Interaction Notes

Note 484

17 April 1991

Transient Scattering Length and Cross Section

Carl E. Baum
Phillips Laboratory

ABSTRACT

By generalizing the use of complex vector magnitude for incident and scattered fields to norms, ng in time domain can be similarly scalarized. The paper explores the use of appropriate norms for scattering length (or cross section) for transient scattering. This allows one to optimize transient fields for maximum scattering.

I. Introduction

Historically electromagnetic scattering and its practical application, radar, have developed using frequency domain concepts [8, 10]. Even here, however, transient characteristics have appeared associated with bandwidth. Beginning in the 1960s, associated with the electromagnetic pulse (EMP) program, much progress has been made in transient electromagnetic theory and associated antenna design, pulser design, and interaction/scattering analysis. (See some reviews [1, 9] to name a few.) It seems appropriate and constructive to apply some of the concepts and technology here to the more general radar scattering problem. In fact, this application has been developing over some time now (an example being the singularity expansion method (SEM) for target identification).

A common concept used in radar is that of a scattering area or cross section traditionally defined by [10]

$$A \equiv 4\pi r^2 \frac{\left|\tilde{\vec{E}}_f(j\omega)\right|^2}{\left|\tilde{\vec{E}}_{inc}(j\omega)\right|^2}$$

$$\vec{E}_f \equiv \text{far field at distance } r$$

$$\vec{E}_{inc} \equiv \text{incident field at target}$$

$$\sim \equiv \text{Laplace transform (two sided)} \qquad (1.1)$$

$$s \equiv \Omega + j\omega \equiv \text{complex frequency}$$

$$\gamma \equiv \frac{s}{c} \equiv \text{propagation constant}$$

This area relates the energy density scattered in a particular direction $\vec{1}_r$ at a distance r to that incident on the target from some direction $\vec{1}_1$. Note that A is in general also a function of the polarization of the incident field (taken as a plane wave). Note that this definition applies to both monostatic ($\vec{1}_r = -\vec{1}_1$ or backscattering) and bistatic $\left(\vec{1}_r \neq -\vec{1}_1\right)$ scattering.

Looking at this definition, let us convert it to something more convenient for our purposes, a scattering length, defined by

$$\ell \equiv \sqrt{4\pi}\; r \frac{\left|\tilde{\vec{E}}_f(j\omega)\right|}{\left|\tilde{\vec{E}}_{inc}(j\omega)\right|} = A^{1/2} \qquad (1.2)$$

This is now first order (scales with) the scattered far field. Later this will sometimes be generalized to a vector scattering length when the scattered field, at least in a maximal sense, has a particular orientation $\vec{1}_\ell$ at the observer with

$$\vec{\ell} = \ell \vec{1}_\ell \,, \quad A = \vec{\ell} \cdot \vec{\ell}^{\,*} \,, \quad \left|\vec{1}_\ell\right| = 1 \qquad (1.3)$$

noting that $\vec{1}_\ell$ may be complex.

One can construct a scattering operator relating the far scattered field to the incident field via the free-space dyadic Green's function used in an integral equation to find the currents on the scatterer, and in turn the scattered fields. This can in turn be used to expand the scattering, for frequency and time domains, in terms of SEM parameters (natural frequencies, etc.) [4, 5]. For our present purposes we merely need to define a scattering dyad (or 2 x 2 matrix) which converts the incident field into a scattered far field as

$$\tilde{\vec{E}}_f(\vec{r},s) = \frac{e^{-\gamma r}}{4\pi r}\,\tilde{\overline{\overline{\Lambda}}}(s)\cdot\tilde{\vec{E}}_{inc}(s) \tag{1.4}$$

Note that $\tilde{\overline{\overline{\Lambda}}}$ has the dimension of length (meters, like ℓ). The factor of $e^{-\gamma r}/(4\pi r)$ is carried through from the leading term in the Green's function for the far field [4, 5], but whether one includes the $1/(4\pi)$ is arbitrary. Not expressed, but included in $\tilde{\overline{\overline{\Lambda}}}$ are $\vec{1}_1$ and $\vec{1}_r$, which for our present purposes will be considered fixed, and often taken for backscattering.

In this form our scattering length is

$$\ell \equiv \frac{1}{\sqrt{4\pi}}\frac{\left|\tilde{\overline{\overline{\Lambda}}}(j\omega)\cdot\tilde{\vec{E}}_{inc}(j\omega)\right|}{\left|\tilde{\vec{E}}_{inc}(j\omega)\right|} \tag{1.5}$$

This is of course a function of frequency and polarization of the incident field. Noting that the complex magnitude is the same as the 2-norm, we might ask how large ℓ might be. For a fixed frequency ω this is

$$\begin{aligned}\ell_{\max}(j\omega) &= \frac{1}{\sqrt{4\pi}}\left\|\tilde{\overline{\overline{\Lambda}}}(j\omega)\right\|_{2v}\\ &= \frac{1}{\sqrt{4\pi}}\left[\lambda_{\max}\left(\tilde{\overline{\overline{\Lambda}}}^{\dagger}(j\omega)\cdot\tilde{\overline{\overline{\Lambda}}}(j\omega)\right)\right]^{\frac{1}{2}}\end{aligned} \tag{1.6}$$

$\dagger$ = adjoint = $*T$ = conjugate transpose

$\lambda_{\max}$ = maximum eigenvalue (real and non negative)

where 2v means 2-norm in the vector (or matrix) sense [3]. The polarization of the incident field which achieves this can be considered in some sense optimal and the polarization of the scattered field in this condition can be used to define $\vec{1}_\ell$ for a given ω. Maximizing over all ω we can define

$$\ell_{\max} \equiv \sup_{\omega}\,\ell_{\max}(j\omega)$$

from the above. This is like an ∞-norm (peak) in a function sense (∞f) (function of ω) and a 2-norm in the matrix sense (2v).

II. Scattering in Time Domain Via Norms

For some arbitrary (physical realizable) incident waveform in time domain, one may ask how to generalize this concept of scattering length (or area). First note that the scattered waveform is not in general the same as the incident waveform, but is related from (1.4) as

$$\vec{E}_f(\vec{r}, t) = \frac{1}{4\pi r} \tilde{\Lambda}(t) \overset{o}{\cdot} \vec{E}_{inc}\left(t - \frac{r}{c}\right)$$

$$o \equiv \text{convolution with respect to time} \tag{2.1}$$

noting that the far field is a function of retarded time.

Second generalize the concept of scattering length from (1.2) to time domain as

$$\ell \equiv \sqrt{4\pi}\ r \frac{\left\| \vec{E}_f(t) \right\|}{\left\| \vec{E}_{inc}(t) \right\|} = A^{\frac{1}{2}} \tag{2.2}$$

where the norm to be used is as yet unspecified. Now, not only is ℓ a function of $\vec{1}_1$, $\vec{1}_p$, and $\vec{1}_r$, but also of the actual detailed incident waveform with its multiplicity of shape-describing parameters instead of just the single parameter, frequency.

Substituting from (2.1) we have

$$\ell \equiv \frac{1}{\sqrt{4\pi}} \frac{\left\| \tilde{\Lambda}(t) \overset{o}{\cdot} \vec{E}_{inc}\left(t - \frac{r}{c}\right) \right\|}{\left\| \vec{E}_{inc}(t) \right\|} \tag{2.3}$$

so that for a given $\tilde{\Lambda}(t)\, o$ scattering matrix operator, the scattering length is formulated in terms of the various possible incident waveforms.

Of all the possible norms one may use, there are desirable properties one may wish to impose based on various physical properties. One may think of these as "natural" norms. The first property to impose is time invariance, in the sense of time-invariant physical systems. This is a time-translation symmetry in which we require

$$\left\| \vec{E}_{inc}\left(t - \frac{r}{c}\right) \right\| = \left\| \vec{E}_{inc}(t) \right\| \tag{2.4}$$

i.e., it does not matter where the waveform is shifted on the time axis as far as the norm value is concerned. This allows (2.3) to take the r-independent form

$$\ell \equiv \frac{1}{\sqrt{4\pi}} \frac{\left\| \tilde{\Lambda}(t) \overset{o}{\cdot} \vec{E}_{inc}(t) \right\|}{\left\| \vec{E}_{inc}(t) \right\|} \tag{2.5}$$

A second property one may impose is time reversal symmetry, i.e.

$$\left\| \vec{E}_{inc}(-t) \right\| = \left\| \vec{E}_{inc}(t) \right\| \tag{2.6}$$

In this sense the "size" of the pulse is independent of which way one looks at it along the time axis.

Note that the scattered field is characterized by a particular direction of propagation $\vec{1}_r$ (measurable) while the polarization can be anything perpendicular to this. In defining coordinates for measurements, while one direction $\vec{1}_r$ is natural, the two coordinates orthagonal to this can be arbitrarily rotated about $\vec{1}_r$. Similarly the incident field has a characteristic direction of propagation $\vec{1}_1$, but the polarization $\vec{1}_p$ need only be perpendicular to this. For the case of backscattering $\left(\vec{1}_r = -\vec{1}_1\right)$ this rotation of the transverse coordinates applies simultaneously to incident and scattered waves. So a third property of a "natural" norm might be taken as invariance to rotation of the transverse (polarization) coordinates. Note that for our far-field scattering problem the vectors are essentially two-dimensional (two components orthogonal to the propagation direction) and so only two-dimensional rotational invariance is postulated here.

In (2.5), for the present let us assume that the norms in numerator and denominator are the same for simplicity. In that we can use these norms to account for various efficiencies in launching the incident field from a transmitting antenna, and receiving the scattered field (in the presence of noise), one may choose different norms for these two. However, one will still need to be concerned with some normalization so that the units come out the same. Letting the two norms be the same then (2.5) is suggestive of the norm of an operator [3, 7] in that

$$\left\|\tilde{\Lambda}(t) \circ\right\| \equiv \sup_{\left\|\vec{E}_{inc}(t)\right\| \neq 0} \frac{\left\|\tilde{\Lambda}(t) \overset{\circ}{,} \vec{E}_{inc}(t)\right\|}{\left\|\vec{E}_{inc}(t)\right\|} \tag{2.7}$$

This allows us to find a maximum scattering length via

$$0 \leq \ell \leq \ell_{\max} = \frac{1}{\sqrt{4\pi}} \left\|\tilde{\Lambda}(t) \circ\right\| \tag{2.8}$$

An interesting question concerns what $\vec{E}_{inc}$ achieves this $\ell_{\max}$, this being a possible definition of the optimum incident waveform. Note that in (2.5) our choice for our norm of $\vec{E}_{inc}$ also should be such that all finite values of the norm correspond to physically realizable waveforms (this perhaps being only an approximation). Let us then define a waveform efficiency as

$$\eta \equiv \frac{\ell}{\ell_{\max}} \tag{2.9}$$

For each norm one can determine the effectiveness of a given waveform in realizing the maximum scattering length.

III. Norms for Scattering

A. p-Norm of Vector Waveforms

Let us now consider some norms that might be appropriate for scattering length in time domain. Begin with the p-norm of a vector function [3] given by

$$\left\| \vec{E}(t) \right\|_p \equiv \left\{ \int_{-\infty}^{\infty} \sum_{n=1}^{2} \left| E_n(t) \right|^p dt \right\}^{\frac{1}{p}} \tag{3.1}$$

where we have assumed the waveform to have two components (normal to the propagation direction) as discussed previously. Here the waveform applies to both incident and far scattered fields.

Noting the commutativity of summation and integration in (3.1) the p-norm is clearly invariant to time translation and time reversal for all p. However, this is not in general invariant to coordinate rotation about the propagation axis. For a "natural" norm we would like the norm, in effect the length of the vector, to be independent of the coordinate rotation around the propagation axis. Physically, the waveforms are real vectors and the length represents a field strength. The only p-norm which preserves this rotational invariance is the 2-norm which thus seems a "natural" norm, in particular for the vector aspects.

B. 2-norm of Vector Waveforms and Scattering Operators

Appendix A considers the 2-norm in detail. Summarizing

$$\left\| \vec{E}(t) \right\|_2 = \left\{ \int_{-\infty}^{\infty} \vec{E}(t) \cdot \vec{E}(t) \; dt \right\}^{\frac{1}{2}}$$

$$\left\| \tilde{\Lambda}(t) \; o \right\|_2 = \left[\lambda_{max} \left(\left(\tilde{\tilde{\Lambda}}(j\omega_{max}) \right)^{\dagger} \cdot \left(\tilde{\tilde{\Lambda}}(j\omega_{max}) \right) \right) \right]^{\frac{1}{2}}$$

$$\vec{\ell}_{max}^{(2)} = \frac{1}{\sqrt{4\pi}} \tilde{\tilde{\Lambda}}(j\omega_{max}) \cdot \vec{u}_{max} \; , \; \left| \vec{\ell}_{max}^{(2)} \right| = \ell_{max}^{(2)} \tag{3.2}$$

where ω_{max} maximizes the maximum eigenvalue for all frequencies and $\vec{u}_{max}$ is the corresponding eigenvector (of the Hermitian product).

C. m-Norm of Vector Waveforms and Scattering Operators

Let us now define a norm based on the maximum electric-field magnitude. This has physical significance from the point of view of electrical breakdown. One can also argue from the concept of coordinate-rotation-invariance (around the propagation direction) for a plane wave (a property of a "natural" norm) that only a vector magnitude can be used. Viewed another way consider for any fixed t that $\vec{E}(t)$ has an orientation angle (say $\psi(t)$) around the propagation axis. Then a real vector $\vec{E}(t)$ can be considered in terms of its magnitude and angle. In an angle-independent sense this leaves magnitude. This relates the norm to an ∞-norm (peak) over angle, similar to the ∞-norm over $-\infty < t < \infty$.

Let us then define the m-norm as

$$\|\vec{E}(t)\|_m \equiv \Big\| \, \|\vec{E}(t)\|_{2v} \, \Big\|_{\infty f}$$

$$= \sup_t \left|\vec{E}(t)\right|$$

$$= \left\{ \left\| \vec{E}(t) \cdot \vec{E}(t) \right\|_{\infty f} \right\}^{\frac{1}{2}} \tag{3.3}$$

This is clearly a norm [3, 7] with scalars factoring in magnitude sense and the sum rule (norm of sum ≤ sum of norms) applying. This norm is discussed in some detail in Appendix B. Also here we find the m-norm of the scattering operator to have the bound

$$\|\tilde{\Lambda}(t) \circ\|_m \leq \Big\| \, \|\tilde{\Lambda}(t)\|_{2v} \, \Big\|_{1f} \tag{3.4}$$

with equality at least for a restricted set of scattering operators.

A graphical illustration of this norm is given in Figure 1. Consider some vector waveform $\vec{E}(t)$ with magnitude and angle (about the propagation direction $\vec{1}_r$ for a scattered wave) both as functions of time. One can consider this waveform as a function of time or space by noting the connection between distance in the $\vec{1}_r$ direction and a snapshot for constant time by looking in space in the $-\vec{1}_r$ direction interpreted as retarded time τ in spatial units $c\tau$ (<< r) as indicated in the figure. Consider a circular cylinder of radius $\|\vec{E}(t)\|_m$ (electric-field units) with axis along the propagation direction. The waveform is contained within this cylinder, "touching" it at one point at least. Note that the coordinates for this field (transverse to $\vec{1}_r$) are specified by the orthogonal directions $\vec{1}_h$ and $\vec{1}_v$ for the two electric-field components.

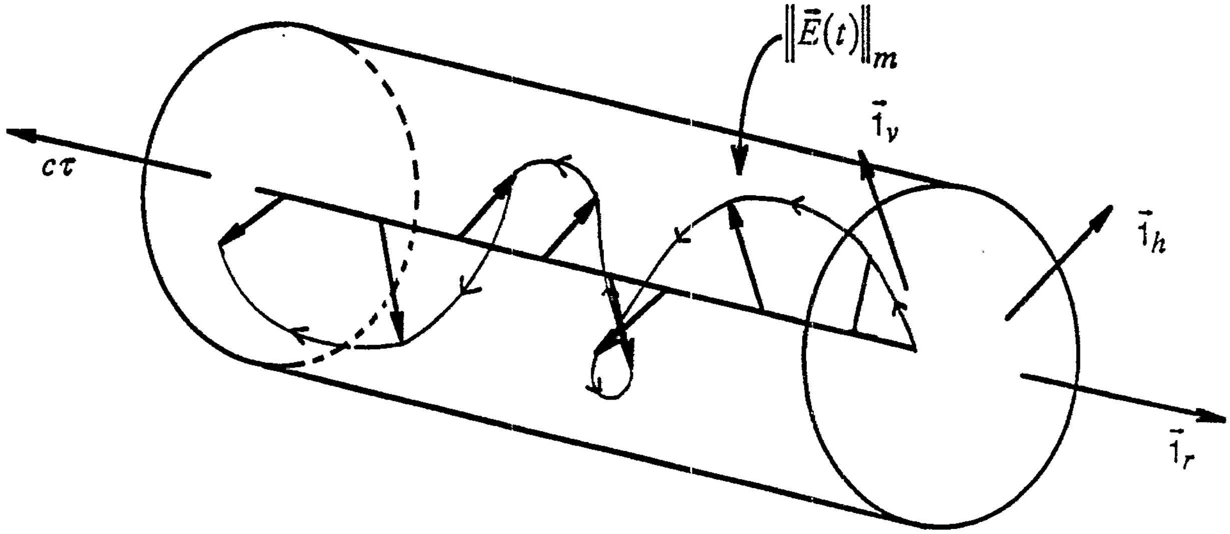

Figure 1. Illustration of m-Norm.

IV. 2-Norm Scattering Length for Scatterer Characterized by a Single Pole Pair

As in Appendix A the scatterer is characterized by a single pole pair as

$$\tilde{\tilde{\Lambda}}(s) = \vec{c}_1[s-s_1] + \vec{c}^*\left[s-s_1^*\right]^{-1}$$
$$s_1 = \Omega_1 + j\omega_1 \,,\; \Omega_1 < 0 \tag{4.1}$$

with $|\Omega_1|$ assumed small as necessary. Then let us consider scattering length for various incident waves $\vec{E}(t)$. Our scattering length is

$$\ell^{(2)} = \frac{1}{\sqrt{4\pi}} \frac{\left\|\tilde{\tilde{\Lambda}}(t) \circ \vec{E}(t)\right\|_2}{\left\|\vec{E}(t)\right\|_2} \tag{4.2}$$

with a superscript to indicate the norm. A subscript can indicate the kind of exciting waveform. The maximum scattering length for small $|\Omega_1|$ is from (A.5)

$$\ell_{\max}^{(2)} = \frac{1}{\sqrt{4\pi}}\left\|\tilde{\tilde{\Lambda}}(t)\, \circ\right\|_2 = \frac{1}{\sqrt{4\pi}}\left\|\tilde{\tilde{\Lambda}}(j\omega_{\max})\right\|_{2\nu}$$
$$\simeq \frac{1}{\sqrt{4\pi}}[-\Omega_1]^{-1}\left[\lambda_{\max}\left(\vec{c}_1^{\dagger}\cdot\vec{c}_1\right)\right]^{\frac{1}{2}}$$
$$\omega_{\max} \simeq \omega_1 \tag{4.3}$$
$$\tilde{\tilde{\Lambda}}(j\omega_1) \simeq [-\Omega_1]^{-1}\,\vec{c}_1$$

From (A.6) the maximizing eigenvector is

$$\vec{c}_1^{\dagger}\cdot\vec{c}_1\cdot\vec{u}_{\max} = \lambda_{\max}\,\vec{u}_{\max} \,,\; |\vec{u}_{\max}| = 1 \tag{4.4}$$

giving

$$\vec{\ell}_{\max}^{(2)} = \frac{1}{\sqrt{4\pi}}\,\tilde{\tilde{\Lambda}}(j\omega_{\max})\cdot\vec{u}_{\max} \tag{4.5}$$

A. Step-Function Incident Field

First let the incident field take the form of a step rise with an exponential decay as

$$\vec{E}(t) = E_o\ \vec{1}_p\ e^{\Omega_o t}u(t)\ ,\ \tilde{\vec{E}}(s) = E_o\ \vec{1}_p[s-\Omega_o]^{-1}$$

$$\Omega_o < 0\ ,\ \vec{1}_p\ real\ ,\ \vec{1}_p \cdot \vec{1}_p = 1 \tag{4.6}$$

$$\left\|\vec{E}(t)\right\|_2 = E_o\left[-2\Omega_o\right]^{\frac{1}{2}}$$

This is a special case of (A.2) and (A.3). Note that the exponential decay, even if small, is needed for the 2-norm to exist.

Now find the scattering norm in the same manner that is used in (A.11) by closing the contour in the left half plane as

$$\left\|\tilde{\Lambda}(t) \circ \vec{E}(t)\right\|_2^2 = \frac{1}{2\pi j}\int_{B_r} \tilde{\vec{E}}(-s) \cdot \tilde{\tilde{\Lambda}}(-s) \cdot \tilde{\tilde{\Lambda}}(s) \cdot \tilde{\vec{E}}(s)ds$$

$$= \tilde{\vec{E}}(-\Omega_o) \cdot \tilde{\tilde{\Lambda}}^T(-\Omega_o) \cdot \tilde{\tilde{\Lambda}}(\Omega_o) \cdot \vec{1}_p\, E_o \tag{4.7}$$

$$+\ 2\text{Re}\left[\tilde{\vec{E}}(-s_1) \cdot \tilde{\tilde{\Lambda}}^T(-s_1) \cdot \vec{c}_1 \cdot \tilde{\vec{E}}(s_1)\right]$$

Assuming that the operator is highly resonant, then let

$$|\Omega_1| << |\omega_1|\ ,\ |\Omega_o| << |\omega_1| \tag{4.8}$$

giving

$$\left\|\tilde{\Lambda}(t) \circ \vec{E}(t)\right\|_2^2$$

$$\simeq E_o^2\left\{\left[-2\Omega_o\right]^{-1}\ \vec{1}_p \cdot \frac{2}{\omega_1}\ \text{Im}\left[-\vec{c}_1^{\,T}\right] \cdot \frac{2}{\omega_1}\ \text{Im}\left[-\vec{c}_1\right] \cdot \vec{1}_p\right.$$

$$\left. +\ 2\text{Re}\left[\frac{1}{-j\omega_1}\ \vec{1}_p \cdot \vec{c}_1^{\,\dagger}\ \left[-2\Omega_1\right]^{-1} \cdot \vec{c}_1 \cdot \vec{1}_p\ \frac{1}{j\omega_1}\right]\right\}$$

$$= E_o^2\left\{2\Omega_o^{-1}\ \omega_1^{-2}\ \vec{1}_p \cdot \text{Im}\left[\vec{c}_1^{\,T}\right] \cdot \text{Im}\left[\vec{c}_1\right] \cdot \vec{1}_p\right. \tag{4.9}$$

$$\left. +\ \left[-\Omega_1\right]^{-1}\ \omega_1^{-2}\ \vec{1}_p \cdot \text{Re}\ \left[\vec{c}_1^{\,T} \cdot \vec{c}_1\right] \cdot \vec{1}_p\right\}$$

By making $\vec{c}_1 / s_1$ imaginary so that $\tilde{\tilde{\Lambda}}(0)$ is zero (corresponding to the property of a real scatterer) the first of the terms goes away giving

$$\left\| \vec{\Lambda}(t) \; ? \; \vec{E}(t) \right\|_2^2 \simeq \vec{1}_p \cdot \mathrm{Re}\left[\vec{c}_1^{\,T} \cdot \vec{c}_1\right] \cdot \vec{1}_p [-\Omega_1]^{-1} \, \omega_1^{-2} \tag{4.10}$$

Alternately one could subtract off a constant dyad (small) in (4.1) to give zero DC content and obtain the same result. Note that the real part of a Hermitian matrix is symmetric so that $\vec{1}_p$ can be chosen as a real eigenvector to maximize this scattering norm.

The scattering length for a step-function incident wave is

$$\ell_s^{(2)} = 0 \tag{4.11}$$

by letting $\Omega_o \to 0-$. The waveform efficiency is similarly

$$\eta_s^{(2)} = \frac{\ell_s^{(2)}}{\ell_{\max}^{(2)}} = 0 \tag{4.12}$$

B. Damped sinusoidal incident field

Choosing the incident waveform as

$$\vec{E}(t) = \frac{E_o}{2} \left\{ \vec{1}_p e^{s_o t} + \vec{1}_p^{\,*} e^{s_o^* t} \right\} u(t)$$

$$\vec{E}(s) = \frac{E_o}{2} \left\{ \vec{1}_p [s - s_o]^{-1} + \vec{1}_p^{\,*} \left[s - s_o^*\right]^{-1} \right\} \tag{4.13}$$

$$s_o = \Omega_o + j\omega_o \; , \; \Omega_o < 0 \; , \; \left|\vec{1}_p\right| = 1$$

then using the results of Appendix A the various norms are evaluated. For present purposes let s_o be nearly matched to s_1 in the sense of (A.12).

The scattering length from (A.14) is

$$\ell^{(2)} \simeq \frac{1}{\sqrt{4\pi}} \left[\vec{1}_p \cdot \vec{c}_1^{\,\dagger} \cdot \vec{c}_1 \cdot \vec{1}_p\right]^{\frac{1}{2}} \left\{1 + \frac{\Omega_o}{\Omega_1}\right\}^{\frac{1}{2}} \left|s_o + s_1^*\right|^{-1} \tag{4.14}$$

So the waveform efficiency is

$$\eta^{(2)} \equiv \frac{\ell^{(2)}}{\ell_{\max}^{(2)}} \simeq \left\{ \frac{\vec{1}_p \cdot \vec{c}_1^{\,\dagger} \cdot \vec{c}_1 \cdot \vec{1}_p}{\lambda_{\max}\left(\vec{c}_1^{\,\dagger} \cdot \vec{c}_1\right)} \right\}^{\frac{1}{2}} \left\{1 + \frac{\Omega_o}{\Omega_1}\right\}^{\frac{1}{2}} \frac{-\Omega_1}{\left|s_o + s_1^*\right|} \tag{4.15}$$

Choosing

$$\vec{1}_p = \vec{u}_{\max} \;,\; s_o = j\omega_1 \tag{4.16}$$

gives

$$\begin{aligned} \vec{\ell}^{(2)}_{\max} &= \frac{1}{\sqrt{4\pi}} \left[-\Omega_1\right]^{-1} \vec{c}_1 \cdot \vec{1}_p \\ \eta^{(2)} &= 1 \end{aligned} \tag{4.17}$$

For s_o near s_1 the more general result in (4.15) can be used. As a special case consider

$$s_o = s_1 \;,\; \vec{1}_p = \vec{u}_{\max} \tag{4.18}$$

This is the case that the incident-field and operator poles are matched giving a second order pole pair for the scattered field. With optimum polarization this gives a waveform efficiency

$$\eta^{(2)} \simeq \frac{1}{\sqrt{2}} \tag{4.19}$$

So one does not need an undamped incident wave to reasonably approach the maximum.

V. m-Norm Scattering Length for Scatterer Characterized by a Single Pole Pair

Again the scatterer is characterized by a single pole pair as

$$\tilde{\vec{\vec{\Lambda}}}(s) = \vec{\vec{c}}_1[s-s_1]^{-1} + \vec{\vec{c}}_1^*\left[s-s_1^*\right]^{-1}$$

$$\vec{\vec{\Lambda}}(t) = \left[\vec{\vec{c}}_1\, e^{s_1 t} + \vec{\vec{c}}_1^*\, e^{s_1^* t}\right] u(t) \tag{5.1}$$

$$s_1 = \Omega_1 + j\omega_1\ ,\ \Omega_1 < 0\ ,\ |\Omega_1| << |\omega_1|$$

Our scattering length is

$$\ell^{(m)} = \frac{1}{\sqrt{4\pi}} \frac{\left\|\vec{\vec{\Lambda}}(t) \circ \vec{E}(t)\right\|_m}{\left\|\vec{E}(t)\right\|_m} \tag{5.2}$$

The maximum scattering length for the restricted case of $\vec{\vec{c}}_1$ symmetric and a constant times a real dyad gives from (B.20) (for small $|\Omega_1|$)

$$\ell_{\max}^{(m)} = \frac{1}{\sqrt{4\pi}} \left\|\vec{\vec{\Lambda}}(t) \circ\right\|_m \simeq \frac{1}{\pi}\sqrt{\frac{4}{\pi}} \left[-\Omega_1\right]^{-1} \left\|\vec{\vec{c}}_1\right\|_{2\nu}$$

$$\vec{\ell}_{\max}^{(m)} = \ell_{\max}^{(2)}\ \vec{1}_{\max} \tag{5.3}$$

$$\vec{1}_{\max} = \text{eigenvector of } \vec{\vec{c}}_1 \text{ for maximum eigenvalue magnitude}$$

with maximizing waveform a square wave of period $2\pi / \omega_1$.

A. Step-Function Incident Field

Let the incident field take the form

$$\vec{E}(t) = E_o\ \vec{1}_p\ e^{\Omega_o t}\ u(t)\ ,\ \tilde{\vec{E}}_o(s) = E_o\ \vec{1}_p\ [s-\Omega_o]^{-1}$$

$$\Omega_o \le 0\ ,\ \vec{1}_p \text{ real}\ ,\ \vec{1}_p \cdot \vec{1}_p = 1 \tag{5.4}$$

$$\left\|\vec{E}(t)\right\|_m = E_o$$

In this case the m-norm exists for all non-positive Ω_o.

For the scattering we have

$$\tilde{\tilde{\Lambda}}(s) \cdot \tilde{\tilde{E}}(s) = E_o\left\{\vec{c}_1[s-s_1]^{-1} + \vec{c}_1^*\left[s-s_1^*\right]^{-1}\right\} \cdot \vec{1}_p[s-s_o]^{-1}$$

$$\vec{\Lambda}(t) \circ \vec{E}(t) = E_o\left\{\frac{e^{s_1 t}-e^{s_o t}}{s_o-s_1}\,\vec{c}_1 \cdot \vec{1}_p + \frac{e^{s_1^* t}-e^{s_o t}}{s_o-s_1^*}\,\vec{c}_1 \cdot \vec{1}_p\right\} u(t) \tag{5.5}$$

For the case of small Ω_o and Ω_1 we have

$$\vec{\Lambda}(t) \circ \vec{E}(t) \simeq \frac{E_o}{-j\omega_1}\left\{\left[e^{j\omega_1 t}-1\right]\vec{c}_1 \cdot \vec{1}_p - \left[e^{-j\omega_1 t}-1\right]\vec{c}_1^* \cdot \vec{1}_p\right\} u(t)$$

$$= -\frac{2E_o}{\omega_1}\sin\left(\frac{\omega_1 t}{2}\right)\left\{e^{\frac{j\omega_1 t}{2}}\vec{c}_1 \cdot \vec{1}_p + e^{\frac{-j\omega_1 t}{2}}\vec{c}_1^* \cdot \vec{1}_p\right\} u(t) \tag{5.6}$$

$$= -\frac{4E_o}{\omega_1}\sin\left(\frac{\omega_1 t}{2}\right)\mathrm{Re}\left[e^{\frac{j\omega_1 t}{2}}\vec{c}_1 \cdot \vec{1}_p\right] u(t)$$

Forming the m-norm let

$$\psi \equiv \frac{\omega_1 t}{2}$$

$$\alpha e^{j\beta} \equiv \frac{\vec{1}_p \cdot \vec{c}_1^T \cdot \vec{c}_1 \cdot \vec{1}_p}{\vec{1}_p \cdot \vec{c}_1^\dagger \cdot \vec{c}_1 \cdot \vec{1}_p} = \frac{\left[\vec{c}_1 \cdot \vec{1}_p\right] \cdot \left[\vec{c}_1 \cdot \vec{1}_p\right]}{\left|\vec{c}_1 \cdot \vec{1}_p\right|^2} \tag{5.7}$$

$$0 \le \alpha \le 1$$

giving

$$\left\|\vec{\Lambda}(t) \circ \vec{E}(t)\right\|_m = \frac{4E_o}{\omega_1}\left|\vec{c}_1 \cdot \vec{1}_p\right| \mathrm{X}$$

$$\mathrm{X}^2 = \sup_{\psi}\frac{\sin^2(\psi)}{2}\mathrm{Re}\left[\alpha e^{j(2\psi+\beta)}+1\right] \tag{5.8}$$

$$= \sup_{\psi}\frac{\sin^2(\psi)}{2}\left\{\alpha\cos(2\psi+\beta)+1\right\}$$

By setting $\alpha = 1$, $\beta = \pm \pi$ we obtain the largest X as 1. This corresponds to $\tilde{c}_1 \cdot \vec{1}_p$ being $\pm j$ times a real vector. For $\alpha = 0$ (circular scattered polarization) X is $1/\sqrt{2}$.

The scattering length for a step-function incident wave is

$$\ell_s^{(m)} \simeq \sqrt{\frac{4}{\pi}}\,\frac{E_o}{\omega_1}\left|\tilde{c}_1 \cdot \vec{1}_p\right| \mathrm{X} \tag{5.9}$$

Using the maximum scattering length for the restricted $\tilde{c}_1$ as in (5.3) gives a waveform efficiency

$$\eta_s^{(m)} = \frac{\ell_s^{(m)}}{\ell_{\max}^{(m)}} \simeq \pi\,\frac{-\Omega_1}{\omega_1}\,\frac{\left|\tilde{c}_1 \cdot \vec{1}_p\right|}{\left\|\tilde{c}_1\right\|_{2\nu}}\,\mathrm{X} \tag{5.10}$$

With $\vec{1}_p$ chosen as $\vec{1}_{\max}$ we have

$$\eta_s^{(m)} \simeq \pi\,\frac{-\Omega_1}{\omega_1} \tag{5.11}$$

B. Damped Sinusoidal Incident Field

Taking the incident waveform as

$$\vec{E}(t) = \frac{E_o}{2}\left\{\vec{1}_p e^{s_o t} + \vec{1}_p^{\,*} e^{s_o^* t}\right\} u(t)$$

$$\tilde{\vec{E}}(s) = \frac{E_o}{2}\left\{\vec{1}_p\left[s-s_o\right]^{-1} + \vec{1}_p^{\,*}\left[s-s_o^*\right]^{-1}\right\} \tag{5.12}$$

$$s_o = \Omega_o + j\omega_o \; , \; \Omega_o < 0 \; , \; \left|\vec{1}_p\right| = 1$$

then from (B.4) we have

$$\left\|\vec{E}(t)\right\|_m = E_o\left\{\frac{1+\left|\vec{1}_p \cdot \vec{1}_p\right|}{2}\right\}^{\frac{1}{2}} \tag{5.13}$$

Let s_o be nearly matched to s_1 as in (5.1) and (A.12).

For the pole-pair scatterer as in (B.16) we have

$$\tilde{\tilde{\Lambda}}(s) \cdot \tilde{\vec{E}}(s) = \frac{E_o}{2}\left\{\tilde{c}_1\left[s-s_1\right]^{-1} + \tilde{c}_1^{\,*}\left[s-s_1^*\right]^{-1}\right\} \cdot \left\{\vec{1}_p\left[s-s_o\right]^{-1} + \vec{1}_p^{\,*}\left[s-s_o^*\right]^{-1}\right\} \tag{5.14}$$

With close matching of the poles

$$\tilde{\vec{\Lambda}}(s) \cdot \tilde{\vec{E}}(s) \simeq \frac{E_o}{2}\left\{\vec{c}_1 \cdot \vec{1}_p [s-s_1]^{-1} [s-s_o]^{-1} + \vec{c}_1^* \cdot \vec{1}_p^* [s-s_1]^{-1} [s-s_o]^{-1}\right\}$$

$$\vec{\Lambda}(t) \circ \vec{E}(t) \simeq \frac{E_o}{2}\left\{\frac{e^{s_1 t} - e^{s_o t}}{s_o - s_1}\, \vec{c}_1 \cdot \vec{1}_p + \frac{e^{s_1^* t} - e^{s_o^* t}}{s_o^* - s_1^*}\, \vec{c}_1^* \cdot \vec{1}_p^*\right\} u(t) \tag{5.15}$$

For simplicity let

$$\omega_o = \omega_1 \; , \; \Omega_o = 0 \tag{5.16}$$

giving

$$\vec{\Lambda} \circ \vec{E}(t) \simeq \frac{E_o}{2}[-\Omega_1]^{-1}\left\{e^{j\omega_1 t}\left[e^{\Omega_1 t} - 1\right]\vec{c}_1 \cdot \vec{1}_p + e^{-j\omega_1 t}\left[e^{\Omega_1 t} - 1\right]\vec{c}_1^* \cdot \vec{1}_p\right\} \tag{5.17}$$

The peak (m-norm) is (similar to (B.4))

$$\begin{aligned}
\left\|\vec{\Lambda}(t) \circ \vec{E}(t)\right\|_m &\simeq \sup_t \frac{E_o}{2}[-\Omega_1]^{-1} \left|e^{j\omega_1 t}\vec{c}_1 \cdot \vec{1}_p + e^{-j\omega_1 t}\vec{c}_1^* \cdot \vec{1}_p^*\right| \\
&= \sup_t \frac{E_o}{2}[-\Omega_1]^{-1} \Big\{e^{j2\omega_1 t}\vec{1}_p \cdot \vec{c}_1^T \cdot \vec{c}_1 \cdot \vec{1}_p \\
&\quad + 2\, \vec{1}_p^* \cdot \vec{c}_1^\dagger \cdot \vec{c}_1 \cdot \vec{1}_p + e^{-j2\omega_1 t}\vec{1}_p^* \cdot \vec{c}_1^\dagger \cdot \vec{c}_1^* \cdot \vec{1}_p^*\Big\}^{\frac{1}{2}} \\
&= \frac{E_o}{\sqrt{2}}[-\Omega_1]^{-1} \left|\vec{c}_1 \cdot \vec{1}_p\right| \left\{1 + \frac{\vec{1}_p \cdot \vec{c}_1^T \cdot \vec{c}_1 \cdot \vec{1}_p}{\left|\vec{c}_1 \cdot \vec{1}_p\right|^2}\right\}^{\frac{1}{2}}
\end{aligned} \tag{5.18}$$

The scattering length for this matched case is

$$\begin{aligned}
\ell_m^{(m)} &\simeq \frac{1}{\sqrt{4\pi}} \frac{\left\|\vec{\Lambda}(t) \circ \vec{E}(t)\right\|_m}{\left\|\vec{E}(t)\right\|_m} \\
&= \frac{1}{\sqrt{4\pi}}\left|\vec{c}_1 \cdot \vec{1}_p\right|[-\Omega_1]^{-1}\left\{\frac{1 + \dfrac{\left|\vec{1}_p \cdot \vec{c}^T \cdot \vec{c} \cdot \vec{1}_p\right|}{\left|\vec{c} \cdot \vec{1}_p\right|^2}}{1 + \left|\vec{1}_p \cdot \vec{1}_p\right|}\right\}^{\frac{1}{2}}
\end{aligned} \tag{5.19}$$

Considering the previous case of $\tilde{c}_1$ a constant times a real dyad with $\vec{1}_p$ as a real maximizing eigenvector gives

$$\ell_m^{(m)} \simeq \frac{1}{\sqrt{4\pi}} \left[-\Omega_1\right]^{-1} \|\tilde{c}_1\|_{2\nu} \, , \; \vec{\ell}_m^{(m)} = \ell_m^{(m)} \, \vec{1}_p \tag{5.20}$$

which gives a waveform efficiency

$$\eta_m^{(m)} = \frac{\ell_m^{(m)}}{\ell_{\max}^{(m)}} \simeq \frac{\pi}{4} \tag{5.21}$$

This is near 1 showing the relative efficiency of matched sinusoidal and square waveforms.

VI. Concluding Remarks

We can now see that the use of norms to extend scattering length or cross section into time domain yields some useful results. It allows us to compare various incident waveforms for maximizing these parameters. For highly resonant scatterers this optimal waveform is basically a sinusoid matched to the resonant frequency with attention payed to optimum incident polarization.

The present discussion considers the problem of scattering in terms only of the scattering length as a fundamental parameter. This analysis can be extended by including other transfer functions such as those of transmitting and receiving antennas as well as data processing problems (such as signal-to-noise ratio) in optimizing the design of the entire transmission and receiving system.

Appendix A. 2-Norm

The 2-norm of our vector waveforms takes the various forms [3]

$$\left\| \vec{E}(t) \right\|_2 = \left\{ \int_{-\infty}^{\infty} \vec{E}(t) \cdot \vec{E}(t) \; dt \right\}^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} \left| \tilde{\vec{E}}(j\omega) \right|^2 \; d\omega \right\}^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \left\| \tilde{\vec{E}}(j\omega) \right\|_{2\omega} \tag{A.1}$$

$$= \left\{ \frac{1}{2\pi j} \int_{B_r} \tilde{\vec{E}}(-s) \cdot \tilde{\vec{E}}(s) \; ds \right\}^{\frac{1}{2}}$$

The 2-norm then applies in both time and frequency (Parseval theorem).

Consider a waveform

$$\vec{E}(t) = E_o \; \text{Re}\left[\vec{1}_p e^{s_o t} \, u(t) \right]$$

$$= \frac{E_o}{2} \left\{ \vec{1}_p e^{s_o t} + \vec{1}_p^{\,*} \; e^{s_o^* t} \right\} u(t)$$

$$\tilde{\vec{E}}(s) = \frac{E_o}{2} \left\{ \vec{1}_p \left[s - s_o \right]^{-1} + \vec{1}_p^{\,*} \left[s - s_o^* \right]^{-1} \right\} \tag{A.2}$$

$$s_o = \Omega_o + j\omega_o \; , \; \Omega_o < 0 \; , \; \left| \vec{1}_p \right| = 1$$

This damped sinusoidal waveform will often be considered as highly resonant $\left(|\Omega_o| << |\omega_o| \right)$. Note that the polarization $\vec{1}_p$ can be in general complex giving elliptical spiral polarization [5]. The peak field magnitude is roughly E_o depending on $\vec{1}_p$. Note that as $s \to 0$ the formula in (A.2) does not give zero (as required for a radiated waveform). One can correct for this, but in general it is not important for present purposes. Note the turn-on at zero time for convenience, the actual turn-on being a function of one's choice of incident and scattered waveform coordinates [4, 5].

Following the procedure in [2] for scalar waveforms our damped sinusoidal vector waveform can be evaluated for 2-norm by contour deformation with the residues in the left half s-plane as

$$\left\|\vec{E}(t)\right\|_2^2 = \frac{E_o^2}{4}\left\{\vec{1}_p \cdot \left[\vec{1}_p[-2s_o]^{-1} + \vec{1}_p^*[-2\Omega_o]\right]^{-1} + \vec{1}_p^* \cdot \left[\vec{1}_p[-2\Omega_o]^{-1} + \vec{1}_p^*\left[-2s_o^*\right]^{-1}\right]\right\}$$

$$= \frac{E_o^2}{4}\,\mathrm{Re}\left[\vec{1}_p \cdot \left[\vec{1}_p[-s_o]^{-1} + \vec{1}_p^*[-\Omega_o]^{-1}\right]\right] \tag{A.3}$$

$$= \frac{E_o^2}{4}\left\{\mathrm{Re}\left[-\vec{1}_p \cdot \vec{1}_p\ s_o^{-1}\right] - \Omega_o^{-1}\right\}$$

Note for highly resonant waveforms the second term is dominant giving

$$\left\|\vec{E}(t)\right\|_2 \simeq \frac{E_o}{2}\,[-\Omega_o]^{\frac{1}{2}} \tag{A.4}$$

which is conveniently independent of $\vec{1}_p$.

The scattering operator has the 2-norm

$$\left\|\vec{\Lambda}(t)\circ\right\|_2 = \sup_{\left\|\vec{E}(t)\right\|_2 \neq 0} \frac{\left\|\vec{\Lambda}(t) \circ \vec{E}(t)\right\|_2}{\left\|\vec{E}(t)\right\|_2}$$

$$= \sup_{\left\|\tilde{\vec{E}}(j\omega)\right\|_{2\omega} \neq 0} \frac{\left\|\tilde{\vec{\vec{\Lambda}}}(j\omega) \cdot \tilde{\vec{E}}(j\omega)\right\|_{2\omega}}{\left\|\tilde{\vec{E}}(j\omega)\right\|_{2\omega}}$$

$$\equiv \left\|\tilde{\vec{\vec{\Lambda}}}(j\omega)\right\|_{2\omega} = \sup_{\omega}\left\|\tilde{\vec{\vec{\Lambda}}}(j\omega)\right\|_{2\nu} \text{ (matrix sense)}$$

$$= \left\|\tilde{\vec{\vec{\Lambda}}}(j\omega_{\max})\right\|_{2\nu} \tag{A.5}$$

$$= \sup_{\omega}\left[\lambda_{\max}\left(\left(\tilde{\vec{\vec{\Lambda}}}(j\omega_{\max})\right)^{\dagger} \cdot \left(\tilde{\vec{\vec{\Lambda}}}(j\omega)\right)\right)\right]^{\frac{1}{2}}$$

$$= \left[\lambda_{\max}\left(\left(\tilde{\vec{\vec{\Lambda}}}(j\omega_{\max})\right)^{\dagger} \cdot \left(\tilde{\vec{\vec{\Lambda}}}(j\omega_{\max})\right)\right)\right]^{\frac{1}{2}}$$

Thus the operator norm is just the maximum value achieved by the 2-norm of its frequency-domain matrix form. Furthermore the frequency $\omega_{\max}$ at which this maximum is achieved is the frequency which should dominate the field waveform. Let the right eigenvector (arbitrary magnitude) of the Hermitian matrix be $\vec{u}_{max}$ as

$$\left(\tilde{\Lambda}(j\omega_{max})\right)^{\dagger} \cdot \left(\tilde{\Lambda}(j\omega_{max})\right) \cdot \vec{u}_{max} = \lambda_{max}\ \vec{u}_{max} \tag{A.6}$$

with $\vec{u}^{*}_{max}$ as the left eigenvector with

$$|\vec{u}_{max}| = 1 \tag{A.7}$$

In this form we can choose our maximizing waveform by selecting

$$\vec{1}_p \equiv \vec{u}_{max}\ ,\ \Omega_o \rightarrow 0 - \tag{A.8}$$

Assuming a bounded $\tilde{\Lambda}(j\omega)$ with appropriate smoothness near ω_{max}, this choice of incident waveform crowds the important frequencies around $\pm\ \omega_{max}$ in (A.5) giving the supremum value in the limiting case as in (A.8). As a practical matter $\Omega_o \neq 0$ since that would give an infinite 2-norm to both incident and scattered waveforms. Then we have

$$\ell^{(2)}_{max} = \frac{1}{\sqrt{4\pi}} \left\| \tilde{\Lambda}(t) \cdot o \right\|_2$$

$$= \frac{1}{\sqrt{4\pi}} \left[\lambda_{max}\left(\left(\tilde{\Lambda}(j\omega_{max})\right)^{\dagger} \cdot \left(\tilde{\Lambda}(j\omega_{max})\right)\right)\right]^{\frac{1}{2}} \tag{A.9}$$

$$\vec{\ell}^{(2)}_{max} = \frac{1}{\sqrt{4\pi}}\ \tilde{\Lambda}(j\omega_{max}) \cdot \vec{u}_{max}$$

Now let the scatterer be characterized by a single pole pair as

$$\tilde{\Lambda}(s) = \vec{c}_1 \left[s - s_1\right]^{-1} + \vec{c}^{*}_1 \left[s - s^{*}_1\right]^{-1}$$

$$s_1 = \Omega_1 + j\omega_1\ ,\ \Omega_1 < 0 \tag{A.10}$$

$$\tilde{\Lambda}(t) = \left[\vec{c}_1 e^{s_1 t} + \vec{c}^{*}_1 e^{s^{*}_1 t}\right] u(t)$$

which will again be often considered highly resonant $\left(|\Omega_1| << |\omega_1|\right)$. The dyadic coefficients take various special forms from SEM considerations [4, 5]. Note the turn-on at zero time for present convenience.

The scattered waveform now takes the form $\tilde{\Lambda}(t) \circ \vec{E}(t)$. From (A.2) this has a 2-norm which can also be evaluated by contour integration closed in the left half plane. Following the procedure for scalar waveforms in [2] we now have

$$\left\| \tilde{\Lambda}(t) \circ \vec{E}(t) \right\|_2^2 = \frac{1}{2\pi j} \int_{Br} \tilde{\vec{E}}(-s) \cdot \tilde{\Lambda}^T(-s) \cdot \tilde{\Lambda}(s) \cdot \tilde{\vec{E}}(s) ds$$

$$= 2\text{Re} \left[\tilde{\vec{E}}(-s_o) \tilde{\Lambda}^T(-s_o) \cdot \tilde{\Lambda}(s_o) \cdot \vec{1}_p \frac{E_o}{2} \right. \tag{A.11}$$

$$\left. + \tilde{\vec{E}}(-s_1) \cdot \tilde{\Lambda}^T(-s_1) \cdot \vec{c}_1 \cdot \tilde{\vec{E}}(s_1) \right]$$

This is an exact result, but a simpler approximate result is found by considering the case of nearly matched highly resonant incident waveform and scattering operator. Specifically let

$$\begin{aligned} |s_o - s_1| &<< |s_o| , |s_1| \\ \left|-s_o - s_1^*\right| &<< |s_o| , |s_1| \\ 0 < -\Omega_o &<< |s_o| , |s_1| \\ 0 < -\Omega_1 &<< |s_o| , |s_1| \end{aligned} \tag{A.12}$$

Then we have

$$\left\| \tilde{\Lambda}(t) \circ \vec{E}(t) \right\|_2^2$$

$$\simeq \frac{E_o^2}{2} \text{Re} \left[[-2\Omega_o]^{-1} \vec{1}_p^* \cdot \vec{c}_1^\dagger \left[-s_o - s_1^*\right]^{-1} \cdot \vec{c}_1 [s_o - s_1]^{-1} \cdot \vec{1}_p \right.$$

$$\left. + \left[-s_1 - s_o^*\right]^{-1} \vec{1}_p^* \cdot \vec{c}_1^\dagger [-2\Omega_1]^{-1} \cdot \vec{c}_1 [s_1 - s_o]^{-1} \cdot \vec{1}_p \right]$$

$$= \frac{E_o^2}{4} \vec{1}_p^* \cdot \vec{c}_1^\dagger \cdot \vec{c}_1 \cdot \vec{1}_p \, \text{Re} \left[[s_o - s_1]^{-1} \left[\Omega_o^{-1} \left[s_o + s_1^*\right]^{-1} - \Omega_1^{-1} \left[s_o^* + s_1\right]^{-1} \right] \right]$$

$$= -\frac{E_o^2}{4} \vec{1}_p^* \cdot \vec{c}_1^\dagger \cdot \vec{c}_1 \cdot \vec{1}_p \frac{\Omega_o + \Omega_1}{\Omega_o \Omega_1} \left| s_o + s_1^* \right|^{-2}$$

$$= \frac{E_o^2}{4} \vec{1}_p^* \cdot \vec{c}_1^\dagger \cdot \vec{c}_1 \cdot \vec{1}_p \left\{ \left[-\Omega_o\right]^{-1} + \left[-\Omega_1\right]^{-1} \right\} \left| s_o + s_1^* \right|^{-2} \tag{A.13}$$

Now form

$$\frac{\left\| \vec{\Lambda}(t) \odot \vec{E}(t) \right\|_2}{\left\| \vec{E}(t) \right\|_2} \simeq \left[\vec{1}_p^* \cdot \vec{c}_1^\dagger \cdot \vec{c}_1 \cdot \vec{1}_p \right]^{\frac{1}{2}} \left\{ 1 + \frac{\Omega_o}{\Omega_1} \right\}^{\frac{1}{2}} \left| s_o + s_1^* \right|^{-1} \tag{A.14}$$

Then choosing (as in (A.8)

$$s_o = j\omega_o = j\omega_1 = j\omega_{\max} \ , \ \vec{1}_p = \vec{u}_{\max}$$

makes (A.14) take the supremum result for the operator norm in (A.5).

Appendix B. m-Norm

As in section III define the m-norm or maximum norm of a vector function as

$$\left\|\vec{f}(t)\right\|_m \equiv \left\| \left\|\vec{E}(t)\right\|_{2v} \right\|_{\infty f} = \sup_t \left|\vec{E}(t)\right|$$

$$= \left\{\left\|\vec{E}(t) \cdot \vec{E}(t)\right\|_{\infty f}\right\}^{\frac{1}{2}} \tag{B.1}$$

with v indicating vector (or matrix) sense and f indicating function (or operator) sense [3]. Note that the vector norm is taken before the function norm (noncommutative). The norm properties are satisfied in that

$$\left\|\vec{f}(t)\right\|_m = 0 \text{ iff } \vec{f}(t) \equiv \vec{0}$$

$$\left\|\alpha\vec{f}(t)\right\|_m = |\alpha| \left\|\vec{f}(t)\right\|_m$$

$$\left\|\vec{f}_1(t) + \vec{f}_2(t)\right\|_m = \left\| \left\|\vec{f}_1(t) + \vec{f}_2(t)\right\|_{2v} \right\|_{\infty f}$$

$$\leq \left\| \left\|\vec{f}_1(t)\right\|_{2v} + \left\|\vec{f}_2(t)\right\|_{2v} \right\|_{\infty f} \tag{B.2}$$

$$\leq \left\| \left\|\vec{f}_1(t)\right\|_{2v} \right\|_{\infty f} + \left\| \left\|\vec{f}_2(t)\right\|_{2v} \right\|_{\infty f}$$

$$= \left\|\vec{f}_1(t)\right\|_m + \left\|\vec{f}_2(t)\right\|_m$$

Consider the damped sinusoidal waveform as before

$$\vec{E}(t) = \frac{E_o}{2} \left\{\vec{1}_p \, e^{s_o t} + \vec{1}_p^{\,*} \, e^{s_o^* t}\right\} u(t)$$

$$s_o = \Omega_o + j\omega_o \ , \ \Omega_o < 0 \ , \ \left|\vec{1}_p\right| = 1 \tag{B.3}$$

For a highly resonant waveform ($\Omega_o \to 0-$) we have

$$\left\|\vec{E}(t)\right\|_m \simeq \sup_t \frac{E_o}{2} \left\{\vec{1}_p \cdot \vec{1}_p e^{j2\omega_o t} + 2 + \vec{1}_p^{\,*} \cdot \vec{1}_p^{\,*} e^{-j2\omega_o t}\right\}^{\frac{1}{2}}$$

$$= E_o \left\{\frac{1 + \left|\vec{1}_p \cdot \vec{1}_p\right|}{2}\right\}^{\frac{1}{2}} \tag{B.4}$$

Thus the peak field magnitude varies between $E_o / \sqrt{2}$ and E_o depending on the complex polarization vector $\vec{1}_p$. This shows some of the difference between linear and circular polarization.

The associated operator norm is

$$\left\| \tilde{\Lambda}(t) \circ \right\|_m = \sup_{\|\vec{f}(t)\|_m \neq 0} \frac{\left\| \tilde{\Lambda}(t) \circ \vec{f}(t) \right\|_m}{\left\| \vec{f}(t) \right\|_m}$$

$$= \sup_{\|\vec{f}(t)\|_m \neq 0} \frac{\left\| \int_0^\infty \tilde{\Lambda}(t') \cdot \vec{f}(t-t') dt' \right\|_m}{\left\| \vec{f}(t) \right\|_m} \tag{B.5}$$

For this purpose write (noting $\tilde{\Lambda}(t)$ is real)

$$\tilde{\Lambda}^T_{(t)} \cdot \tilde{\Lambda}(t) = \xi_{max}(t)\, \vec{1}_{max}(t)\, \vec{1}_{max}(t) + \xi_{min}(t) \vec{1}_{min}(t) \vec{1}_{min}(t)$$

$$\xi_{max}(t) \geq \xi_{min}(t) \geq 0 \tag{B.6}$$

$$\left| \vec{1}_{max} \right| = \left| \vec{1}_{min} \right| = 1$$

and note that this is a real symmetric dyad (and therefore Hermitian) with real eigenvectors [6]. With this we have

$$\left\| \tilde{\Lambda}(t) \right\|_{2\nu} = \xi_{max}^{\frac{1}{2}}(t) \tag{B.7}$$

For simplicity let $\tilde{\Lambda}(t)$ be zero for negative time for present purposes. In actuality it can be causal and non zero for some negative time depending on choice of time reference or coordinate reference in (1.4).

Then following the procedure in [7] for the similar scalar case we have

$$\left\| \int_0^\infty \tilde{\Lambda}(t') \cdot \vec{f}(t-t') dt \right\|_m \leq \int_0^\infty \left\| \tilde{\Lambda}(t') \cdot \vec{f}(t-t') \right\|_m dt' \leq \int_0^\infty \left\| \Lambda(t') \right\|_m \left\| \vec{f}(t-t') \right\|_m dt'$$

$$= \int_0^\infty \left\| \Lambda(t') \right\|_{2\nu} \left\| \vec{f}(t-t') \right\|_m dt' = \int_0^\infty \xi_{max}^{\frac{1}{2}}(t') \left\| \vec{f}(t) \right\|_m dt' = \left\| \vec{f}(t) \right\|_m \int_0^\infty \xi_{max}^{\frac{1}{2}}(t')\, dt' \tag{B.8}$$

noting in the above that the m-norm is over t (not t'). This establishes the inequality

$$\left\|\tilde{\Lambda}(t)\circ\right\|_{m} \le \int_{o}^{\infty} \xi_{\max}^{\frac{1}{2}}(t')\,dt' = \int_{o}^{\infty} \left\|\tilde{\Lambda}(t')\right\|_{2v}\,dt'$$

$$= \left\| \left\|\tilde{\Lambda}(t)\right\|_{2v} \right\|_{1f} \tag{B.9}$$

Note the mixture of norms in this result. The "inside" 2-norm remains from (B.1) but the "outside" ∞-norm is replaced by a 1-norm, a generalization of and consistent with the scalar results in [7].

In order to obtain an equality for the operator norm consider some restrictions on the operator. Let $\tilde{\Lambda}$ be symmetric (as in the case of backscattering, by reciprocity) so that it is also Hermitian with real eigenvalues and eigenvectors [6] giving

$$\tilde{\Lambda}(t) = \zeta_{\max}(t)\,\vec{1}_{\max}(t)\,\vec{1}_{\max}(t) + \zeta_{\min}(t)\,\vec{1}_{\min}(t)\,\vec{1}_{\min}(t)$$

$$\tilde{\Lambda}^{T}(t) = \tilde{\Lambda}(t)$$

$$\zeta_{\max}^{(t)} = \xi_{\max}(t)\;,\; \zeta_{\min}^{2}(t) = \xi_{\min}(t) \tag{B.10}$$

$$\left|\zeta_{\max}(t)\right| \ge \left|\zeta_{\min}(t)\right| \ge 0$$

where the real eigenvalues can now have either sign. Further assume that $\vec{1}_{\max}$ is a constant vector (not a function of time) giving

$$\tilde{\Lambda}(t) = \zeta_{\max}(t)\,\vec{1}_{\max}\,\vec{1}_{\max} + \zeta_{\min}(t)\,\vec{1}_{\min}\,\vec{1}_{\min} \tag{B.11}$$

This can be achieved for certain cases of symmetry of the scatterer with respect to the observer direction $\left(-\vec{1}_1\right)$ [5] provided there is no interchange of roles between $\vec{1}_{\max}$ and $\vec{1}_{\min}$, i.e., the particular direction of $\vec{1}_{\max}$ has the dominant eigenvalue for all t. A thin-wire scatterer (negligible radius) meets this last requirement as does a perfectly conducting disk (zero thickness) seen edge on.

Now choose $\vec{f}(t)$ in a special way as

$$\vec{f}(t-t') \equiv \begin{cases} +\,\vec{1}_{\max} & \text{if } \zeta_{\max}(t') > 0 \\ 0 & \text{if } \zeta_{\max}(t') = \zeta_{\min}(t') = 0 \\ -\,\vec{1}_{\max} & \text{if } \zeta_{\min}(t') < 0 \end{cases} \tag{B.12}$$

This gives

$$\left\| \vec{f}(t) \right\|_m = \left\| \left| \vec{f}(t) \right| \right\|_\infty = \left\| 1 \right\|_\infty = 1$$

$$\left\| \int_o^\infty \vec{\Lambda}(t') \cdot \vec{f}(t-t') \, dt' \right\|_m = \left\| \int_o^\infty \left| \zeta_{max}(t') \right| \vec{1}_{max} \, dt' \right\|_m \tag{B.13}$$

$$= \int_o^\infty \left| \zeta_{max}(t') \right| dt' = \left\| \zeta_{max}(t) \right\|_{1f} = \left\| \left\| \vec{\Lambda}(t) \right\|_{2\nu} \right\|_{1f}$$

Since this special chose of $\vec{f}$ is one possible choice in (B.5) we have

$$\left\| \vec{\Lambda}(t) \, o \right\| \geq \left\| \left\| \vec{\Lambda}(t) \right\|_{2\nu} \right\|_{1f} \tag{B.14}$$

Combining with (B.9) gives for our restricted form of scattering operator in (B.11) the result

$$\left\| \vec{\Lambda}(t) \, o \right\|_m = \left\| \left\| \vec{\Lambda}(t) \right\|_{2\nu} \right\|_{1f} \tag{B.15}$$

Let the operator be characterized by a single pole pair as

$$\tilde{\vec{\Lambda}}(s) = \vec{c}_1 [s - s_1]^{-1} + \vec{c}_1^{\,*} [s - s_1]^{-1}$$

$$s = \Omega_1 + j\omega_1 \;,\; \Omega_1 < 0 \;,\; |\Omega_1| << |\omega_1| \tag{B.16}$$

$$\vec{\Lambda}(t) = \left[\vec{c}_1 \, e^{s_1 t} + \vec{c}_1^{\,*} \, e^{s_1^* t} \right] u(t)$$

Let $\vec{\Lambda}(t)$ be symmetric (as in backscattering) so that $\vec{c}_1$ is symmetric. Furthermore let

$$\vec{c}_1 = \vec{c}_1^{\,T} = \vec{c}_o e^{j\upsilon} \;,\; \vec{c}_o \text{ real}$$

$$\left\| \vec{c}_1 \right\|_2 = \left\| \vec{c}_o \right\|_2 \tag{B.17}$$

giving

$$\vec{\Lambda}(t) = 2e^{\Omega_1 t} \cos\left(\omega_1 t + \upsilon\right) u(t) \, \vec{c}_o \tag{B.18}$$

corresponding to the conditions in (B.11). Then we have

$$\zeta_{max}(t) = 2e^{\Omega_1 t} \cos(\omega_1 t + \upsilon) \, u(t) \, \|\vec{c}_1\|_{2\nu}$$

$$\|\vec{\Lambda}(t) \circ \|_m = \int_0^\infty |\zeta_{max}(t)| \, dt \tag{B.19}$$

$$\vec{\Lambda}(t) \cdot \vec{1}_{max} = \zeta_{max}(t) \, \vec{1}_{max}$$

Assuming Ω_1 small one can average over the magnitude of the cosine as $2/\pi$ giving

$$\|\vec{\Lambda}(t) \circ \|_m \simeq \frac{4}{\pi} \, [-\Omega_1]^{-1} \, \|\vec{c}_1\|_{2\nu} \tag{B.20}$$

The maximizing waveform from (B.12) is a square wave polarized in the $\vec{1}_{max}$ direction with period $2\pi / \omega_1$, i.e. of frequency matched to the scatterer.

For $\vec{c}_1$ more general than in (B.17), then (B.9) indicates that (B.20) represents an upperbound.

References

1. C. E. Baum, Emerging Technology for Transient and Broad-Band Analysis and Synthesis of Antennas and Scatterers, Interaction Note 300, November 1976, and Proc. IEEE, November 1976, pp. 1598-1616.

2. C. E. Baum, Transfer of Norms Through Black Boxes, Interaction Note 462, October 1987 and Proc. EMC Symposium, Zurich, March, 1989, pp. 157-162.

3. C. E. Baum, Norms of Vectors of Time-Domain Signals Passing Through Filters and Norm Limiters at Subshields, Interaction Note 469, January 1988, and Proc. EMC Symposium, Zurich, March 1991, pp. 589-594.

4. C. E. Baum, Scattering, Reciprocity, Symmetry, EEM, and SEM, Interaction Note 475, May 1989.

5. C. E. Baum, SEM Backscattering, Interaction Note 476, July 1989.

6. J. Nitsch, C. E. Baum, and R. J. Sturm, Commutative Tubes in Multiconductor-Transmission-Line Theory, Interaction Note 480, January 1990.

7. C. E. Baum, Norms of Time-Domain Functions and Convolution Operators, Mathematics Note 86, December 1985, and chap. 2, pp. 31-55, in H. N. Kritikos and D. L. Jaggard (eds.), Recent Advances in Electromagnetic Theory, Springer-Verlag, 1990.

8. J. J. Bowman, T.B.A. Senior, and P.L.E. Uslenghi, Electromagnetic and Acoustic Scattering by Simple Shapes, Hemisphere Publishing Corp., 1987.

9. L. B. Felsen (ed.), Transient Electromagnetic Fields, Springer-Verlag, 1976.

10. M. I. Skolnik, Radar Handbook, Second Edition, McGraw Hill, 1990.