p. 2

Interaction Notes

Note 476

19 July 1989

SEM Backscattering

Carl E. Baum

Weapons Laboratory

ABSTRACT

The paper considers the properties of the backscattering residue matrix corresponding to the SEM poles or natural frequencies of a scatterer. For the case of non degenerate natural modes this matrix becomes a single dyad which is characterized by a complex two-component vector. Introducing various symmetries in the scatterer, combined with choice of observer location, gives other special properties to the residue matrix.

Interaction Notes

Note 476

19 July 1989

SEM Backscattering

Carl E. Baum

Weapons Laboratory

ABSTRACT

The paper considers the properties of the backscattering residue matrix corresponding to the SEM poles or natural frequencies of a scatterer. For the case of non degenerate natural modes this matrix becomes a single dyad which is characterized by a complex two-component vector. Introducing various symmetries in the scatterer, combined with choice of observer location, gives other special properties to the residue matrix.

I. Introduction

A previous paper [4] has considered the general electromagnetic scattering problem with some implications of reciprocity and symmetry, and applied these to the forms that appear in the eigenmode expansion method (EEM) and singularity expansion method (SEM). Looking at the far-field scattering, of an incident plane wave one can define a scattering matrix (in complex frequency domain) which exhibits a symmetry (reciprocity) between the incident and scattered fields. In the case of backscattering this matrix is symmetric in the usual case of defined orthogonal horizontal and vertical polarizations. This paper further develops the backscattering results for the SEM representation.

Our starting point is (5.18) from [4] which reads

$$\vec{1}_m \cdot \tilde{\vec{E}}_f(\vec{r},s) = -\frac{E_o}{4\pi r} e^{-\gamma r} \mu_o \sum_{\alpha} s_\alpha \tilde{f}(s_\alpha) \left\langle \vec{j}_{s_\alpha}(\vec{r}_s); \frac{\partial}{\partial s} \tilde{\tilde{Z}}_t(\vec{r}_s,\vec{r}_s\,';s)\Big|_{s=s_\alpha}; \vec{j}_{s_\alpha}(\vec{r}_s\,') \right\rangle^{-1}$$

$$\vec{1}_m \cdot \tilde{\vec{C}}_{f_\alpha}\left(\vec{1}_r,\vec{1}_1\right) \cdot \vec{1}_p [s - s_\alpha]^{-1}$$

$$+ \text{ singularities of } \tilde{f}(s)$$

$$+ \text{ possible entire function}$$

$$\tilde{\vec{C}}_{f_\alpha}\left(\vec{1}_r,\vec{1}_1\right) = \tilde{\vec{C}}_{r_\alpha}\left(\vec{1}_r\right) \tilde{\vec{C}}_\alpha\left(\vec{1}_1\right)$$

$$\tilde{\vec{C}}_\alpha\left(\vec{1}_1\right) = \left\langle \vec{1}_1 e^{-\gamma_\alpha \vec{1}_r \cdot \vec{r}_s\,'}; \vec{j}_{s_\alpha}(\vec{r}_s\,') \right\rangle$$

$$\tilde{\vec{C}}_{r_\alpha}\left(\vec{1}_r\right) = \left\langle \vec{1}_r e^{\gamma_\alpha \vec{1}_r \cdot \vec{r}_s}; \vec{j}_{s_\alpha}(\vec{r}_s\,') \right\rangle$$

$$\eta_{f_\alpha}\left(\vec{1}_1,\vec{1}_p;\vec{1}_r,\vec{1}_m\right) = -s_\alpha \mu_o \left\langle \vec{j}_{s_\alpha}(\vec{r}_s); \frac{\partial}{\partial s} \tilde{\tilde{Z}}_t(\vec{r}_s,\vec{r}_s\,';s)\Big|_{s=s_\alpha}; \vec{j}_{s_\alpha}(\vec{r}_s\,') \right\rangle^{-1} \vec{1}_m \cdot \tilde{\vec{C}}_{f_\alpha}\left(\vec{1}_r,\vec{1}_1\right) \cdot \vec{1}_p$$

$$\vec{1}_m \cdot \tilde{\vec{C}}_{f_\alpha}\left(\vec{1}_r,\vec{1}_1\right) \cdot \vec{1}_p = \left[\vec{1}_m \cdot \left\langle \vec{1}_r e^{\gamma_\alpha \vec{1}_r \cdot \vec{r}_s}; \vec{j}_{s_\alpha}(\vec{r}_s) \right\rangle\right] \left[\vec{1}_p \cdot \left\langle \vec{1}_1 e^{-\gamma_\alpha \vec{1}_1 \cdot \vec{r}_s\,'}; \vec{j}_{s_\alpha}(\vec{r}_s\,') \right\rangle\right]$$

$$= \left\langle \vec{1}_m e^{\gamma_\alpha \vec{1}_r \cdot \vec{r}_s}; \vec{j}_{s_\alpha}(\vec{r}_s) \right\rangle \left\langle \vec{1}_p e^{-\gamma_\alpha \vec{1}_1 \cdot \vec{r}_s\,'}; \vec{j}_{s_\alpha}(\vec{r}_s\,') \right\rangle \tag{1.1}$$

where

$$\vec{1}_1 \equiv \text{direction of incidence}$$

$$\vec{1}_p \equiv \text{polarization of incident electtric field}$$

$$\vec{1}_1 \cdot \vec{1}_p = 0$$

$$\overleftrightarrow{1}_1 \equiv \overleftrightarrow{1} - \vec{1}_1 \vec{1}_1 \equiv \text{incident transverse dyad}$$

$$\vec{1}_r \equiv \text{scattering direction}$$

$$\vec{1}_m \equiv \text{direction for measuring (sampling) scattered electric field}$$

$$\vec{1}_r \cdot \vec{1}_m = 0 \quad (\text{constraint on } \vec{1}_m)$$

$$\overleftrightarrow{1}_r \equiv \overleftrightarrow{1} - \vec{1}_r \vec{1}_r \equiv \text{scattered transverse dyad}$$

$$r \equiv \text{distance from observer of scattered field to scatterer(center of minimum circumscribing sphere)}$$

$$E_o \equiv \text{scaling constant (V / m) for incident field at scatterer}$$

$$f(t) \equiv \text{waveform of incident field}$$

$$s \equiv \Omega + j\omega \equiv \text{Laplace – transform (two sided) variable}$$

$$\equiv \text{complex frequency}$$

$$\overleftrightarrow{Z}_t \equiv \text{Kernel of E – field (or impedance) integral equation}$$

$$\vec{j}_{s_\alpha} \equiv \text{natural mode of scatterer}$$

$$s_\alpha \equiv \text{natural frequency of scatterer}$$

$$\tilde{\eta}_{f_\alpha} \equiv \text{far coupling coefficient}$$

(1.2)

Noting the important symmetry (reciprocity) relationship

$$\vec{C}_{r_\alpha}\left(\vec{1}_r\right) = \vec{C}_\alpha\left(-\vec{1}_r\right)$$

$$\vec{C}_{f_\alpha}\left(\vec{1}_r, \vec{1}_1\right) = \vec{C}_\alpha\left(-\vec{1}_r\right)\vec{C}_\alpha\left(\vec{1}_1\right) \tag{1.3}$$

then the single vector function $\vec{C}_\alpha$ can be used to characterize the symmetric dyadic residue.

As noted in [4] the integrals defining $\vec{C}_\alpha$, $\vec{C}_{r_\alpha}$, and the three-term symmetric-product denominator in (1.1) are taken over the surface S of the scatterer and involve surface-current-density natural modes. This, however, is not a restriction in that volume integrals are also allowed with volume current-density natural modes. It is only necessary that the media comprising the scatterer be reciprocal, i.e., have symmetric constitutive-parameter matrices.

For backscattering we have

$$\vec{1}_r \equiv -\vec{1}_1$$

$$\tilde{C}_{b_\alpha}\left(\vec{1}_1\right) \equiv \tilde{C}_{f_\alpha}\left(-\vec{1}_1, \vec{1}_1\right) = \vec{C}_\alpha\left(\vec{1}_1\right)\vec{C}_\alpha\left(\vec{1}_1\right) \tag{1.4}$$

which is evidently a symmetric dyad (required by reciprocity). Furthermore introducing the standard unit vectors as in fig. 1.1

$$\vec{1}_h = horizontal\ polarization$$

$$\vec{1}_v = \vec{1}_h x \vec{1}_1 = \vec{1}_r x \vec{1}_h = \text{"}vertical\text{"}\ polarization$$

$$\vec{1}_v x \vec{1}_h = \vec{1}_1$$

$$\vec{1}_1 x \vec{1}_v = \vec{1}_h \tag{1.5}$$

Here horizontal is usually taken as parallel to the horizon and vertical is interpreted loosely. Taken in the sequence $\vec{1}_v, \vec{1}_h, \vec{1}_1$ this is a right handed system with $\vec{1}_1$ as the direction from the observer to the scatterer (direction of incidence). Later considering the h, v plane with positive sense of rotation from $\vec{1}_h$ toward $\vec{1}_v$ then $\left(\vec{1}_h, \vec{1}_v, -\vec{1}_1\right)$ *or* $\left(\vec{1}_h, \vec{1}_v, \vec{1}_r\right)$ form a right handed system.

Defining

$$W_\alpha \equiv w_\alpha^2 \equiv -s_\alpha \mu_o \left\langle \vec{j}_{s_\alpha}(\vec{r}_s); \frac{\partial}{\partial s} \tilde{\tilde{Z}}_t(\vec{r}_s, \vec{r}_s', s)\Big|_{s=s_\alpha}; \vec{j}_{s_\alpha}(\vec{r}_s') \right\rangle^{-1} \tag{1.6}$$

we can in turn define

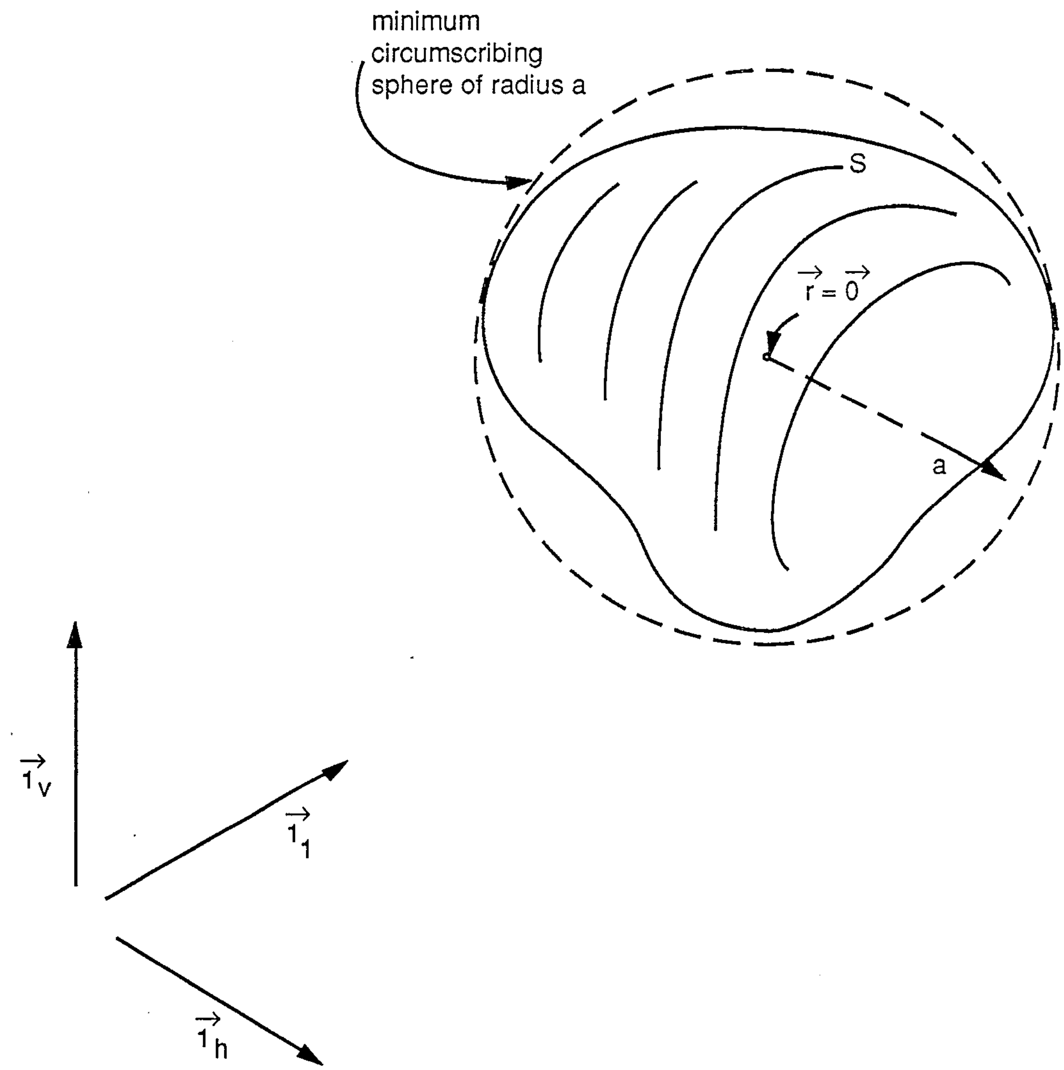


Fig. 1.1. Backscattering from General Scatterer

$$
\begin{aligned}
\vec{c}_\alpha(\vec{1}_1) &\equiv w_\alpha \vec{C}_\alpha(\vec{1}_1) \\
\vec{c}_{r_\alpha}(\vec{1}_r) &\equiv w_\alpha \vec{C}_{r_\alpha}(\vec{1}_r) = \vec{c}_\alpha(-\vec{1}_r) \\
\vec{c}_{f_\alpha}(\vec{1}_r, \vec{1}_1) &\equiv W_\alpha \vec{C}_{f_\alpha}(\vec{1}_r, \vec{1}_1) \\
&= \vec{c}_{r_\alpha}(\vec{1}_r)\vec{c}_\alpha(\vec{1}_1) \\
\vec{c}_{b_\alpha}(\vec{1}_1) &\equiv W_\alpha \vec{C}_{b_\alpha}(\vec{1}_1) \\
&= \vec{c}_\alpha(\vec{1}_1)\vec{c}_\alpha(\vec{1}_1)
\end{aligned}
\tag{1.7}
$$

This for is what is experimentally observable in (1.1) using far-field measurements. For symmetry one might as well normalize the two vectors in the dyad (particularly for backscattering) the same way. The relative normalization of the two is arbitrary since it is actually the dyadic product which is observable in the far field.

This form for $\vec{c}_{b_\alpha}$ is convenient in that it is effectively the backscattering matrix for SEM measurement purposes. Rewriting (1.1) we have for backscattering

$$
\begin{aligned}
\vec{1}_m \cdot \tilde{\vec{E}}_f(\vec{r},s) &= \frac{E_o e^{-\gamma r}}{4\pi r} \sum_\alpha \tilde{f}(s_\alpha)[s-s_\alpha]^{-1} \vec{1}_m \cdot \vec{c}_{b_\alpha}(\vec{1}_1) \cdot \vec{1}_p \\
&\quad + \text{singularities of } \tilde{f}(s) \\
&\quad + \text{possible entire function} \\
&= \frac{e^{-\gamma r}}{4\pi r} \sum_\alpha [s-s_\alpha]^{-1} \vec{1}_m \cdot \vec{c}_{b_\alpha}(\vec{1}_1) \cdot \tilde{\vec{E}}^{(inc)}(\vec{0}, s_\alpha) \\
&\quad + \text{singularities of } \tilde{f}(s) \\
&\quad + \text{possible entire function} \\
\tilde{\vec{E}}^{(inc)}(\vec{r},s) &= E_o \tilde{f}(s) \vec{1}_p e^{-\gamma \vec{1}_1 \cdot \vec{r}}
\end{aligned}
\tag{1.8}
$$

So allowing for the delay $e^{-\gamma r}$ and $1/(4\pi r)$ expansion to the observer then the pole terms of the backscattering matrix are just $[s-s_\alpha]^{-1}\tilde{c}_{b_\alpha}(\vec{1}_1)$. Allowing for the $1/(4\pi r)$ each such term has dimension meters. The reside matrix $\tilde{c}_{b_\alpha}$ then has dimensions m/s giving $\tilde{c}_{b_\alpha}$ dimensions (m/s)$^{1/2}$.

In terms of coupling coefficients we have [4]

$$\eta_{f_\alpha}(\vec{1}_1,\vec{1}_p,\vec{1}_r,\vec{1}_m) = \eta_\alpha(\vec{1}_1,\vec{1}_p)\eta_{r_\alpha}(\vec{1}_r,\vec{1}_m) \qquad \text{(far coupling coefficient)}$$

$$\eta_\alpha(\vec{1}_1,\vec{1}_p) = -\frac{W_\alpha}{s_a\mu_o}\vec{1}_p\cdot\vec{C}_\alpha(\vec{1}_1) \qquad \text{(coupling coefficient)}$$

$$= -\frac{w_\alpha}{s_\alpha\mu_o}\vec{1}_p\cdot\vec{c}_\alpha(\vec{1}_1)$$

$$\eta_{r_\alpha}(\vec{1}_r,\vec{1}_m) = -s_\alpha\mu_o\vec{1}_m\cdot\vec{C}_{r_\alpha}(\vec{1}_r) \qquad \text{(recoupling coefficient)}$$

$$= -s_\alpha\mu_o\vec{1}_m\cdot\vec{C}_\alpha(-\vec{1}_r)$$

$$= -\frac{s_\alpha\mu_o}{w_\alpha}\vec{1}_m\cdot\vec{c}_\alpha(-\vec{1}_r)$$

$$\eta_{f_\alpha}(\vec{1}_1,\vec{1}_p;\vec{1}_r,\vec{1}_m) = \vec{1}_m\cdot\vec{c}_\alpha(-\vec{1}_r)\vec{c}_\alpha(\vec{1}_1)\cdot\vec{1}_p$$

$$= W_\alpha\vec{1}_m\cdot\tilde{C}_{f_\alpha}(\vec{1}_r,\vec{1}_1)\cdot\vec{1}_p \tag{1.9}$$

For backscattering this becomes

$$\eta_{f_\alpha}(\vec{1}_1,\vec{1}_p;-\vec{1}_1,\vec{1}_m) = \vec{1}_m\cdot\tilde{c}_{b_\alpha}(\vec{1}_1)\cdot\vec{1}_p \tag{1.10}$$

where $\vec{1}_m$ *and* $\vec{1}_p$ assume values of $\vec{1}_h$ *and* $\vec{1}_v$ in all combinations.

Now interpret our backscattering residue dyad as a 2 x 2 matrix noting that $\tilde{c}_\alpha(\vec{1}_1)$ has no component in the $\vec{1}_1$ direction. Labelling the two components of vectors (transverse) to $\vec{1}_1$ by subscripts h and v we have

$$\vec{c}_{b_\alpha}\left(\vec{1}_1\right) = \vec{c}_\alpha\left(\vec{1}_1\right)\vec{c}_\alpha\left(\vec{1}_1\right) = \begin{pmatrix} c_{h_\alpha}\left(\vec{1}_1\right) \\ c_{v_\alpha}\left(\vec{1}_1\right) \end{pmatrix} \begin{pmatrix} c_{h_\alpha}\left(\vec{1}_1\right) \\ c_{v_\alpha}\left(\vec{1}_1\right) \end{pmatrix}$$

$$= \begin{pmatrix} c^2_{h_\alpha}\left(\vec{1}_1\right) & c_{h_\alpha}\left(\vec{1}_1\right) \\ c_{v_\alpha}\left(\vec{1}_1\right) & c^1_{v_\alpha}\left(\vec{1}_1\right) \end{pmatrix}$$

$$= \begin{pmatrix} \vec{1}_h \cdot \vec{c}_{b_\alpha}\left(\vec{1}_1\right) \cdot \vec{1}_h & \vec{1}_\eta \cdot \vec{c}_{b_\alpha}\left(\vec{1}_1\right) \cdot \vec{1}_v \\ \vec{1}_v \cdot \vec{c}_{b_\alpha}\left(\vec{1}_1\right) \cdot \vec{1}_h & \vec{1}_v \cdot \vec{c}_{b_\alpha}\left(\vec{1}_1\right) \cdot \vec{1}_v \end{pmatrix}$$

$$= \left(c_{b_{n,m}}\left(\vec{1}_1\right)\right)_\alpha$$

$$= \begin{pmatrix} \eta_{f_\alpha}\left(\vec{1}_1, \vec{1}_h; -\vec{1}_1, \vec{1}_h\right) & \eta_{f_\alpha}\left(\vec{1}_1, \vec{1}_v; -\vec{1}_1, \vec{1}_h\right) \\ \eta_{f_\alpha}\left(\vec{1}_1, \vec{1}_h; -\vec{1}_1, \vec{1}_v\right) & \eta_{f_\alpha}\left(\vec{1}_1, \vec{1}_v; -\vec{1}_1, \vec{1}_v\right) \end{pmatrix} \tag{1.11}$$

The various forms of this matrix are of course symmetric by reciprocity.

In this two-dimensional h, v space we have the identity (transverse identity)

$$\vec{1}_1 = \vec{1}_r = \vec{1} - \vec{1}_1\vec{1}_1 = \vec{1} - \vec{1}_r\vec{1}_r = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.12}$$

with rotation by $\pi/2$ in the positive sense (from $\vec{1}_h$ toward $\vec{1}_v$) as

$$\vec{1}_r \times \quad = -\vec{1}_1 \times \quad = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bullet \tag{1.13}$$

Rotation by an angle Ψ in the positive sense is

$$\begin{pmatrix} \cos(\Psi) & -\sin(\Psi) \\ \sin(\Psi) & \cos(\Psi) \end{pmatrix} \bullet \tag{1.14}$$

II. Representation of $\vec{c}_\alpha(\vec{1}_1)$

Assuming that at the natural frequency s_α of interest there is only one natural mode $\vec{j}_{s_\alpha}$ (no model degeneracy), then the observer has a backscattering residue dyad $\vec{c}_{b_\alpha}$ characterized by a single vector $\vec{c}_\alpha$. This is in general a complex valued vector. Restricting our attention to the two-dimensional h, v plane there are two vector components, each complex valued. This can also be thought of as four real numbers. The magnitude is straight forward as

$$\left|\vec{c}_\alpha(\vec{1}_1)\right|^2 = \vec{c}_\alpha(\vec{1}_1)\cdot\vec{c}_\alpha^{\,*}(\vec{1}_1) = \left|c_{h_\alpha}(\vec{1}_1)\right|^2 + \left|c_{v_\alpha}(\vec{1}_1)\right|^2$$

$$= \mathrm{Re}^2\left[c_{h_\alpha}(\vec{1}_1)\right] + \mathrm{Im}^2\left[c_{h_\alpha}(\vec{1}_1)\right] + \mathrm{Re}^2\left[c_{v_\alpha}(\vec{1}_1)\right] + \mathrm{Im}^2\left[c_{v_\alpha}(\vec{1}_1)\right] \tag{2.1}$$

This vector is an observable characteristic of the scatterer so its orientation or direction is also of interest. This is traditionally characterized by a polarization ellipse [14]. In such a context an incoming electromagnetic wave (monochomatic, i.e. single ω) is viewed as an electric vector rotating in the h, v plane with the tip of the electric vector forming an ellipse. By establishing the lengths of the major and minor axes of the ellipse and orientation of the major axis in the h, v plane this incoming wave is characterized except for an arbitrary phase (in effect 3 of 4 real numbers established). Viewed as a complex vector this incoming wave has the same form as $\vec{c}_\alpha$.

Then analogous to the traditional use $\vec{c}_\alpha$ can also be so characterized by a polarization ellipse. As in fig. 2.1, let us introduce an angle ψ in the positive sense in the h, v plane. This polarization ellipse is contained in many articles. A convenient form in [7] is applied in our case as

$$\vec{c}_\alpha = c_{h_\alpha}\vec{1}_h + c_{h_v}\vec{1}_v = \begin{pmatrix} c_{h_\alpha} \\ c_{v_\alpha} \end{pmatrix} = \begin{pmatrix} c_h \\ c_v \end{pmatrix}_\alpha$$

$$= a_\alpha \begin{pmatrix} \cos(\psi_\alpha) & -\sin(\psi_\alpha) \\ \sin(\psi_\alpha) & \cos(\psi_\alpha) \end{pmatrix} \cdot \begin{pmatrix} \cos(\tau_\alpha) \\ j\sin(\tau_\alpha) \end{pmatrix} e^{j\Delta_\alpha}$$

$$a_\alpha, \psi_\alpha, \tau_\alpha, \Delta_\alpha \quad \textit{all real scalars} \tag{2.2}$$

Here the result is cast in terms of one magnitude and three angles. Note in the polarization ellipse the "overall" phase angle Δ_α is not included, but can be specified by a number. Also all these parameters (including $\vec{1}_h$ *and* 1_v) are functions of $\vec{1}_1$ which is now suppressed for convenience.

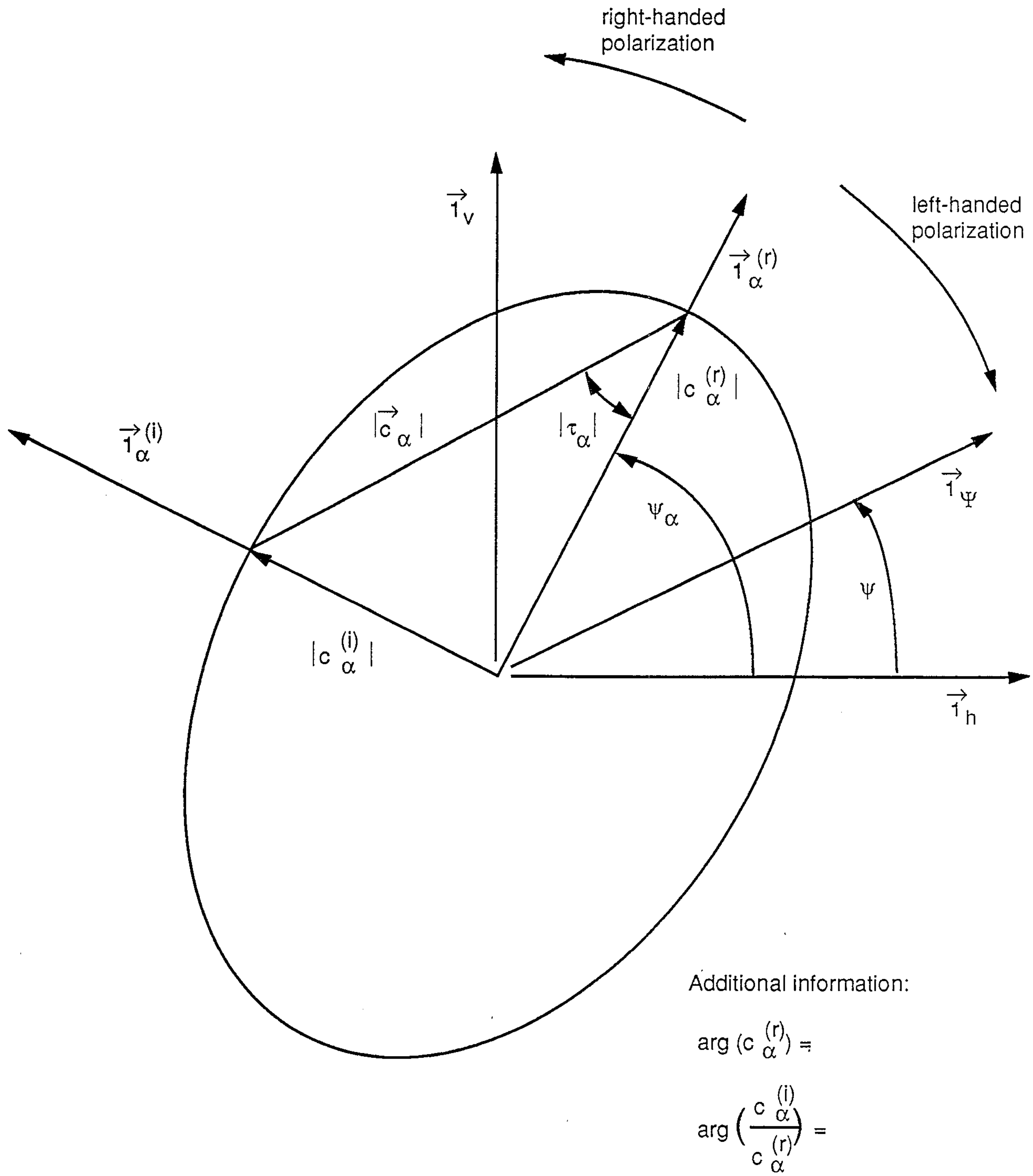


Fig. 2.1. Polarization Ellipse for $\vec{c}_\alpha$

To visualize this better, define a real direction $\vec{1}_\alpha^{(r)}$ in the h, v plane such that $\left|\vec{1}_\alpha^{(r)} \cdot \vec{c}_\alpha\right|$ is maximized. This will determine the orientation of the major axis of the ellipse. Consider the radial unit vector in the h, v plane

$$\vec{1}_\Psi = \cos(\psi)\vec{1}_h + \sin(\psi)\vec{1}_h = \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \end{pmatrix} \tag{2.3}$$

and the corresponding angular unit vector

$$\vec{1}_\psi = -\sin(\psi)\vec{1}_h + \cos(\psi)\vec{1}_v = \begin{pmatrix} -\sin(\psi) \\ \cos(\psi) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \vec{1}_\Psi = -\vec{1}_1 \times \vec{1}_\Psi \tag{2.4}$$

So now maximize

$$\left|\vec{1}_\Psi \cdot \vec{c}_\alpha\right|^2 = \vec{1}_\Psi \cdot \vec{c}_\alpha^{\,*}\vec{c}_\alpha \cdot \vec{1}_\Psi$$

$$= \left|c_{h_\alpha}\right|^2 \cos^2(\psi) + \left[c_{h_\alpha} c_{h_\alpha}^{*} + c_{h_\alpha}^{*} c_{v_\alpha}\right]\cos(\psi)\sin(\psi)$$

$$+ \left|c_{v_\alpha}\right|^2 \sin^2(\psi) \tag{2.5}$$

and use this to define $\vec{1}_\alpha \; as \; \vec{1}_\Psi \; and \; \psi_\alpha \; as \; \psi$ corresponding to the maximum. Note the Hermetian dyad $\vec{c}_\alpha^{\,*}\vec{c}_\alpha$ for which we are finding the maximum projection along a real unit vector. Differentiating with respect to ψ and setting to zero gives (after a little algebra) the well-known result

$$\tan(2\psi_\alpha) = \frac{c_{h_\alpha} c_{v_\alpha}^{*} + c_{h_\alpha}^{*} c_{v_\alpha}}{\left|c_{h_\alpha}\right|^2 - \left|c_{v_\alpha}\right|^2} = \frac{2\,\mathrm{Re}\left[c_{h_\alpha} c_{v_\alpha}^{*}\right]}{\left|c_{h_\alpha}\right|^2 - \left|c_{v_\alpha}\right|^2} \tag{2.6}$$

Over the range of $-\pi < \psi_\alpha < \pi$ there are 4 solutions, in general, of (2.6) corresponding to 2 opposite directions for for the major axis and 2 opposite directions for the minor axis. To resolve this form

$$c_\alpha^{(r)} \equiv \vec{1}_\alpha^{(r)} \cdot \vec{c}_\alpha$$

$$c_\alpha^{(i)} \equiv \vec{1}_\alpha^{(i)} \cdot \vec{c}_\alpha$$

$$\vec{1}_\alpha^{(r)} \equiv \begin{pmatrix} \cos(\psi_\alpha) \\ \sin(\psi_\alpha) \end{pmatrix}$$

$$\vec{1}_\alpha^{(i)} \equiv \begin{pmatrix} \cos\left(\psi_\alpha + \frac{\pi}{2}\right) \\ \sin\left(\psi_\alpha + \frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} -\sin(\psi_\alpha) \\ \cos(\psi_\alpha) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \vec{1}_\alpha^{(r)} \tag{2.7}$$

and constrain

$$\left|c_\alpha^{(r)}\right| \geq \left|c_\alpha^{(i)}\right| \tag{2.8}$$

to select a major axis for $\vec{1}_\alpha^{(r)}$ and a minor axis for $\vec{1}_\alpha^{(i)}$. It remains but to choose which of two values of ψ_α separated by π to specify. Note that reversing $\vec{1}_\alpha^{(r)}$ (rotation by π) changes the complex number $c_\alpha^{(r)}$ *to* $-c_\alpha^{(r)}$, so one could choose that ψ_α which made, say, $\mathrm{Re}\left[c_\alpha^{(r)}\right] \geq 0$. The usual convention is to choose $0 \leq \Psi_\alpha < \pi$, as in the example in fig. 2.1. Note that in a measurement situation absolute phase may not be available so that one may only have, say, $\left|c_\alpha^{(r)}\right|$.

Also, note that the sign of $\vec{c}_\alpha$ is ambiguous since the dyad $\vec{c}_\alpha\,\vec{c}_\alpha$ is what we obtain from the scattering measurement, this being the same as $(-\vec{c}_\alpha)\,(-\vec{c}_\alpha)$. The usual convention is then not a limitation. Even if we have absolute phase by knowing r sufficiently accurately, or have relative phases, referencing $\vec{c}_\alpha$ to the phase of some other selected $\vec{c}_{\alpha_o}$, there is still this sign ambiguity inherent in the dyad.

Referring back to the form in (2.2), this can be now put in the form

$$\vec{c}_\alpha = \begin{pmatrix} \cos(\psi) & -\sin(\psi_\alpha) \\ \sin(\psi_\alpha) & \cos(\psi_\alpha) \end{pmatrix} \cdot \begin{pmatrix} c_\alpha^{(r)} \\ c_\alpha^{(i)} \end{pmatrix} = c_\alpha^{(r)} \vec{1}_\alpha^{(r)} + c_\alpha^{(i)} \vec{1}_\alpha^{(i)}$$

$$|\vec{c}_\alpha|^2 = \left|c_\alpha^{(r)}\right|^2 + \left|c_\alpha^{(i)}\right|^2$$

$$\frac{c_\alpha^{(i)}}{c_\alpha^{(r)}} = j\tan(\tau_\alpha), \quad -\frac{\pi}{4} \le \tau_\alpha \le \frac{\pi}{4}$$

$$\frac{\text{minor axis}}{\text{major axis}} = \left|\frac{c_\alpha^{(i)}}{c_\alpha^{(r)}}\right|$$

$$c_\alpha^{(r)} = a_\alpha \cos(\tau_\alpha) e^{j\Delta_\alpha}$$

$$c_\alpha^{(i)} = j a_\alpha \sin(\tau_\alpha) e^{j\Delta_\alpha}$$

$$\arg\left(\frac{c_\alpha^{(i)}}{c_\alpha^{(r)}}\right) = \pm\frac{\pi}{2} \quad \left(\text{for } c_\alpha^{(i)} \ne 0\right) \tag{2.9}$$

This form of $\vec{c}_\alpha$ has it represented as two complex numbers times orthogonal real unit vectors. The two complex numbers are orthogonal also in the complex-plane sense (have imaginary ratio). So this decomposition is like a double complex number (with components designated by "r" and "i" superscripts).

Note that there is an IEEE standard for the sense of rotation of a polarization based on radiation and reception by a helical antenna, right or left handedness being defined in the usual screw-thread sense [14]. Right handed elliptical polarization has for an incoming plane wave the electric vector in the h, v plane rotating in the sense of increasing ψ. Left handed elliptical polarization has the sense of decreasing ψ. Noting time dependence e^{st} as $e^{j\omega t}$ with ω positive (or $s_\alpha = \Omega_\alpha + j\omega_\alpha$ with ω_α positive) then we have

$$\arg\left(\frac{c_\alpha^{(i)}}{c_\alpha^{(r)}}\right) = \begin{Bmatrix} +\frac{\pi}{2} \Rightarrow \text{left handed elliptical polarization} \\ -\frac{\pi}{2} \Rightarrow \text{right handed elliptical polarization} \end{Bmatrix} \tag{2.10}$$

For special cases we have

$$\left|c_{\alpha}^{(i)}\right| = \left|c_{\alpha}^{(r)}\right| \Rightarrow \text{circular polarization}$$

$$c_{\alpha}^{(i)} = 0 \Rightarrow \text{linear polarization} \tag{2.11}$$

One can also use this information to represent the polarization on what is called the Poincare sphere [7,14].

III. Properties of $\tilde{c}_{b_\alpha}$ for Non-Degenerate Modes

Now look at some of the properties of $\tilde{c}_{b_\alpha}$ assuming that at the s_α of interest the scatterer has only one natural mode, this being the typically encountered case. Since $\tilde{c}_{b_\alpha}$ can be represented as a single dyad then as discussed previously we need only know $\vec{c}_\alpha$, a vector in the h, v plane which is characterized by two complex numbers c_{h_α} *and* c_{v_α}, or equivalently four real numbers. This can be compared to the usual backscattering matrix which is a 2 x 2 matrix in the h, v plane (as in (3.8) of [4]). Applying reciprocity makes the matrix symmetric which means it requires in general three complex numbers (or equivalently six real numbers) to characterize it.

This single vector $\vec{c}_\alpha$ which characterizes the residue scattering matrix leads to an interesting result. From (1.11) such a backscattering matrix is singular as

$$\det\left(\tilde{c}_{b_\alpha}\left(\vec{1}_1\right)\right) = \det\left(\left(c_{b_{n,m}}\left(\vec{1}_1\right)\right)\right) = \det\left(\vec{c}_\alpha\left(\vec{1}_1\right)\vec{c}_\alpha\left(\vec{1}_1\right)\right) = 0 \text{ for all } \vec{1}_1 \tag{3.1}$$

This is a necessary condition for a single natural mode characterizing $\tilde{c}_{b_\alpha}$ and can be used as a test on data (allowing for noise). Having found a zero-determinant residue matrix then (1.11) can be used to construct $\vec{c}_\alpha$.

Note that $\tilde{c}_{b_\alpha}$ is already in diagonal form. One normalized eigenvector (right and left) is $\vec{c}_\alpha / \left[\vec{c}_\alpha \cdot \vec{c}_\alpha\right]^{1/2}$ with eigenvalue $\vec{c}_\alpha \cdot \vec{c}_\alpha$. The second eigenvalue is zero and corresponding eigenvector is orthogonal to $\vec{c}_\alpha$.

Now $\tilde{c}_{b_\alpha}$ *and* $\vec{c}_\alpha$ are associated with s_α *and* $\vec{j}_{s_\alpha}$. Having found these we also have $s_\alpha^*, \vec{j}_{s_\alpha}^*, \tilde{c}_{b_\alpha}^*$, *and* $\vec{c}_\alpha^*$ since we are dealing with the Laplace transform of real-valued time functions and operators. For each s_α not on the real axis of the s plane there is another (a separate value of α) that we find automatically together with the corresponding $\vec{c}_\alpha^*$.

For the case that s_α is on the negative real axis we have real $\vec{j}_{s_\alpha}$, real $\vec{c}_\alpha$, real $\tilde{c}_{b_\alpha}$ and real W_α. However W_α can be real or imaginary depending on the sign of W_α. From section 2 we then have

$$\vec{c}_\alpha\left(\vec{1}_1\right) = c_\alpha^{(r)}\left(\vec{1}_1\right)\vec{1}_\alpha\left(\vec{1}_1\right), \text{real or imaginary for all } \vec{1}_1 \tag{3.2}$$

Since this comes from the residue matrix we have

$$\overleftrightarrow{c}_{b_\alpha}\left(\vec{1}_1\right) = \vec{c}_\alpha\left(\vec{1}_1\right)\vec{c}_\alpha\left(\vec{1}_1\right) = c_\alpha^{(r)^2}\left(\vec{1}_1\right)\vec{1}_\alpha^{(r)}\left(\vec{1}_1\right)\vec{1}_\alpha^{(r)}\left(\vec{1}_1\right)$$

$$c_\alpha^{(r)^2}\left(\vec{1}_1\right) = \vec{c}_\alpha\left(\vec{1}_1\right)\cdot\vec{c}_\alpha\left(\vec{1}_1\right) \text{ (real eigenvalue)}$$

$$\vec{1}_\alpha^{(r)}\left(\vec{1}_1\right) = \left[\vec{c}_\alpha\left(\vec{1}_1\right)\cdot\vec{c}_\alpha\left(\vec{1}_1\right)\right]^{-1/2}\vec{c}_\alpha\left(\vec{1}_1\right) \text{ (real eigenvector)} \tag{3.3}$$

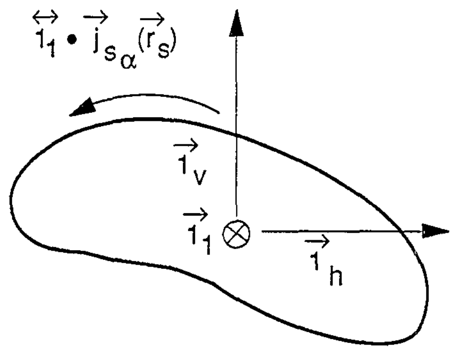
Note also in this case of real s_α that $\vec{c}_\alpha$ is characterized by two real numbers (or one real and one imaginary) as contrasted to two complex numbers.

Since $\vec{c}_\alpha$ is an integral over the natural mode $\vec{j}_{s_\alpha}$, we can use the orientation of to tell something about the orientation of $\vec{j}_{s_\alpha}$ and hence of the scatterer. From (1.1) and (1.7) we have

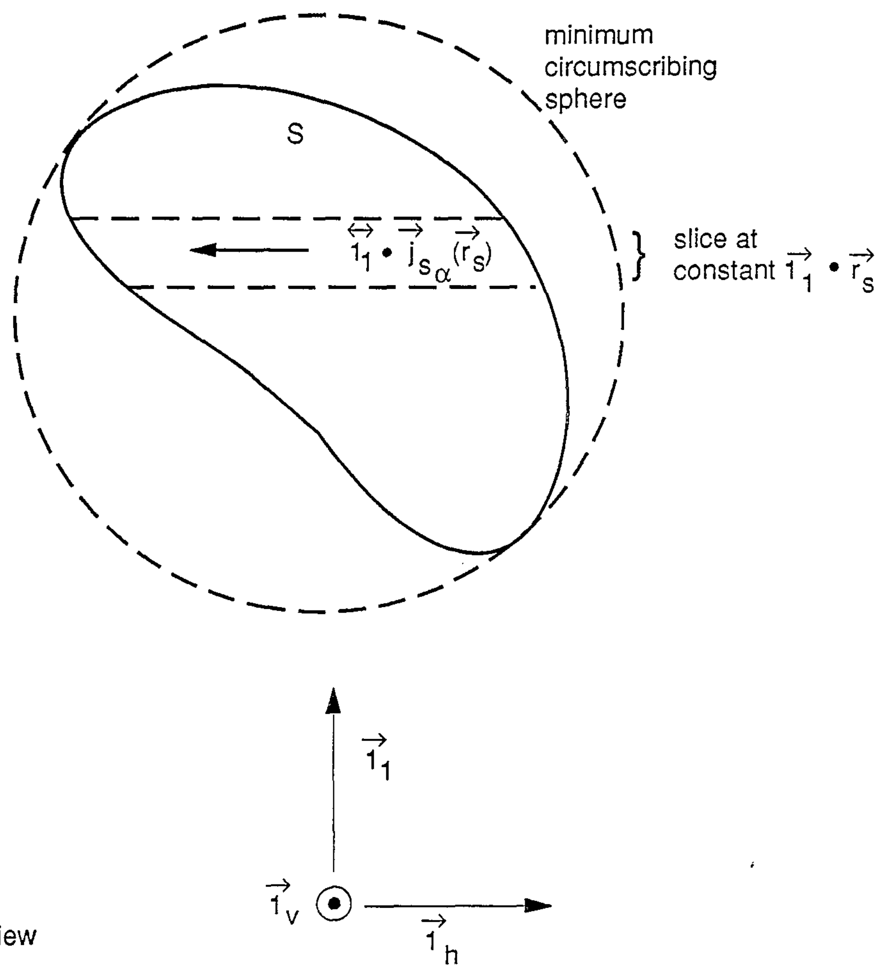
$$\begin{aligned}
\vec{c}_\alpha\left(\vec{1}_1\right) &= w_\alpha\left\langle \vec{1}_1 e^{-\gamma_\alpha \vec{1}_1\cdot\vec{r}_s\,'} ; \vec{j}_{s_\alpha}\left(\vec{r}_s\,'\right)\right\rangle \\
&= w_\alpha \vec{1}_1\left\langle e^{-\gamma_\alpha \vec{1}_1\cdot\vec{r}_s\,'} ; \vec{j}_{s_\alpha}\left(\vec{r}_s\,'\right)\right\rangle \\
&= w_\alpha\left\langle e^{-\gamma_\alpha \vec{1}_1\cdot\vec{r}_s\,'} , \vec{1}_1\cdot\vec{j}_{s_\alpha}\left(\vec{r}_s\,'\right)\right\rangle
\end{aligned} \tag{3.4}$$

Noting especially the last form, only components of $\vec{j}_{s_\alpha}$ orthogonal to $\vec{1}_1$ contribute to $\vec{c}_\alpha$.

Consulting fig. 3.1, look at some thin slice at constant $\vec{1}_1\cdot\vec{r}_s\,'$ through the scatterer. While for a perfectly conducting scatterer $\vec{1}_1\cdot\vec{j}_{s_\alpha}$ is locally tangential (parallel) to the scatterer, it can vary in phase over this slice. In some cases of interest there is a predominant orientation of $\vec{1}_1\cdot\vec{j}_{s_\alpha}$ over the slice in some particular direction. Integrating (summing) over all such slices there is the phase and amplitude factor $e^{-\gamma_\alpha \vec{1}_1\cdot\vec{r}_s\,'}$ which weights the contributions from the different slices. Provided there is a predominant orientation to the $\vec{c}_\alpha$ contributions from each slice, then this predominant orientation will appear in $\vec{c}_\alpha$. One general kind of scatterer to which such considerations are simply applicable is a long slender conductor. Then the low-order natural modes can be thought of as currents running along the body. Then if the body is straight so the currents can be thought of as passing in the same direction, the projection of direction is given by $\vec{c}_\alpha$ at the observer. The observer then has the scatterer orientation as projected on the h, v plane.



A. Cross-section view at slice



B. Plan view

Fig. 3.1. Relation of $\vec{c}_\alpha(\vec{1}_1)$ to the Natural Mode

Looking at fig. 3.2 one can see how $\vec{c}_\alpha$ can be used in some cases to tell shapes and orientations of scatterers. Suppose one is backscattering from a typical aircraft with side-on illumination. Consider, for example, one of the lowest order symmetric modes [2, 3, 11]. From the side $\vec{c}_\alpha$ will be dominated by fuselage currents and so will point (in the sense of $\vec{1}_\alpha^{(r)}$) approximately parallel to the fuselage. So in this case $\vec{c}_\alpha$ is related to the pitch angle (nose up or down) of the aircraft. For comparison consider some higher order symmetric natural frequency which might be associated with the vertical stabilizer (at least approximately, depending on details of the scattering shape). In such a case one could have $\vec{c}_\alpha$ more vertical.

Note that with side-on illumination the dominant antisymmetric resonances (associated with wings and horizontal stabilizers) will be suppressed, being small for $\vec{1}_1$ perpendicular to the symmetry plane of the aircraft provided that the antisymmetric currents are predominantly parallel to $\vec{1}_1$.

Excessive presence of antisymmetric $\vec{c}_\alpha$ in the backscattering then tells something about the yaw angle of the aircraft (nose side to side) and/or roll angle of the aircraft, depending on the orientation of the $\vec{c}_\alpha$ for the low-order antisymmetric modes.

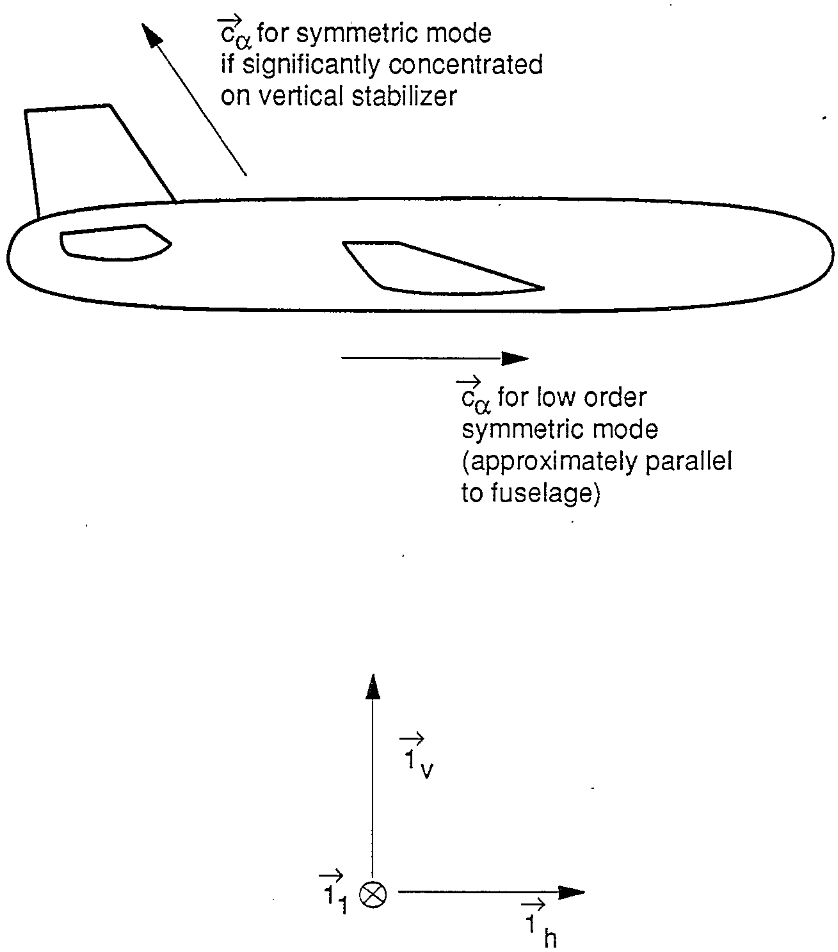


Fig. 3.2. Typical Aircraft Viewed from Side

IV. Observer on Symmetry Plane of Scatterer

Now let the scatterer have a symmetry plane [1, 3]. This separates the natural modes into two kinds, labelled symmetric (subscript sy) and antisymmetric (subscript as). As in fig. 4.1, let there be a symmetry plane P through the scatterer with unit normal $\vec{1}_P$. Then we have a reflection dyad

$$\vec{\vec{R}}_P = \vec{\vec{1}} - 2\,\vec{1}_P\vec{1}_P \tag{4.1}$$

which reflects coordinates through the symmetry plane as

$$\vec{r}_m = \vec{\vec{R}}_P \cdot \vec{r} \tag{4.2}$$

The point symmetry group is

$$R_P = \{(R_P),(1)\} \quad , \quad (R_P)^2 = (1) \quad (identity) \tag{4.3}$$

Symmetric and antisymmetric natural modes are defined by their reflection properties as

$$\vec{j}_{s_{\substack{sy\\as}},\alpha'}\left(\vec{r}_{s_m}\right) = \pm\vec{\vec{R}}_P \cdot \vec{j}_{s_{\substack{sy\\as}},\alpha'}\left(\vec{r}_s\right) \tag{4.4}$$

where the natural-mode index is now partitioned as

$$\alpha = \begin{pmatrix} sy & \\ & ,\alpha' \\ as & \end{pmatrix} \tag{4.5}$$

For a general observer the backscattering residue matrices will exhibit this symmetry as $\vec{1}_1$ is reflected through P. (See some discussion concerning this in [4].) For present purposes, however, confine the observer position to lie on P so that

$$\vec{1}_1 \cdot \vec{1}_P = 0 \tag{4.6}$$

Then we have the result that the $\vec{c}_\alpha$ are all either perpendicular to or parallel to P (as well as being perpendicular to $\vec{1}_P$). This follows from the fact that on P a symmetric electric field is parallel to P while an antisymmetric electric field is perpendicular to P [1, 3]. Thus we have

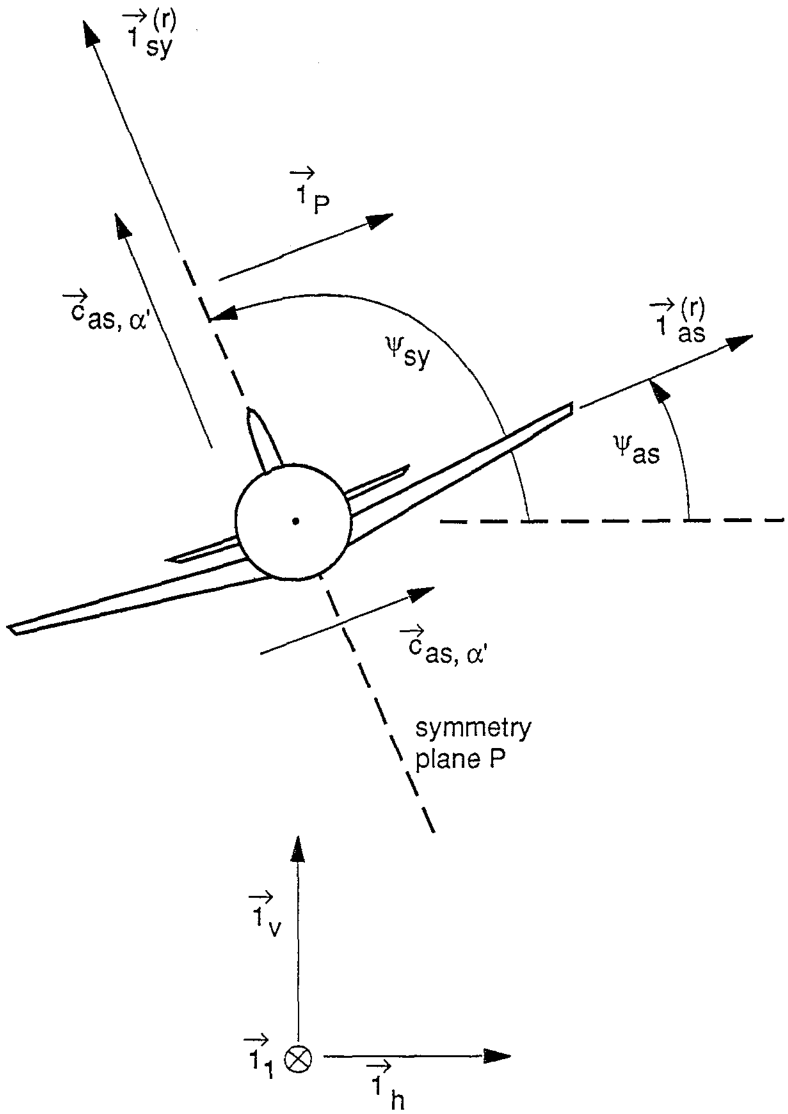


Fig. 4.1. Symmetry Plane Through Scatterer and Observer

$$\vec{c}_{sy,\alpha'} = c^{(r)}_{sy,\alpha'} \ \vec{1}^{(r)}_{sy} \quad , \quad \vec{1}^{(r)}_{sy} = -\vec{1}_1 \times \vec{1}_P$$

$$\vec{c}_{as,\alpha'} = c^{(r)}_{as,\alpha'} \ \vec{1}^{(r)}_{as} \quad , \quad \vec{1}^{(r)}_{as} = \vec{1}_P \tag{4.7}$$

Thus the $\vec{c}_\alpha$ are all now linearly polarized and are characterized in general by one complex and one real number (or equivalently three real numbers). Furthermore there are only two polarization vectors to cover all (assuming no modal degeneracy). Thus the observer can determine from measurements of the $\vec{c}_\alpha$ the presence of such a scatterer symmetry plane passing through itself (at least in a necessary-condition sense).

Referred to our basis in the h, v plane we have angles ψ_{sy} and ψ_{as} for ψ_α describing the linear polarizations via

$$\vec{1}^{(r)}_{sy} = \cos(\psi_{sy})\vec{1}_h + \sin(\psi_{sy})\vec{1}_v$$

$$\vec{1}^{(r)}_{as} = \cos(\psi_{as})\vec{1}_h + \sin(\psi_{as})\vec{1}_v$$

$$\psi_{sy} = \psi_{as} \pm \frac{\pi}{2} \tag{4.8}$$

Considering a typical aircraft with nose-on or tail-on illumination, then of course ψ_{as} is the roll angle.

V. Body of Revolution

Suppose now our scatterer has an axis of revolution as illustrated in fig. 5.1. This means that the scatterer can be rotated about this axis by some angle, say ϕ, with no observable change. This is C_∞ symmetry with group elements.

$$
\begin{aligned}
&(C_\infty)_\phi \equiv \text{rotation by } \phi \\
&(C_\infty)_\phi^L = (C_\infty)_{L\phi} \equiv \text{rotation by } L\phi \\
&(C_\infty)_o = (C_\infty)_{2\pi} = (C_\infty)_{2\pi L'} = (1) \equiv \text{identity} \\
&\qquad L' \equiv \text{an integer } (+, -, \mathit{or}\ 0)
\end{aligned}
\tag{5.1}
$$

This symmetry assures that the backscattering to the observer will be independent of ϕ. Note the 2π periodicity so that increasing ϕ by 2π does not give a new group element.

Now consider a plane P containing this axis and the observer location. Let us further assume that this is a symmetry plane. This result is not implied by the symmetry axis if one allows the scatterer to contain anisotropic (yet still reciprocal) materials. If the scatterer is perfectly conducting then C_∞ symmetry implies reflection symmetry as in (4.3), but which we now term R_a symmetry ("a" denoting the fact that this plane contains the symmetry axis) with

$$
R_a = \{(R_a), (1)\} \ , \ (R_a)^2 = (1)
\tag{5.2}
$$

Note that this differs from other notations [12] which term this R_v symmetry, but this conflicts with the standard horizontal and vertical (h and v) designation for polarization of the transmitted and received electric fields. In this context one might also use R_t (instead of R_h) to denote a symmetry plane perpendicular to (or "transverse" to) the symmetry axis.

If anisotropic materials are allowed let us require that the associated constitutive-parameter matrices also reflect with the appropriate symmetry with respect to P [1]. With this restriction the scatterer still has R_a symmetry, which when adjoined to the C_∞ symmetry, gives $C_{\infty a}$ symmetry and all planes passing through the axis are symmetry planes. This high order of symmetry has elements

$$
C_{\infty a} = \left\{(C_\infty)_\phi, (C_\infty)_\phi (R_a) \mid \phi \ \mathit{real}\right\}
\tag{5.3}
$$

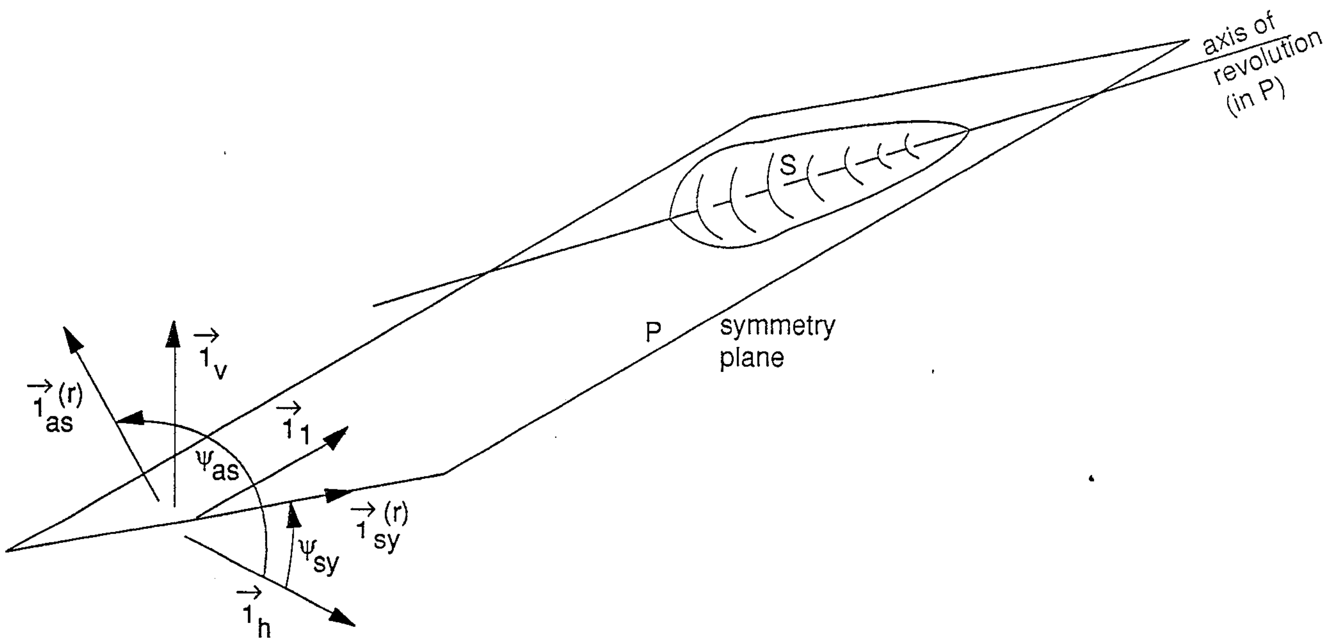


Fig. 5.1. Body of Revolution with Symmetry Plane Through Observer

For every ϕ there are then two kinds of group elements: rotation by ϕ and a combination of rotation and reflection. Note that $C_{\infty a}$ and (R_a) do not in general commute. However, we have

$$(C_\infty)_\phi (R_a) = (R_a)(C_\infty)_{-\phi} = (R_a)(C_\infty)_{2\pi-\phi} \tag{5.4}$$

where the defining symmetry plane is taken on $\phi = 0$.

Referring back to section 4 then the effect of the symmetry plane is to make the $\vec{c}_\alpha$ be some symmetric and some antisymmetric. So the observer will see these two sets as in (4.7) except for perhaps some shift to relative to ψ_{as} as

$$\psi_{sy} = \psi_{as} - \frac{\pi}{2} \tag{5.5}$$

for the orientation of P in the h, v plane in fig. 4.1. This relates to the choice of convention for polarization angles in section 2.

Now as the scatterer is rotated through some angle ϕ, the observer see no change, i.e. $\vec{c}_{b_\alpha}$ is unchanged. As one rotates the scatterer and the associated natural modes $\vec{j}_{s_\alpha}$ of the scatterer, and hence the associated $\vec{c}_\alpha$, this may raise some concern. However, the resolution is simple by noting that there is a modal degeneracy for such a body of revolution. The ϕ dependence of the currents on the body is expandable as a Fourier series giving cos (mϕ) and sin (mϕ) terms. The index m belongs to certain natural frequencies s_α. There are then two natural modes for each s_α (except for m = 0 when there is but one). These two modes have separate $\vec{c}_\alpha(\vec{1}_1)$. There are then two different α index sets for the same s_α.

For a particular s_α let us consider symmetric (sy) and antisymmetric (as) excitation. Given the fact that a symmetric incident field must give only a symmetric backscattered field, and similarly in the antisymmetric case, then we have

$$\begin{aligned}\vec{c}_{b_\alpha}(\vec{1}_1) &= \vec{c}_{sy,\alpha'}(\vec{1}_1)\vec{c}_{sy,\alpha'}(\vec{1}_1) + \vec{c}_{as,\alpha'}(\vec{1}_1)\vec{c}_{as,\alpha'}(\vec{1}_1) \\ &= c_{b_{sy,\alpha'}}(\vec{1}_1)\vec{1}_{sy}(\vec{1}_1)\vec{1}_{sy}(\vec{1}_1) + c_{b_{as,\alpha'}}(\vec{1}_1)\vec{1}_{as}(\vec{1}_1)\vec{1}_{as}(\vec{1}_1) \\ \alpha &= \begin{pmatrix} sy \\ & ,\alpha' \\ as \end{pmatrix}\end{aligned} \tag{5.6}$$

This is a diagonal form for $\vec{c}_{b_\alpha}$ with eigenvalues $c_{b_{sy,\alpha'}}$ *and* $c_{b_{as,\alpha'}}$, corresponding to symmetric and antisymmetric excitaiton, respectively. From measurements this can be reconstructed by diagonalizing the measured $\vec{c}_{b_\alpha}$.

Note that

$$\det\left(\vec{c}_{b_\alpha}\left(\vec{1}_1\right)\right) = \det\left(\left(c_{b_{n,m}}\left(\vec{1}_1\right)\right)\right) = c_{b_{sy,\alpha'}}\left(\vec{1}_1\right)c_{b_{as,\alpha'}}\left(\vec{1}_1\right)$$

$$= c_{b_{h,h}}^{(\alpha)}\left(\vec{1}_1\right)c_{b_{v,v}}^{(\alpha)}\left(\vec{1}_1\right) - c_{b_{h,v}}^{(\alpha)^2}\left(\vec{1}_1\right) \tag{5.7}$$

which is in general non zero, in contradistinction to the case of a single $\vec{c}_\alpha$ as in (3.1).

Noting that

$$tr\left(\vec{c}_{b_\alpha}\left(\vec{1}_1\right)\right) = tr\left(\left(c_{b_{n,m}}\left(\vec{1}_1\right)\right)\right) = c_{b_{sy,\alpha'}} + c_{b_{as,\alpha'}}$$

$$= c_{b_{h,h}}\left(\vec{1}_1\right) + c_{b_{v,v}}\left(\vec{1}_1\right) \tag{5.8}$$

then with (5.7) both eigenvalues are readily determined. The eigenvectors are normalized and real as $\vec{1}_{sy,\alpha'}\,and\,\vec{1}_{as,\alpha'}$. The scatering residue matrix is now characterized by two complex numbers (the eigenvalues) and one real angle ψ_{sy} (since by (5.5) ψ_{as} is constrained). This is equivalently five real numbers. This is still one less than the three complex (or six real) numbers needed to generally characterize a scattering matrix in the h, v plane.

For the results of (5.6) to apply it is merely necessary that the two natural modes belonging to s_α produce two linearly independent $\vec{c}_\alpha$ (i.e. non parallel). These two vectors then span the h, v plane (i.e. any vector can be expressed as a linear combination of the two). The $\vec{1}_{sy}$ *and* $\vec{1}_{as}$ are merely a convenient diagonalizing basis. As can be seen in special cases (e.g. a symmetry plane perpendicular to the symmetry axis (R_t symmetry) with this plane also through the observer), the two $\vec{c}_\alpha$ can become parallel giving a representation involving a single dyad.

For the case of m=0 (no ϕ variation in cylindrical coordinates based on the axis of rotation) there is only one natural mode for each s_α. The results of section 4 then apply and $\vec{c}_{b_\alpha}$ is characterized by a single vector $\vec{c}_\alpha$ which is either parallel to P or perpendicular to it.

Note that the symmety axis of the scatterer is assumed to not pass through the observer for these results. Otherwise P is not uniquely specified.

VI. Observer on Symmetry Axis of Scatterer

Continuing our consideration of symmetry implications in backscattering let us now orient the scatterer such that its symmetry axis points at the observer. As indicated in fig. 6.1A our body of revolution is described in a cylindrical coordinate system $(\psi,\ \phi,\ z)$ where the z axis is the symmetry axis passing through the observer so that

$$\vec{1}_z = \vec{1}_1\ ,\ \vec{1}_z = \vec{1}_1 \tag{6.1}$$

Now as a body of revolution the group elements in (5.1) have matrix (dyadic) representations [12]

$$(C_\infty)_\phi \to (C_{n,m}(\phi)) \equiv \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \quad \equiv rotation\ by\ \phi$$

$$(C_\infty)_\phi^L \to (C_{n,m}(\phi))^L \equiv \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}^L = \begin{pmatrix} \cos(L\phi) & -\sin(L\phi) \\ \sin(L\phi) & \cos(L\phi) \end{pmatrix}$$

$$= (C_{n,m}(L\phi)) \qquad \equiv rotation\ by\ L\phi$$

$$(C_\infty)_0 \to (C_{n,m}(0)) = (C_{n,m}(2\pi)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \vec{1}_z \equiv \vec{1}_x, \vec{1}_x + \vec{1}_y \vec{1}_y$$

$$= \vec{1}_h \vec{1}_h + \vec{1}_v \vec{1}_v \equiv identity \tag{6.2}$$

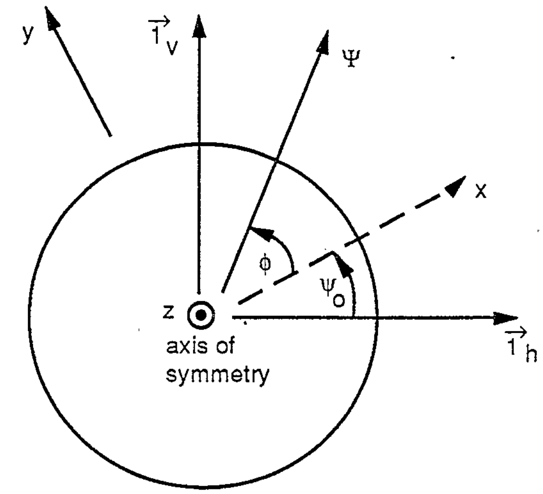
Note in fig. 6.1A that $\phi = 0$ can be taken at any angle, say ψ_o, in the h, v plane. This is a two dimensional representation since we are not considering any symmetries (such as R_t) involving the z coordinate.

Note that the rotation matrix is real and unitary since

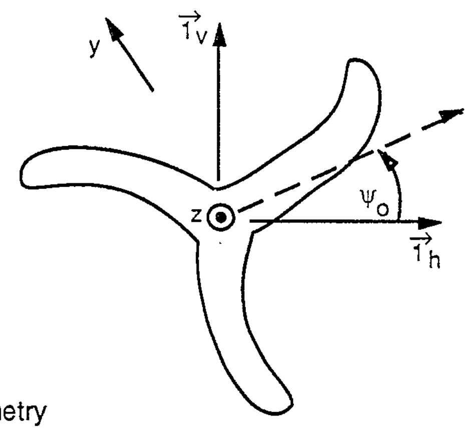
$$(C_{n,m}(\phi))^T = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} = (C_{n,m}(\phi))^{-1}$$

$$= (C_{n,m}(-\phi)) \tag{6.3}$$

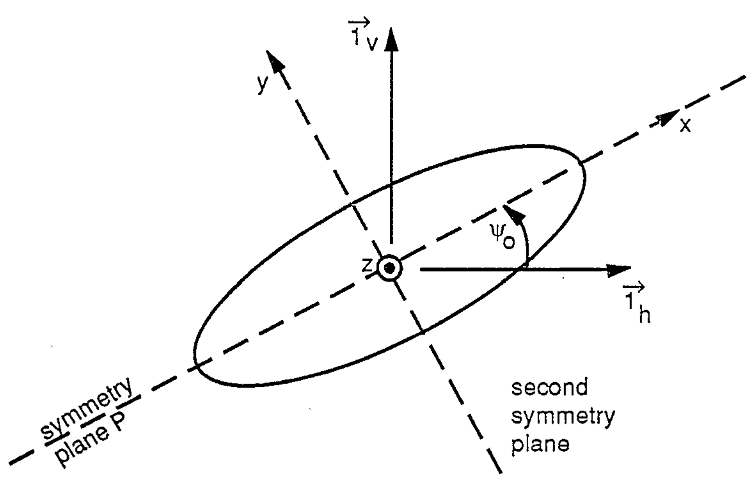
Let us not yet assume that there is R_a symmetry (axial symmetry planes). What can be inferred from the C_∞ symmetry? Now if we consider some incident polarization (real) characterized by some unit vector, say $\vec{1}_o$, in the h, v plane and scatter axially the field we obtain an amplitude and polarization



A. Body of Revolution



B. C_3 Symmetry



C. C_{2a} Symmetry

Fig. 6.1. On-Axis Backscattering

characterized by $\vec{c}_{b_\alpha} \cdot \vec{1}_o$. Rotating the scattering by ϕ gives backscatter $\left(C_{n,m}(\phi)\right) \cdot \vec{c}_{b_\alpha} \cdot \vec{1}_o$. Now reverse the order. Rotate and then scatter by $\vec{c}_{b_\alpha}$ giving $\vec{c}_{b_\alpha} \cdot \left(C_{n,m}(\phi)\right) \cdot \vec{1}_o$. Since $\vec{1}_o$ is arbitrary we have

$$\left(C_{n,m}(\phi)\right) \cdot \vec{c}_{b_a} = \vec{c}_{b_\alpha} \cdot \left(C_{n,m}(\phi)\right) \tag{6.4}$$

i.e., the rotation and scattering <u>commute</u>. Viewed another way this is a coordinate rotation, in one case before and another after the scattering.

To see the constraints this imposes on $\vec{c}_{b_\alpha}$ write (6.4) out in components as

$$\begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \cdot \begin{pmatrix} c_{b_{1,1}} & c_{b_{1,2}} \\ c_{b_{2,1}} & c_{b_{2,2}} \end{pmatrix}_\alpha = \begin{pmatrix} c_{b_{1,1}} \cos(\phi) - c_{b_{2,1}} \sin(\phi) & c_{b_{1,2}} \cos(\phi) - c_{b_{2,2}} \sin(\phi) \\ c_{b_{1,1}} \sin(\phi) + c_{b_{2,1}} \cos(\phi) & c_{b_{1,2}} \sin(\phi) + c_{b_{2,2}} \cos(\phi) \end{pmatrix}_\alpha$$

$$\begin{pmatrix} c_{b_{1,1}} & c_{b_{1,2}} \\ c_{b_{2,1}} & c_{b_{2,2}} \end{pmatrix}_\alpha \cdot \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} = \begin{pmatrix} c_{b_{1,1}} \cos(\phi) + c_{b_{1,2}} \sin(\phi) & c_{b_{1,1}} \sin(\phi) + c_{b_{1,2}} \cos(\phi) \\ c_{b_{2,1}} \cos(\phi) + c_{b_{2,2}} \sin(\phi) & c_{b_{2,1}} \sin(\phi) + c_{b_{2,2}} \cos(\phi) \end{pmatrix}_\alpha \tag{6.5}$$

Equating each of the matrix elements in the two forms of the matrix products we have

$$\left[c_{b_{1,1}}^{(\alpha)} - c_{b_{2,2}}^{(\alpha)} \right] \sin(\phi) = 0$$

$$\left[c_{b_{1,2}}^{(\alpha)} + c_{b_{2,1}}^{(\alpha)} \right] \sin(\phi) = 0 \tag{6.6}$$

For $\sin(\phi) \neq 0$ we then have

$$c_{b_{1,1}}^{(\alpha)} = c_{b_{2,2}}^{(\alpha)}$$

$$c_{b_{2,1}}^{(\alpha)} = c_{b_{1,2}}^{(\alpha)}$$

$$\vec{c}_{b_\alpha} = c_{b_{1,1}}^{(\alpha)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_{b_{1,2}}^{(\alpha)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{6.7}$$

so that the rotation invariance allows $\vec{c}_{b_\alpha}$ to be the linear combination of an identity and a rotation.

Now constrain the scatterer to be reciprocal, i.e. comprised of reciprocal media, in addition to symmetry. Then we have

$$c_{b_{1,2}}^{(\alpha)} = c_{b_{2,1}}^{(\alpha)} = 0 \quad , \quad \tilde{c}_{b_\alpha} = \tilde{c}_{b_\alpha}^T$$

$$\tilde{c}_{b_\alpha} = \tilde{c}_{b_\alpha} \tilde{1}_z \tag{6.8}$$

i.e., just a complex constant times the identify. This requires just one complex number (or two real numbers) to specify, no angles being needed. Note that while the on-axis C_∞ discussion has been centered around scattering residue matrices it applies to scattering matrices at arbitrary complex frequencies as well.

Note that we have not assumed a symmetry plane containing the axis, i.e. R_a symmetry for this result to hold, as has been done previously [6]. It only relies on C_∞ symmetry and reciprocity. One might ask if this necessarily implies R_a symmetry, but it does not. This can be seen through examples. Let the scatterer contain anisotropic materials (say conducting in some preferred direction). Let the currents spiral around the symmetry axis. Approximations involving N arms on, say, a conical spiral scatterer meet this condition.

Consider now some other symmetries with respect to the z axis for on-axis backscattering. Begin as in fig. 6.1 B with C_N symmetry, the illustration being for C_3 symmetry. As in [5, 12] we have

$$C_N = \left\{ (C_N)_1, (C_N)_2, \ldots, (C_N)_{N-1}, (1) \right\}$$

$$(C_N)_n \equiv \textit{rotation by } \frac{2\pi n}{N} \ (\textit{positive } \phi \textit{ direction})$$

$$(C_N)_n = (C_N)_1^n$$

$$(C_N)_1^N = (C_N)_N = (1) \tag{6.9}$$

This group is a subgroup of C_∞. It has a two-dimensional matrix representation like (6.2) as

$$(C_N)_1 \rightarrow \left(C_{n,m}\left(\frac{2\pi}{N}\right)\right) = \begin{pmatrix} \cos\left(\frac{2\pi}{N}\right) & -\sin\left(\frac{2\pi}{N}\right) \\ \sin\left(\frac{2\pi}{N}\right) & \cos\left(\frac{2\pi}{N}\right) \end{pmatrix} \equiv \text{rotation by } \frac{2\pi}{N}$$

$$(C_N)_n \rightarrow \left(C_{n,m}\left(\frac{2\pi n}{N}\right)\right) = \begin{pmatrix} \cos\left(\frac{2\pi n}{N}\right) & -\sin\left(\frac{2\pi n}{N}\right) \\ \sin\left(\frac{2\pi n}{N}\right) & \cos\left(\frac{2\pi n}{N}\right) \end{pmatrix} = \left(C_{n,m}\left(\frac{2\pi}{N}\right)\right)^n \equiv \text{rotation by } \frac{2\pi n}{N}$$

$$(C_N)_0 \rightarrow \left(C_{n,m}(0)\right) = \left(C_{n,m}(2\pi)\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \vec{\vec{1}}_z = \vec{1}_x\vec{1}_x + \vec{1}_y\vec{1}_y = \vec{1}_h\vec{1}_h + \vec{1}_v\vec{1}_v \tag{6.10}$$

As indicated in fig. 6.1 B one might associate some angle ψ_o (corresponding to $\phi = 0$ and some plane which is in general not a symmetry plane). Let this scatterer have C_N symmetry. Then (6.4) is replaced by

$$\left(C_{n,m}\left(\frac{2\pi n}{N}\right)\right) \cdot \vec{\vec{c}}_{b_\alpha} = \vec{\vec{c}}_{b_\alpha} \cdot \left(C_{n,m}\left(\frac{2\pi n}{N}\right)\right) \tag{6.11}$$

i.e., ϕ for rotation symmetry is now discrete. Considering (6.5) and the result (6.6) let us require

$$\sin\left(\frac{2\pi n}{N}\right) \neq 0 \; \textit{for at least one of } n = 1, 2, \ldots, N \tag{6.12}$$

This requirement is satisfied for $N \geq 3$. As can be seen for N=1 (n=1) and N=2 (n=1 and n=2) the sine function is always zero. From this it follows as in (6.8) that

$$\vec{\vec{c}}_{b_\alpha} = c_{b_\alpha} \vec{\vec{1}}_z \; \textit{for } C_N \textit{ symmetry with } N \geq 3 \tag{6.13}$$

Note the example in fig. 6.1 B does not have R_a symmetry; it is not required for this result.

Note that C_2 is not adequate for this result. As a special case in fig. 6.1 C let C_2 symmetry be adjoined by R_a symmetry, giving C_{2a} symmetry. The existence of one symmetry plane (say defining ψ_o) implies a second (at $\psi_o + \pi/2$). However, as illustrated, the extension of the scatterer in the direction of these two symmetry planes need not be the same. Then the results of section IV apply, giving two different eigenvalues to $\vec{\vec{c}}_{b_\alpha}$ with unit eigenvectors oriented by ψ_o and $\psi_o + \pi/2$.

VII. Concluding Remarks

So it appears that the backscattering matrix can be cast in the form of a residue matrix which is closely related to the scatterer properties. For non-degenerate natural modes this is a dyad which characterized by a single two-component complex vector. For cases of symmetry in the scatterer and observer location various additional properties appear in the $\vec{c}_{b_\alpha}$.

Some of these results are reminiscent of low-frequency scattering [13]. The polarizability tensors are real valued for perfect conductors. The scattering matrix, being real valued, is characterized by three real (not complex) numbers. This is a reduction comparable to that for the case of a symmetry plane on the $\vec{c}_\alpha$ as in section IV.

A previous paper [4] has considered the bistatic form of the SEM residue matrices, noting various symmetries in coupling to the incident field and recoupling to the scattered field based on reciprocity, as well as some effects of a symmetry plane. Various other symmetries can be explored in the scattering residue matrices, including effects of reciprocity and scatterer symmetry such as is done for the usual scattering matrices [8, 9, 10]. Hopefully further simplifications will result.

References

1. C. E. Baum, Interaction of Electromagnetic Fields With an Object Which Has an Electromagnetic Symmetry Plane, Interaction Note 63, March 1971.

2. C. A. Lin and J. T. Cordaro, Singularity Expansion Method Parameter Measurement, Interaction Note 409, May 1981, and Determination of the SEM Parameters for an Aircraft Model from the Transient Surface Current, Electromagnetics, pp. 65-75, 1983.

3. C. E. Baum, A Priori, Application of Results of Electromagnetic Theory to the Analysis of Electromagnetic Interaction Data, Interaction Note 444, February 1985, and Radio Science, 1987, pp. 1127-1136.

4. C. E. Baum, Scattering, Reciprocity, Symmetry, EEM, and SEM, Interaction Note 475, May 1989.

5. C. E. Baum, Combining RF Sources Using C_N Symmetry, Circuit and Electromagnetic System Design, Note 37, June 1989.

6. S. L. Borison, Diagonal Representation of the Radar Scattering Matrix for an Axially Symmetric Body, IEEE Trans. Antennas and Propagation, pp. 176-177, 1965.

7. W. -M. Boerner, M. B. El-Arini, C. -Y. Chan, and P. M. Mastoris, Polarization Dependence in Electromagnetic Inverse Problems, IEEE Trans. Antennas and Propagation, pp. 262-271, 1981.

8. G. E. Heath, Bistatic Scattering Reflection Asymmetry, Polarization Reversal Asymmetry, Polarization Reversal Reflection Symmetry, IEE Trans. Antennas and Propagation, pp. 429-434, 1981.

9. G. E. Heath, Properties of the Linear Polarization Bistatic Scattering Matrix, IEEE Trans. Antennas and Propagation, pp. 523-525, 1981.

10. M. Davidovitz and W. -M. Boerner, Reduction of Bistatic Scattering Matrix Measurements for Inversely Symmetric Radar Targets, IEEE Trans. Antennas and Propagation, pp. 237-242, 1983.

11. C. D. Taylor, Calculation of Natural Resonances for Perpendicular Crossed Wires Parallel to an Imperfect Ground, Electromagnetics, pp. 41-64, 1983.

12. M. Hamermesh, Group Theory and Its Application to Physical Problems, Addison-Wesley, 1962.

13. R. E. Kleinman, Low Frequency Electromagnetic Scattering, in P. L. E. Uslenghi (ed.), Electromagnetic Scattering, Academic Press, 1978.

14. J. D. Kraus, Antennas, 2nd. ed., McGraw Hill, 1988.