INTERACTION NOTES

NOTE 466

JUNE 1988

A Bound on EMP Coupling

Larry K. Warne and Kenneth C. Chen

Sandia National Laboratories, Albuquerque, NM 87185

ABSTRACT

A bound on the energy coupled to a load by means of an antenna or aperture subjected to a step function EMP is given by the polarizabilities of the antenna or aperture. Examples of coupling bounds for a slender electric dipole antenna, a circular aperture, and a narrow slot aperture having depth are given.

ACKNOWLEDGMENT

The authors wish to thank T. E. Koontz, Sandia National Laboratories, who prepared the numerical figures. The authors also wish to thank Dr. K. S. H. Lee, Dikewood Division, Kaman Sciences Corp., for helpful discussions regarding coupling bounds.

I. INTRODUCTION

A recent paper [1] expounded the fact that the total cross section of an antenna integrated over all wavelengths is given by its polarizabilities. The proof relies only on the optical theorem [2], the fact that the scattering process must satisfy the causality principle [3], and the fact that the low frequency limit of the far zone scattered field can be expressed in terms of the antenna polarizabilities.

This short note points out that this result can be used to bound the energy coupled to a load by means of an antenna or aperture subjected to a step function EMP. Examples involving a slender electric dipole antenna, a circular aperture, and a narrow slot aperture having depth are given.

II. STEP FUNCTION EMP BOUND

The total cross section of an object σ_t integrated over all wavelengths is given [1] by

$$\int_0^\infty \sigma_t \, d\lambda = \pi^2 V(P_{11} + M_{22}) , \tag{1}$$

where V is the object volume. The polarizability tensors, **P** and **M**, are defined by the static dipole moments of the object

$$p = \epsilon_o VP . E_o \tag{2}$$

and

$$\mathbf{m} = \mathbf{VM} \cdot \mathbf{H}_o , \tag{3}$$

where the incident fields are taken as

$$\mathbf{E}_o = E_o \, \mathbf{e}_1 \tag{4}$$

and

$$\mathbf{H}_o = E_o \, \mathbf{e}_2 / \eta_o , \tag{5}$$

where the direction of propagation is coincident with the unit vector $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$ and $\eta_o = \sqrt{\mu_o / \epsilon_o}$ is the free space impedance.

The integral of the antenna's absorption cross section or effective area A_e can thus be bounded [1] by

$$\int_0^\infty A_e \, d\lambda \leq \pi^2 \, V(P_{11} + M_{22}) . \tag{6}$$

Consider the usual double exponential EMP

$$\mathbf{E}_o(t) = E_o \, (e^{-\alpha t} - e^{-\beta t}) \, u(t) \, \mathbf{e}_1 , \tag{7}$$

where α and β are real constants controlling the fall and rise times and u(t) is the unit step function

$$u(t) = 0, \; t < 0 , \tag{8}$$

$$= 1, t \geq 0 . \tag{9}$$

Taking the Fourier transform of (7) by means of

$$E(\omega) = \int_{-\infty}^{\infty} E(t)\, e^{i\omega t}\, dt , \tag{10}$$

gives

$$\mathbf{E_o}(\omega) = \frac{(\beta - \alpha)}{(\alpha - i\omega)(\beta - i\omega)} E_o\, \mathbf{e_1} . \tag{11}$$

Now letting the rise time approach zero ($\beta \to \infty$) and the fall time approach infinity ($\alpha \to 0$) equation (7) becomes the step function

$$\mathbf{E_o}(t) = E_o\, u(t)\, \mathbf{e_1} , \tag{12}$$

with transform

$$\mathbf{E_o}(\omega) = \frac{i}{\omega} E_o\, \mathbf{e_1} . \tag{13}$$

Notice that the magnitude of (13) is larger than that of (11) for all ω.

Using the energy theorem [4] (Rayleigh's theorem)

$$\int_{-\infty}^{\infty} E^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |E(\omega)|^2 d\omega \tag{14}$$

and the fact that the effective area (in the frequency domain) is the area that intercepts the same incident power as that absorbed (or received) by the actual load, the energy received W_{rec} can be written as

$$W_{rec} = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_e(\omega)\,\frac{1}{\eta_o}\,|E(\omega)|^2\,d\omega\ . \tag{15}$$

For the step function (13), this becomes

$$W_{rec} = \frac{E_o^2}{\pi\eta_o}\int_0^{\infty} A_e\,\frac{d\omega}{\omega^2} = \frac{1}{2}\,\epsilon_o E_o^2\,\frac{1}{\pi^2}\int_0^{\infty} A_e\,d\lambda\ . \tag{16}$$

Therefore, from (6), we obtain the desired result

$$W_{rec} \leq \frac{1}{2}\,\epsilon_o E_o^2\,V(P_{11} + M_{22})\ . \tag{17}$$

Notice that the right side of (17) is just the static energy required to polarize the object, or equivalently, the energy stored in the static fields of the polarized object.

The result (17) can of course be averaged over all polarization angles, however, for our purposes it is more desirable to adjust the incident polarization so that the right hand side of (17) is maximized.

III. EXAMPLES

In the following examples, the amplitude E_o of the EMP is taken as the typical

$$E_o = 50 \text{ kV/m} . \tag{18}$$

1. Slender Electric Dipole

Consider a center loaded electric dipole of length $\ell = 2h$ coincident with unit vector $\mathbf{e}_1$ and having radius a (h >> a). The electric dipole moment dominates and is maximized when the load represents a short circuit at zero frequency. The first order Hallén solution for the dipole moment gives [5]

$$VP_{11} \sim \frac{\ell^3 \pi}{6(\frac{\Omega}{2} + \ell n\ 2 - \frac{7}{3})} , \tag{19}$$

where the antenna fatness parameter is $\Omega = 2\ \ell n(2h/a)$. The energy bound (17) in this case gives

$$W_{rec} \leq \frac{1}{2} \epsilon_o E_o^2\ VP_{11} \cdot \tag{20}$$

Figure 1 shows a plot of the right hand side of (20) as a function of ℓ for $\Omega = 6$ and 10.

2. Apertures

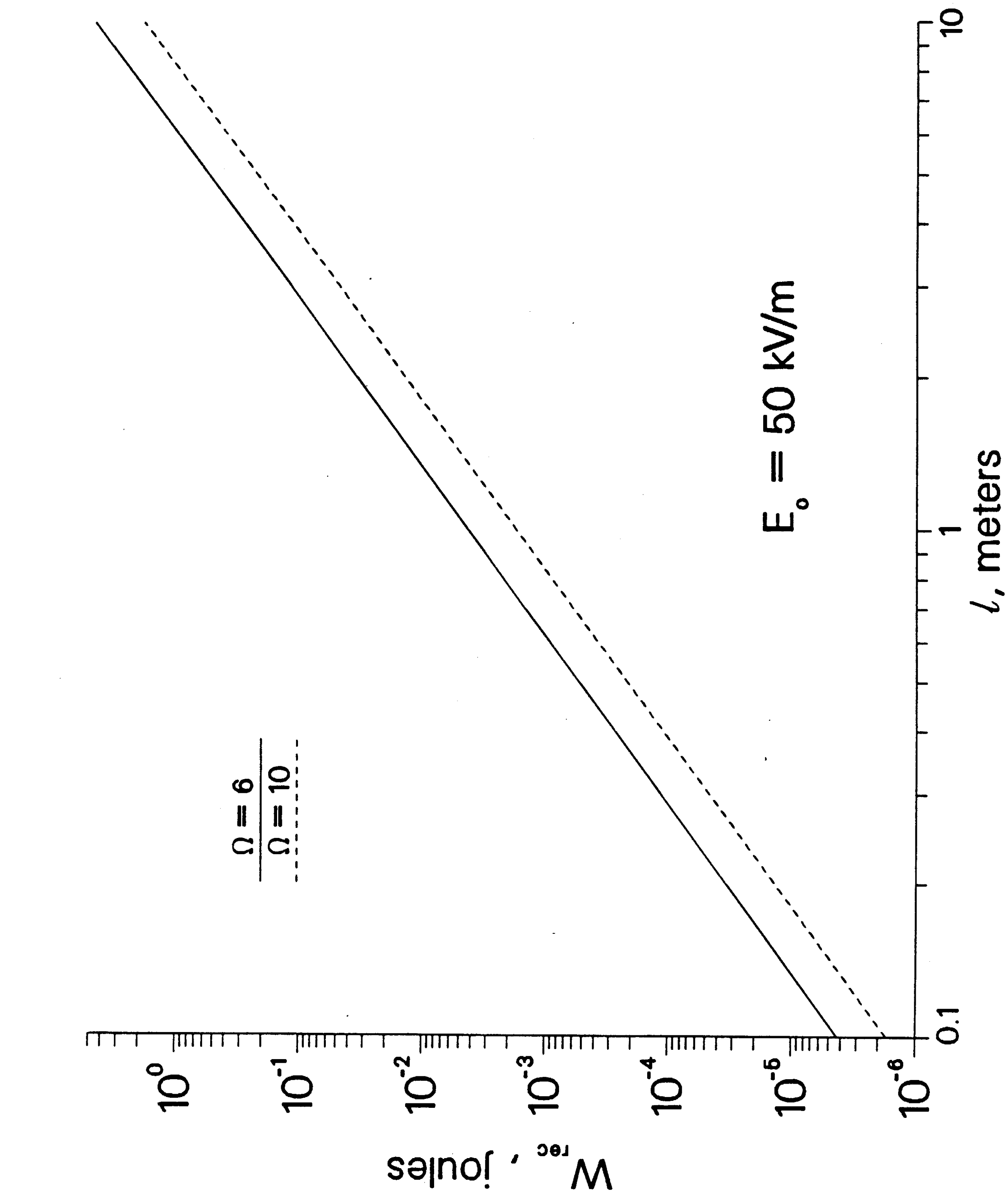

Figure 1. Maximum received energy for a center loaded slender electric dipole of length $\ell = 2h$ and radius a ($\Omega = 2\ \ell n(2h/a)$), subjected to an incident step function EMP plane wave of amplitude E_o.

The optical theorem can also be derived in a half space as noted in the Appendix. This fact is worthy of note in applying the preceding results to apertures for which Babinet's principle [2] does not strictly apply (for example, apertures having depth or cavity backed apertures). The scattered field in the specularly reflected direction is used to determine the total cross section in this case. The electric field component of the scattered field, used in the optical theorem, is polarized in the direction of the electric field of the specularly reflected wave.

The polarizabilities of an aperture [6], α_e and α_m, on the transmitted side of the aperture, are introduced by means of the dipole moments as

$$p = 2\,\epsilon_o\,\alpha_e \cdot E_{sc} \ , \text{ transmitted side} \ , \qquad (21)$$

$$m = -2\,\alpha_m \cdot H_{sc} \ , \text{ transmitted side} \ , \qquad (22)$$

where E_{sc} and H_{sc} are the fields at the location of the aperture, on the incident side, with the aperture shorted (external fields). The dipole moments on the incident side of the aperture are

$$p = -2\,\epsilon_o\,\alpha_e \cdot E_{sc} \ , \text{ incident side} \ , \qquad (23)$$

$$m = 2\,\alpha_m \cdot H_{sc} \ , \text{ incident side} \ , \qquad (24)$$

where, for an aperture in an infinitely thin ground plane with no backing cavity, etc. (Babinet's principle applies), the polarizabilities in (23) and (24) are identical to those entering (21) and (22). Making use of the fact that

$$\mathbf{n} \times \boldsymbol{\alpha}_e = \boldsymbol{\alpha}_e \times \mathbf{n} = 0 \tag{25}$$

and

$$\mathbf{n} \cdot \boldsymbol{\alpha}_m = \boldsymbol{\alpha}_m \cdot \mathbf{n} = 0 \ , \tag{26}$$

where **n** is the unit normal to the conducting ground plane, and the optical theorem in the incident half space, equation (1) becomes

$$\int_0^\infty \sigma_t \, d\lambda = \pi^2 \, 4(\alpha_{m_{22}} - \alpha_{e_{11}}) \ . \tag{27}$$

The integral of the effective area is thus bounded by

$$\int_0^\infty A_e \, d\lambda \leq \pi^2 \, 4(\alpha_{m_{22}} - \alpha_{e_{11}}) \ . \tag{28}$$

The energy received is therefore bounded by

$$W_{rec} \leq \frac{1}{2} \epsilon_o E_o^2 \, 4(\alpha_{m_{22}} - \alpha_{e_{11}}) \ , \tag{29}$$

where, in (27), (28), and (29), the components of the polarizability tensor are taken to be those corresponding to the incident side of the aperture if Babinet's principle should fail to apply.

3. Circular Aperture

A circular aperture, of diameter b, in an infinitely thin ground plane, has polarizabilities

[6] given by

$$\alpha_{e_{11}} = (\mathbf{n} \cdot \mathbf{e}_1)^2 \frac{b^3}{12} \tag{30}$$

and

$$\alpha_{m_{22}} = |\mathbf{n} \times \mathbf{e}_2|^2 \frac{b^3}{6} \ . \tag{31}$$

The energy bound is thus maximized at normal incidence $\mathbf{n} \cdot \mathbf{e}_1 = 0$ for which

$$W_{rec} \leq \epsilon_o E_o^2 \frac{b^3}{3} \ . \tag{32}$$

Figure 2 shows a plot of the right hand side of (32).

Note that the polarizabilities entering (29) must in principle include the proximity effects of backing cavities, etc. However, because these are purely static quantities and because of studies involving backup ground planes and backup cylindrical cavities [6], the changes in the polarizabilities may typically be ignored if clearance exists behind the aperture on the order of the aperture dimensions. Furthermore, these proximity effects typically reduce the polarizabilities [6].

4. Narrow Slot Aperture Having Depth

Figure 3 shows the geometry of the slot aperture with length ℓ, depth d, and width w. The slot aperture is assumed to be very narrow ($\ell >> w$). The unit vector $\mathbf{t}$ is tangential to the conducting plane and directed along the length dimension of the slot.

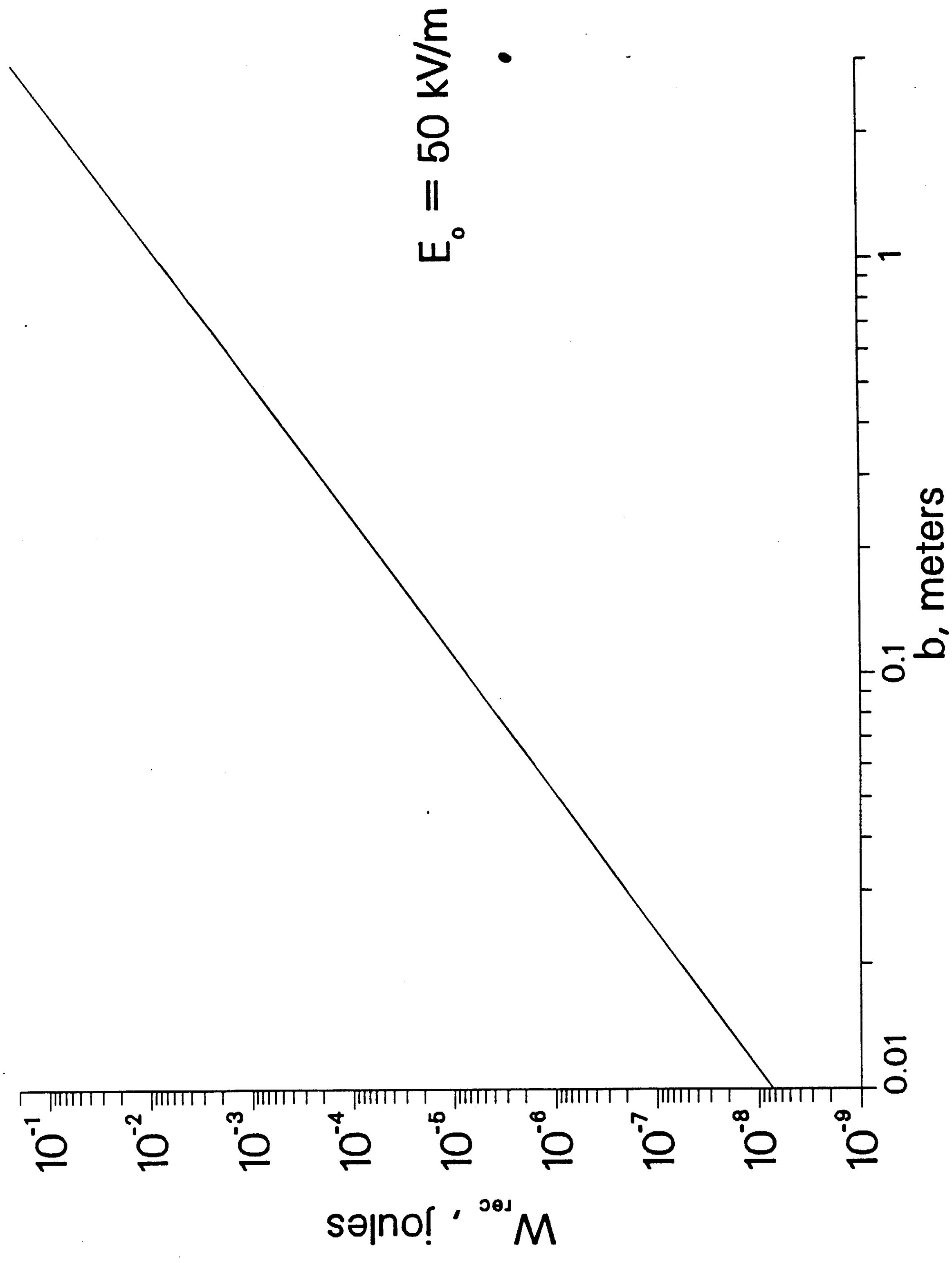

Figure 2. Maximum received energy for a circular aperture of diameter b in an infinitely thin conducting plane, subjected to an incident step function EMP plane wave of amplitude E_o.

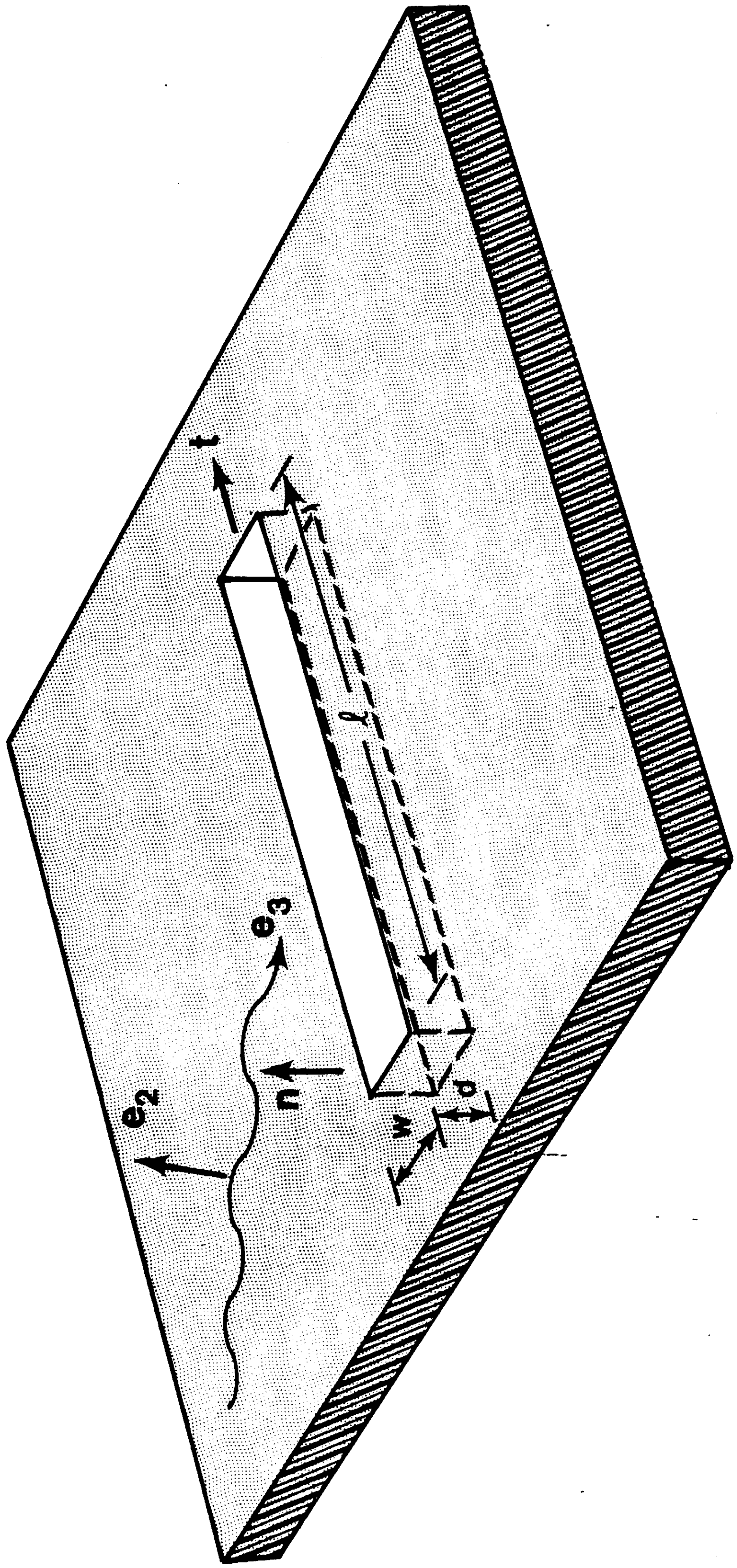

Figure 3. A narrow slot aperture of length ℓ, depth d, and width w. The unit vectors normal to the conducting plane **n** and tangential to the conducting plane along the length dimension of the slot **t** are depicted.

The "thick" case ($\ell >> w,d$) has an axial polarizability given [5] by Hallén's first order theory

$$\alpha_{m_{22}} \sim (\mathbf{t} \cdot \mathbf{e}_2)^2 \, \frac{\ell^3 \pi}{24(\frac{\Omega}{2} + \ell n \; 2 - \frac{7}{3})} \, , \tag{33}$$

where the fatness parameter of the "thick" slot is given approximately [5] by

$$\Omega \approx 2 \, \ell n(4\ell/w) + \pi \, d/w \ . \tag{34}$$

Note that the polarizability given by equation (33) is approximately the same on either side of the aperture in the "thick" case.

The contribution to the polarizabilities resulting from the transverse field components [5] is approximately

$$\alpha_{m_{22}} \approx [(\mathbf{n} \times \mathbf{t}) \cdot \mathbf{e}_2]^2 \, \pi \, (\tfrac{w}{4})^2 \, \ell \, , \text{ incident side} \ , \tag{35}$$

$$\approx [(\mathbf{n} \times \mathbf{t}) \cdot \mathbf{e}_2]^2 \, \pi \left[\frac{w}{4} \, e^{-\frac{\pi d}{2w}}\right]^2 \ell \, , \text{ transmitted side} \ , \tag{36}$$

$$\alpha_{e_{11}} \approx (\mathbf{n} \cdot \mathbf{e}_1)^2 \, \pi \, (\tfrac{w}{4})^2 \, \ell \, , \text{ incident side} \ , \tag{37}$$

$$\approx (\mathbf{n} \cdot \mathbf{e}_1)^2 \, \pi \left[\frac{w}{4} \, e^{-\frac{\pi d}{2w}}\right]^2 \ell \, , \text{ transmitted side} \ . \tag{38}$$

The contribution resulting from equations (35) and (37) is dominated ($\ell >> w$) by the axial magnetic contribution (33). These transverse components are therefore ignored as in the case of the slender electric dipole.

The "deep" case ($\ell, d >> w$) results, in general, in different axial magnetic polarizabilities on either side of the slot. For example, approximations that, in fact, hold for all d ($0 \leq d < \infty$) have been given [5] as

Incident Side

$$\alpha_{m_{22}} \approx (\mathbf{t} \cdot \mathbf{e}_2)^2 \frac{8\ell^3}{\pi^3} \left[\frac{1}{\Omega_o + \frac{2\ell}{w} \tanh(\frac{\pi d}{2\ell})} + \frac{1}{\Omega_o + \frac{2\ell}{w} \coth(\frac{\pi d}{2\ell})} \right], \tag{39}$$

Transmitted Side

$$\alpha_{m_{22}} \approx (\mathbf{t} \cdot \mathbf{e}_2)^2 \frac{16\,\ell^4 / (\pi^3 w)}{[\Omega_o \cosh(\frac{\pi d}{2\ell}) + \frac{2\ell}{w} \sinh(\frac{\pi d}{2\ell})]} \frac{1}{[\Omega_o \sinh(\frac{\pi d}{2\ell}) + \frac{2\ell}{w} \cosh(\frac{\pi d}{2\ell})]}, \tag{40}$$

where $\Omega_o = 2\,\ell n(4\ell/w)$.

Ignoring the very small transverse polarizabilities, the energy bound becomes

$$W_{rec} \leq \epsilon_o E_o^2 \, 2\alpha_{m_{22}} (\text{incident side}) \; . \tag{41}$$

Figures 4, 5, 6, and 7 show examples of the right hand side of (41) using polarizability (39) with incident polarization $|\mathbf{t} \cdot \mathbf{e_2}| = 1$.

Note that aperture depth tends to reduce, even further, the proximity effect of backing cavities, etc., on the incident polarizabilities.

IV. CONCLUSIONS

It has been demonstrated, by use of the bound on the integral of the effective area over all frequencies [1], that the maximum energy absorbed by an object subjected to a step function EMP is bounded by the static energy required to polarize the object. (The energy spectrum of the step function EMP bounds the spectrum of EMP described by the double exponential over all frequencies.) This static energy is, of course, given in terms of the static dipole moments of the object or its polarizabilities. It has also been demonstrated that this bound can be used on apertures for which Babinet's principle fails to apply (apertures having depth and cavity backed apertures), the requirement being simply to use the polarizabilities on the incident side of the aperture.

APPENDIX. OPTICAL THEOREM IN A HALF SPACE

Figure 8 shows the geometry, consisting of an aperture in a thick plane with a backing

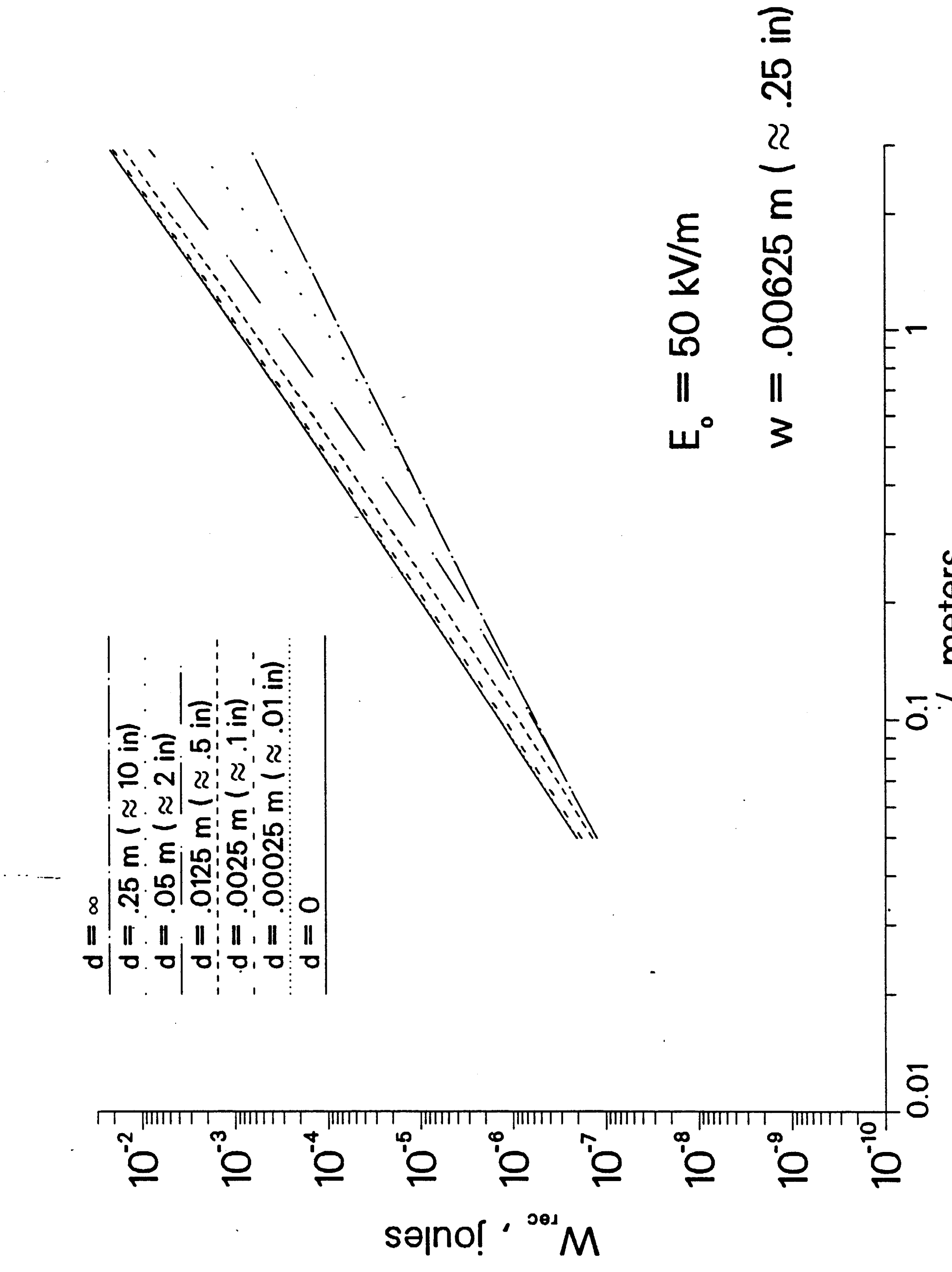

Figure 4. Maximum received energy for a narrow slot aperture of width w ≈ 0.25 in, subjected to a step function EMP plane wave of amplitude E_o.

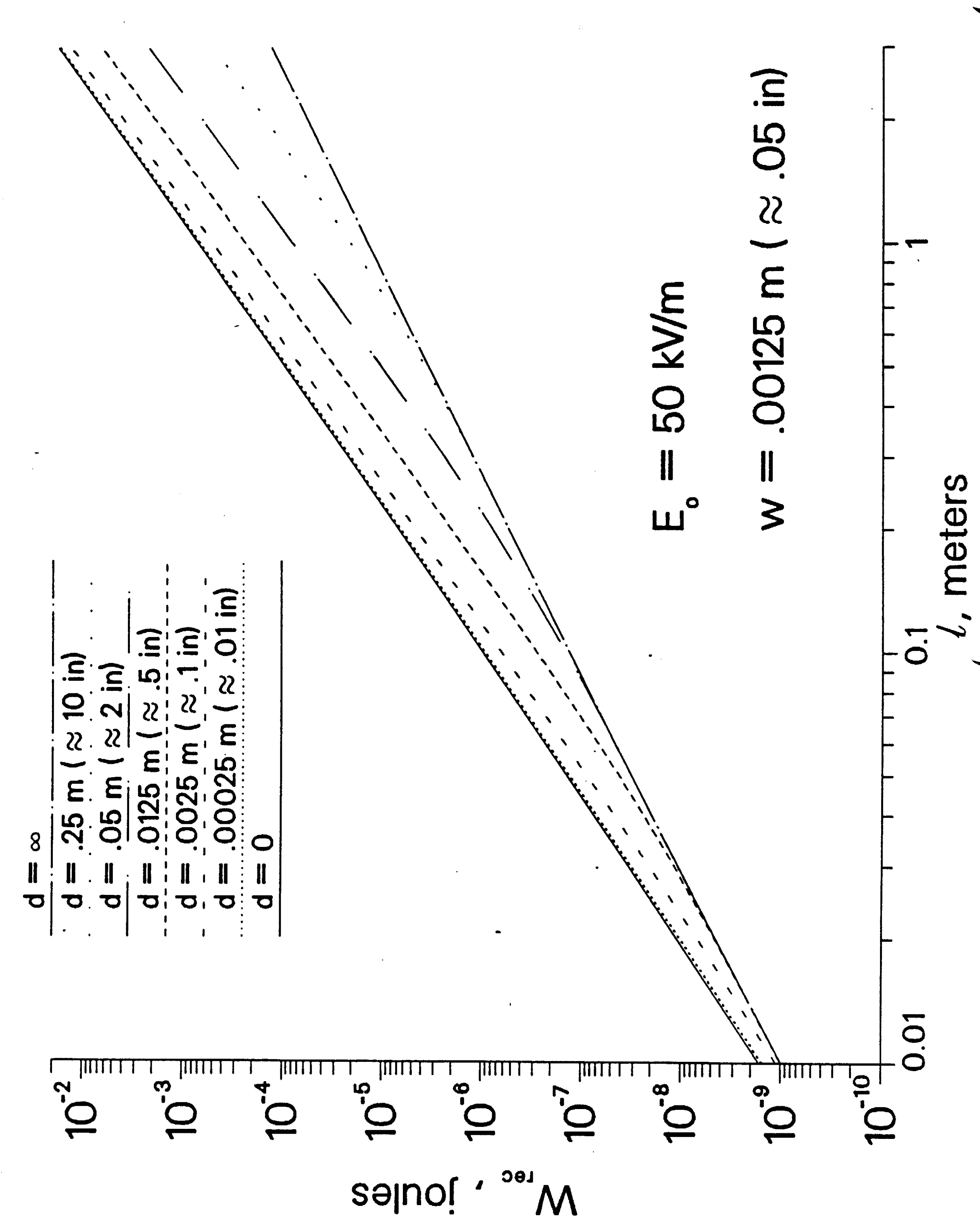

Figure 5. Maximun received energy for a narrow slot aperture of width w ≈ 0.05 in.

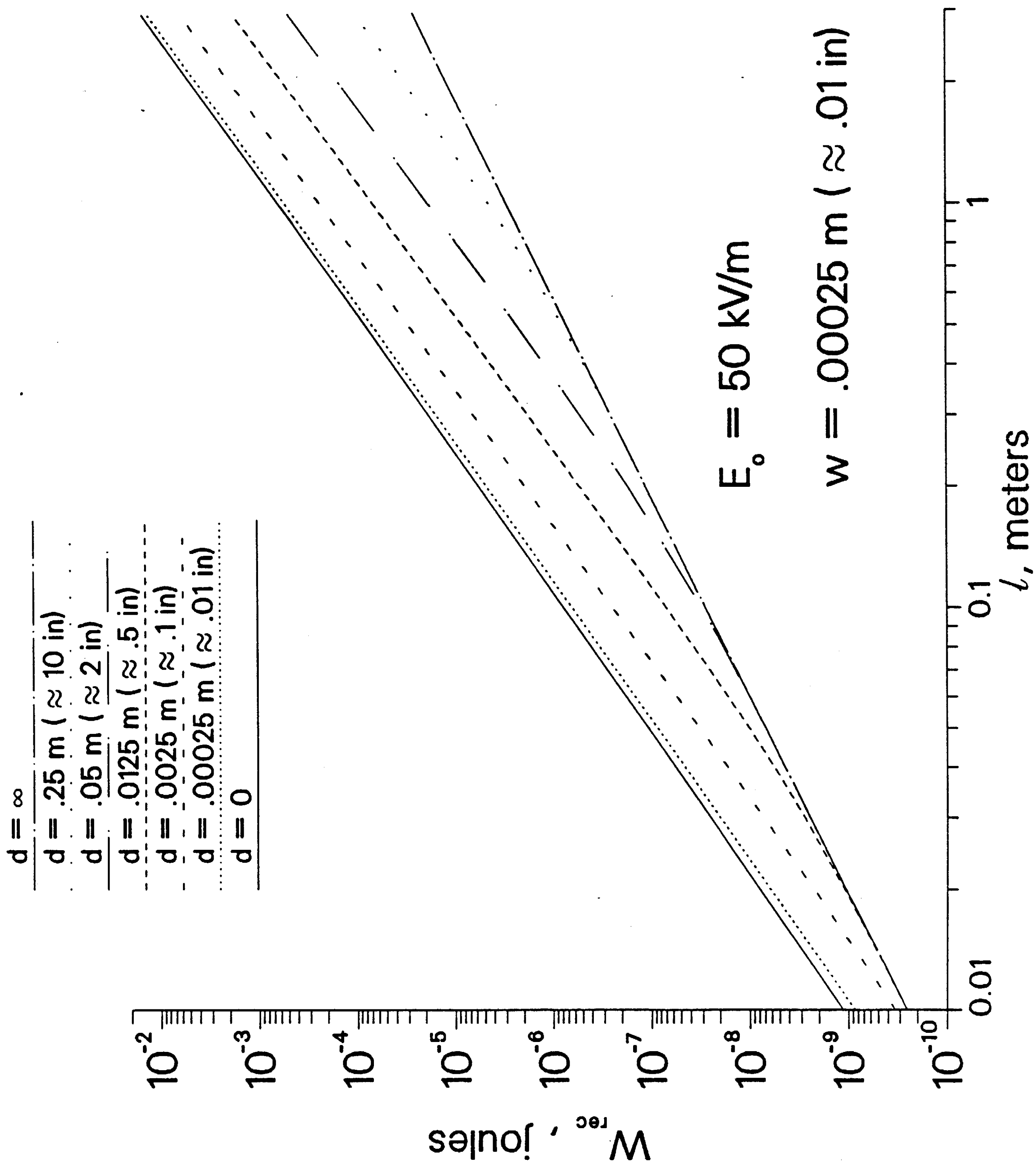

Figure 6. Maximum received energy for a narrow slot aperture of width w ≈ 0.01 in.

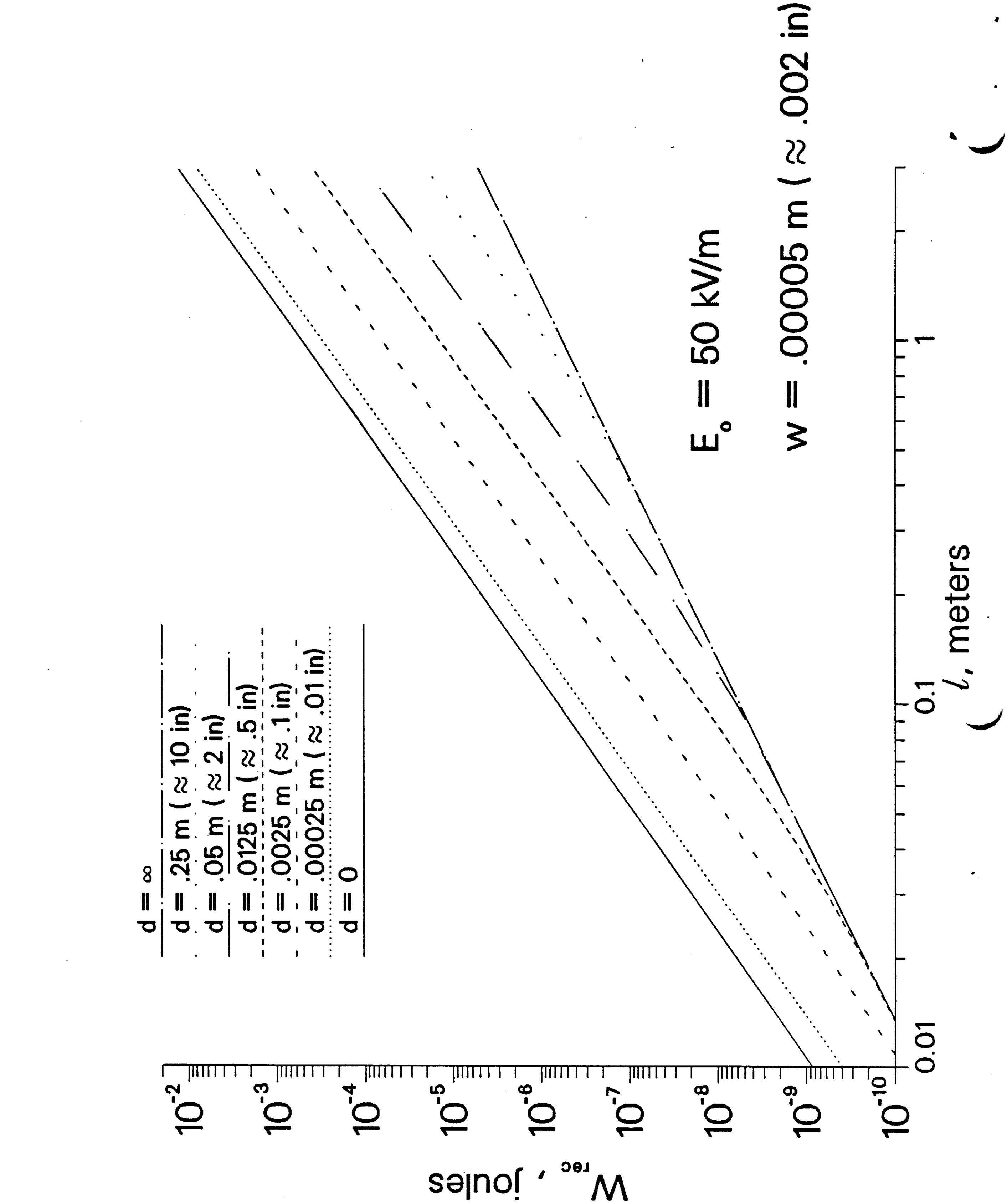

Figure 7. Maximum received energy for a narrow slot aperture of width $w \approx 0.002$ in.

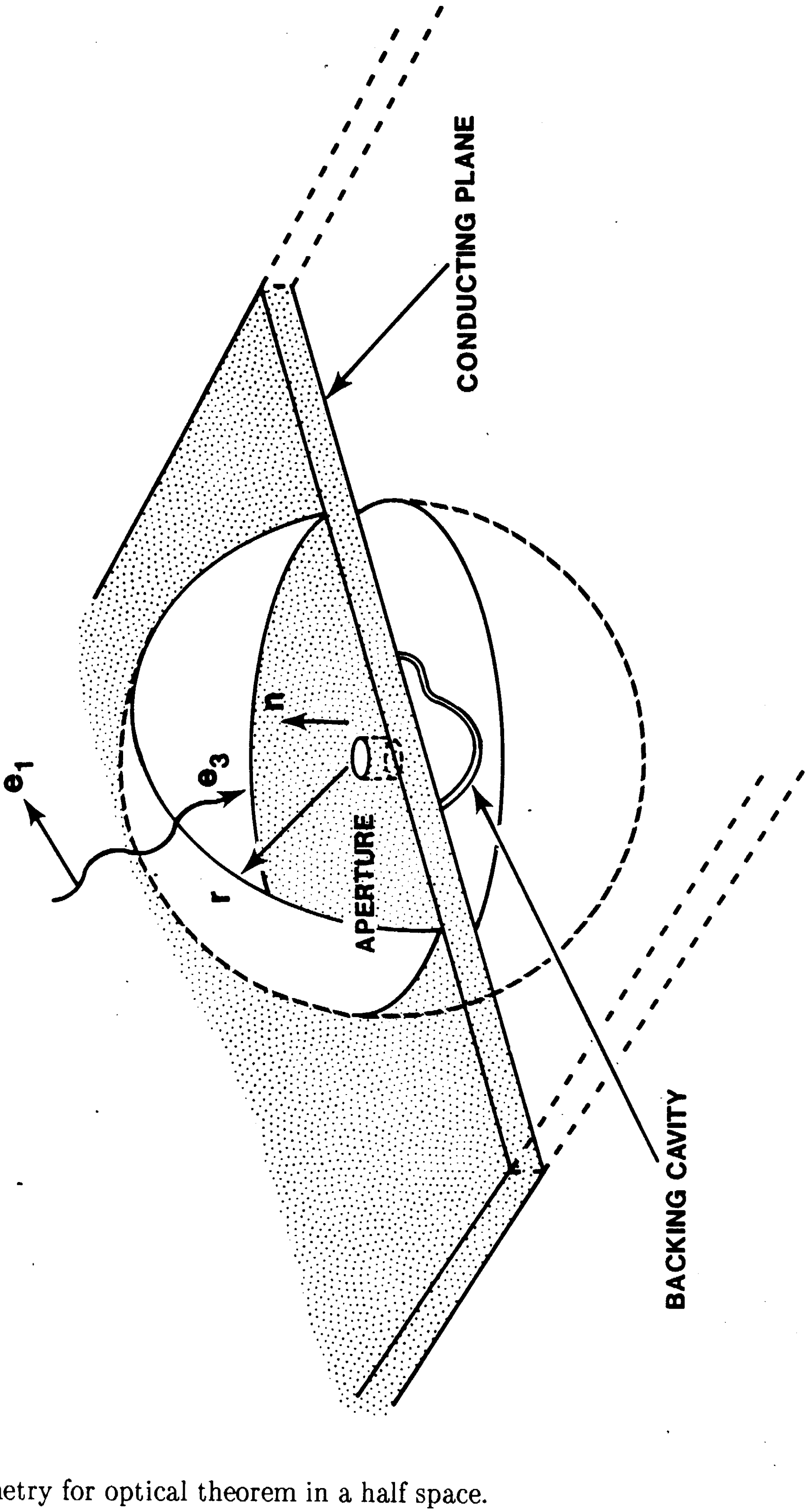

Figure 8. Geometry for optical theorem in a half space.

cavity. A plane time harmonic electromagnetic wave (time dependence $e^{-i\omega t}$) impinges on a cavity backed aperture. The incident wave field components can be written as

$$\mathbf{E}_{inc} = \mathbf{e}_1 \, e^{ik(\mathbf{e}_3 \cdot \mathbf{r})} , \tag{42}$$

$$\mathbf{H}_{inc} = \mathbf{e}_3 \times \mathbf{E}_{inc}/\eta_o , \tag{43}$$

where $\mathbf{r}$ is the position vector and k is the wavenumber.

A reflected plane wave is generated by the conducting plane with fields

$$\mathbf{E}_{ref} = \mathbf{e}_1^{(ref)} \, e^{ik(\mathbf{e}_3^{(ref)} \cdot \mathbf{r})} , \tag{44}$$

$$\mathbf{H}_{ref} = \mathbf{e}_3^{(ref)} \times \mathbf{E}_{ref}/\eta_o , \tag{45}$$

where $\mathbf{e}_1^{(ref)} \times \mathbf{e}_2^{(ref)} = \mathbf{e}_3^{(ref)}$ are unit vectors associated with the reflected wave. These two waves constitute the short circuit field

$$\mathbf{E}_{sc} = \mathbf{E}_{inc} + \mathbf{E}_{ref} , \tag{46}$$

$$\mathbf{H}_{sc} = \mathbf{H}_{inc} + \mathbf{H}_{ref} . \tag{47}$$

Note that we can write

$$\mathbf{e}_1^{(ref)} = -\mathbf{e}_1 + 2\,\mathbf{n}(\mathbf{n}\cdot\mathbf{e}_1) \tag{48}$$

and

$$\mathbf{e}_3^{(ref)} = \mathbf{e}_3 - 2\,\mathbf{n}(\mathbf{n}\cdot\mathbf{e}_3)\ , \tag{49}$$

where **n** is the unit normal to the conducting plane, directed into the incident half space, as shown in Figure 8.

In addition to the incident and reflected waves, there is a scattered field, which, in the far zone ($kr \to \infty$), becomes

$$\mathbf{E}_{scatt} = iF(\mathbf{e}_r)\,\frac{e^{ikr}}{kr}\ , \tag{50}$$

$$\mathbf{H}_{scatt} = \mathbf{e}_r \times \mathbf{E}_{scatt}/\eta_o\ , \tag{51}$$

where $\mathbf{e}_r$ is a unit vector in the radial direction.

Following the usual procedure for the optical theorem [2], the Poynting vector is written as the sum

$$\mathbf{S} = \mathbf{S}_{sc} + \mathbf{S}_{scatt} + \mathbf{S}'\ , \tag{52}$$

where

$$\mathbf{S}_{sc} = \frac{1}{2}\,\mathbf{E}_{sc} \times \mathbf{H}_{sc}^*\ , \tag{53}$$

$$S_{scatt} = \frac{1}{2} E_{scatt} \times H^*_{scatt} \tag{54}$$

and

$$S' = \frac{1}{2} (E_{sc} \times H^*_{scatt} + E_{scatt} \times H^*_{sc}) \ , \tag{55}$$

where * denotes complex conjugate. Integrating the real part of (52) over the sphere at infinity yields

$$-P_{abs} = P_{sc} + P_{scatt} + P' \ , \tag{56}$$

where P_{abs} is the power absorbed by the aperture (and backing cavity),

$$P_{sc} = \mathrm{Re} \int_S S_{sc} \cdot e_r \, dS = 0 \ , \tag{57}$$

where (57) vanishes because the plane boundary is assumed to be perfectly conducting,

$$P_{scatt} = \mathrm{Re} \int_S S_{scatt} \cdot e_r \, dS \tag{58}$$

and

$$P' = \mathrm{Re} \int_S S' \cdot e_r \, dS \ . \tag{59}$$

Note that, because all field components vanish inside and behind the conducting plane, S can be taken as the hemisphere at infinity in the incident half space. Using (57) gives

$$P_{abs} + P_{scatt} = -P' \ . \tag{60}$$

The total cross section of the aperture defined by

$$\sigma_t = (P_{abs} + P_{scatt})/\mathrm{Re}(\mathbf{S}_{inc} \cdot \mathbf{e}_3) \ , \tag{61}$$

where

$$\mathbf{S}_{inc} = \frac{1}{2} \mathbf{E}_{inc} \times \mathbf{H}^*_{inc} = \frac{1}{2\eta_o} \mathbf{e}_3 \ , \tag{62}$$

is therefore

$$\sigma_t = -2\eta_o P' \ . \tag{63}$$

The integrand of P′ includes the four terms

$$(\mathbf{E}_{inc} \times \mathbf{H}^*_{scatt}) \cdot \mathbf{e}_r = \frac{f_1}{kr} e^{ikr\varphi^{(inc)}} \ , \tag{64}$$

$$(\mathbf{E}_{ref} \times \mathbf{H}^*_{scatt}) \cdot \mathbf{e}_r = \frac{f_1^{(ref)}}{kr} e^{ikr\varphi^{(ref)}} \ , \tag{65}$$

$$(\mathbf{E}_{scatt} \times \mathbf{H}^*_{inc}) \cdot \mathbf{e}_r = \frac{f_2}{kr} e^{-ikr\varphi^{(inc)}} \ , \tag{66}$$

and

$$(\mathbf{E}_{scatt} \times \mathbf{H}^*_{ref}) \cdot \mathbf{e}_r = \frac{f_2^{(ref)}}{kr} e^{-ikr\varphi^{(ref)}} \ , \tag{67}$$

where

$$f_1(\mathbf{e_r}) = \frac{-i}{\eta_o} \mathbf{e_1} \cdot \mathbf{F}^*(\mathbf{e_r})\, e^{-ikr} \quad , \tag{68}$$

$$f_1^{(ref)}(\mathbf{e_r}) = \frac{-i}{\eta_o} \mathbf{e}_1^{(ref)} \cdot \mathbf{F}^*(\mathbf{e_r})\, e^{-ikr} \quad , \tag{69}$$

$$f_2(\mathbf{e_r}) = \frac{i}{\eta_o} [(\mathbf{e_3} \cdot \mathbf{e_r})(\mathbf{e_1} \cdot \mathbf{F}) - (\mathbf{e_1} \cdot \mathbf{e_r})(\mathbf{e_3} \cdot \mathbf{F})]\, e^{ikr} \quad , \tag{70}$$

$$f_2^{(ref)}(\mathbf{e_r}) = \frac{i}{\eta_o} [(\mathbf{e}_3^{(ref)} \cdot \mathbf{e_r})(\mathbf{e}_1^{(ref)} \cdot \mathbf{F})$$

$$- (\mathbf{e}_1^{(ref)} \cdot \mathbf{e_r})(\mathbf{e}_3^{(ref)} \cdot \mathbf{F})]\, e^{ikr} \quad , \tag{71}$$

$$\varphi^{(inc)}(\mathbf{e_r}) = \mathbf{e_r} \cdot \mathbf{e_3} \quad , \tag{72}$$

and

$$\varphi^{(ref)}(\mathbf{e_r}) = \mathbf{e_r} \cdot \mathbf{e}_3^{(ref)} \quad . \tag{73}$$

The following stationary phase asymptotic approximations are used to evaluate P′:

<u>Stationary Point Contribution [2]</u>

$$\frac{1}{kr} \int_S f(\mathbf{e_r})\, e^{-ikr\varphi(\mathbf{e_r})}\, dS \sim \frac{2\pi i}{k^2} [f(\mathbf{e_s})\, e^{-ikr} - f(-\mathbf{e_s})\, e^{ikr}] \tag{74}$$

<u>Boundary Contribution [7]</u>

$$\frac{1}{kr}\int_S f(\mathbf{e_r})\, e^{-ikr\varphi(\mathbf{e_r})}\, dS \sim \frac{-i}{k^2}\int_{\partial S} f(\mathbf{e_r})\, \frac{\mathbf{n}\ .\ \nabla\varphi}{r\,|\nabla\varphi|^2}\, e^{-ikr\varphi}\, \frac{d\ell}{r}\ , \tag{75}$$

where ∇ is the gradient operator on the surface S, $\mathbf{n}$ is the unit normal to the plane conducting boundary directed into the incident half space, and ∂S is the boundary of the hemispherical surface S formed by the intersection with the conducting boundary, and

$$\varphi(\mathbf{e_r}) = \mathbf{e_r} \cdot \mathbf{e_s}\ , \tag{76}$$

where the stationary points are $\mathbf{e_r} = \pm\, \mathbf{e_s}$.

The boundary contribution (75) to P′ can be ignored because on the boundary we have $f_1 = f_1^{(ref)}$, $f_2 = f_2^{(ref)}$, $\varphi^{(inc)} = \varphi^{(ref)}$, $\mathbf{n} \times \nabla\varphi^{(inc)} = \mathbf{n} \times \nabla\varphi^{(ref)}$, and $\mathbf{n}\ .\ \nabla\varphi^{(inc)} = -\,\mathbf{n}\ .\ \nabla\varphi^{(ref)}$; combined with the fact that the boundary integral is of lower order than the stationary contribution anyway. Furthermore, the stationary points $\mathbf{e_r} = \mathbf{e_3}, -\,\mathbf{e}_3^{(ref)}$ do not contribute, because they lie outside the surface of integration (it is assumed that $\mathbf{e_3}\ .\ \mathbf{n} < 0$). Therefore application of (74) yields

$$P' = -\frac{\pi}{k^2}\,\mathrm{Im}\,[f_1(-\,\mathbf{e_3})\, e^{-ikr} - f_2(-\,\mathbf{e_3})\, e^{ikr}$$

$$-\, f_1^{(ref)}(\mathbf{e}_3^{(ref)})\, e^{ikr} + f_2^{(ref)}(\mathbf{e}_3^{(ref)})\, e^{-ikr}]\ , \tag{77}$$

where Im denotes imaginary part. The total cross section therefore becomes

$$\sigma_t = \frac{4\pi}{k^2} \mathrm{Re}[\mathbf{e}_1^{(ref)} \cdot \mathbf{F}(\mathbf{e}_3^{(ref)})] \ . \tag{78}$$

Using the Hilbert transform relation for the scattered field [1] gives

$$\int_0^\infty \sigma_t(\lambda)\, d\lambda = -4\pi^3 c^3 \omega^{-3}\, \mathrm{Im}[\mathbf{e}_1^{(ref)} \cdot \mathbf{F}(\mathbf{e}_3^{(ref)})] \ , \ \omega \to 0 \ , \tag{79}$$

where c is the velocity of light. The low frequency limit of the scattered field amplitude **F** may be evaluated in terms of the aperture static dipole moments by means of [1]

$$\mathbf{E}_{scatt} = -\frac{\mu_o \omega^2 k}{4\pi} [\mathbf{e}_3^{(ref)} \times (\mathbf{e}_3^{(ref)} \times \mathbf{p})$$

$$+ \frac{1}{c} \mathbf{e}_3^{(ref)} \times \mathbf{m}] \frac{e^{ikr}}{kr} \ . \tag{80}$$

REFERENCES

[1] K. S. H. Lee, "Relations between electric and magnetic polarizabilities and other related quantities," Radio Science, Vol. 22, No. 7, pp 1235–1238, Dec. 1987.

[2] M. Born and E. Wolf, Principles of Optics, Pergamon: New York, 1980, pp 657–659.

[3] L. D. Landau and E. M. Lifshitz, Electrodynamics of Continuous Media, Pergamon: New York, 1960, pp 24, 259.

[4] R. W. Bracewell, The Fourier Transform and Its Applications, McGraw–Hill: New York, 1978, p 112.

[5] L. K. Warne, K. C. Chen, and T. E. Koontz, "Equivalent Polarizabilities of Apertures Having Depth," to be published as AFWL Interaction Note.

[6] K. S. H. Lee (editor), EMP Interaction: Principles, Techniques, and Reference Data, Hemisphere: New York, 1986, pp 184–188, 439–441, 519–522.

[7] N. Bleistein, Mathematical methods for Wave Phenomena, Academic Press: New York, 1984, pp 82–88.