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Realization of Sublayer Relative Shielding Order in Electromagnetic Topology

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Abstract

A fundamental problem in qualitative electromagnetic topology is the construction of the interaction sequence diagram given a preassigned shielding between all pairs of primary sublayers. Idealizing this into one of relative shielding order makes this problem amenable to graph theoretic treatment. A constructive characterization of the relative shielding order matrix for primary sublayers of an electromagnetic topology defined to the level of layers and sublayers is given. In addition, all possible sublayer topologies with relative shielding order at most 5 are explicitly given.

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A fundamental problem in qualitative electromagnetic topology is the construction of the interaction sequence diagram given a preassigned shielding between all pairs of primary sublayers. Idealizing this into one of relative shielding order makes this problem amenable to graph theoretic treatment. A constructive characterization of the relative shielding order matrix for primary sublayers of an electromagnetic topology defined to the level of layers and sublayers is given. In addition, all possible sublayer topologies with relative shielding order at most 5 are explicitly given.

I. Introduction

Electromagnetic (EM) topology is a mathematical abstraction of the electromagnetic design of systems and is primarily concerned with the electromagnetic interaction process [1,2,3,5]. Given system requirements, one can hopefully find an appropriate electromagnetic topology and assign design specifications to the parts of the system. For an electromagnetic topology defined to the level of layers and sublayers we abstract this by introducing a relative shielding order (the number of subshields crossed in passing between two sublayers) applied to selected sublayers–the primary sublayers–and ask whether an electromagnetic topology exists with these specifications.

In [2] it was shown that for the special case that the relative shielding order between all pairs of primary sublayers was constant R the results were quite simple. In particular, R could only be even and the case $R = 2$ was of practical significance. In this paper we characterize the relative shielding order matrix for the primary sublayers of an EM topology when the primary sublayers are extremal (leaves of the dual graph), when the primary sublayers include the extremal sublayers together with specified internal sublayers, when all sublayers are primary and when the primary sublayers are arbitrary.

Corresponding to the EM topology is the interaction sequence diagram. This diagram is essentially the dual graph of the electromagnetic topology and, as such, may be analyzed from the graph-theoretic vantage. For an EM topology defined to the level of layers and sublayers, the dual graph is particularly nice since it is a tree and our relative shielding order matrix is the distance-matrix applied to selected vertices–the primary vertices–of this tree. We then characterize the relative shielding order matrix for the primary sublayers of an EM topology by solving the analogous problem for the dual graph. Our characterization gives a recursive procedure for constructing the interaction sequence diagram and, hence, an EM topology.

Having determined necessary and sufficient conditions for the realization of EM topologies subject to various constraints on the primary sublayers, illustrative examples are given. In particular, if the maximum relative shielding order $R_{\max}$ among the primary sublayers is given, all possible trees and associated EM topologies, including differences introduced by inversion, are exhibited. This is treated for $R_{\max}$ at most 5 for which the trees and dual EM topologies are not overly complicated.

For electromagnetics, see [2,5] for a discussion of EM topology defined to the level of layers and sublayers, the relative shielding order matrix for primary sublayers and inversion of the interaction sequence diagram at a vertex and see [3] for a discussion of EM topology defined to the level of elementary volumes, which is beyond the scope of this paper.

For graph theory see [4] for a discussion of trees and distance matrices.

II. Realization of Specified Sublayer Relative Shielding Order Matrices

Recall that an electromagnetic topology is given by partitioning Euclidean space into a set $\{V_\lambda\}$ of nested volumes called layers where each V_λ is composed of one or more subvolumes $\{V_{\lambda,\ell}\}$ called sublayers and by separating layers V_λ and $V_{\lambda+1}$ by disjoint closed surfaces $S_{\lambda;\lambda+1}$ called shields where each $S_{\lambda;\lambda+1}$ is composed of one or more closed surfaces $\{S_{\lambda,\ell;\lambda+1,\ell'}\}$ called subshields [1,2,3,5]. We construct the interaction sequence diagram from the EM topology by placing a vertex in each sublayer and joining two sublayers with an edge provided they share a common subshield. (See Figure 1.) Note that the above construction is equivalent to the usual formulation of the interaction sequence diagram as we identify each edge with the appropriate subshield in place of subdividing each edge and then identifying the new vertex with the appropriate subshield.

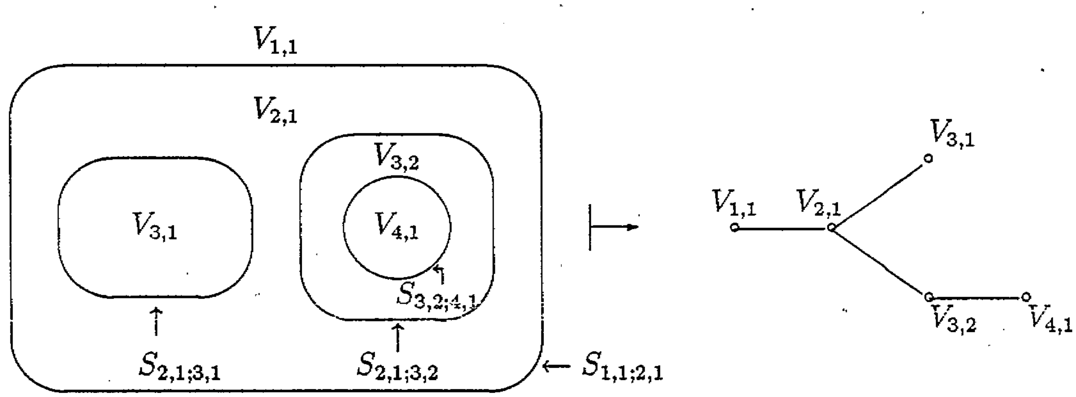


Figure 1.

For example, sublayers $V_{1,1}$, $V_{3,1}$ and $V_{4,1}$ are said to be extremal in the EM topology since they are leaves (endvertices) in the interaction sequence diagram.

Recall that the distance $d_T(v,w)$ between vertices v and w of the interaction sequence diagram T is the number of edges traversed in traveling from one vertex to the other. We assume that the weights of all subshields have been normalized to 1 so that the relative shielding order between sublayers in the EM topology is the distance between the corresponding vertices in the interaction sequence diagram.

We designate certain sublayers of the EM topology as primary sublayers $\Delta^{(1)}$ and the corresponding vertices of the interaction sequence diagram as primary vertices and note that the relative shielding order matrix among primary sublayers is the distance-matrix among the corresponding vertices of the interaction sequence diagram. (See Table 1 where we refer to Figure 1.)

	$V_{1,1}$	$V_{3,1}$	$V_{4,2}$	
$V_{1,1}$	0	2	3	For primary sublayers
$V_{3,1}$	2	0	3	$\Delta^{(1)} = \{V_{1,1}, V_{3,1}, V_{4,2}\}$;
$V_{4,2}$	3	3	0	the extremal sublayers.

and

	$V_{1,1}$	$V_{2,1}$	$V_{3,1}$	$V_{3,2}$	$V_{4,1}$	
$V_{1,1}$	0	1	2	2	3	
$V_{2,1}$	1	0	1	1	2	For primary sublayers
$V_{3,1}$	2	1	0	2	3	$\Delta^{(1)} = \{V_{1,1}, V_{2,1}, V_{3,1}, V_{3,2}, V_{4,1}\}$;
$V_{3,2}$	2	1	2	0	1	all sublayers.
$V_{4,1}$	3	2	3	1	0	

Table 1.

Relative Shielding Order Among Primary Sublayers.

Let $\Upsilon_{m\times m}$ denote the set of $m \times m$ symmetric matrices whose entries are nonnegative integers and which are zero precisely on the main diagonal. For $D \in \Upsilon_{m\times m}$ and distinct $1 \leq i_1, \ldots, i_s \leq m$, we call the $s \times s$ matrix $D_{i_1}, \ldots, i_s$ obtained from the $i_1^{\text{st}}, \ldots, i_s^{\text{th}}$ rows and columns of D a *principal* $s \times s$ submatrix of D. For example, if D is given below

$$D = \begin{bmatrix} 0 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 0 & 2 & 3 \\ 2 & 1 & 2 & 0 & 1 \\ 3 & 2 & 3 & 1 & 0 \end{bmatrix}$$

then

$$D_{1,2,4} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

is a principal 3×3 submatrix of D obtained from the first, second and fourth rows and columns of D.

A. Primary Sublayers are Extremal Sublayers

In this subsection we characterize the symmetric matrices that are relative shielding order matrices of an EM topology where the primary sublayers are extremal by characterizing the symmetric matrices that are distance-matrices for the leaves of a tree.

For $D = (d_{i,j}) \in \Upsilon_{m \times m}$, we say D is *leaf-realizable* iff there exists a tree T with precisely m leaves labelled $\{1, \ldots, m\}$ satisfying $d_T(i,j) = d_{i,j}$ for $1 \leq i, j \leq m$.

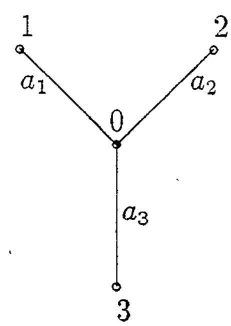
Theorem 1. Let $D = (d_{i,j}) \in \Upsilon_{3 \times 3}$. The following are equivalent.

(a) D is leaf-realizable.

(b) $b_p \in Z_p^+ = \{1,2,3,\ldots\}$ for $p \in \{1,2,3\}$ where

$$b_p = \begin{cases} (d_{i,j} + d_{i,k} - d_{j,k})/2 & \text{for } p = i \\ (d_{p,i} + d_{p,q} - d_{q,i})/2 & \text{for } \{p,q\} = \{j,k\}. \end{cases}$$

Proof. D is leaf-realizable by a tree T iff tree T is as below



where $a_1 = d_T(0,1)$, $a_2 = d_T(0,2)$ and $a_3 = d_T(0,3)$ are positive integers satisfying

$$\begin{array}{rcrcrcl} a_1 & + & a_2 & & & = & d_{1,2} \\ a_1 & & & + & a_3 & = & d_{1,3} \\ & & a_2 & + & a_3 & = & d_{2,3}\ . \end{array} \tag{1}$$

Gauss-Jordan Elimination shows system (1) has the unique solution

$$\begin{aligned} a_1 &= (d_{1,2} + d_{1,3} - d_{2,3})/2 \\ a_2 &= (d_{1,2} - d_{1,3} + d_{2,3})/2 \\ a_3 &= (-d_{1,2} + d_{1,3} + d_{2,3})/2\ . \end{aligned}$$

(a) $\Longrightarrow$ (b) System (1) is consistent and $a_p = b_p$ for $p \in \{1,2,3\}$.

(b) $\Longrightarrow$ (a) Since $b_p = a_p$ for $p \in \{1,2,3\}$, system (1) is consistent. Then $(b_1, b_2, b_3) \in (\mathbf{Z}^+)^3$, the set of 3-tuples of positive integers, is the solution of system (1). Construct T as above so that D is leaf-realizable by T. □

Remark. Condition (b) expresses that the branches a_1, a_2, a_3 of the tree have positive integral length.

Corollary 1.1. If $D \in \Upsilon_{3\times3}$ is leaf-realizable by a tree T then T is unique.

Proof. Tree T is characterized by a_1, a_2, a_3, which are unique. □

Theorem 2. Let $D = (d_{i,j}) \in \Upsilon_{4\times4}$. The following are equivalent.

(a) D is leaf-realizable.

(b) For some $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$ we have

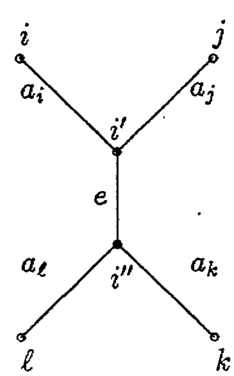
(i) $d_{i,k} + d_{j,\ell} = d_{i,\ell} + d_{j,k}$,

(ii) $b_p \in \mathbf{Z}^+$ for $p \in \{1, 2, 3, 4\}$ where

$$b_p = \begin{cases} (d_{p,j} + d_{p,k} - d_{j,k})/2 & \text{for } p \in \{i, \ell\} \\ (d_{p,i} + d_{p,\ell} - d_{i,\ell})/2 & \text{for } p \in \{j, k\}, \end{cases}$$

(iii) $d_{i,\ell} - b_i - b_\ell \in N = \{0, 1, 2, \ldots\}$.

Proof. D is leaf-realizable by a tree T iff for some $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$ tree T is as below



where $a_i = d_T(i,i')$, $a_j = d_T(i',j)$, $a_k = d_T(i'',k)$, $a_\ell = d_T(i'',\ell)$ are positive integers and $e = d_T(i',i'')$ is a nonnegative integer satisfying

$$\begin{array}{ccccccccccc} a_i & + & a_j & & & & & & & = & d_{i,j} \\ a_i & & & + & a_k & & & + & e & = & d_{i,k} \\ a_i & & & & & + & a_\ell & + & e & = & d_{i,\ell} \\ & & a_j & + & a_k & & & + & e & = & d_{j,k} \\ & & a_j & & & + & a_\ell & + & e & = & d_{j,\ell} \\ & & & & a_k & + & a_\ell & & & = & d_{k,\ell} \end{array} \qquad (2)$$

Gauss-Jordan Elimination shows system (2) has the unique solution

$$\begin{aligned} a_i &= (d_{i,j} + d_{i,\ell} - d_{j,\ell})/2 \\ a_j &= (d_{i,j} - d_{i,\ell} + d_{j,\ell})/2 \\ a_k &= (d_{i,k} - d_{i,\ell} + d_{k,\ell})/2 \\ a_\ell &= (-d_{i,k} + d_{i,\ell} + d_{k,\ell})/2 \\ e &= (-d_{i,j} + d_{i,k} + d_{j,\ell} - d_{k,\ell})/2 \end{aligned}$$

when consistent, while system (2) is consistent iff $d_{i,k} + d_{j,\ell} = d_{i,\ell} + d_{j,k}$.

(a) $\Longrightarrow$ (b) Since system (2) is consistent, b(i) holds. Now $a_p = b_p$ for $p \in \{j,k\}$. Since b(i) holds, $d_{i,k} - d_{j,k} = d_{i,\ell} - d_{j,\ell}$ and $a_i = b_i$ while $d_{j,\ell} - d_{j,k} = d_{i,\ell} - d_{i,k}$ and $a_\ell = b_\ell$. Then $a_p = b_p$ for $p \in \{1,2,3,4\}$ and b(ii) holds. Since $a_i = b_i$ and $a_\ell = b_\ell$, $e = d_{i,\ell} - b_i - b_\ell$ and b(iii) holds.

(b) $\Longrightarrow$ (a) Since b(i) holds, system (2) is consistent and, as above, $b_p = a_p$ for $p \in \{1,2,3,4\}$ so that $d_{i,\ell} - b_i - b_\ell = e$. Then $(b_i, b_j, b_k, b_\ell, e) \in (\mathbf{Z}^+)^4 \times \mathbf{N}$, the set of 5-tuples of nonnegative integers, the first four of which are positive, is the solution of system (2). Construct tree T as above and note that D is leaf-realizable by T by using b(i) to show that $d_T(j,k) = d_{j,k}$. $\square$

Remark. Condition b(i) expresses the consistency of the system, condition b(ii) expresses that the branches a_i, a_j, a_k, a_ℓ have positive integral length and condition b(iii) expresses that e has nonnegative integral length.

Corollary 2.1. If $D \in \Upsilon_{4\times 4}$ is leaf-realizable by a tree T then T is unique.

Proof. Given the labels of the leaves, T is characterized by a_i, a_j, a_k, a_ℓ, e, which are unique. □

Denote the tree whose leaves are precisely $i_1, \ldots, i_s$ by $T\{i_1, \ldots, i_s\}$. For a tree T containing leaves $i_1, \ldots, i_s$, let $T_{i_1,\ldots,i_s}$ denote the subtree of T whose leaves are precisely $i_1, \ldots, i_s$.

We give now the main result of this subsection.

Theorem 3. $D = (d_{i,j}) \in \Upsilon_{m\times m}$ is leaf-realizable iff all principal 4×4 submatrices of D are leaf-realizable.

Proof. ($\Longrightarrow$) Examine the tree for the appropriate subtree.
($\Longleftarrow$) Let D be a counterexample to the theorem with m as small as possible. Necessarily $m \geq 5$. Then $D' = D_{1,\ldots,m-1}$ satisfies the hypotheses of the theorem and, by the minimality of m, is leaf-realizable by T'.

For $1 \leq i, j, k, k' \leq m-1$, note that $T\{i,j,k,m\}_{i,j,m} = T\{i,j,k',m\}_{i,j,m} \equiv T^*_{i,j}$ and $T\{i,j,k,m\}_{i,j,k} = T'_{i,j,k}$ by Theorems 1, 2 (Easy to arrange equality). Define $x(i,j,k)$ as the vertex of degree at least 3 in $T\{i,j,k,m\}$ that is closest to m and define

$$\begin{aligned}\alpha(i,j,k) &= d_{T\{i,j,k,m\}}(m, x(i,j,k))\,,\\ \alpha(i,j) &= \min\{\alpha(i,j,k) : k \in \{1,\ldots,m-1\} - \{i,j\}\} \text{ and}\\ \alpha &= \min\{\alpha(i,j) : 1 \leq i < j \leq m-1\}\,.\end{aligned}$$

Let $\alpha = \alpha(i,j,k)$ and let $\ell \in \{1,\ldots,m-1\}-\{i,j\}$. Consequently, $d_{T\{i,j,\ell,m\}}(m, x(i,j,\ell)) \geq \alpha$ and $x(i,j,\ell)$ is an internal vertex of the (i,j)-path in T'. Let $T = T' \cup T^*_{i,j}$. Then

$$\begin{aligned} d_T(m,\ell) &= d_{T^*_{i,j}}(m, x(i,j,\ell)) + d_{T'}(x(i,j,\ell),\ell) \\ &= d_{T\{i,j,\ell,m\}}(m, x(i,j,\ell)) + d_{T\{i,j,\ell,m\}}(x(i,j,\ell),\ell) \\ &= d_{T\{i,j,\ell,m\}}(m,\ell) = d_{m,\ell} \,. \end{aligned}$$

Hence, D is leaf-realizable by T. □

Corollary 3.1. $D = (d_{i,j}) \in \Upsilon_{m\times m}$ is leaf-realizable iff for all $1 \leq i_1 < i_2 < i_3 < i_4 \leq m$ some $\{i,j,k,\ell\} = \{i_1,i_2,i_3,i_4\}$ satisfies

(i) $d_{i,k} + d_{j,\ell} = d_{i,\ell} + d_{j,k}$,

(ii) $b_p \in \mathbf{Z}^+$ for $p \in \{1,2,3,4\}$ where

$$b_p = \begin{cases} (d_{p,j} + d_{p,k} - d_{j,k})/2 & \text{for } p \in \{i,\ell\} \\ (d_{p,i} + d_{p,\ell} - d_{i,\ell})/2 & \text{for } p \in \{j,k\}, \end{cases}$$

(iii) $d_{i,\ell} - b_i - b_\ell \in \mathbf{N}$

Proof. Use Theorems 2,3. □

Remark. See the remark following Theorem 2 for a discussion of conditions (i), (ii), (iii).

Corollary 3.2. If $D \in \Upsilon_{m\times m}$ is leaf-realizable by a tree T then T is unique.

Proof. Use Corollary 2.1 and the proof of Theorem 3. □

Remark. The proof of Theorem 3 is constructive and gives an algorithm for finding the tree T that leaf-realizes $D \in \Upsilon_{m\times m}$.

Leaf-Realizable Algorithm

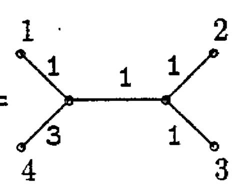
1. Set $T_4 = T\{1,2,3,4\}$.
2. For $\ell \leq m$, choose $1 \leq i,j,k \leq \ell - 1$ with $d_{T\{i,j,k,\ell\}}(x,\ell)$ as small as possible where x is the vertex of degree at least 3 closest to ℓ in $T\{i,j,k,\ell\}$.
3. Set $T_\ell = T_{\ell-1} \cup T\{i,j,k,\ell\}$.
4. Stop.

Example 1. We illustrate the algorithm for D below

$D =$

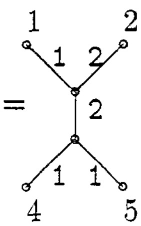
	1	2	3	4	5	6
1	0	3	3	4	4	4
2	3	0	2	5	5	5
3	3	2	0	5	5	5
4	4	5	5	0	2	4
5	4	5	5	2	0	4
6	4	5	5	4	4	0

By Theorem 2,

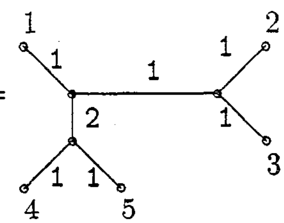
$T_4 = T\{1,2,3,4\} =$ 

(edge weights w denote path lengths)

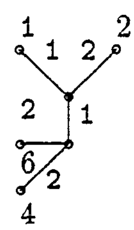
For $1 \leq i, j, k \leq 4$, the minimum value of $d_{T\{i,j,k,5\}}(x, 5)$ is 1 for

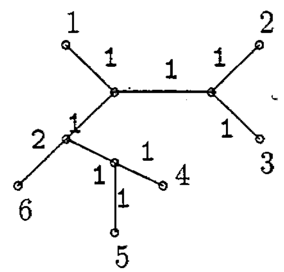
$T\{1,2,4,5\} =$  by Theorem 2

so that

$T_5 =$  .

For $1 \leq i, j, k \leq 5$, the minimum value of $d_{T\{i,j,k,6\}}(x, 6)$ is 2 for

$T\{1,2,4,6\} =$  by Theorem 2 so that

$T_6 =$ 

□

We now show that Theorem 3 is best possible in that the minimum size of the testing submatrices is 4×4.

Example 2. For $m \geq 4$ and even $R \geq 4$, let D be the symmetric matrix defined below.

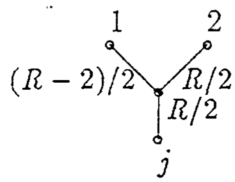
$D =$

	1	2	3	4	$\cdots$	m
1	○	$R-1$	$R-2$	$R-2$	$\cdots$	$R-2$
2		○	$R-1$	$R-1$	$\cdots$	$R-1$
3			○	R	$\cdots$	R
4				○	$\cdots$	R
$\vdots$					$\ddots$	$\vdots$
m						○

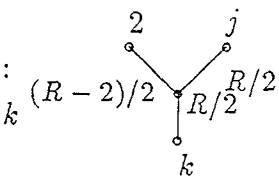
← All nondiagonal entries are R.

All principal 3×3 submatrices of D are leaf-realizable:

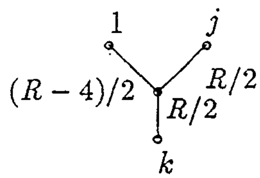
$\{1, 2, j\}$: $j \geq 3$



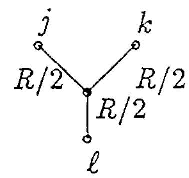
$\{2, j, k\}$: $3 \leq j < k$



$\{1, j, k\}$: $3 \leq j < k$



$\{j, k, \ell\}$: $3 \leq j < k < \ell$



However, D is not leaf-realizable, since $D_{1,2,3,4}$ is not leaf-realizable, otherwise as in the proof of Theorem 2 with $i = 1$, we have:

$(j, k, \ell) = (2, 3, 4)$ implies $a_2 = R/2$, $a_3 = R/2$ so that $d_{2,3} \geq R$, a contradiction. (Similarly, $(j, k, \ell) = (2, 4, 3)$.)

$(j, k, \ell) = (3, 2, 4)$ implies $a_3 = R/2$, $a_4 = (R - 2)/2$ but then $a_2 = R/2$ so that $e = 0$ and $d_{3,4} = R - 1$, a contradiction. (Similarly, $(j, k, \ell) = (3, 4, 2)$.)

$(j, k, \ell) = (4, 2, 3)$ implies $a_3 = (R - 2)/2$, $a_4 = R/2$ but then $a_2 = R/2$ and $d_{2,4} \geq R$, a contradiction. (Similarly, $(j, k, \ell) = (4, 3, 2)$.) □

B. Primary Sublayers Include Extremal Sublayers and Specified Internal Sublayers

In this subsection we characterize the symmetric matrices that are relative shielding order matrices of an EM topology where the primary sublayers include the extremal sublayers by characterizing the symmetric matrices that are distance matrices for at least the leaves of a tree.

For $D = (d_{i,j}) \in \Upsilon_{4\times4}$ and $L \subseteq \{1,2,3,4\}$, we say D is *L-realizable* iff there exists a tree T with labelled vertices 1, 2, 3, 4 so that $L \subseteq V_1(T) \subseteq \{1,2,3,4\}$ and satisfying $d_T(i,j) = d_{i,j}$ for $i,j \in \{1,2,3,4\}$. Here $V_1(T)$ denotes the leaves of T.

Theorem 4. Let $D = (d_{i,j}) \in \Upsilon_{4\times4}$ and $L \subseteq \{1,2,3,4\}$. The following are equivalent.

(a) D is L-realizable.

(b) For some $\{i,j,k,\ell\} = \{1,2,3,4\}$ we have

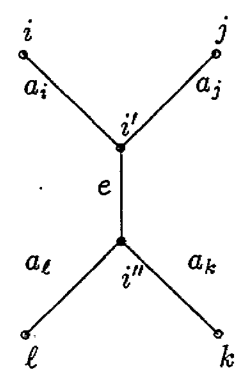
(i) $d_{i,k} + d_{j,\ell} = d_{i,\ell} + d_{j,k}$,

(ii) $b_p \in \mathbf{N}$ for $p \in \{1,2,3,4\} - L$ and $b_p \in \mathbf{Z}^+$ for $p \in L$ where

$$b_p = \begin{cases} (d_{p,j} + d_{p,k} - d_{j,k})/2 & \text{for } p \in \{i,\ell\} \\ (d_{p,i} + d_{p,\ell} - d_{i,\ell})/2 & \text{for } p \in \{j,k\}, \end{cases}$$

(iii) $d_{i,\ell} - b_i - b_\ell \in \mathbf{N}$.

Proof. D is L-realizable by a tree T iff for some $\{i,j,k,\ell\} = \{1,2,3,4\}$ tree T is as below.



where $a_i = d_T(i,i')$, $a_j = d_T(i',j)$, $a_k = d_T(i'',k)$, $a_\ell = d_T(i'',\ell)$ and $e = d_T(i',i'')$ are nonnegative integers satisfying

$$\begin{array}{ccccccccccc} a_i & + & a_j & & & & & & & = & d_{i,j} \\ a_i & & & + & a_k & & & + & e & = & d_{i,k} \\ a_i & & & & & + & a_\ell & + & e & = & d_{i,\ell} \\ & & a_j & + & a_k & & & + & e & = & d_{j,k} \\ & & a_j & & & + & a_\ell & + & e & = & d_{j,\ell} \\ & & & & a_k & + & a_\ell & & & = & d_{k,\ell} \end{array} \tag{3}$$

with $a_p \in \mathbf{Z}^+$ for $p \in L$. Gauss-Jordan Elimination shows system (3) has the unique solution

$$\begin{aligned} a_i &= (d_{i,j} + d_{i,\ell} - d_{j,\ell})/2 \\ a_j &= (d_{i,j} - d_{i,\ell} + d_{j,\ell})/2 \\ a_k &= (d_{i,k} - d_{i,\ell} + d_{k,\ell})/2 \\ a_\ell &= (-d_{i,k} + d_{i,\ell} + d_{k,\ell})/2 \\ e &= (-d_{i,j} + d_{i,k} + d_{j,\ell} - d_{k,\ell})/2 \end{aligned}$$

when consistent, while system (3) is consistent iff $d_{i,k} + d_{j,\ell} = d_{i,\ell} + d_{j,k}$.

(a) $\Longrightarrow$ (b) Since system (3) is consistent, b(i) holds. Now $a_p = b_p$ for $p \in \{j,k\}$. Since b(i) holds, $d_{i,k} - d_{j,k} = d_{i,\ell} - d_{j,\ell}$ and $a_i = b_i$ while $d_{j,\ell} - d_{j,k} = d_{i,\ell} - d_{i,k}$ and $a_\ell = b_\ell$. Then $a_p = b_p$ for $p \in \{1,2,3,4\}$ and b(ii) holds. Since $a_i = b_i$ and $a_\ell = b_\ell$, $e = d_{i,\ell} - b_i - b_\ell$ and b(iii) holds.

(b) $\Longrightarrow$ (a) Since b(i) holds, system (3) is consistent and, as above, $b_p = a_p$ for $p \in \{1,2,3,4\}$ so that $d_{i,\ell} - b_i - b_\ell = e$. Then $(b_i, b_j, b_k, b_\ell, e) \in \mathbf{N}^5$, the set of 5-tuples of nonnegative integers, is the solution of system (3). Construct tree T as above and note that D is L-realizable by T by using b(i) to show that $d_T(j,k) = d_{j,k}$ and using b(ii) to insure $L \subseteq V_1(T)$. □

Remark. Condition b(i) expresses the consistency of the system, condition b(ii) expresses that the branches a_i, a_j, a_k, a_ℓ have nonnegative integral lengths and condition b(iii) expresses that e has nonnegative integral length.

Corollary 4.1. If $D \in \Upsilon_{4\times 4}$ is L-realizable by a tree T then T is unique.

Proof. Given the labels, T is characterized by a_i, a_j, a_k, a_ℓ, e, which are unique. $\square$

For $D = (d_{i,j}) \in \Upsilon_{(m+r)\times(m+r)}$, we say D is *(m,r)-realizable* iff there exists a tree T with precisely m leaves labelled $\{1,\ldots,m\}$ and (at least) r internal vertices labelled $\{m+1,\ldots,m+r\}$ satisfying $d_T(i,j) = d_{i,j}$ for $1 \le i,j \le m+r$ and we say D is *r-linear* iff for all $m+1 \le k \le m+r$ and for all $1 \le i \le m$ there exists $1 \le j \le m$ satisfying $d_{i,j} = d_{i,k} + d_{k,j}$. In words, D is r-linear iff for each internal vertex $m+1 \le k \le m+r$ and each leaf $1 \le i \le m$ there exists another leaf $1 \le j \le m$ with vertex k on the (i,j)-path in the tree.

For $D \in \Upsilon_{(m+r)\times(m+r)}$ and $E = D_{i_1,\ldots,i_s}$ where $1 \le i_1 < \cdots < i_s \le m+r$, let $L(E) = \{i_1,\ldots,i_s\} \cap \{1,\ldots,m\}$.

We give now the main result of this subsection.

Theorem 5. $D = (d_{i,j}) \in \Upsilon_{(m+r)\times(m+r)}$ is (m,r)-realizable iff D is r-linear and all principal 4×4 submatrices E of D are $L(E)$-realizable.

Proof. ($\Longrightarrow$) Examine the tree for the appropriate path and for the appropriate subtree and note that leaves of the tree are leaves of the subtree.

($\Longleftarrow$) Let D be a counterexample to the theorem with $m+r$ as small as possible. Necessarily $m+r \ge 5$ and $r \ge 1$. Then $D' = D_{1,\ldots,m+r-1}$ satisfies the hypotheses of the theorem and, by the minimality of $m+r$, is $(m,r-1)$-realizable by T'. Let $d_{i,j} = d_{i,m+r} + d_{m+r,j}$ and, using this, let T be obtained from T' by labelling the appropriate vertex of the (i,j)-path P in T' with label $m+r$. Let $\ell \in \{1,\ldots,m+r\} - \{i,j,m+r\}$. Then $T\{i,j,\ell,m+r\}_{i,j,\ell} = T_{i,j,\ell}$ and $T\{i,j,\ell,m+r\}_{i,j,m+r} = T_{i,j,m+r}$. (Easy to arrange equality.) For ℓ on P, the tree $T\{i,j,\ell,m+r\}$ is the (i,j)-path P with the labelled vertices ℓ and $m+r$ so that

$$d_T(\ell, m+r) = d_{\ell,m+r}$$

while for ℓ not on P and y the vertex of degree 3 in $T\{i,j,\ell,m+r\}$ we have

$$\begin{aligned} d_T(\ell,m+r) &= d_T(\ell,y) + d_T(y,m+r) \\ &= d_{T\{i,j,\ell,m+r\}}(\ell,y) + d_{T\{i,j,\ell,m+r\}}(y,m+r) \\ &= d_{T\{i,j,\ell,m+r\}}(\ell,m+r) = d_{\ell,m+r}\ . \end{aligned}$$

Hence, D is (m,r)-realizable by T. □

Corollary 5.1. $D = (d_{i,j}) \in \Upsilon_{(m+r)\times(m+r)}$ is (m,r)-realizable iff for all $1 \le i_1 < i_2 < i_3 < i_4 \le m+r$ some $\{i,j,k,\ell\} = \{i_1,i_2,i_3,i_4\}$ satisfies

(i) $d_{i,k} + d_{j,\ell} = d_{i,\ell} + d_{j,k}$,

(ii) $b_p \in \mathbf{N}$ for $p \in \{i_1,i_2,i_3,i_4\} - \{1,\ldots,m\}$ and $b_p \in \mathbf{Z}^+$ for $p \in \{i_1,i_2,i_3,i_4\} \cap \{1,\ldots,m\}$ where

$$b_p = \begin{cases} (d_{p,j} + d_{p,k} - d_{j,k})/2 & \text{for } p \in \{i,\ell\} \\ (d_{p,i} + d_{p,\ell} - d_{i,\ell})/2 & \text{for } p \in \{j,k\}, \end{cases}$$

(iii) $d_{i,\ell} - b_i - b_\ell \in \mathbf{N}$.

Proof. Use Theorems 4, 5. □

Remark. See the remark following Theorem 4 for a discussion of conditions (i), (ii), (iii).

Remark. The proof of Theorem 5 is constructive and gives an algorithm for finding the tree T that (m,r)-realizes $D \in \Upsilon_{(m+r)\times(m+r)}$.

(m, r)-Realizable Algorithm

1. Set $T_m = T\{1, \ldots, m\}$.
2. For $\ell \leq m + r$, choose $1 \leq i, j \leq m$ with $d_{i,j} = d_{i,\ell} + d_{\ell,j}$.
3. Set $T_\ell = T_{\ell-1}$ where the appropriate vertex has been labelled ℓ.
4. Stop.

We now show that Theorem 5 is best possible in that the minimum size of the testing submatrices is 4×4.

Example 3. Let T be a tree whose longest path has length 2ℓ and with precisely m leaves labelled $\{1, \ldots, m\}$ and (at least) $r - 1$ internal vertices labelled $\{m+1, \ldots, m+r-1\}$ so that some longest path has endvertices 1, 2 and center $m+1$. Let D be the symmetric matrix defined below.

	1	2	$\cdots$	$m+1$	$\cdots$	$m+r-1$	$m+r$
1	○	2ℓ	$\cdots$	ℓ	$\cdots$		ℓ
2		○	$\cdots$	ℓ	$\cdots$		ℓ
$\vdots$			$\ddots$				$\vdots$
$m+1$				○	$\cdots$		2
$\vdots$					$\ddots$		$\vdots$
$m+r-1$						○	
$m+r$							○

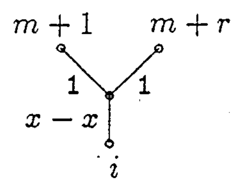
Columns $m+1$, $m+r$ are identical except that $d_{m+1,m+r} = 2$ and $D_{1,\ldots,m+r-1}$ is the distance matrix for the labelled vertices of T. Since $m+r$ acts as $m+1$, D is r-linear. All principal

3×3 submatrices E of D are $L(E)$-realizable:

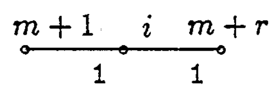
$\{i,j,k\} \subseteq \{1,\ldots,m+r-1\}$: Consider the subtree of T with leaves contained in $\{i,j,k\}$.

$\{i,j,m+r\} \subseteq \{1,\ldots,m+r\}$: Consider $\{i,j,m+1\}$ and argue as above.
where $m+1 \notin \{i,j\}$.

$\{i,m+1,m+r\}$: For $d_{i,m+1} = d_{i,m+r} = x \geq 2$, consider



the tree with leaves $\{i,m+1,m+r\}$. While for $d_{i,m+1} = d_{i,m+r} = 1$, consider



the path of length 2 with leaves $\{m+1,n\}$. Note that $i \notin \{1,\ldots,m\}$.
However, D is not (m,r)-realizable since any tree realization must contain a cycle, a contradiction. □

C. All Sublayers are Primary Sublayers

In this subsection we characterize the symmetric matrices that are relative shielding order matrices of an EM topology where all sublayers are primary sublayers by characterizing the symmetric matrices that are distance-matrices for a tree.

For $D = (d_{i,j}) \in \Upsilon_{n\times n}$ and $1 \le i \le n$, let $d(i) = |\{1 \le j \le n : d_{i,j} = 1\}|$ and let $L(D) = \{1 \le i \le n : d(i) = 1\}$.

For $D = (d_{i,j}) \in \Upsilon_{n\times n}$, we say D is *tree-realizable* iff there exists a tree T with precisely n vertices labelled $\{1, \ldots, n\}$ satisfying $d_T(i,j) = d_{i,j}$ for $1 \le i, j \le n$ and we say D is *linear* iff $|L(D)| \ge 2$ and for all $k \notin L(D)$ and for all $i \in L(D)$ there exists $j \in L(D)$ satisfying $d_{i,j} = d_{i,k} + d_{k,j}$.

For $D \in \Upsilon_{n\times n}$ and $E = D_{i_1,\ldots,i_s}$ where $1 \le i_1 < \cdots < i_s \le n$, let $L(E) = \{i_1, \ldots, i_s\} \cap L(D)$.

We give now the main result of this section.

Theorem 6. $D = (d_{i,j}) \in \Upsilon_{n\times n}$ is tree-realizable iff D is linear and all principal 4×4 submatrices E of D are $L(E)$-realizable.

Proof. ($\Longrightarrow$) Examine the tree for the appropriate path and for the appropriate subtree and note that leaves of the tree are leaves of the subtree.

($\Longleftarrow$) Let D be a counterexample to the theorem with n as small as possible. Necessarily $n \ge 5$. Let $D' = D_{1,\ldots,i-1,i+1,\ldots,n}$ where $i \in L(D)$ and $d_{i,i'} = 1$. Then $i' \notin L(D)$ or the tree $T\{i, i', k, \ell\}$ of order at least 4 contains adjacent leaves i, i', a contradiction.

Now $|L(D')| = 1$ implies $|L(D)| = 2$ and $d(i') \ge 3$. Let $L(D) = \{i, j\}$ and $d_{i',i} = d_{i',i''} = d_{i',i'''} = 1$. Then $T\{i, i', i'', i'''\}$ is a tree of order 4 isomorphic to $K_{1,3}$ (the tree with precisely 3 leaves and 1 center vertex) and we must have

$$d_{i,i''} = d_{i,i'''} = d_{i'',i'''} = 2 \; . \tag{1}$$

Then one of $i'', i''' \notin L(D) = \{i, j\}$. Let $i'' \notin L(D)$ so that $d_{i,j} = d_{i,i''} + d_{i'',j}$ and by (1), $i''' \neq j$. Then $i''' \notin L(D) = \{i, j\}$ so that $d_{i,j} = d_{i,i'''} + d_{i''',j}$. Then $T\{i, i'', i''', j\}$ is a path with endvertices i, j and distinct internal vertices i'', i''', contradicting (1). Hence, $|L(D')| \geq 2$.

Let $k \notin L(D') \cup \{i\}$ (so $k \notin L(D)$) with $d_{i,j} = d_{i,k} + d_{k,j}$ for $j \in L(D)$.

Now $i' \in L(D')$ implies $i' \neq k$ while $i' \notin L(D)$ so that $i' \neq j$. Since $d_{i,i'} = 1$, $T\{i, i', j, k\}$ is a path with endvertices i, j and distinct internal vertices i', k and we must have $d_{i',j} = d_{i',k} + d_{k,j}$. Assume $i' \notin L(D')$ so that $d(i') \geq 3$. Observe that $d_{i,i''} = 2$ for all $i'' \neq i$ with $d_{i',i''} = 1$. Since no tree of order at least 4 contains adjacent leaves, $i' \neq j$.

Now $i'' = j$ with $d_{i',i''} = 1$ implies $i' = k$ or $T\{i, i', j, k\}$ is a path of length 2 with endvertices i, j and distinct internal vertices i', k, a contradiction. Then there exists $i''' \neq i, j, k$ with $d_{i',i'''} = 1$ since $d(i') \geq 3$. Now $i''' \in L(D')$ (so $i''' \in L(D)$) implies $T\{i, i''', j, k\}$ is a tree with leaves i, i''', j and internal vertex k is on the (i''', j)-path there since $d_{i,i'''} = 2$. Assume $i''' \notin L(D')$ (so $i''' \notin L(D)$) so that $d_{j,j'} = d_{j,i'''} + d_{i''',j'}$ for $j' \in L(D)$. Now $j' = i$ implies $T\{i, i''', j, k\}$ is a path of length 2 with endvertices i, j and distinct internal vertices i''', k, a contradiction. Assume $i'' \neq j$ for all $d_{i',i''} = 1$.

Now $i'' \in L(D')$ (so $i'' \in L(D)$) with $d_{i',i''} = 1$ and $i'' \neq i$ implies $i'' \neq i, j, k$ since $k \notin L(D')$ and $T\{i, i'', j, k\}$ is a tree with leaves i, i'', j and k on the (i'', j)-path there since $d_{i,i''} = 2$. Assume $i'' \notin L(D')$ for all $d_{i',i''} = 1$ with $i'' \neq i$ (so $i'' \notin L(D)$).

Now $d_{i,j} = d_{i,i''} + d_{i'',j}$ for all $d_{i',i''} = 1$ implies there exist distinct $i'', i''' \neq i, j$ with $d_{i',i''} = d_{i',i'''} = 1$ since $d(i') \geq 3$ and $T\{i, i'', i''', j\}$ is a path with endvertices i, j and distinct internal vertices i'', i''' each distance 2 from i, a contradiction. Hence, $d_{i,j} \neq d_{i,i''} + d_{i'',j}$ for some $d_{i',i''} = 1$ so that $i'' \neq i, j, k$. Since $i'' \notin L(D')$ (so $i'' \notin L(D)$) we have $d_{j,j'} = d_{j,i''} + d_{i'',j'}$ for $j' \in L(D)$. Then $j' \neq i$ (so $j' \in L(D')$), $j' \neq i'', k$ and $T\{i'', j, j', k\}$ is a path with endvertices j, j' and internal vertices i'', k and we must have $d_{j,j'} = d_{j,k} + d_{k,j'}$.

Then D' satisfies the hypotheses of the theorem and, by the minimality of n, is tree-realizable by T'. Construct T from T' by adding vertex i and edge ii'.

For any distinct $j,k \in \{1,\ldots,n\} - \{i,i'\}$, i' is on the (i,j)-path in $T\{i,i',j,k\}$ since $d_{i,i'} = 1$ and we must have

$$\begin{aligned} d_{i,j} &= d_{i,i'} + d_{i',j} \\ &= 1 + d_{i',j} \\ &= 1 + d_{T'}(i',j) \\ &= d_T(i,j) \ . \end{aligned}$$

Hence, D is tree-realizable by T. □

Corollary 6.1. $D = (d_{i,j}) \in \Upsilon_{n\times n}$ is tree-realizable iff for all $1 \le i_1 < i_2 < i_3 < i_4 \le n$ some $\{i,j,k,\ell\} = \{i_1,i_2,i_3,i_4\}$ satisfies

(i) $d_{i,k} + d_{j,\ell} = d_{i,\ell} + d_{j,k}$,

(ii) $b_p \in \mathbf{N}$ for $p \in \{i_1,i_2,i_3,i_4\} - L(D)$ and $b_p \in \mathbf{Z}^+$ for $p \in \{i_1,i_2,i_3,i_4\} \cap L(D)$ where

$$b_p = \begin{cases} (d_{p,j} + d_{p,k} - d_{j,k})/2 & \text{for } p \in \{i,\ell\} \\ (d_{p,i} + d_{p,\ell} - d_{i,\ell})/2 & \text{for } p \in \{j,k\}, \end{cases}$$

(iii) $d_{i,\ell} - b_i - b_\ell \in \mathbf{N}$.

Proof. Use Theorems 4, 6. □

Remark. See the remark following Theorem 4 for a discussion of conditions (i), (ii), (iii).

Remark. The proof of Theorem 6 is constructive and gives an algorithm for finding the tree T that tree-realizes $D \in \Upsilon_{n\times n}$.

Tree-Realizable Algorithm

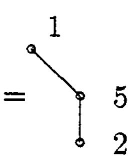
1. Set $T_3 = T\{i_1, i_2, i_3\}$ a path of length 2.
2. For $\ell \le n$, choose $i_k \in \{i_1, \ldots, i_{\ell-1}\}$, $i_\ell \in \{1, \ldots, n\} - \{i_i, \ldots, i_{\ell-1}\}$ with $d_{i_k, i_\ell} = 1$.
3. Set $T_\ell = T_{\ell-1}$ plus the edge $i_k i_\ell$.
4. Stop.

Example 4. We illustrate the algorithm for D below.

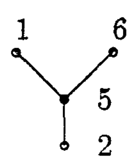
$D =$

	1	2	3	4	5	6
1	0	2	4	3	1	2
2	2	0	2	1	1	2
3	4	2	0	1	3	4
4	3	1	1	0	2	3
5	1	1	3	2	0	1
6	2	2	4	3	1	0

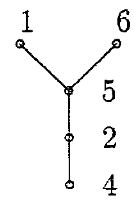
Let

$T_3 = T\{1, 2, 5\} =$ 

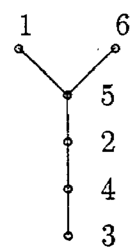
First $d_{5,6} = 1$, so that

$T_4 =$ 

Next $d_{2,4} = 1$, so that

$T_5 =$ 

Finally $d_{3,4} = 1$, so that

$T_6 =$ 

□

We now show that Theorem 6 is best possible in that the minimum size of the testing submatrices is 4×4.

Example 5. Let T be a tree of order $n-1$ with longest path $1, 2, \ldots, 2m+1$ where $m \geq 3$ and let D be the symmetric matrix defined below.

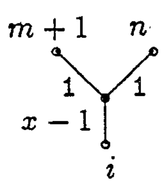
	1	2	$\cdots$	$m+1$	$\cdots$	$2m+1$	$\cdots$	$n-1$	n
1	○	1	$\cdots$	m	$\cdots$	$2m$	$\cdots$		m
2		○	$\cdots$	$m-1$	$\cdots$	$2m-1$	$\cdots$		$m-1$
$\vdots$			$\ddots$						$\vdots$
$m+1$				○	$\cdots$	m	$\cdots$		2
$\vdots$					$\ddots$				$\vdots$
$2m+1$						○	$\cdots$		
$\vdots$							$\ddots$		
$n-1$								○	
n									○

Columns $m+1, n$ are identical except that $d_{m+1,n} = 2$ and $D_{1,\ldots,n-1}$ is the distance matrix for T. Since $L(D)$ is precisely the leaves of T and n acts as $m+1$, D is linear. All principal 3×3 submatrices E of D are $L(E)$-realizable:

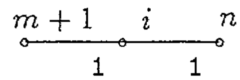
$\{i,j,k\} \subseteq \{1,\ldots,n-1\}$: Consider the subtree of T with leaves contained in $\{i,j,k\}$.

$\{i,j,n\} \subseteq \{1,\ldots,n\}$: Consider $\{i,j,m+1\}$ and argue as above.
where $m+1 \notin \{i,j\}$

$\{i,m+1,n\}$: For $d_{i,m+1} = d_{i,n} = x \geq 2$, consider



the tree with leaves $\{i,m+1,n\}$. While for $d_{i,m+1} = d_{i,n} = 1$, consider



the path of length 2 with leaves $\{m+1, n\}$. Note that $i \notin L(D)$.

However, D is not tree-realizable since any tree realization must contain the cycle m, $m+1$, $m+2$, n, m, a contradiction. □

D. Primary Sublayers are Arbitrary Sublayers

In this subsection we characterize the symmetric matrices that are relative shielding order matrices of an EM topology where the primary sublayers are arbitrary by characterizing the symmetric matrices that are distance-matrices of a set of vertices for a tree.

For $D = (d_{i,j}) \in \Upsilon_{pxp}$, we say D is *realizable* iff there exists a tree T containing p vertices labelled $\{1, \ldots, p\}$ satisfying $d_T(i,j) = d_{i,j}$ for $1 \le i, j \le p$.

Remark. With no loss of generality, we may assume that all leaves of T are labelled vertices in that we may rid the EM topology of excess secondary sublayers.

We give now the main result of this subsection.

Theorem 7. $D = (d_{i,j}) \in \Upsilon_{pxp}$ is realizable iff all principal 4×4 submatrices of D are realizable.

Proof. ($\Longrightarrow$) Examine the tree for the appropriate subtree.

($\Longleftarrow$) Let D be a counterexample to the theorem with p as small as possible. Necessarily

$p \geq 5$. Now $d_{i,j} \neq d_{i,k} + d_{k,j}$ for all $1 \leq i,j,k \leq p$ implies that any realizable matrix is. in fact, leaf-realizable and, consequently, D is leaf-realizable by Theorem 3, a contradiction. Choose $d_{i,j} = d_{i,k} + d_{k,j}$ with $d_{i,j}$ as large as possible. Then $D' = D_{1,\ldots,k-1,k+1,\ldots,p}$ satisfies the hypotheses of the theorem and, by the minimality of p, is realizable by T' with vertices i, j as leaves. Use $d_{i,j} = d_{i,k} + d_{k,j}$ to construct T from T' by labelling the appropriate vertex on the (i,j)-path in T' with label k. For $\ell \in \{1, \ldots, p\} - \{i,j,k\}$, note that $T\{i,j,k,\ell\}_{i,j,k} = T_{i,j,k}$ and $T\{i,j,k,\ell\}_{i,j,\ell} = T_{i,j,\ell}$ by Theorems 1, 2. (Easy to arrange equality.)

For $T\{i,j,k,\ell\}$ a path then $T\{i,j,k,\ell\}$ is an (i,j)-path with internal vertices k, ℓ by the maximality of $d_{i,j}$. Clearly,

$$d_T(k,\ell) = d_{T\{i,j,k,\ell\}}(k,\ell) = d_{k,\ell}\ .$$

For $T\{i,j,k,\ell\}$ not a path, let x be the vertex of degree 3 in $T\{i,j,k,\ell\}$. Note that x, but not k, is on the (i,j)-path in T. Then

$$\begin{aligned} d_T(k,\ell) &= d_T(k,x) + d_T(x,\ell) \\ &= d_{T\{i,j,k,\ell\}}(k,x) + d_{T\{i,j,k,\ell\}}(x,\ell) \\ &= d_{T\{i,j,k,\ell\}}(k,\ell) = d_{k,\ell}\ . \end{aligned}$$

Hence, D is realizable by T. □

Remark. The proof of Theorem 7 is constructive and gives an algorithm for finding the tree T that realizes $D \in \Upsilon_{p \times p}$.

Realizable Algorithm

1. Set $T_m = T\{i_1, \ldots, i_m\}$ the tree of leaves of D.
2. For $\ell \le p$, choose $i, j \in \{i_1, \ldots, i_m\}$ with $d_{i,j} = d_{i,i_\ell} + d_{j,i_\ell}$.
3. Set $T_\ell = T_{\ell-1}$ where the appropriate vertex has been labelled i_ℓ.
4. Stop.

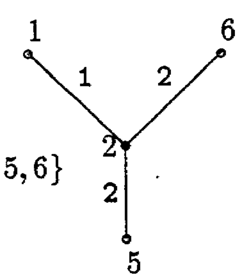
Example 6. We illustrate the algorithm for D below.

$D =$

	1	2	3	4	5	6
1	0	1	2	2	3	3
2	1	0	1	1	2	2
3	2	1	0	2	3	1
4	2	1	2	0	1	3
5	3	2	3	1	0	4
6	3	2	1	3	4	0

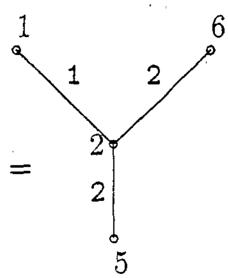
Note that 1, 5, 6 are the leaves of D. Let

$T_3 = \{1, 5, 6\}$

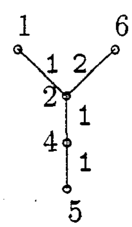


(edge weights w denote path lengths)

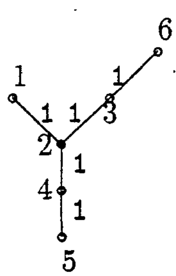
First $d_{1,5} = d_{1,2} + d_{2,5}$, so that

$T_4 =$ 

Next $d_{1,5} = d_{1,4} + d_{4,5}$, so that

$T_5 =$ 

Finally $d_{5,6} = d_{3,5} + d_{5,6}$, so that

$T_6 =$ 

□

Remark. Any of examples 2, 3, 5 show that Theorem 7 is best possible in that the minimum size of the testing submatrices is 4×4.

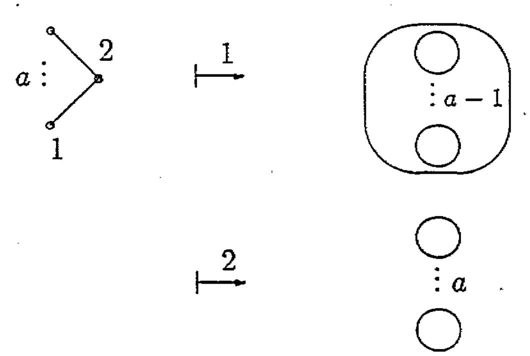
III. Construction of All Sublayer Electromagnetic Topologies with $R_{\max}$ at Most 5

In this section we exhibit all EM topologies with the properties that the primary sublayers include all the extremal sublayers, possibly some of the internal sublayers and that the relative shielding order matrix among primary sublayers has largest entry $R_{\max}$ of at most 5 by examining all trees of diameter (length of longest path) at most 5 and constructing the EM topology by inversion at a vertex in the tree. Note that only inversion at a leaf of the tree, which is always a primary sublayer, results in a connected system. Observe that a system is connected iff there exists one subshield which contains all other subshields and all but one sublayer. In our diagrams it will not be necessary to distinguish primary and secondary sublayers.

In what follows, we use the obvious symmetries of the tree to construct the electromagnetic topology by inversion of the tree at a labelled vertex. We leave the trivial cases $R_{\max} = 0, 1$ to the interested reader.

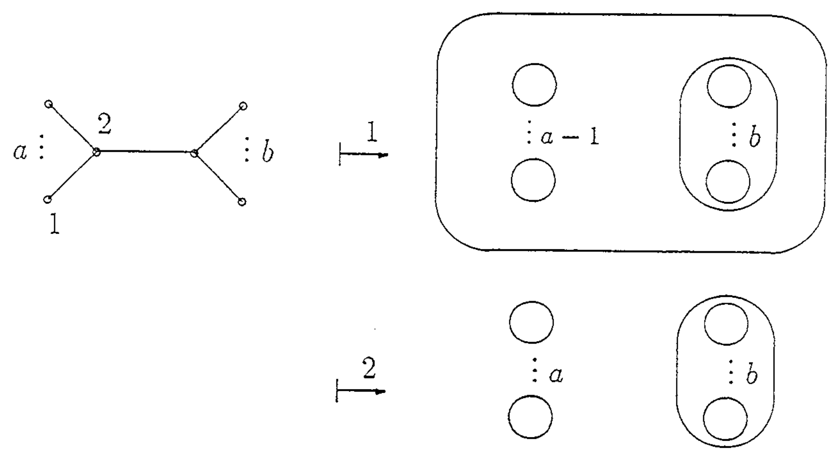
A. $R_{\max} = 2$

Any tree of diameter 2 is as below for some $a \geq 2$.



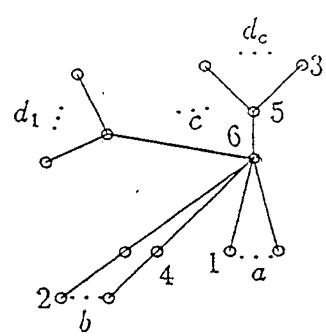
B. $R_{\max} = 3$

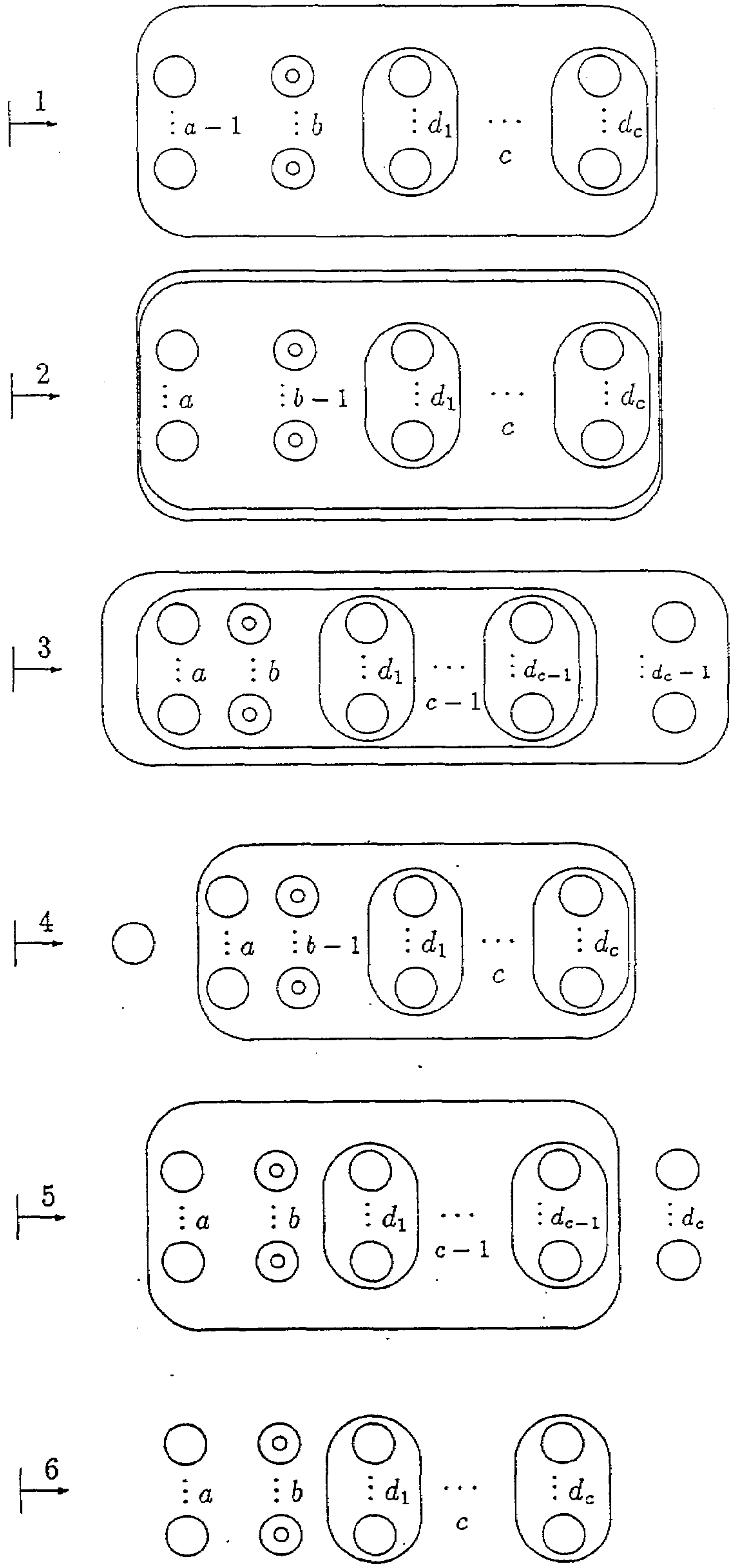
Any tree of diameter 3 is as below for some $a, b \geq 1$.



C. $R_{\max} = 4$

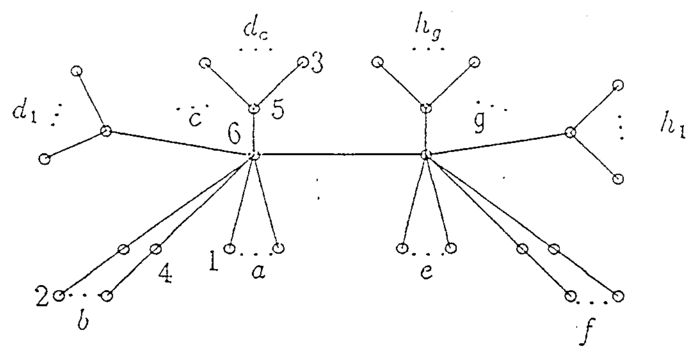
Any tree of diameter 4 is as below for some $a, b, c, d_1, \ldots, d_c$ where $d_i \geq 2$ for $1 \leq i \leq c$ and $b + c \geq 2$.

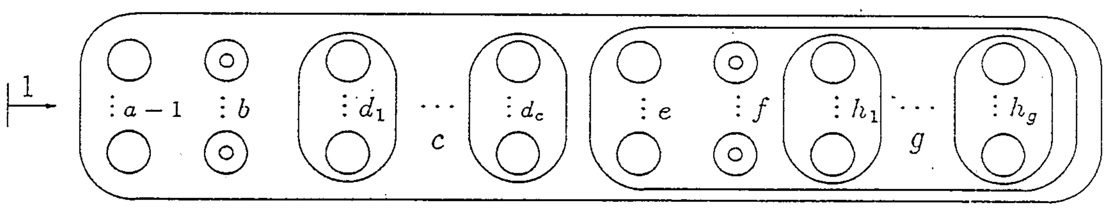


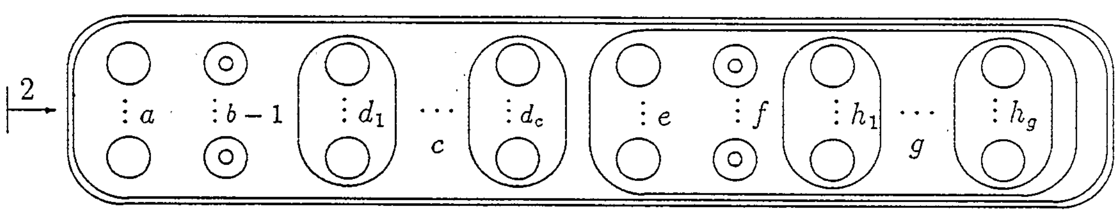

1
a − 1
b
d1
c
dc
2
a
b − 1
d1
c
dc
3
a
b
d1
c − 1
dc−1
dc − 1
4
a
b − 1
d1
c
dc
5
a
b
d1
c − 1
dc−1
dc
6
a
b
d1
c
dc

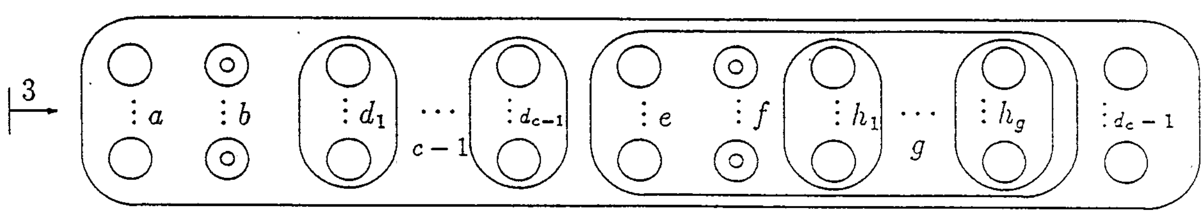
D. $R_{\max} = 5$

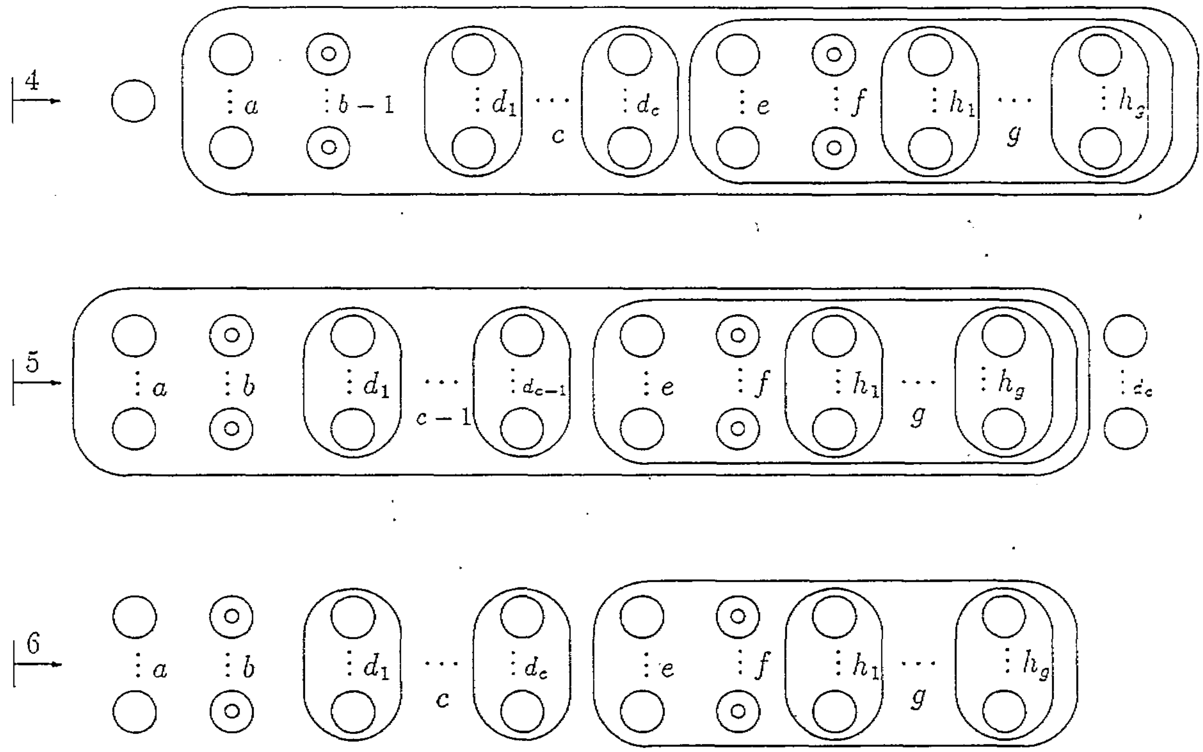
Any tree of diameter 5 is as below for some $a, b, c, d_1, \ldots, d_c, e, f, g, h_1, \ldots, h_g$ where $d_i, h_j \geq 2$ for $1 \leq i \leq c$, $1 \leq j \leq g$ and $b + c + f + g \geq 2$.











Note that appropriate choice of the parameters allows for construction of a small system with the desired properties.

IV. Summary

We have characterized the relative shielding order matrix among primary sublayers for an electromagnetic topology defined to the level of layers and sublayers. Moreover, our characterization gives a recursive procedure for constructing the interaction sequence diagram and, hence, an EM topology. For small relative shielding order the EM topologies and dual graphs are not overly complicated and are exhibited.

In the results of the present paper we have found that there are cases of specified relative shielding order matrices which are not realizable. An important related problem is to determine when such a matrix can be appropriately repaired so as to become a realizable relative shielding order matrix. By this we mean that certain matrix elements may be increased (more shielding) and so achieve at least the desired shielding performance. Ideally this would be done in some optimal manner involving minimum repair. We have some results concerning this to report in a future paper.

Since electromagnetic systems can acquire considerable complexity, also of interest is a similar analysis of the realizability and repairability of the relative shielding order matrix for an EM topology defined to the level of elementary volumes. This will produce a dual graph which is not a tree and complicate matters considerably.

References

1. C. E. Baum, Electromagnetic Topology: A Formal Approach to the Analysis and Design of Complex Electronic Systems, Interaction Note 400, September 1980, also as *Proc. EMC Symposium*, Zurich, pp. 209-214 (1981).

2. C. E. Baum, Sublayer Sets and Relative Shielding Order in Electromagnetic Topology, Interaction Note 416, April 1982, also as *Electromagnetics* vol. 2, no. 4 (1982).

3. C. E. Baum, On the Use of Electromagnetic Topology for the Decomposition of Scattering Matrices for Complex Physical Structures, Interaction Note 454, July 1985.

4. R. J. Gould, *Graph Theory*, Benjamin Cummings (1988).

5. K.S.H. Lee (ed.), *EMP Interaction: Principles, Techniques and Reference Data*, Hemisphere Publishing (1986).