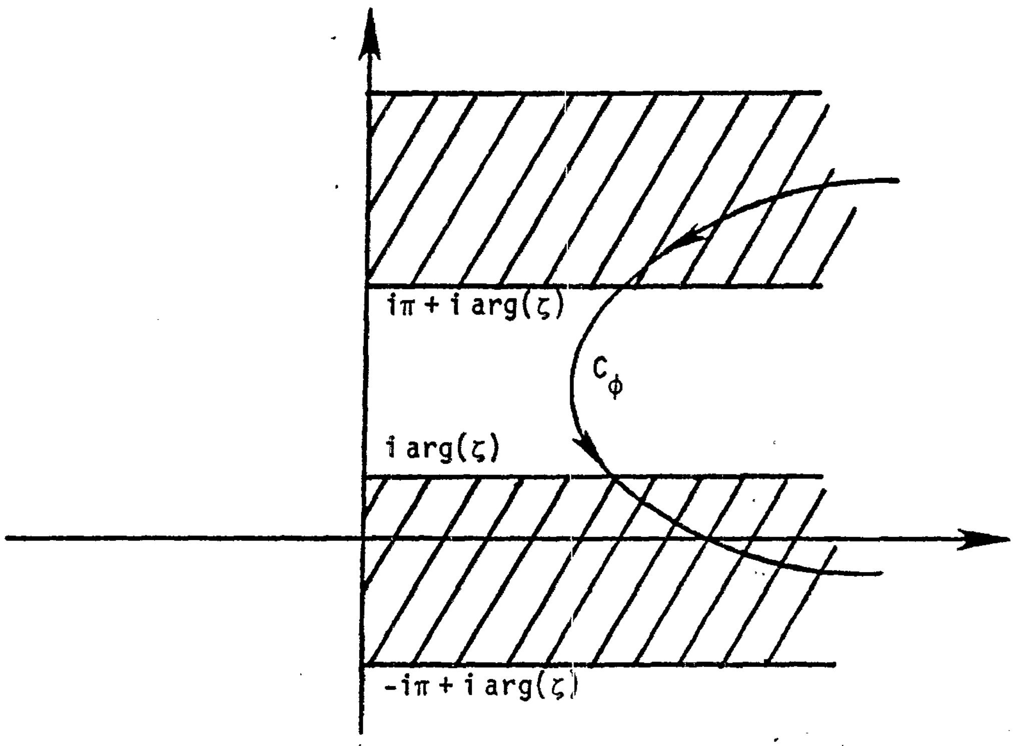


Figure 7. The ntour for the φ Integration in the φ-Plane.

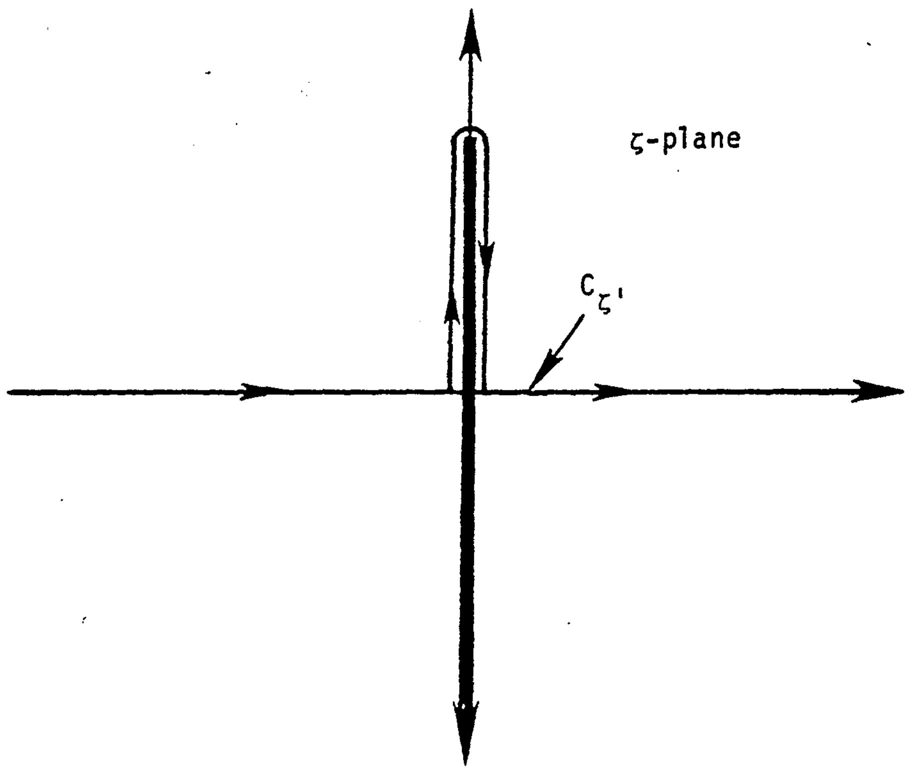


Figure 8. Contour Path of Integration ($C_{\zeta'}$) in the ζ-Plane.

The contour has been deformed to C'_ζ as shown in Figure 8 since the only singular point in the upper half plane is the branch point at $\zeta = i\ \sigma/2\varepsilon$. It can be seen that the contribution from the contour in the left half plane is the complex conjugate of the contribution from the contour in the right half plane. It is possible to write Equation 45 as

$$I(z,t) = \frac{4}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} R_e\left\{-\int_0^{\frac{i\sigma}{2\varepsilon}} + \int_0^{\infty} \left[\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2\right]^{-1}\right\}$$

$$\left\{J_0\left[a\sqrt{\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2}/c\right] H_0^{(1)}\left[a\sqrt{\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2}/c\right]\right\}^{-1} J_0(\zeta\tau_1)\zeta d\zeta \ . \quad (46)$$

Note the similarity between Equation 46 and the corresponding expression for the antenna current in free space. The branch cut integration in the ζ-plane from 0 to $i\sigma/2\varepsilon$ and the presence of exponential factor $e^{-\sigma/2\varepsilon}$ are the two major differences in the two cases.

Other methods for reducing Equation 33 to Equation 46 include the following:

a. First perform k-integration, then transform ω to ζ integration as given by Morgan[4] for the lossless case.

b. First transform k to ζ integration, then perform ω-integration as given by Latham and Lee[5] for the lossless case.

In conclusion, the antenna current $\mathcal{I}(z,t)$ is given by Equation 32 with I(z,t) given by Equation 46.

SECTION 5
SIMPLIFICATIONS OF THE EXPLICIT SOLUTION

Equation 46 can be separated into three parts. The first part, I_1, is the branch cut integration and is the dominant contribution to the exterior current for late time. The second part, I_2, is the integration along the real axis and contributes to the exterior current.[2] The third part, I_3, is the contribution due to poles located at zeros of J_o. Defining $\tau = \tau_1 c/a$ and $\alpha = \sigma a/2\varepsilon c$, it is possible to express I_1, I_2, and I_3 as follows:

$$I_1 = \frac{4}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} R_e\Bigg[\int_0^{\frac{\sigma}{2\varepsilon}} \left[\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2\right]^{-1}$$

$$\left\{J_0\left[a\sqrt{\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2}/c\right] H_0^{(1)}\left[a\sqrt{\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2}/c\right]\right\}^{-1} I_0(\zeta\tau_1)\zeta d\zeta\Bigg]$$

$$= \frac{4}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} \int_0^{\alpha} \frac{d\eta}{\eta} I_0\left(\tau\sqrt{\alpha^2 - \eta^2}\right) \left\{J_0(\eta)^2 + Y_0(\eta)^2\right\}^{-1} \qquad (47)$$

$$I_2 = \frac{4}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} R_e\Bigg[\int_0^{\infty} \left[\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2\right]^{-1}$$

$$\left\{J_0\left[a\sqrt{\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2}/c\right] H_0^{(1)}\left[a\sqrt{\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2}/c\right]\right\}^{-1} J_0(\zeta\tau_1)\zeta d\zeta\Bigg]$$

$$= \frac{4}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} \int_{\alpha}^{\infty} \frac{d\eta}{\eta} J_0\left(\sqrt{\eta^2 - \alpha^2\tau}\right)\left\{J_0(\eta)^2 + Y_0(\eta)^2\right\}^{-1} \qquad (48)$$

and

$$I_3 = -\frac{4}{\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} \sum_{j=1}^{\infty} \frac{J_0\left[\sqrt{\eta_j^2 - \alpha^2\tau}\right]}{\eta_j J_0'(\eta_j) Y_0(\eta_j)} \tag{49}$$

where η_j satisfies $J_0(\eta,j) = 0$.

Equation 49 denotes interior wave resonant modes and will be omitted in the ensuing discussion. Therefore,

$$I(z,t) = I_1 + I_2 \tag{2}$$

with I_1 and I_2 given by Equations 47 and 48.

Since J_0 and Y_0 in Equation 48 converge slowly when Equation 48 is used for numerical integration, an approximate formula for I_2 is derived in Appendix B and is given by

$$I_2 \approx \frac{2}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} \left\{ \ln \frac{\sqrt{\pi^2 + \left[\ln\left(\frac{\alpha_0}{2}\right) + \gamma\right]^2}}{-\left[\ln\left(\frac{\alpha_0}{2}\right) + \gamma\right]} K_0\left(\tau\sqrt{\alpha^2 - \alpha_0^2}\right) \right.$$

$$\left. + \pi^2 \int_{\alpha_0}^{\infty} \frac{d\eta}{\eta} \frac{I_0(\eta)}{K_0(\eta)} \frac{K_0\left(\tau\sqrt{\alpha^2 + \eta^2}\right)}{\{[K_0(\eta)]^2 + \pi^2[I_0(\eta)]^2\}} \right\} . \tag{50}$$

Comments on numerical accuracy are given in Appendix B. When replacing I_2 in Equation 2 by Equation 50, Equation 3 is obtained as given in Section 1.

Finally, let $\sigma = 0$, or $\alpha = 0$. Equations 48 and 49 correctly reduce to these for the lossless case[1,4].

SECTION 6
AN ASYMPTOTIC FORMULA

An asymptotic formula derived in Appendix D is examined here for free space and late-time limiting cases. Equation 2 for the exterior curent can be concisely written as

$$I(z,t) = \frac{4}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} \int_0^\infty J_0\left(\tau\sqrt{\eta^2 - \alpha^2}\right)\left\{[J_0(\eta)]^2 + [Y_0(\eta)]^2\right\}^{-1} \frac{d\eta}{\eta} \quad . \qquad (51)$$

Appendix D shows that an asymptotic evaluation of Equation 51 gives

$$I(z,t) \sim \frac{2}{\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} I_0(\alpha\tau) \arctan\left[\frac{-\pi}{\ell n\left(\frac{\alpha}{\tau}\right) + \frac{K_0(\alpha\tau)}{I_0(\alpha\tau)} - \ell n\, 2 + \gamma}\right] \quad . \qquad (6)$$

When the bracket inside arctan is negative, the functional value of arctan is chosen such that Equation 6 is a continuous function of τ.

Two special cases can now be examined. Case one is the late time formula, and case two is the free space formula. To obtain the late time formula, let $\tau \to \infty$. As a result,

$$\lim_{\zeta\to 0} I_0(\zeta) = 1$$

and

$$\lim_{\zeta\to 0} K_0(\zeta) = -\ell n \frac{\zeta}{2} - \gamma = \ell n\, \zeta + \ell n\, 2 - \gamma \quad .$$

Then Equation 6 can be written as

$$I(z,t) \sim \frac{2}{\zeta_0 (2\pi\alpha\tau)^{\frac{1}{2}}} e^{-\frac{\sigma t}{2\varepsilon} + \alpha\tau} \arctan\left[\frac{\pi}{\ln\left(\frac{\tau}{\alpha}\right) + \ln 2 - \gamma}\right] . \tag{52}$$

This agrees with the formula derived in Appendix E.

The other special case is to determine the limit as $\sigma \to 0$. It is easy to see

$$I(z,t) \sim \frac{2}{\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} I_0(\alpha\tau) \arctan\left[\frac{-\pi}{\ln\left(\frac{\alpha}{\tau}\right) + \frac{K_0(\alpha\tau)}{I_0(\alpha\tau)} - \ln 2 + \gamma}\right] \tag{6}$$

Using the limits, Equation 6 is reduced to

$$I(z,t) = \frac{2}{\zeta_0} \arctan\left[\frac{\pi}{2 \ln(\tau)}\right] . \tag{53}$$

Substituting Equation 53 into Equation 1 gives

$$\mathscr{I}(z,t) = \frac{2V}{\zeta_0} \frac{\partial}{\partial t} \left\{\arctan\left[\frac{\pi}{2 \ln(\tau)}\right]\right\} . \tag{54}$$

This agrees with the case for free space[6] except for the difference between a unit step input voltage versus a unit impulse voltage given here.

SECTION 7
CONCLUSIONS

The transient response of an infinite cylindrical antenna in a dissipative medium has been addressed. New findings presented here include:

(1) A rigorous treatment of the transient response of an infinite cylindrical antenna is provided in exact integral form and in a simple, accurate formula.

(2) A transmission line analog solution is given and rigorous transmission line solution discussed.

(3) A general asymptotic method is further extended to evaluate more complicated integrals.

(4) A new definite integral is obtained.

REFERENCES

1. T. T. Wu, "Transient Response of a Dipole Antenna," J. Math. Phys., 2, pp. 892-894, 1961.

2. R. E. Collin, and F. J. Zucker, Antenna Theory, Vol. 1, pp. 347-351, McGraw Hill, New York, 1969.

3. P. O. Brundell, "Transient Electromagnetic Waves Around a Cylindrical Transmitting Antenna," Ericsson Technics, 16(1), pp. 137-162, 1960.

4. S. P. Morgan, "Transient Response of a Dipole Antenna," J. Math. Phys., 3, pp. 564-565, 1962.

5. R. W. Latham, and K.S.H. Lee, "Transient Properties of an Infinite Cylindrical Antenna," Radio Science, 5, No. 4, pp. 715-723, 1970.

6. K. C. Chen, "Transient Response of an Infinite Cylindrical Antenna," IEEE Trans. on Antenna and Propagation, AP 31, No. 1, pp. 170-172, January 1983.

7. E. F. Vance, Coupling to Cables, pp. 11-86, John Wiley & Sons, New York, December 1974.

8. K. C. Chen, "Transient Response of an Infinite Wire in a Dissipative Medium," Interaction Note 453, October 1985.

9. G. N. Watson, A Treatise on the Theory of Bessel Functions, p. 192, Cambridge University Press, London, 1966.

10. G. F. Carrier, M. Krook, and C. E. Pearson, Functions of a Complex Variable, p. 237, McGraw Hill, New York, 1966.

APPENDIX A

DERIVATION OF THE GOVERNING EQUATIONS

The basic governing equations are Maxwell's equations:

$$\nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t} \tag{A1}$$

$$\mu^{-1} \nabla \times \mathcal{B} = \varepsilon_0 \frac{\partial \mathcal{E}}{\partial t} + \sigma \mathcal{E} + \mathcal{J} \tag{A2}$$

$$\nabla \cdot \mathcal{B} = 0 \tag{A3}$$

$$\varepsilon \nabla \cdot \mathcal{E} = \rho \quad . \tag{A4}$$

The equation of continuity is

$$\nabla \cdot \mathcal{J} + \frac{\partial \rho}{\partial t} + \frac{\sigma}{\varepsilon} \rho = 0 \ . \tag{A5}$$

Introduce

$$\mathcal{B} = \nabla \times \mathcal{A} \quad . \tag{A6}$$

By Equation A3

$$\nabla \times \left(\mathcal{E} + \frac{\partial \mathcal{A}}{\partial t}\right) = 0 \quad .$$

Introduce

$$\mathcal{E} = -\frac{\partial \mathcal{A}}{\partial t} - \nabla \phi \ . \tag{A7}$$

Substitution of Equations A6 and A7 into Equations A3 and A4 gives

$$\mu^{-1} \nabla \times \nabla \times \mathcal{A} = \varepsilon \frac{\partial}{\partial t} \left(-\frac{\partial \mathcal{A}}{\partial t} - \nabla \phi\right) - \sigma \frac{\partial \mathcal{A}}{\partial t} - \sigma \nabla \phi + \mathcal{J} \tag{A8}$$

$$\varepsilon \nabla \cdot \left(-\frac{\partial \mathcal{A}}{\partial t} - \nabla \phi\right) = \rho \quad . \tag{A9}$$

Let

$$\chi = \nabla \cdot \mathscr{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} + \mu\sigma\phi \quad . \tag{A10}$$

Equations A8 and A9 reduce to

$$\nabla^2 \mathscr{A} - \frac{1}{c^2}\frac{\partial^2 \mathscr{A}}{\partial t^2} - \sigma\mu \frac{\partial \mathscr{A}}{\partial t} = -\mu \mathscr{J} + \nabla\chi \tag{A11}$$

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \sigma\mu \frac{\partial \phi}{\partial t} = - \frac{\rho}{\varepsilon} - \frac{\partial \chi}{\partial t} \quad . \tag{A12}$$

Choosing the Lorentz gauge, which has $\chi = 0$, Equation A10 becomes

$$\nabla \cdot \mathscr{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} + \mu\sigma\phi = 0 \tag{A13}$$

and Equation A11 and A12 are the lossy wave equations.

APPENDIX B

DERIVATION OF AN ALTERNATIVE FORMULA FOR NUMERICAL INTEGRATION

The expression for I_2 is the contribution of the integral (Equation 48) along the real axis. This appendix gives an alternaive formula with K_o in the numerator and I_o and K_o in the denominator to avoid the oscillatory behavior of J_o and Y_o via contour integration. This formula has the real arguments in the modified Bessel function. Consider

$$I_2 = \frac{4}{\pi\zeta_o} e^{-\frac{\sigma t}{2\varepsilon}} R_e\left[\int_0^\infty \left[\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2\right]^{-1}\right.$$

$$\left.\left\{J_o\left[a\sqrt{\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2}/c\right] H_o^{(1)}\left[a\sqrt{\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2}/c\right]\right\}^{-1} J_o(\zeta\tau_1)\zeta d\zeta\right] \qquad (B1)$$

$$= \frac{4}{\pi\zeta_o} e^{-\frac{\sigma t}{2\varepsilon}} R_e\left[\int_\alpha^\infty \frac{d\eta}{\eta}\left[J_o(\eta)\, H_o^{(1)}(\eta)\right]^{-1} J_o\left(\tau\sqrt{\eta^2 - \alpha^2}\right)\right] . \qquad (B2)$$

Equation B2 is obtained via the substitution of

$$\eta = \frac{a}{c}\sqrt{\zeta^2 + \left(\frac{\sigma}{2\varepsilon}\right)^2} .$$

Using the identities (Ref. 9)

$$J_o(\eta) = I_o(\eta\, e^{-\frac{1}{2}\pi i})$$
$$H_o^{(1)}(\eta) = -\frac{i2}{\pi} K_o(\eta\, e^{-\frac{1}{2}\pi i}) \qquad (B3)$$

in I_2 gives

$$I_2 = \frac{4}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} R_e\left[\frac{i\pi}{2}\int_\alpha^\infty \frac{d\eta}{\eta}\left\{I_0\left(\eta\, e^{-\frac{\pi i}{2}}\right) K_0\left(\eta\, e^{-\frac{\pi i}{2}}\right)\right\}^{-1} I_0\left(\tau\, e^{-\frac{\pi i}{2}}\sqrt{\eta^2 - \alpha^2}\right)\right] . \qquad \text{(B4)}$$

Again, the identity (Ref. 9)

$$i\pi\, I_0\left(\tau\, e^{-\frac{\pi i}{2}}\sqrt{\eta^2 - \alpha^2}\right) = K_0\left(\tau\, e^{-\frac{\pi i}{2}}\sqrt{\eta^2 - \alpha^2}\right) - K_0\left(\tau\, e^{\frac{\pi i}{2}}\sqrt{\eta^2 - \alpha^2}\right)$$

converts I_2 into

$$I_2 = \frac{2}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} R_e\left[\left[\int_\alpha^\infty \frac{d\eta}{\eta}\left\{I_0\left(\eta\, e^{-\frac{\pi i}{2}}\right) K_0\left(\eta\, e^{-\frac{\pi i}{2}}\right)\right\}^{-1} K_0\left(\tau\, e^{-\frac{\pi i}{2}}\sqrt{\eta^2 - \alpha^2}\right)\right.\right.$$

$$\left. - \int_\alpha^\infty \frac{d\eta}{\eta}\left\{I_0\left(\eta\, e^{-\frac{\pi i}{2}}\right) K_0\left(\eta\, e^{-\frac{\pi i}{2}}\right)\right\}^{-1} K_0\left(\tau\, e^{\frac{\pi i}{2}}\sqrt{\eta^2 - \alpha^2}\right)\right] . \qquad \text{(B5)}$$

Deforming the contour for the first integral in Equation B5 to the solid line in the upper half plane as shown in Figure B1, the first integral can be shown to be

$$\frac{i2}{\pi}\int_{\alpha_0}^{\alpha} \frac{d\eta}{\eta}\left\{J_0(\eta)\, H_0^{(1)}(\eta)\right\}^{-1} K_0\left(\tau\sqrt{\alpha^2 - \eta^2}\right)$$

$$- \frac{2}{\pi}\int_0^{-\frac{\pi}{2}} d\theta \left\{J_0(\alpha_0\, e^{i\theta})\, H_0^{(1)}(\alpha_0\, e^{i\theta})\right\}^{-1} K_0\left(\tau\sqrt{\alpha^2 - \alpha_0^2\, e^{i2\theta}}\right)$$

$$+ \int_{\alpha_0}^\infty \frac{d\eta}{\eta}\left\{J_0(\eta)\, K_0(\eta)\right\}^{-1} K_0\left(\tau\sqrt{\alpha^2 + \eta^2}\right) \qquad \text{(B6)}$$

Notice α_0 cannot be zero, because if it is, the first term in Equation B6 diverges. This is the same integral as the infinite antenna impedance at the gap[2].

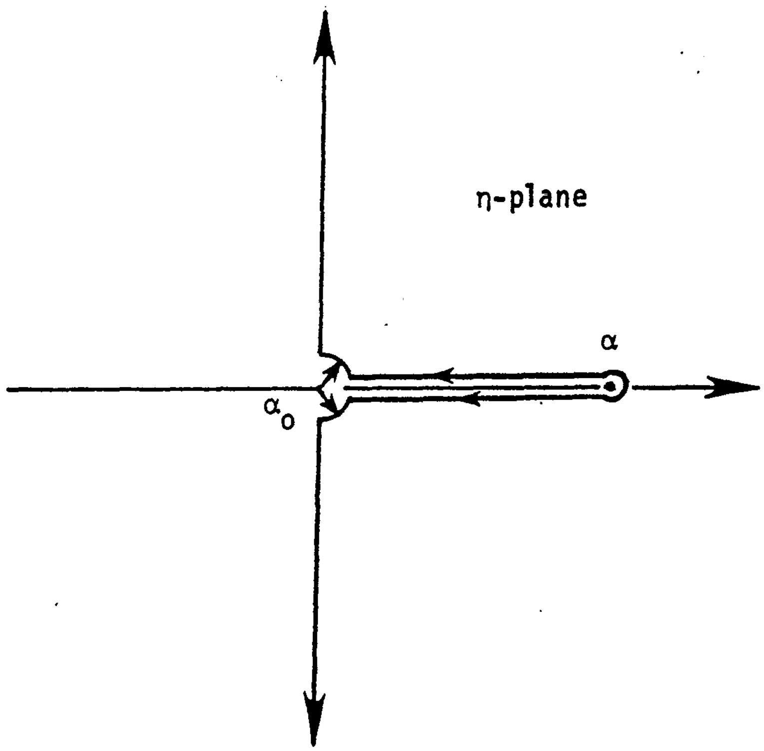


Figure B1. Contour Deformation in the η-Plane.

The second integral in Equation B5 is integrated with the contour shown as solid lines in the lower half plane. It is approximated by

$$
\begin{aligned}
&- \frac{i2}{\pi} \int_{\alpha_0}^{\alpha} \frac{d\eta}{\eta} \left\{ J_0(\eta)\, H_0^{(1)}(\eta) \right\}^{-1} K_0\left(\tau \sqrt{\alpha^2 - \eta^2}\right) \\
&+ \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\theta \left\{ J_0(\alpha_0\, e^{i\theta})\, H_0^{(1)}(\alpha_0\, e^{i\theta}) \right\}^{-1} K_0\left(\tau \sqrt{\alpha^2 - \alpha_0^2\, e^{i2\theta}}\right) \\
&- \int_{\alpha_0}^{\infty} \frac{d\eta}{\eta} \left\{ I_0(\eta)[K_0(\eta) + i\pi\, I_0(\eta)] \right\}^{-1} K_0\left(\tau \sqrt{\alpha^2 + \eta^2}\right) \; . \qquad \text{(B7)}
\end{aligned}
$$

Substitution of Equations B6 and B7 into Equation B5 gives

$$
\begin{aligned}
I_2 = \frac{2}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} &\left\{ 2\, R_e \int_0^{\frac{\pi}{2}} d\theta\; J_0(\alpha_0\, e^{i\theta})\, H_0^{(1)}(\alpha_0\, e^{i\theta}) \right\}^{-1} K_0\left(\tau \sqrt{\alpha^2 - \alpha_0^2\, e^{i2\theta}}\right) \\
&+ \pi^2 \int_{\alpha_0}^{\infty} \frac{d\eta}{\eta} \frac{I_0(\eta)}{K_0(\eta)} \frac{K_0\left(\tau \sqrt{\alpha^2 + \eta^2}\right)}{\left\{ [K_0(\eta)]^2 + \pi^2 [I_0(\eta)]^2 \right\}} \; . \qquad \text{(B8)}
\end{aligned}
$$

For $\alpha_0 << \alpha$, I_2 can be approximated as

$$
\begin{aligned}
I_2 \approx \frac{2}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} &\left\{ \ell n \frac{\sqrt{\pi^2 + \left[\ell n\left(\frac{\alpha_0}{2}\right) + \gamma\right]^2}}{-\left[\ell n\left(\frac{\alpha_0}{2}\right) + \gamma\right]} K_0\left(\tau \sqrt{\alpha^2 - \alpha_0^2}\right) \right. \\
&\left. + \pi^2 \int_{\alpha_0}^{\infty} \frac{d\eta}{\eta} \frac{I_0(\eta)}{K_0(\eta)} \frac{K_0\left(\tau \sqrt{\alpha^2 + \eta^2}\right)}{\left\{ [K_0(\eta)]^2 + \pi^2 [I_0(\eta)]^2 \right\}} \right\} \qquad \text{(B9)}
\end{aligned}
$$

For $\alpha_o = 10^{-5}$ and $\alpha_o < \alpha/100$, the approximate I_2 can be shown to be accurate to at least 10^{-3}.

Table B1 provides a comparison of I_2 calculated by Equations B2 and B9 versus logarithmically spaced values of τ. Both Equations B2 (or 48) and B9 were numerically integrated for this comparison. Each integral was evaluated piecewise over adjoining finite intervals. An adaptive order 40 Gaussian quadrature was used to integrate each interval with a required error convergence criterion of 10^{-8}. The oscillatory behavior of the Bessel function in Equation B2, however, limited the convergence of larger piecewise limits in the integration. Although the convergence of Equation B2 is also a function of τ, at least three decimal place accuracy was achieved for τ values in Table B1. Equation B9 was computed to within a $\pm 10^{-6}$ accuracy for the same τ values.

Table B1. NUMERICAL COMPARISON OF I_2 GIVEN BY EQUATIONS B2 AND B9 FOR $\alpha = 5 \times 10^{-3}$; VALUES GIVEN IN MILLIAMPERES. ENTRIES SHOWN ARE $I_2 \, e^{\frac{\sigma t}{2\varepsilon} - \alpha\tau}$

τ Logarithmically Spaced	I_2 (per Eq. 48 or B2)	I_2 (per Eq. B9)
1.0000	7.4323	7.45992
1.2956	6.1396	6.14856
1.6785	5.1037	5.10201
2.1746	4.2524	4.25891
2.8173	3.5771	3.57232
3.6494	3.0092	3.00711
4.7287	2.5395	2.53670
6.1263	2.1408	2.14103
7.9370	1.8043	1.80488
10.2829	1.5159	1.51669
13.3220	1.2696	1.26761
17.2595	1.0493	1.05091
22.3607	0.86041	0.861476
28.9696	0.69541	0.695537
37.5318	0.55015	0.550412
48.6246	0.42473	0.424314
62.9961	0.31644	0.316996
81.6151	0.22525	0.225534
105.7371	0.15193	0.152032
136.9887	0.095352	0.0952896
177.4768	0.054363	0.0543744
229.9316	0.027515	0.0274889
297.8899	0.011876	0.0118855
385.9339	0.004201	0.0041990
500.0000	0.001143	0.0011425

APPENDIX C

EVALUATION OF DEFINITE INTEGRALS

This appendix evaluates two definite integrals as follows:

$$\int_0^\alpha \frac{\eta I_1\left(\tau\sqrt{\alpha^2 - \eta^2}\right)}{\sqrt{\alpha^2 - \eta^2}} \ln \eta \, d\eta \tag{C1}$$

and

$$\int_\alpha^\infty \frac{\eta I_1\left(\tau\sqrt{\alpha^2 - \eta^2}\right)}{\sqrt{\alpha^2 - \eta^2}} \ln \eta \, d\eta \ . \tag{C2}$$

They are used in Appendix D.

The first integral can be evaluated term by term when I_1 is expanded in a series form to give

$$\sum_{m=0}^{\infty} \frac{\left(\frac{\tau}{2}\right)^{1+2m}}{m!(m+1)!} \int_0^\alpha \eta(\alpha^2 - \eta^2)^m \ln \eta \, d\eta$$

$$= \frac{1}{4} \sum_{m=0}^{\infty} \frac{\left(\frac{\tau}{2}\right)^{1+2m}}{m!(m+1)!} \int_0^{\alpha^2} y^m \ln(\alpha^2 - y) \, dy$$

$$= \frac{1}{4} \sum_{m=0}^{\infty} \frac{\left(\frac{\tau}{2}\right)^{1+2m}}{[(m+1)!]^2} \left[\ln(\alpha^2) - \sum_{k=1}^{m+1} \frac{1}{m - k + 2}\right] \alpha^{2m+2} \tag{C3}$$

$$= \frac{1}{2\tau} \left\{[I_0(\alpha\tau) - 1][\ln(\alpha^2)] - K_0(\alpha\tau) - I_0(\alpha\tau)\left[\ln\left(\frac{\alpha\tau}{2}\right) + \gamma\right]\right\}$$

$$= \frac{I_0(\alpha\tau)}{\tau} \left\{\frac{1}{2}\left[\ln\left(\frac{\alpha}{\tau}\right) - \ln 2 - \gamma - \frac{K_0(\alpha\tau)}{I_0(\alpha\tau)}\right]\right\} - \frac{\ln \alpha}{\tau} \ . \tag{C4}$$

The step from Equation C3 to C4 is the identification of modified Bessel functions.

The second integral of interest (Equation C2) can be deduced as follows:

$$\int_\alpha^\infty \frac{\eta J_1\left(\tau\sqrt{\eta^2 - \alpha^2}\right)}{\sqrt{\eta^2 - \alpha^2}} \ln \eta \, d\eta = \frac{1}{2}\int_0^\infty J_1(\tau\eta_1) \ln(\eta_1^2 + \alpha^2) \, d\eta_1$$

$$= \frac{\ln \alpha}{\tau} + \frac{1}{\tau}\int_0^\infty \frac{J_0(\tau\eta_1)\eta_1 d\eta_1}{\eta_1^2 + \alpha^2} \tag{C5}$$

$$= \frac{\ln \alpha}{\tau} + \frac{K_0(\alpha\tau)}{\tau} \; . \tag{C6}$$

Integration by parts is used in obtaining Equation C5, and Equation C6 follows from an identity in Reference 9.

Finally, combining Equations C4 and C6 gives

$$\int_0^\infty \frac{\eta J_1\left(\tau\sqrt{\eta^2 - \alpha^2}\right)}{\sqrt{\eta^2 - \alpha^2}} \ln \eta \, d\eta = \frac{I_0(\alpha\tau)}{\tau}\left\{\frac{1}{2}\left[\ln\left(\frac{\alpha}{\tau}\right) + \ln 2 - \gamma + \frac{K_0(\alpha\tau)}{I_0(\alpha\tau)}\right]\right\} \; . \tag{C7}$$

APPENDIX D

ASYMPTOTIC EVALUATION OF I_1, I_2 AND I

Consider the definite integrals (Equations C7, C4, and C6) given in Appendix C. It is possible to write them as

$$\int_0^\infty \frac{\eta J_1\left(\tau\sqrt{\eta^2 - \alpha^2}\right)}{\sqrt{\eta^2 - \alpha^2}} \ln \eta \, d\eta = \langle \ln \eta \rangle_{\eta=0} \int_0^\infty \frac{\eta J_1\left(\tau\sqrt{\eta^2 - \alpha^2}\right)}{\sqrt{\eta^2 - \alpha^2}} d\eta$$

$$= \langle \ln \eta \rangle_{\eta=0} \frac{I_0(\alpha\tau)}{\tau} \tag{D1}$$

$$\langle \ln \eta \rangle_{\eta=0} = \frac{1}{2}\left[\ln\left(\frac{\alpha}{\tau}\right) + \ln 2 - \gamma + \frac{K_0(\alpha\tau)}{I_0(\alpha\tau)}\right] . \tag{D2}$$

$$\int_\alpha^\infty \frac{\eta J_1\left(\tau\sqrt{\eta^2 - \alpha^2}\right)}{\sqrt{\eta^2 - \alpha^2}} \ln \eta \, d\eta = \langle \ln \eta \rangle_{\eta=\alpha} \int_\alpha^\infty \frac{\eta J_1\left(\tau\sqrt{\eta^2 - \alpha^2}\right)}{\sqrt{\eta^2 - \alpha^2}} d\eta$$

$$= \langle \ln \eta \rangle_{\eta=\alpha} \frac{1}{\tau} \tag{D3}$$

$$\langle \ln \eta \rangle_{\eta=\alpha} = \ln \alpha + K_0(\alpha\tau) . \tag{D4}$$

Finally,

$$\int_0^\alpha \frac{\eta I_1\left(\tau\sqrt{\alpha^2 - \eta^2}\right)}{\sqrt{\alpha^2 - \eta^2}} \ln \eta \, d\eta = \langle \ln \eta \rangle\Big|_{\eta=0} \frac{I_0(\alpha\tau)}{\tau} - \langle \ln \eta \rangle\Big|_{\eta=\alpha} \frac{1}{\tau} . \tag{D5}$$

I_1, I_2, and I are now evaluated as follows:

$$I_1 = \frac{4}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} \int_0^{\alpha} I_0\left(\tau\sqrt{\alpha^2 - \eta^2}\right)\left\{[J_0(\eta)]^2 + [Y_0(\eta)]^2\right\}^{-1} \frac{d\eta}{\eta}$$

$$\sim \frac{4}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} \int_0^{\alpha} I_0\left(\tau\sqrt{\alpha^2 - \eta^2}\right) \frac{1}{1 + \left(\frac{2}{\pi}\right)^2 \left(- \ell n \frac{\eta}{2} + \gamma\right)^2} \frac{d\eta}{\eta} \qquad \text{(D6)}$$

$$= \frac{2}{\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} \left\{\arctan\left[\frac{-\pi/2}{\ell n\left(\frac{\alpha}{2}\right) + \gamma}\right] + \tau \int_0^{\alpha} \arctan\left[\frac{-\pi/2}{\ell n\left(\frac{\eta}{2}\right) + \gamma}\right] \frac{I_1\left(\tau\sqrt{\alpha^2 - \eta^2}\right)}{\sqrt{\alpha^2 - \eta^2}} \eta \, d\eta\right\} \qquad \text{(D7)}$$

A small argument approximation for J_o and Y_o is used in obtaining Equation D6 and integration by parts is used in deriving Equation D7.

Similarly, I_2 can be shown to be

$$I_2 = \frac{4}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} \int_{\alpha}^{\infty} J_0\left(\tau\sqrt{\eta^2 - \alpha^2}\right)\left\{[J_0(\eta)]^2 + [Y_0(\eta)]^2\right\}^{-1} \frac{d\eta}{\eta}$$

$$\sim \frac{2}{\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} \left\{\arctan\left[\frac{\pi/2}{\ell n\left(\frac{\alpha}{2}\right) + \gamma}\right] + \tau \int_{\alpha}^{\infty} \arctan\left[\frac{-\pi/2}{\ell n\left(\frac{\eta}{2}\right) + \gamma}\right] \frac{J_1\left(\tau\sqrt{\eta^2 - \alpha^2}\right)}{\sqrt{\eta^2 - \alpha^2}} \eta \, d\eta\right\} . \qquad \text{(D8)}$$

Finally, adding I_1 and I_2 gives

$$I = \frac{4}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} \int_0^{\infty} J_0\left(\tau\sqrt{\eta^2 - \alpha^2}\right)\left\{[J_0(\eta)]^2 + [Y_0(\eta)]^2\right\}^{-1} \frac{d\eta}{\eta}$$

$$\sim \frac{2}{\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} \tau \int_0^{\infty} \arctan\left[\frac{-\pi/2}{\ell n\left(\frac{\eta}{2}\right) + \gamma}\right] \frac{J_1\left(\tau\sqrt{\eta^2 - \alpha^2}\right)}{\sqrt{\eta^2 - \alpha^2}} \eta \, d\eta \qquad \text{(D9)}$$

Write I_1, I_2, and I in the same manner as was done for Equations D7, D8, and D9.

$$I \sim \frac{2}{\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} \tau \arctan\left[\frac{-\pi/2}{<\ell n\,\eta>_{\eta=0} - \ell n\, 2 + \gamma}\right] \int_0^\infty \frac{J_1\left(\tau\sqrt{\eta^2 - \alpha^2}\right)}{\sqrt{\eta^2 - \alpha^2}}\, \eta\, d\eta$$

$$= \frac{2}{\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} I_0(\alpha\tau) \arctan\left[\frac{-\pi}{\ell n\left(\frac{\alpha}{\tau}\right) + \frac{K_0(\alpha\tau)}{I_0(\alpha\tau)} - \ell n\, 2 + \gamma}\right] \qquad \text{(D10)}$$

where Equation D2 has been used for $<\ell n\ \eta>_{\eta=0}$.

Similarly

$$I_2 \sim \frac{2}{\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}} \left[\arctan\left[\frac{\pi}{2(\ell n\ \alpha - \ell n\, 2 + \gamma)}\right] + \arctan\left\{\frac{-\pi}{2[\ell n\ \alpha + K_0(\alpha\tau) - \ell n\, 2 + \gamma]}\right\}\right] \qquad \text{(D11)}$$

$$I_1 \sim I - I_2 \qquad \text{(D12)}$$

A numerical comparison of I_2 given by Equation B9 and the approximate formula D11 and I_2, and a numerical comparison of I_1 given by Equation 47 and the approximate formula D12, are given in Table D1 for $\alpha = 10^{-2}$, in Table D2 for $\alpha = 10^{-3}$, in Table D3 for $\alpha = 10^{-4}$, and in Table D4 for $\alpha = 10^{-5}$. All entries shown in Tables D1 through D4 are $(I_{1,2})\left(e^{\frac{\sigma t}{2\varepsilon} - \alpha\tau}\right)$.

Table D1. NUMERICAL COMPARISON FOR I_1 AND I_2 WITH THE APPROXIMATE FORMULAS VERSUS τ FOR $\alpha = 10^{-2}$

τ	I_1		I_2	
	Eq. 47	Eq. D12	Eq. B9	Eq. D11
1.00	1.48418	1.68825	7.21887	9.94406
1.25	1.48050	1.68406	6.06987	5.81217
1.50	1.47683	1.67988	5.28421	5.21478
1.75	1.47317	1.67571	4.70857	4.73442
2.00	1.46953	1.67157	4.26584	4.34126
2.25	1.46589	1.66743	3.91290	4.01406
2.50	1.46228	1.66331	3.62368	3.73759
2.75	1.45867	1.65921	3.38145	3.50084
3.00	1.45508	1.65512	3.17496	3.29567
3.50	1.44794	1.64699	2.84000	2.95719
4.00	1.44085	1.63891	2.57818	2.68863
5.00	1.42683	1.62293	2.19090	2.28676
7.50	1.39267	1.58398	1.61645	1.68549
10.00	1.35974	1.54639	1.28768	1.34115
12.00	1.33425	1.51727	1.10648	1.15185
15.00	1.29739	1.47511	0.90936	0.94652
20.00	1.23940	1.40870	0.69050	0.71938
30.00	1.13519	1.28899	0.44135	0.46175
50.00	0.96585	1.09340	0.21604	0.22888
75.00	0.80924	0.91114	0.10182	0.10987
100.00	0.69529	0.77756	0.05156	0.05670
150.00	0.54624	0.60159	0.01467	0.01675
250.00	0.39782	0.42592	0.00143	0.00174
500.00	0.26483	0.27258	0.00001	0.00001
1000.00	0.17847	0.17906	0.00000	0.00000

Table D2. NUMERICAL COMPARISON FOR I_1 AND I_2 WITH THE APPROXIMATE FORMULAS VERSUS τ FOR $\alpha = 10^{-3}$

τ	I_1		I_2	
	Eq. 47	Eq. D12	Eq. B9	Eq. D11
1.00	1.16690	1.16689	7.82042	7.16366
1.25	1.16661	1.16660	6.67460	6.41367
1.50	1.16632	1.16631	5.89185	5.82103
1.75	1.16602	1.16602	5.31890	5.34459
2.00	1.16573	1.16573	4.87866	4.95476
2.25	1.16544	1.16544	4.52803	4.63045
2.50	1.16515	1.16515	4.24098	4.35656
2.75	1.16486	1.16486	4.00077	4.12212
3.00	1.16457	1.16456	3.79619	3.91906
3.50	1.16399	1.16398	3.46469	3.58429
4.00	1.16341	1.16340	3.20595	3.31892
5.00	1.16225	1.16224	2.82389	2.92230
7.50	1.15935	1.15935	2.25853	2.32978
10.00	1.15647	1.15647	1.93501	1.99027
12.00	1.15417	1.15417	1.75611	1.80297
15.00	1.15074	1.15073	1.56014	1.59841
20.00	1.14504	1.14504	1.33890	1.36843
30.00	1.13378	1.13378	1.07533	1.09603
50.00	1.11175	1.11174	0.80466	0.81821
75.00	1.08510	1.08508	0.62687	0.63682
100.00	1.05939	1.05937	0.51673	0.52485
150.00	1.01068	1.01063	0.38089	0.38718
250.00	0.92309	0.92297	0.23929	0.24405
500.00	0.75068	0.75033	0.09997	0.10316
1000.00	0.53685	0.53603	0.02493	0.02644

Table D3. NUMERICAL COMPARISON FOR I_1 AND I_2 WITH THE APPROXIMATE FORMULAS VERSUS τ FOR $\alpha = 10^{-4}$

τ	I_1		I_2	
	Eq. 47	Eq. D12	Eq. B9	Eq. D11
1.00	0.88575	0.88575	8.10965	7.45230
1.25	0.88573	0.88573	6.96429	6.70307
1.50	0.88571	0.88570	6.18198	6.11107
1.75	0.88569	0.88568	5.60944	5.63518
2.00	0.88566	0.88566	5.16960	5.24584
2.25	0.88564	0.88564	4.81936	4.92198
2.50	0.88562	0.88562	4.53267	4.64850
2.75	0.88560	0.88559	4.29281	4.41445
3.00	0.88558	0.88557	4.08857	4.21176
3.50	0.88553	0.88553	3.75772	3.87768
4.00	0.88549	0.88548	3.49960	3.61295
5.00	0.88540	0.88539	3.11868	3.21750
7.50	0.88518	0.88517	2.55579	2.62747
10.00	0.88496	0.88495	2.23437	2.29005
12.00	0.88478	0.88477	2.05697	2.10424
15.00	0.88451	0.88451	1.86302	1.90168
20.00	0.88407	0.88407	1.64468	1.67458
30.00	0.88319	0.88319	1.38579	1.40679
50.00	0.88143	0.88142	1.21276	1.13551
75.00	0.87923	0.87923	0.94939	0.95939
100.00	0.87705	0.87704	0.84270	0.85075
150.00	0.87270	0.87270	0.71014	0.71617
250.00	0.86410	0.86409	0.56702	0.57131
500.00	0.84314	0.84313	0.40525	0.40808
1000.00	0.80345	0.80344	0.27178	0.27380

Table D4. NUMERICAL COMPARISON FOR I_1 AND I_2 WITH THE APPROXIMATE FORMULAS VERSUS τ FOR $\alpha = 10^{-5}$

τ	I_1		I_2	
	Eq. 47	Eq. D12	Eq. B9	Eq. D11
1.00	0.71279	0.71278	8.28342	7.62602
1.25	0.71278	0.71278	7.13811	6.87687
1.50	0.71278	0.71278	6.35586	6.28494
1.75	0.71278	0.71278	5.78337	5.80911
2.00	0.71278	0.71278	5.34358	5.41982
2.25	0.71278	0.71277	4.99338	5.09602
2.50	0.71278	0.71277	4.70673	4.82259
2.75	0.71277	0.71277	4.46692	4.58859
3.00	0.71277	0.71277	4.26271	4.38594
3.50	0.71277	0.71276	3.93195	4.05194
4.00	0.71277	0.71276	3.67390	3.78729
5.00	0.71276	0.71275	3.29312	3.39198
7.50	0.71274	0.71274	2.73055	2.80228
10.00	0.71272	0.71272	2.40941	2.46514
12.00	0.71271	0.71270	2.23221	2.27954
15.00	0.71269	0.71268	2.03856	2.07727
20.00	0.71265	0.71265	1.82066	1.85061
30.00	0.71258	0.71258	1.56252	1.58358
50.00	0.71244	0.71243	1.29977	1.31357
75.00	0.71226	0.71225	1.12871	1.13877
100.00	0.71208	0.71208	1.02316	1.03127
150.00	0.71173	0.71172	0.89253	0.89861
250.00	0.71102	0.71101	0.75242	0.75674
500.00	0.70924	0.70924	0.59572	0.59863
1000.00	0.70572	0.70571	0.46763	0.46953

APPENDIX E

ASYMPTOTIC FORMULA FOR LARGE τ

For large τ, I_1 dominates the transient currents; therefore, we will derive a very late time approximation based on I.

Two approximations are used to simplify the calculation. The first approximation is

$$I_0(\eta\tau) \sim \frac{e^{\eta\tau}}{(2\pi\eta\tau)^{\frac{1}{2}}} \quad . \tag{E1}$$

The second approximation is, as $\eta \to 0$

$$J_0(\eta) \sim 1 \quad , \quad Y_0(\eta) \sim \frac{-2}{\pi}\left(\ell n \frac{\eta}{2} + \gamma\right) \quad . \tag{E2}$$

Equation E6 can be used to approximate $I_0(\eta\tau)$ in Equation E1 when $\alpha\tau$ is large; this is the basic condition for the resulting asymptotic formula to be valid.

Using these two approximations and letting $\eta = \sqrt{(2\alpha-u)u}$ I_1 given in Equation E1 becomes:

$$I_1 \sim \frac{4}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}+\alpha\tau} \int_0^{\alpha} \frac{e^{-u\tau}(\alpha - u)/(2\alpha - u)\,\frac{du}{u}}{[2\pi\tau(\alpha - u)]^{\frac{1}{2}}\left\{1 + \left(\frac{2}{\pi}\right)^2\left[\ell n\, \frac{1}{2}\sqrt{u(2\alpha - u)} + \gamma\right]^2\right\}} \quad . \tag{E3}$$

Since the contribution to Equation E3 comes mostly from $u \sim 0$, we approximate $a - u \sim \alpha$, $2\alpha - u \sim 2\alpha$ to give

$$I_1 \sim \frac{2}{\pi\zeta_0} e^{-\frac{\sigma t}{2\varepsilon}+\alpha\tau} \int_0^{\infty} \frac{e^{-u\tau}\,\frac{du}{u}}{(2\pi\alpha\tau)^{\frac{1}{2}}\left[1 + \left(\frac{2}{\pi}\right)^2\left\{\frac{1}{2}\,\ell n\, \alpha - \frac{1}{2}\,\ell n\, 2 + \frac{1}{2}\,\ell n\, u + \gamma\right\}^2\right]} \quad . \tag{E4}$$

Integration by parts converts Equation E4 into

$$I_1 \sim \frac{2}{\zeta_0 (2\pi\alpha\tau)^{\frac{1}{2}}} e^{-\frac{\sigma t}{2\varepsilon} + \frac{\sigma a}{2\varepsilon c}\tau} \int_0^\infty du\ e^{-u\tau} \arctan\left[\frac{-\pi}{\ell n\ \alpha - \ell n\, 2 + \ell n\ u + 2\gamma}\right]. \qquad \text{(E5)}$$

Invoking the averaging formula

$$\int_0^\infty du\ e^{-u\tau}\ \ell n\ u = \frac{1}{\tau}(-\ell n\ \tau - \gamma) \qquad \text{(E6)}$$

and the fact that arctan $\left[\frac{-\pi}{\ell n\ \alpha - \ell n\, 2 + \ell n\ u + 2\gamma}\right]$ as a slowly varying function of $\ell n\ u$ provides

$$I(z,t) \sim I_1 \sim \frac{2}{\zeta_0 (2\pi\alpha\tau)^{\frac{1}{2}}} e^{-\frac{\sigma t}{2\varepsilon} + \alpha\tau} \arctan\left[\frac{\pi}{\ell n\left(\frac{\tau}{\alpha}\right) + \ell n\, 2 - \gamma}\right]. \qquad \text{(E7)}$$

APPENDIX F

RIGOROUS TRANSMISSION LINE THEORY

Consider the diffusion limit of Equation 6 and a rigorous transmission line solution. We expand Equation 6 for large $\alpha\tau$ to give Equation E7 and impose the diffusion limit of $t >> z/c$. Then

$$\frac{\sigma}{2\varepsilon} t - \alpha\tau = \frac{\sigma}{2\varepsilon}\left(t - \sqrt{t^2 - z^2/c^2}\right) \sim \frac{\sigma\mu}{4t} z^2 = \frac{z^2}{2\delta^2}$$

$$\delta = \sqrt{\frac{2t}{\sigma\mu}}$$

and

$$\ell n \frac{\tau}{\alpha} \sim 2 \ \ell n \frac{\delta}{a} >> 1 \quad .$$

As a result,

$$I(z,t) \sim 2\varepsilon \left(\frac{\pi}{\mu\sigma t}\right)^{\frac{1}{2}} \frac{e^{-z^2/2\delta^2}}{2 \ \ell n\left(\frac{\delta}{a}\right) + \ell n 2 - \gamma} \quad . \tag{F1}$$

Proceeding with the simplified Maxwell's equations, we have

$$\mu\left(\sigma E_\rho + \varepsilon \frac{\partial E_\rho}{\partial t}\right) = \frac{\partial B_\theta}{\partial z} \tag{F2}$$

$$\mu\left(\sigma E_z + \varepsilon \frac{\partial E_z}{\partial t}\right) = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\theta) \tag{F3}$$

$$\frac{\partial B_\theta}{\partial t} = \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} \quad . \tag{F4}$$

Near the wire $E_z \sim 0$, Equation F3 is henceforth omitted from the discussion. Furthermore, B_θ can be given in the diffusion limit as

$$B_\theta(\rho,z,t) = \frac{\mu \mathscr{I}(z,t)}{2\pi\rho} e^{-\rho^2/2\delta^2} . \quad \text{(F5)}$$

Substituting Equations F5 and 32 with $V \equiv 1$ into Equation F2 gives

$$E_\rho(\rho,z,t) = \frac{\frac{\partial I}{\partial z}}{2\pi\varepsilon\rho} e^{-\rho^2/2\delta^2} . \quad \text{(F6)}$$

The line voltage can be defined as

$$\mathscr{V}(z,t) = -\int_a^\infty E_\rho d\rho = -\int_a^\infty \frac{\frac{\partial I}{\partial z}(z,t)}{2\pi\varepsilon\rho} e^{-\rho^2/2\delta^2} d\rho$$

$$= \frac{-1}{2\pi\varepsilon} \frac{\partial I(z,t)}{\partial z} E_1\left(\frac{a^2}{2\delta^2}\right) \sim \frac{1}{2\pi\varepsilon} \frac{\partial I(z,t)}{\partial z} \left[2 \ln \frac{\delta}{a} + \ln 2 - \gamma\right] \quad \text{(F7)}$$

where $E_1(a^2/2\delta^2)$ is the exponential integral.

Equation F7 can be rewritten as

$$-\frac{\partial \mathscr{I}}{\partial z}(z,t) = \left(\frac{\partial}{\partial t} + \frac{\sigma}{\varepsilon}\right) \frac{2\pi\varepsilon}{\left(2 \ln \frac{\delta}{a} + \ln 2 - \gamma\right)} \mathscr{V}(z,t) . \quad \text{(F8)}$$

Let

$$C = \frac{2\pi\varepsilon}{2 \ln \frac{\delta}{a} + \ln 2 - \gamma} \quad \text{and} \quad G = \frac{2\pi\sigma}{2 \ln \frac{\delta}{a} + \ln 2 - \gamma} . \quad \text{(F9)}$$

The first of the transmission line equations is obtained as

$$\left(C \frac{\partial}{\partial t} + G\right)\mathscr{V} = -\frac{\partial \mathscr{I}}{\partial z} . \quad \text{(F10)}$$

The second equation can easily be shown from Equation 4 to be

$$L \frac{\partial \mathscr{I}}{\partial t} = - \frac{\partial \mathscr{V}}{\partial z} \tag{F11}$$

with

$$L = \frac{2 \ln \frac{\delta}{a} + \ln 2 - \gamma}{2\pi\mu} \ . \tag{F12}$$

Explicit expressions for $\mathscr{V}(z,t)$ and $\mathscr{I}(z,t)$ for unit impulse input voltage at the gap are

$$\mathscr{V}(z,t) \sim \frac{1}{\sqrt{2\pi}} \left(\frac{z}{\delta t}\right) e^{-z^2/2\delta^2} \tag{F13}$$

and

$$\mathscr{I}(z,t) = \frac{1}{\sqrt{2\pi}} \left[\frac{G\delta}{t} + C\left(- \frac{\delta}{2t^2} + \frac{z^2}{2\delta t^2}\right)\right] e^{-z^2/2\delta^2} \ . \tag{F14}$$