Interaction Notes

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# Norms of Vectors of Time-Domain Signals Passing Through Filters and Norm Limiters at Subshields

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Abstract

This paper extends the use of filters and norm limiters from single time-domain signals to vectors of such signals penetrating subshields. This gives a general format for specifying the performance of protection of electronic systems in the presence of various electromagnetic environments.

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## I. Introduction

In designing protection of electronic systems from various undesirable electromagnetic environments it is necessary to control the signals reaching various parts of the system. The basic concept of electromagnetic topology reduces the necessary set of control locations to closed boundary surfaces (subshields) through which the signals must pass to reach the equipment of interest.

At the subshields there are filters and/or norm limiters which control the penetrating signals. Extending concepts in [6] time-domain norms can be applied to the filters and norm limiters to bound the norm of the vector of penetrating signals. This can be extended to successive subshields as desired in a form similar to the good-shielding approximation [5].

## II. Matrix Filters

As in [6] let us consider a filter which relates two waves as

$$\tilde{V}^{(out)}(s) = \tilde{T}(s)\, \tilde{V}^{(in)}(s) \tag{2.1}$$

$V^{(in)} \equiv$ wave incident on filter (input)

$V^{(out)} \equiv$ wave transmitted through filter (output)

Here we have expressed the basic relation in complex-frequency (two-sided Laplace-transform) domain so that our filter is characterized by a transfer function $\tilde{T}(s)$. In time domain we have

$$\begin{aligned} V^{(out)}(t) &= T(t) \circ V^{(in)}(t) \\ &= \int_0^t T(t - t')\, V(t')\, dt' \\ \circ &\equiv \text{convolution} \end{aligned} \tag{2.2}$$

where our filter has been assumed to be causal.

Note for our purposes $V^{(in)}$ and $V^{(out)}$ are waves of the form $V \pm RI$ where R is a convenient constant resistance and the + or − is chosen in conjunction with the convention on the positive direction for the current I to give waves propagating in the desired direction as in [6]. Furthermore $V^{(out)}$ is assumed to be terminated in a resistance R so that no reflected wave is incident back on the filter from the transmitted-wave side. Similarly $V^{(in)}$ or its reflection from the filter (or both waves) is assumed to be terminated in R on the incident-wave side. This allows $\tilde{T}(s)$ to characterize the physical filter without being complicated by resonances on either side.

Another note [5] generalizes this filter concept to a matrix filter characterized by a matrix of transfer functions. This can be written as

$$(\tilde{V}_n^{(out)}(s)) = (\tilde{T}_{n,m}(s)) \cdot (\tilde{V}_n^{(in)}(s)) \tag{2.3}$$

where now we have vector waves representing say M incident waves $\tilde{V}_m^{(in)}(s)$ and N transmitted waves $\tilde{V}_n^{(out)}(s)$ giving an NxM matrix of transfer functions. In the general formulation in [5] this $(\tilde{T}_{n,m}(s))$ is just one block in a large scattering supermatrix. In time domain we have

$$(V_n^{(out)}(t)) = (T_{n,m}(t)) \circ (V_n^{(in)}(t)) \tag{2.4}$$

giving a matrix transfer convolution operator. (See Appendix A.) Note as in [5] we assume both vector waves are terminated in all their components by a matrix resistance of the form $R(1_{n,m})$ where our combined voltage vectors now have the form $(V_n) \pm R(I_n)$. This termination is to prevent waves other than in (2.3) from returning to the matrix filter and complicating the description of the matrix transfer function.

The matrix transfer function is intended to characterize a set of filters at some subshield [5] and so control all the signals passing through this subshield. Then $V_n^{(out)}$ represents all the signals passing through some subshield into the corresponding sublayer of the system. Similarly $V_m^{(in)}$ represents all the signals from an adjacent sublayer which are exciting our sublayer via the set of filters.

Now as discussed in Appendix A one can bound the time-domain norm of the vector of signals penetrating the subshield as

$$||(V_n^{(out)}(t))|| \leq ||(T_{n,m}(t))\circ|| \; ||(V_n^{(in)}(t))|| \tag{2.5}$$

If we are considering the p-norm we also have

$$||(V_n^{(out)}(t))||_p \leq ||(T_{n,m}(t))\circ||_p \; ||(V_n^{(in)}(t))||_p \tag{2.6}$$

with (using results of Appendix A)

$$\begin{gathered} ||(T_{n,m}(t)) \circ ||_p \leq ||(||T_{n,m}(t) \circ ||_{pf})||_{pv} \\ ||T_{n,m}(t) \circ ||_{pf} \leq ||T_{n,m}(t)||_{1f} \\ ||(T_{n,m}(t)) \circ ||_p \leq ||(||T_{n,m}(t)||_{1f})||_{pv} \end{gathered} \tag{2.7}$$

where pf indicates p-norm in the function sense. Regarding $||T_{n,m}(t)||_{1f}$ as matrix elements various bounds for matrix p-norms (indicated by pv) can be applied such as from (B.14), (B.28), and (B.31).

For the special case that the transfer-function matrix is diagonal (and hence square NxN) we have

$$||(T_{n,m}) \circ ||_p = \max_{1 \le n \le N} ||T_{n,n}(t) \circ ||_{pf}$$

$$\le \max_{1 \le n \le N} ||T_{n,n}(t)||_{1f} \qquad (2.8)$$

with equality for p = 1, ∞. There are other bounds for this case given by

$$||(V_n^{(out)}(t))|| = ||(T_{n,m}(t)) \circ (V_n^{(in)}(t))||$$

$$= ||(T_{n,n}(t) \circ V_n^{(in)}(t))|| \qquad (2.9)$$

In p-norm sense this is

$$||(V_n^{(out)}(t))||_p = ||(T_{n,n}(t) \circ V_n^{(in)}(t))||_p$$

$$= ||(||T_{n,n}(t) \circ V_n^{(in)}(t)||_{pf})||_{pv}$$

$$\le ||(||T_{n,n}(t) \circ ||_{pf} \, ||V_n^{(in)}(t)||_{pf})||_{pv}$$

$$\le ||(||T_{n,n}(t)||_{1f} \, ||V_n^{(in)}(t)||_{pf})||_{pv} \qquad (2.10)$$

III. Norm Limiters

Following [6] let us consider a norm limiter as one which limits a transmitted wave to

$$||V^{(out)}(t)|| \le X \tag{3.1}$$

no matter what the incident wave $V^{(in)}(t)$ is. We can think of this as an ideal norm limiter if it does not distort $V^{(out)}(t)$ until the norm limit X is reached, in particular as

$$||V^{(in)}(t)|| < X$$
$$\Rightarrow V^{(out)}(t) = V^{(in)}(t) \qquad \text{for all } t \tag{3.2}$$

Of course, the norm of interest has to be specified. Again we assume the waves are terminated as discussed in the previous section.

This can be generalized for signals $(V_n^{(out)}(t))$ by requiring (3.1) to apply to each component as

$$||V_n^{(out)}(t)|| \le X_n \qquad \text{for } 1 \le n \le N \tag{3.3}$$

Then in the case of the p-norm we have the simple result

$$\begin{aligned} ||(V_n^{(out)}(t))|| &= ||(||V_n^{(out)}(t)||_{pf})||_{pv} \\ &\le ||(X_n)||_{pv} \\ &= \left\{ \sum_{n=1}^{N} X_n^p \right\}^{\frac{1}{p}} \end{aligned} \tag{3.4}$$

noting that all the $X_n$ are real and non-negative.

There are other forms that norm limiters can take for $(V_n^{(out)}(t))$. In particular analogous to $(\tilde{T}^{*}_{n,m}(s))$ one can consider each $V_n^{(out)}(t)$ as a combination of the $V_m^{(in)}(t)$ and limit the contribution to $V_n^{(out)}(t)$ from each $V_m^{(in)}(t)$ in norm sense. This might be thought of as a matrix norm limiter in that we might constrain

$$||V_n^{(out)}(t)|| \leq \sum_{m=1}^{M} X_{n,m} \qquad (3.5)$$

where the $X_{n,m}$ represent a contribution in norm sense from the $V_m^{(in)}(t)$. Various other forms are also possible. For a diagonal norm limiter we might think of (3.3) in the form of (3.5) as

$$||V_n^{(out)}(t)|| \leq X_{n,n} \qquad (3.6)$$

IV. Combining Filters with Norm Limiters

In [6] we have the canonical problem of a norm limiter with filters on both sides. For a single scalar signal this gives a fairly simple norm relation. Let us apply this scheme to each of N signals as

$$||V_n^{(out)}(t)|| \le ||T_{n,n}^{(2)}(t) \circ || \left\{ \text{lesser of} \left[ X_{n,n} , ||T_{n,n}^{(1)}(t) \circ || \; ||V_n^{(in)}(t)|| \right] \right\} \tag{4.1}$$

Here $V_n^{(in)}$ is passed through a first filter $T_{n,n}^{(1)}$, then a norm limiter (with norm limit $X_n$), and then a second filter $T_{n,n}^{(2)}$. Of course, the type of norm has to be specified.

As we have constructed this case it is basically a diagonal matrix norm-limiter/filter, as indicated by the n,n subscripts on the filters and norm limits. Constructing, for example, the p-norm of the transmitted signal vector we have (using results of appendix A)

$$||(V_n^{(out)}(t))||_p \le \left\{ \sum_{n=1}^{N} ||T_{n,n}^{(2)}(t) \circ ||_{pf}^{p} \left\{ \text{lesser of} \left[ X_{n,n}^{p} , ||T_{n,n}^{(1)}(t) \circ ||_{pf}^{p} ||V_n^{(in)}(t)||_{pf}^{p} \right] \right\} \right\}^{\frac{1}{p}}$$

$$\le \left\{ \sum_{n=1}^{N} ||T_{n,n}^{(2)}(t)||_{1f}^{p} \left\{ \text{lesser of} \left[ X_{n,n}^{p} , ||T_{n,n}^{(1)}(t)||_{1f}^{p} ||V_n^{(in)}(t)||_{pf}^{p} \right] \right\} \right\}^{\frac{1}{p}} \tag{4.2}$$

or in another form

$$||(V_n^{(out)}(t))||_p \le \left|\left| \left( ||T_{n,n}^{(2)}(t) \circ ||_{pf} \left\{ \text{lesser of} \left[ X_{n,n} , ||T_{n,n}^{(1)}(t) \circ ||_{pf} ||V_n^{(in)}(t)||_{pf} \right] \right\} \right) \right|\right|_{pv}$$

$$\le \left|\left| \left( ||T_{n,n}^{(2)}(t)||_{1f} \left\{ \text{lesser of} \left[ X_{n,n} , ||T_{n,n}^{(1)}(t)||_{1f} ||V_n^{(in)}(t)||_{pf} \right] \right\} \right) \right|\right|_{pv} \tag{4.3}$$

Note that the $X_{n,n}$ are to be p-norm limiters in this case.

As discussed in [6] one function of the filters on both sides of the norm limiter is to provide a resistive termination R for waves passing through or reflecting back from the norm limiter. In the present case with N norm limiters such filters help to isolate the N signals from each other at the norm limiters. This allows us to approximate that each norm limiter limits only one signal allowing the results in (4.2) and (4.3).

One can consider even more complex forms of combinations of norm limiters and filters. Taking the basic combination of filter/norm-limiter/filter one could consider $V_n^{(out)}$ as some linear combination of the various $V_m^{(in)}$, each after passing through such a norm-limiter/filter combination. Then $||V_n^{(out)}(t)||$ would be a combination of terms like those in (4.1) and could be bounded by a generalized form of (3.5).

## V. Concluding Remarks

The results here give general forms for the norms of signals penetrating subshields including both filters and norm limiters. By restricting the norm of the vector of penetrating signals one can limit the size of the interference (noise) in the corresponding sublayer, thereby protecting it from damage and/or upset. This then gives a format in which one can establish electromagnetic specifications for subshield specifications to protect the sublayer from various undesirable electromagnetic environments ($EMX_n$).

Note that for these results to be applicable, there are certain assumptions concerning the design of the system as in [5,6]. This involves the termination of signals at the penetrations (from both sides) and isolation of the norm limiters by termination of the signals scattered from them. While it may be possible to relax the requirements somewhat, it should be observed that the protection of electronic systems is still basically a problem of synthesis in which subshields are defined and norms of penetrating signals are controlled. The protection design should include whatever is necessary to accomplish that with a sense of completeness.

## Appendix A: Norms of Vector Functions and Matrix Operators

Various papers have discussed norms of vectors and matrices for use in frequency-domain analysis [1,3,5]. This is extended to functions and operators, particularly for time-domain analysis [2,4,6,7]. In more general situations we can have vectors of functions (say a set of time-domain waveforms on a multiconductor cable). For this case we need to extend our norm concepts to include vectors of functions and matrices of operators.

Consider a vector function as $(f_n(t))$ in which there are N functions $f_n(t)$ for $n = 1,2,\ldots,N$ of a parameter t which will normally be taken real. Often $f_n$ will also be taken real (but this is not essential). While t is suggestive of the parameter time it could be something else such as frequency. Then define norms by the properties

$$||(f_n(t))|| \begin{cases} = 0 & \text{iff } (f_n(t)) \equiv (0_n) \text{ or has zero "measure"} \\ & \text{per the particular norm} \\ > 0 & \text{otherwise} \end{cases} \tag{A.1}$$

$$||\alpha(f_n(t))|| = |\alpha| \; ||(f_n(t))|| \quad , \quad \alpha \equiv \text{a complex scalar}$$

$$||(f_n(t)) + (g_n(t))|| \le ||(f_n(t))|| + (g_n(t))||$$

Define the p-norm for a vector function as

$$||(f_n(t))||_p \equiv \left\{ \int_{-\infty}^{\infty} \sum_{n=1}^{N} |f_n(t)|^p \, dt \right\}^{\frac{1}{p}} \tag{A.2}$$

Note the consistency of this definition with the usual forms for vectors and functions [7]. Let N=1 and the usual function p-norm is produced. Drop the integration over time and the usual vector p-norm is produced. If pv is used to denote the vector p-norm and pf is used to denote the function p-norm we have

$$||(f_n(t))||_p = || \; ||(f_n(t))||_{pv} \; ||_{pf} = || \; (\; ||f_n(t)||_{pf}) \; ||_{pv} \tag{A.3}$$

so the vector function p-norm has the form of norm of a norm.

Take the usual special cases of interest for the p-norm. For the 1-norm we have

$$||(f_n(t))||_1 \equiv \int_{-\infty}^{\infty}\sum_{n=1}^{N} |f_n(t)|\, dt = \sum_{n=1}^{N}\int_{-\infty}^{\infty} |f_n(t)|\, dt \tag{A.4}$$

which can be interpreted as the sum of the rectified integrals of the vector components. For the ∞-norm we have the generalization of the concept of peak magnitude as

$$\begin{aligned} ||(f_n(t))||_\infty &= ||(\,||f_n(t)||_{\infty f})||_{\infty v} \\ &= \max_{1\le n\le N}\ \sup_{-\infty<t<\infty} |f_n(t)| \end{aligned} \tag{A.5}$$

where isolated points of $f_n(t)$ are excluded. This norm is in effect the peak of the peaks.

In the case of the 2-norm we have first

$$\begin{aligned} ||(f_n(t))||_2 &\equiv \left\{\int_{-\infty}^{\infty}\sum_{n=1}^{N} |f_n(t)|^2\, dt\right\}^{\frac{1}{2}} \\ &= \left\{\int_{-\infty}^{\infty} (f_n(t))^* \bullet (f_n(t))\, dt\right\}^{\frac{1}{2}} \\ &= ||(F_n)||_2 \end{aligned}$$

$$F_n \equiv ||f_n(t)||_2 = \left\{\int_{-\infty}^{\infty} |f_n(t)|^2\, dt\right\}^{\frac{1}{2}} \tag{A.6}$$

By the Parseval theorem [7].

$$||f_n(t)||_2 = \frac{1}{\sqrt{2\pi}} ||\tilde{f}(j\omega)||_2$$

$$||\tilde{f}(j\omega)_2 \equiv \left\{ \int_{-\infty}^{\infty} |\tilde{f}(j\omega)|^2 \, d\omega \right\}^{\frac{1}{2}} \tag{A.7}$$

This directly generalizes to vector functions as

$$||(f_n(t))||_2 = \frac{1}{\sqrt{2\pi}} ||(\tilde{f}_n(j\omega))||_2$$

$$||(\tilde{f}_n(j\omega)||_2 \equiv \frac{1}{\sqrt{2\pi}} \left\{ \sum_{n=1}^{N} \int_{-\infty}^{\infty} (\tilde{f}_n(j\omega))^* \cdot (\tilde{f}_n(j\omega)) \, d\omega \right\}^{\frac{1}{2}} \tag{A.8}$$

Here the Laplace transform (two-sided) is

$$\tilde{f}(s) \equiv \int_{-\infty}^{\infty} f(t) \, e^{-st} \, dt$$

$$s = \Omega + j\omega \equiv \text{complex frequency} \tag{A.9}$$

$$f(t) = \frac{1}{2\pi j} \int_{B_r} \tilde{f}(s) \, e^{st} \, ds$$

with the Bromwich contour in the strip of convergence paralleling the $j\omega$ axis. This directly generalizes to vector functions and matrix functions because the Laplace transform is a scalar operator.

Now consider a matrix operator as $(\Lambda_{n,m}(\ ))$ which can be considered element by element as $\Lambda_{n,m}(\ )$ with $n = 1,2, \ldots, N$ and $m = 1,2, \ldots, M$. The elements operate on functions $f_m(t)$, so we can consider that the matrix operator operates on the vector function $(f_n(t))$ with a number of elements compatible with the matrix elements. Symbolically we have in the form of elements

$$F_n(t) = \sum_{m=1}^{M} \Lambda_{n,m}(f_m(t)) \tag{A.10}$$

or in vector and matrix form

$$(F_n(t)) = (\Lambda_{n,m}(\ )) \bullet (f_n(t)) \tag{A.11}$$

indicating the dot product inherent in our definition of the matrix operator. Then define associated matrix operator norms via

$$||(\Lambda_{n,m}(\ ))|| \equiv \sup_{(f_n(t))\not\equiv(0_n)} \frac{||(\Lambda_{n,m}(\ )) \bullet (f_n(t))||}{||f_n(t))||} \tag{A.12}$$

Here the $\Lambda_{n,m}$ can include any kind of linear operation that results in the following properties of an operator norm.

$$||(\Lambda_{n,m}(\ ))|| \begin{cases} = 0 & \text{iff } (\Lambda_{n,m}(\ )) \equiv (0_{n,m}) \text{ or has zero "measure"} \\ & \text{per the particular norm} \\ > 0 & \text{otherwise} \end{cases}$$

$$||\alpha(\Lambda_{n,m}(\ ))|| = |\alpha|\ ||(\Lambda_{n,m}(\ ))||$$

$$||(\Lambda_{n,m}(\ )) + (\Upsilon_{n,m}(\ ))|| \leq ||(\Lambda_{n,m}(\ ))|| + ||(\Upsilon_{n,m}(\ ))|| \tag{A.13}$$

$$||(\Lambda_{n,m}(\ )) \bullet (\Upsilon_{n,m}(\ ))|| \leq ||(\Lambda_{n,m}(\ ))||\ ||(\Upsilon_{n,m}(\ ))|$$

For the case that the operation is convolution let us symbolize a matrix convolution operator as $(\Lambda_{n,m}(t))\underset{\bullet}{\circ}$ where

$$(F_n(t)) \equiv (\Lambda_{n,m}(t)) \underset{\bullet}{\circ} (f_n(t))$$

$$= \int_{-\infty}^{\infty} (\Lambda_{n,m}(t - t')) \bullet (f_n(t'))\, dt'$$

$$= \int_{-\infty}^{\infty} (\Lambda_{n,m}(t')) \bullet (f_n(t - t'))\, dt' \tag{A.14}$$

In terms of the Laplace transform this is

$$(\tilde{F}_n(s)) = (\tilde{\Lambda}_{n,m}(s)) \bullet (\tilde{f}_n(s)) \tag{A.15}$$

with convolution replaced by multiplication in complex-frequency domain. The associated norm of a matrix convolution operator is then defined as

$$||(\Lambda_{n,m}(t)) \circ|| \equiv \sup_{(f_n(t))\not\equiv(0_n)} \frac{||(\Lambda_{n,m}(t)) \underset{\bullet}{\circ} (f_n(t))||}{||(f_n(t))||} \tag{A.16}$$

For the p-norm the results of [7] are readily generalized as

$$||(\Lambda_{n,m}(t)) \circ||_p \equiv \sup_{(f_n(t))\not\equiv(0_n)} \frac{||(\Lambda_{n,m}(t)) \underset{\bullet}{\circ} (f_n(t))||_p}{||(f_n(t))||_p} \tag{A.17}$$

The numerator on the right can be evaluated via (A.3) as

$$\begin{aligned} ||(\Lambda_{n,m}(t)) \underset{\bullet}{\circ} (f_n(t))||_p &= \left|\left|\left(\sum_{m=1}^{M} \Lambda_{n,m}(t) \circ f_m(t)\right)\right|\right|_p \\ &= \left|\left|\left(\left|\left|\sum_{m=1}^{M} \Lambda_{n,m}(t) \circ f_m(t)\right|\right|_{pf}\right)\right|\right|_{pv} \end{aligned} \tag{A.18}$$

The interior pf-norm can be evaluated as

$$\left|\left|\sum_{m=1}^{M} \Lambda_{n,m}(t) \circ f_m(t)\right|\right|_{pf} \le \sum_{m=1}^{M} ||\Lambda_{n,m}(t) \circ f_m(t)||_{pf} \tag{A.19}$$

with equality if all the $f_m(t)$ except one (say for m=m') are identically zero. Next we have

$$||\Lambda_{n,m}(t) \circ f_m(t)||_{pf} \le ||\Lambda_{n,m}(t) \circ||_{pf} \, ||f_m(t)||_{pf} \tag{A.20}$$

With equality for some choice of $f_m(t)$ (say $f'_{m'}(t)$), at least in a limiting sense (corresponding to the supremum in the definition of the operator norm). Note the use of pf now to also indicate the associated operator norm. Building this back up substitute (A.20) in (A.19) to give

$$\left|\left|\sum_{m=1}^{M} \Lambda_{n,m}(t) \circ f_m(t)\right|\right|_{pf} \le \sum_{m=1}^{M} ||\Lambda_{n,m}(t) \circ||_{pf} \, ||f_m(t)||_{pf} \tag{A.21}$$

with equality if both restrictions above are observed, i.e., the only $f_m(t)$ is $f'_{m'}(t)$).

Placing this last result back in (A.18) gives

$$||(\Lambda_{n,m}(t)) \underset{\bullet}{\circ} (f_n(t))||_p \le \left|\left|\left( \sum_{m=1}^{M} ||\Lambda_{n,m}(t) \circ ||_{pf}\, ||f_m(t)||_{pf} \right)\right|\right|_{pv}$$

$$= ||(\Lambda_{n,m}(t) \circ ||_{pf}) \bullet (||f_n(t)||_{pf})||_{pv} \qquad (A.22)$$

with equality for the two restrictions above. This is now in the form of the vector p-norm of a matrix (with elements $||\Lambda_{n,m}(t) \circ ||_{pf}$) times a vector (with elements $||f_n(t)||_{pf}$) . This gives the inequality

$$||(\Lambda_{n,m}(t)) \underset{\bullet}{\circ} (f_n(t))||_p \le ||(||\Lambda_{n,m}(t) \circ ||_{pf})||_{pv}\, ||(||f_n(t)||_{pf})||_{pv}$$

$$= ||(||\Lambda_{n,m}(t) \circ ||_{pf})||_{pv}\, ||(f_n(t))||_p \qquad (A.23)$$

Note the use of pv now to also indicate the associated matrix norm.

Combining (A.17) and (A.23) now gives the fundamental result

$$||(\Lambda_{n,m}(t)) \circ ||_p \le ||(||\Lambda_{n,m}(t) \circ ||_{pf})||_{pv} \qquad (A.24)$$

If now the result

$$||\Lambda_{n,m}(t) \circ ||_{pf} \le ||\Lambda_{n,m}(t)||_{1f} \qquad (A.25)$$

from [7] is substituted in (A.22) the inequality remains now as

$$||(\Lambda_{n,m}(t)) \underset{\bullet}{\circ} (f_n(t))||_p \le ||(||\Lambda_{n,m}(t)||_{1f}) \bullet (||f_n(t)||_{pf})||_{pv}$$

$$\le ||(||\Lambda_{n,m}(t)||_{1f})||_{pv}\, ||(||f_n(t)||_{pf})||_{pv}$$

$$= ||(||\Lambda_{n,m}(t)||_{1f})||_{pv}\, ||(f_n(t))||_p \qquad (A.26)$$

With (A.17) this gives the inequality

$$||(\Lambda_{n,m}(t)) \circ ||_p \le ||(||\Lambda_{n,m}(t)||_{1f})||_{pv} \qquad (A.27)$$

so that the pf-norm, i.e. the associated operator p-norm of the matrix element operators, is replaced by the 1-norm of the corresponding functions.

From appendix B there are various inequalities that one can use for the pv-norm in (A.27). A particularly simple case occurs if $(\Lambda_{n,m}(\ ))$ is square (NxN) and diagonal in the matrix sense. Then we have

$$||(||\Lambda_{n,m}(t)||_{1f})_{pv} = \max_{1\leq n\leq N} ||\Lambda_{n,n}(t)||_{1f} \tag{A.28}$$

giving

$$||(\Lambda_{n,m}(t))\ o||_p \leq \max_{1\leq n\leq N} ||\Lambda_{n,n}(t)||_{1f} \tag{A.29}$$

Going back to (A.18) a diagonal $(\Lambda_{n,m}(t))o$ means that only m=n is included in the sum giving

$$||(\Lambda_{n,m}(t)) \underset{\bullet}{o} (f_n(t))||_p = ||(||\Lambda_{n,n}(t)\ o\ f_n(t)||_{pf})||_{pv} \tag{A.30}$$

Defining

$$||(\Lambda_{n',n'}(t))\ o||_{pf} \equiv \max_{1\leq n\leq N} ||(\Lambda_{n,n}(t))\ o\ ||_{pf} \tag{A.31}$$

then

$$\begin{aligned} ||(\Lambda_{n,m}(t) \underset{\bullet}{o} (f_n(t))||_p &\leq ||(||\Lambda_{n,n}(t)\ o\ ||_{pf}\ ||f_n(t)||_{pf})||_{pv} \\ &\leq ||(||\Lambda_{n',n'}(t)\ o||_{pf}\ ||f_n(t)||_{pf})||_{pv} \\ &= ||\Lambda_{n',n'}(t)\ o||_{pf}\ ||(||f_n(t)||_{pf})||_{pv} \\ &= ||\Lambda_{n',n'}(t)\ o||_{pf}\ ||(f_n(t))||_p \end{aligned} \tag{A.32}$$

By choosing

$$(f_n(t)) \equiv (0,0,\ldots, f_{n'}(t), \ldots, 0) \tag{A.33}$$

with $f_{n'}(t)$ chosen to given equality in (A.20) (with n=m=n') we have equality in (A.32). This gives for diagonal matrix convolution operators

$$||(\Lambda_{n,m}(t))\circ||_p = ||\Lambda_{n',n'}(t)\circ||_{pf}$$

$$\equiv \max_{1\leq n\leq N} ||\Lambda_{n,n}(t)\circ||_{pf} \tag{A.34}$$

Applying results of [7] for special cases of p we then have special results for diagonal matrix convolution operators. For $p = 1,\infty$ we have

$$||(\Lambda_{n,m}(t))\circ||_1 = ||(\Lambda_{n,m}(t))\circ||_\infty$$

$$\equiv \max_{1\leq n\leq N} ||\Lambda_{n,n}(t)||_{1f} \tag{A.35}$$

For $p = 2$ we have

$$||(\Lambda_{n,m}(t))\circ||_2 = \max_{1\leq n\leq N} |\tilde{\Lambda}_{n,n}(j\omega)|_{\max_\omega}$$

$$= \max_{1\leq n\leq N} |\tilde{\Lambda}_{n,n}(j\omega_{max})|$$

$$\leq \max_{1\leq n\leq N} ||\Lambda_{n,n}(t)||_{1f} \tag{A.36}$$

Note in this last result we have maxima (or suprema) over both n and $\omega$.

Appendix B: Vector and Matrix Norm Inequalities

There are various vector and matrix norm inequalities such as those discussed in [1]. Here a few more are presented. As in [7] we have the Hölder inequality

$$|(x_n)\cdot(y_n)| \equiv ||(x_n)\cdot(y_n)||_2^2$$

$$\leq ||(x_n)||_{p_1}\ ||(y_n)||_{p_2} \tag{B.1}$$

$$1 = \frac{1}{p_1} + \frac{1}{p_2}$$

$$p_1 > 1 \quad , \quad p_2 > 1$$

Setting these two vectors equal we have

$$||(x_n)||_{p_1}||(x_n)||_{p_2} \geq ||(x_n)||_2^2 \tag{B.2}$$

There is the special case (in the limit $p_2 \rightarrow \infty$)

$$||(x_n)||_1 ||(x_n)||_\infty \geq ||(x_n)||_2^2 \tag{B.3}$$

By letting the number of vector components tend to $\infty$ these results can be applied to functions as well [7]. Next a general inequality is found from

$$||(x_n)||_p \equiv \left\{ \sum_{n=1}^{N} |x_n|^p \right\}^{\frac{1}{p}}$$

$$\ell n(||(x_n)||_p) = \frac{1}{p}\ell n\left[ \sum_{n=1}^{N} |x_n|^p \right] \tag{B.4}$$

$$= \frac{1}{p}\ell n\left[ \sum_{n=1}^{N} e^{p\ell n(|x_n|)} \right]$$

Differentiating with respect to p we have

$$A_1 \equiv \frac{d}{dp}\ln(||(x_n)||_p = \frac{1}{||(x_n)||_p}\frac{d}{dp}||(x_n)||_p$$

$$= -\frac{1}{p^2}\ln\left[\sum_{n=1}^{N} e^{p\ln(|x_n|)}\right]$$

$$+ \frac{1}{p\sum_{n=1}^{N} e^{p\ln(|x_n|)}}\sum_{n=1}^{N}\ln(|x_n|)\, e^{p\ln(|x_n|)}$$

$$= -\frac{1}{p}\ln(||(x_n)||_p) + \frac{1}{p||(x_n)||_p^p}\sum_{n=1}^{N}\ln(|x_n|)\,|x_n|^p \qquad \text{(B.5)}$$

Then we have

$$A_2 \equiv A_1\, p\, ||(x_n)||_p^p$$

$$= -\ln(||(x_n)||_p)||(x_n)||_p^p \sum_{n=1}^{N} + \ln(|x_n|)\,|x_n|^p$$

$$= \sum_{n=1}^{N} |x_n|^p\left[\ln(|x_n|) - \ln(||(x_n)||_p)\right]$$

$$= \sum_{n=1}^{N} |x_n|^p \ln\left[\frac{|x_n|}{||(x_n)||_p}\right] \qquad \text{(B.6)}$$

Note that

$$||(x_n)||_p^p = \sum_{n=1}^{N} |x_n|^p \geq |x_{n'}|^p \qquad \text{for } 1 \leq n' \leq N$$

$$||(x_n)||_p \geq |x_{n'}| \tag{B.7}$$

$$\ell n\left[\frac{|x_n|}{||(x_n)||_p}\right] \leq 0 \qquad \text{for } 1 \leq n \leq N$$

$$|x_n|^p \ell n\left[\frac{|x_n|}{||(x_n)||_p}\right] \leq 0$$

Strictly this applies for $|x_n|>0$, but as $|x_n|\to 0$ the limit is zero, so the inequality applies as well for $|x_n|=0$. Then we have

$$A_2 \leq 0$$

$$A_1 \leq 0 \qquad \text{since } \; p\,||(x_n)||_p^p \geq 0 \tag{B.8}$$

$$\frac{d}{dp}||(x_n)||_p \leq 0$$

This last result of a negative derivative gives

$$||(x_n)||_q \leq ||(x_n)||_p \qquad \text{for } q \geq p \tag{B.9}$$

Turning to matrices we have

$$||(A_{n,m})||_p \equiv \sup_{(x_n)\neq 0} \frac{||(A_{n,m}) \cdot (x_n)||_p}{||(x_n)||_p} \tag{B.10}$$

for an NxM matrix. Now we have

$$||(A_{n,m}) \cdot (x_n)||_p = \left\{ \sum_{n=1}^{N} \left| \sum_{m=1}^{M} A_{n,m} x_n \right|^p \right\}^{\frac{1}{p}} \tag{B.11}$$

The Hölder inequality (B.1) gives for each n

$$\left| \sum_{m=1}^{M} A_{n,m} x_m \right| \le \left\{ \sum_{m=1}^{M} |A_{n,m}|^{p'} \right\}^{\frac{1}{p'}} \left\{ \sum_{m=1}^{M} |x_m|^{p} \right\}^{\frac{1}{p}}$$

$$= \| (A_{n,m})_n \|_{p'} \, \| (x_n) \|_p$$

$$1 = \frac{1}{p} + \frac{1}{p'} \tag{B.12}$$

$(A_{n,m})_n \equiv$ nth vector with components $A_{n,1}\ ,\ A_{n,2}\ ,\ \cdots\ ,\ A_{n,M}$

Combining with (B.11) gives

$$\| (A_{n,m}) \bullet (x_n) \|_p \le \| ( \| (A_{n,m})_n \|_{p'} ) \|_p \, \| (x_n) \|_p \tag{B.13}$$

and with (B.10) gives

$$\| (A_{n,m}) \|_p \le \| ( \| (A_{n,m})_n \|_{p'} ) \|_p = \| ( \| (A_{n,m})_n \|_{\frac{p}{p-1}} ) \|_p$$

$$= \left\{ \sum_{n=1}^{N} \left\{ \sum_{m=1}^{M} |A_{n,m}|^{\frac{p}{p-1}} \right\}^{p-1} \right\}^{\frac{1}{p}} \tag{B.14}$$

Note the use of the p-norm of a vector with components the p'-norm of $(A_{n,m})_n$. So this bound takes the form of a vector norm of a vector norm.

Special cases include the 1-norm

$$||(A_{n,m})||_1 \leq ||(||(A_{n,m})_n||_\infty)||_1$$

$$= ||(\max_{1\leq m\leq M})|A_{n,m}|)||_1$$

$$= \sum_{n=1}^{N} \max_{1\leq m\leq M} |A_{n,m}| \qquad \text{(B.15)}$$

which compares to the known result [1]

$$||(A_{n,m})||_1 = \max_{1\leq m\leq M} \sum_{n=1}^{N} |A_{n,m}| \equiv \text{maximum column magnitude sum} \qquad \text{(B.16)}$$

For the 2-norm we have

$$||(A_{n,m})||_2 \leq ||(||(A_{n,m})_n||_2)||_2 = \left\{ \sum_{n=1}^{N} \sum_{m=1}^{M} |A_{n,m}|^2 \right\}^{\frac{1}{2}} \qquad \text{(B.17)}$$

which compares to the known result

$$||(A_{n,m})||_2 = \left\{ \lambda_{max}((A_{n,m})^t \cdot (A_{n,m})) \right\}^{\frac{1}{2}} \qquad \text{(B.18)}$$

$$\lambda_m \equiv \text{maximum eigenvalue}$$

Some additional insight is obtained into the 2-norm inequality by observing [8]

$$\left\{ \sum_{n=1}^{N} \sum_{m=1}^{M} |A_{n,m}|^2 \right\}^{\frac{1}{2}} = \left\{ tr((A_{n,m})^t \cdot (A_{n,m})) \right\}^{\frac{1}{2}}$$

$$= \left\{ \sum_{\ell=1}^{M} \lambda_\ell((A_{n,m})^t \cdot (A_{n,m})) \right\}^{\frac{1}{2}}$$

$$= \left\{ \lambda_{max}((A_{n,m})^t \cdot (A_{n,m})) \right\}^{\frac{1}{2}} \qquad \text{(B.19)}$$

$$(A_{n,m})^t \cdot (A_{n,m}) = \text{MxM Hermitian matrix with real non-negative eigenvalues } \lambda_\ell$$

$$t_r \equiv \text{trace} \equiv \text{sum of diagonal elements}$$

giving some estimate of the tightness of such a bound on the 2-norm.

For the ∞-norm we have

$$||(A_{n,m})||_\infty \leq ||(||(A_{n,m})_n||_1)||_\infty$$

$$= \left|\left|\left(\sum_{m=1}^{M} |A_{n,m}|\right)\right|\right|_\infty$$

$$= \max_{1\leq n\leq N} \sum_{m=1}^{M} |A_{n,m}| \qquad \text{(B.20)}$$

However, in this case the known result is

$$||(A_{n,m})||_\infty = \max_{1\leq n\leq N} \sum_{m=1}^{M} |A_{n,m}| \equiv \text{maximum row magnitude sum} \qquad \text{(B.21)}$$

so that in (B.20) we actually have equality.

By applying (B.9) to the vector norms with

$$||(x_n)||_p \leq ||(x_n)||_1 \qquad \text{for } 1 \leq p < \infty \qquad \text{(B.22)}$$

then (B.14) gives

$$||(A_{n,m})||_p \leq \sum_{n=1}^{N}\sum_{m=1}^{M} |A_{n,m}| \qquad \text{for } 1 \leq p < \infty \qquad \text{(B.23)}$$

as a general upper bound on the p-norm of a matrix.

Let the maximum matrix element in magnitude be $A_{n',m'}$ for $(n,m) = (n',m')$. If there are more than one such element, select any one for this purpose. Then in (B.11) select

$$(x_m) = (0,0, \ldots, x_{m'}, \ldots, 0) \qquad \text{(B.24)}$$

i.e., only the m'th component is nonzero. Then we have

$$\left|\left|(A_{n,m}) \cdot (x_n)\right|\right|_p = \left\{\sum_{n=1}^{N} \left|A_{n,m'}\right|^p \left|x_{m'}\right|^p\right\}^{\frac{1}{p}}$$

$$\geq \left\{\left|A_{n',m'}\right|^p \left|x_m\right|^p\right\}^{\frac{1}{p}}$$

$$= \left|A_{n',m'}\right| \, \left|\left|(x_n)\right|\right| \tag{B.25}$$

which with (B.10) gives

$$\left|\left|(A_{n,m})\right|\right|_p \geq \left|A_{n',m'}\right| \equiv \max_{n,m} \left|A_{n,m}\right| \tag{B.26}$$

Alternately we can find an upper bound from (B.14) by replacing $\left|A_{n,m}\right|$ by the maximum magnitude as above giving

$$\left|\left|(A_{n,m})\right|\right|_p \leq \left\{\sum_{n=1}^{N} \left\{\sum_{m=1}^{M} \left|A_{n',m'}\right|^{\frac{p}{p-1}}\right\}^{p-1}\right\}^{\frac{1}{p}}$$

$$= \left|A_{n',m'}\right| \left\{\sum_{n=1}^{N} \left\{\sum_{m=1}^{M} 1\right\}^{p-1}\right\}^{\frac{1}{p}}$$

$$= \left|A_{n',m'}\right| \left\{\sum_{n=1}^{N} M^{p-1}\right\}^{\frac{1}{p}}$$

$$= \left|A_{n',m'}\right| M^{\frac{p-1}{p}} \left\{\sum_{n=1}^{N} 1\right\}^{\frac{1}{p}}$$

$$= N^{\frac{1}{p}} M^{\frac{p-1}{p}} \left|A_{n',m'}\right| \equiv N^{\frac{1}{p}} M^{\frac{p-1}{p}} \left\{\max_{n,m} \left|A_{n,m}\right|\right\} \tag{B.27}$$

Note for a square matrix (NxN) the coefficient of $|A_{n',m'}|$ is just N. Summarizing we have in terms of the maximum matrix element magnitude

$$\max_{n,m} |A_{n,m}| \leq ||(A_{n,m})||_p \leq N^{\frac{1}{p}} M^{\frac{p-1}{p}} \left\{ \max_{n,m} |A_{n,m}| \right\} \tag{B.28}$$

For square matrices (NxN) we have a lower bound by choosing $(X_n)$ in (B.10) as an eigenvector of $(A_{n,m})$ with eigenvalue $\lambda_\ell$ giving

$$(A_{n,m}) \cdot (x_n)_\ell = \lambda_\ell((A_{n,m}))(x_n)_\ell \tag{B.29}$$

substituting this as one case in (B.10) gives

$$||(A_{n,m}||_p \geq |\lambda_\ell((A_{n,m}))| \tag{B.30}$$

Since this is true for all eigenvalues we have

$$||(A_{n,m}||_p \geq |\lambda((A_{n,m}))|_{max} \tag{B.31}$$

where the right side is also referred to as the spectral radius of $(A_{n,m})$. Note that the above holds for all associated matrix norms, not just p-norms.

As in [1,5] for square matrices a block diagonal matrix can be considered as a supermatrix

$$((A_{n,m})_{u,v}) = \begin{pmatrix} (A_{n,m})_{1,1} & & 0 \\ & (A_{n,m})_{2,2} & \\ 0 & & \ddots \; (A_{n,m})_{N,N} \end{pmatrix}$$

$$\equiv (A_{n,m})_{1,1} \oplus (A_{n,m})_{2,2} \oplus \ldots \oplus (A_{n,m})_{N,N}$$

$$\equiv \bigoplus_{u=1}^{N} (A_{n,m})_{u,u} \tag{B.32}$$

where the diagonal blocks are square and of size $N_u x N_u$. Corresponding to this we have supervectors

$$((x_n)_u) \equiv ((x_n)_1, (x_n)_2, \ldots, (x_n)_N) \tag{B.33}$$

with elementary vectors $(x_n)_u$ of size $N_u$. Now the p-norm of a supervector is just

$$||((x_n)_u||_p = ||(||(x_n)_u||_p)||_p = \left\{ \sum_{n=1}^{N} \sum_{n_u=1}^{N_u} |x_{n_u}|^p \right\}^{\frac{1}{p}} \quad \text{(B.34)}$$

i.e., it is the p-norm of an N component vector, each of whose elements is the p-norm of an $N_u$ component vector.

The norm of a supermatrix is now

$$||((A_{n,m})_{u,v})|| \equiv \sup_{((x_n)_u)\neq((0_n)_u)} \frac{||((A_{n,m})_{u,v}) \odot ((x_n)_u)||}{||((x_n)_u)||} \quad \text{(B.35)}$$

For the p-norm of a supermatrix as in (B.29) this reduces to

$$||((A_{n,m})_{u,v})||_p \equiv \sup_{((x_n)_u)\neq((0_n)_u)} \frac{||((A_{n,m})_{u,u} \bullet (x_n)_u)||_p}{||((x_n)_u)||_p} \quad \text{(B.36)}$$

Now the numerator is just the p-norm of an N component vector each of whose elements is the p-norm of an $N_u$ component vector $(A_{n,m})_{u,u} \bullet (x_n)_u$. So we can write

$$||((A_{n,m})_{u,v})||_p \equiv \sup_{((x_n)_u)\neq((0_n)_u)} \frac{||(||(A_{n,m})_{u,u} \bullet (x_n)_u||_p)||_p}{||(||(x_n)_u||_p)||_p} \quad \text{(B.37)}$$

Now the p-norm of the diagonal blocks are defined by

$$||(A_{n,m})_{u,u}||_p \equiv \sup_{((x_n)_u\neq(0_n)_u} \frac{||(A_{n,m})_{u,u} \bullet (x_n)_u||_p}{||(x_n)_u||_p} \quad \text{(B.38)}$$

giving

$$||(A_{n,m})_{u,u} \bullet (x_n)_u||_p \leq ||(A_{n,m})_{u,u}||_p \, ||(x_n)_u||_p \quad \text{(B.39)}$$

Define

$$||(A_{n,m})_{u',u'}||_p \equiv \max_{1 \le u \le N} ||(A_{n,m})_{u,u}||_p \equiv \text{maximum p norm for any diagonal block} \tag{B.40}$$

where if this occurs for more than one u, pick one arbitrarily. We have from (B.34)

$$\begin{aligned} ||((A_{n,m})_{u,v})||_p &\le \sup_{((x_n)_u) \ne ((0_n)_u)} \frac{||(||(A_{n,m})_{u,u}||_p\, ||(x_n)_u||_p)||_p}{||((x_n)_u)||_p} \\ &\le \sup_{((x_n)_u) \ne ((0_n)\ )} \frac{||(||(A_{n,m})_{u',u'}||_p\, ||(x_n)_u||_p)||_p}{||((x_n)_u)||_p} \\ &\le \sup_{((x_n)_u) \ne ((0_n)\ _u)} \frac{||(A_{n,m})_{u',u'}||_p\, ||(||(x_n)_u||_p)||_p}{||((x_n)_u)||_p} \\ &= ||(A_{n,m})_{u',u'}||_p \end{aligned} \tag{B.41}$$

By choosing

$$((x_n)_u) \equiv ((0_n)_1\, , (0_n)_2\, , \ldots, (x_n)_{u'}\, , \ldots, (0_n)_N) \tag{B.42}$$

with $(X_n)_{u'}$ chosen to give equality in (B.39) then equality is achieved in (B.37). Hence we have the very general result

$$||((A_{n,m})_{u,v})||_p = ||(A_{n,m})_{u',u'}||_p \equiv \max_{1 \le u \le N} ||(A_{n,m})_{u,u}|| \tag{B.43}$$

Thus the p-norm of a block diagonal supermatrix is the maximum p-norm of the diagonal blocks. As a special case if $(A_{n,m})$ is a diagonal NxN matrix the above matrix blocks can be considered as 1x1 giving

$$||(A_{n,m})||_p = \max_{1 \le n \le N} |A_{n,n}| \tag{B.44}$$

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