Interaction Notes

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Integral Equations and Polarizability for Magnetic Singularity Identification

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Abstract

This paper considers both volume and surface integral equations for low-frequency quasi-magnetostatic scattering from permeable, highly conducting targets, using appropriate approximations. These are used in turn to obtain formulas for the magnetic polarizability dyadic and the pole terms (natural frequency, residue unit vectors, and scalar coefficient).

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Contents

1. Introduction

As discussed in [3, 18] there is a set of natural frequencies associated with finite-dimensioned, highly conducting (metal) targets at the low frequencies characteristic of diffusion of magnetic fields through the target, or skin depths related to target dimensions. In the s-plane (complex frequency plane) these natural frequencies all lie on the negative real s-axis, and correspond to pure exponential decays in time domain. These can be used for target identification, and this technique is referred to as magnetic singularity identification (MSI) [5, 18]. Since it emphasizes the near-field, low-frequency magnetic scattering it is insensitive to surrounding media of comparatively modest conductivity (e.g., soil, water) provided such media are non-magnetic, i.e., have permeability μ_0.

The general scattering situation is indicated in fig. 1.1. The surrounding uniform medium has permeability μ_0 (free space), permittivity $\varepsilon_1(\geq \varepsilon_0)$, and conductivity $\sigma_1(\geq 0)$, which gives

$$\begin{aligned}
\tilde{\gamma}_1(s) &= \left[s\,\mu_0\left[\sigma_1 + s\,\varepsilon_1\right]\right]^{1/2} \equiv \text{ propagation constant} \\
\tilde{Z}_1(s) &= \left[\frac{s\,\mu_0}{\sigma_1 + s\,\varepsilon_1}\right]^{1/2} \equiv \text{ wave impedance} \\
s &\equiv \Omega + j\omega \equiv \text{ complex frequency or Laplace-transform variable}
\end{aligned} \tag{1.1}$$

Associated with this external medium there is a (scalar) Green's function

$$\tilde{G}_1(\vec{r},\vec{r}\,';s) = \frac{e^{-\tilde{\gamma}_1(s)|\vec{r}-\vec{r}\,'|}}{4\pi\,|\vec{r}-\vec{r}\,'|} \tag{1.2}$$

from which a dyadic form can be constructed, but the above is sufficient for present purposes. As discussed in [3, 18], our interest being in low-frequency, near-field magnetic scattering, the external-medium parameters play little role provided $\mu = \mu_0$, so we might as well thing of $\varepsilon_1 = \varepsilon_0$ and $\sigma_1 = 0$. In this case

$$\begin{aligned}
\tilde{\gamma}_1(s) \to \tilde{\gamma}_0(s) &= s\left[\mu_0\varepsilon_0\right]^{1/2} = \frac{s}{c} \\
\tilde{Z}_1(s) \to Z_0 &= \left[\frac{\mu_0}{\varepsilon_0}\right]^{1/2}
\end{aligned} \tag{1.3}$$

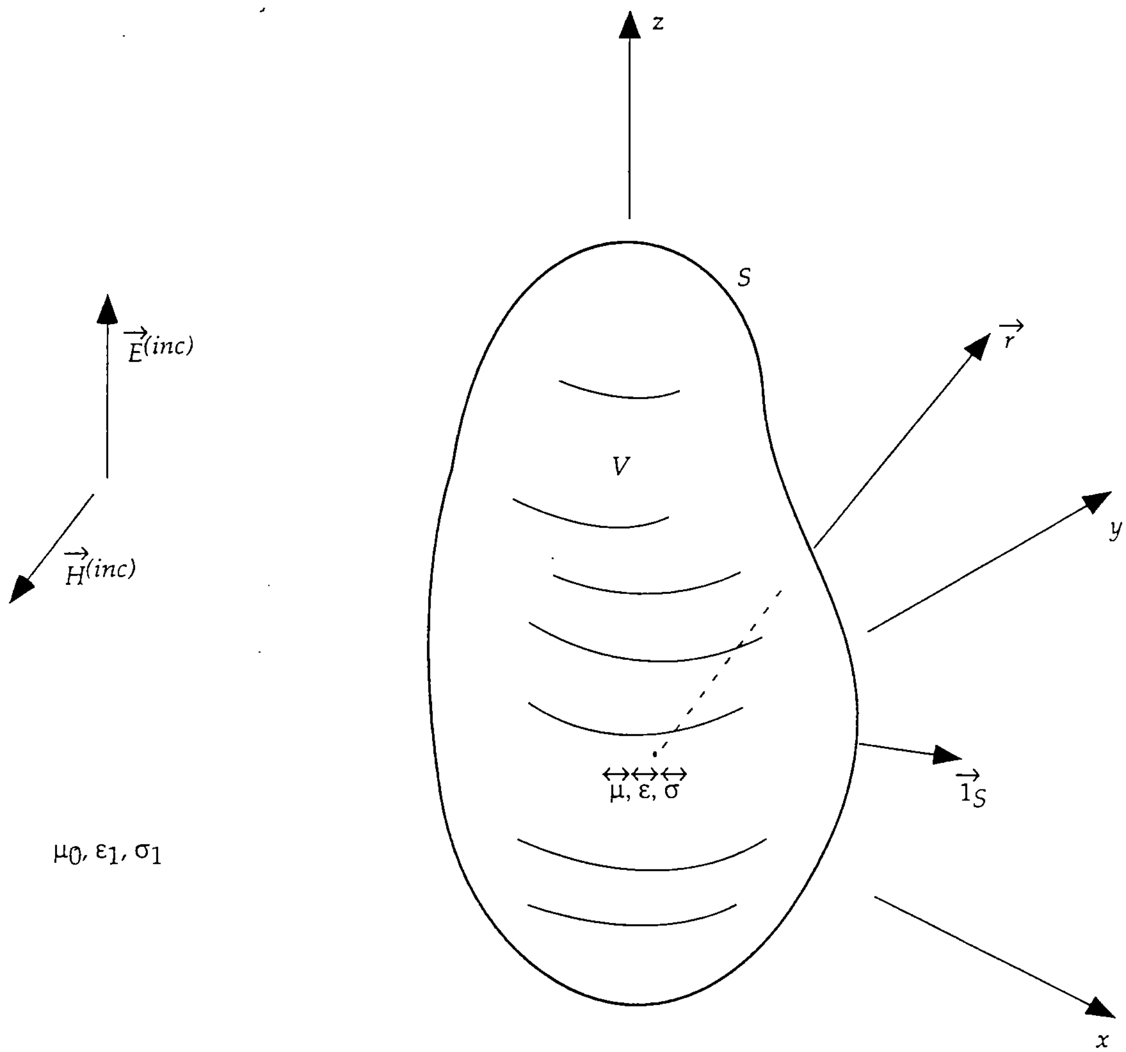


Fig. 1.1. Scatterer in Uniform Isotropic Medium

From an experimental point of view this means that measurements of the target scattering can be made in air, and the proximity of non-metallic/non-magnetic objects is not significant. Furthermore, the wavelengths or skin depths of interest in the external medium are so large compared to target dimensions and observer distances that (1.2) can be replaced by

$$\tilde{G}_1(\vec{r},\vec{r}\,';s) \to G_0(\vec{r},\vec{r}\,') = \frac{1}{4\pi\,|\vec{r}-\vec{r}\,'|} \tag{1.4}$$

from which we also need

$$\nabla' G_0(\vec{r},\vec{r}\,') = -\nabla G_0(\vec{r},\vec{r}\,') = \frac{1}{4\pi}\,\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \tag{1.5}$$

Let us call the foregoing (and related) approximations as MSI approximations.

The target is in general inhomogeneous and anisotropic with

$$\begin{aligned} \overleftrightarrow{\mu}(\vec{r}) &\equiv \text{permeability dyadic} \\ \overleftrightarrow{\varepsilon}(\vec{r}) &\equiv \text{permittivity dyadic} \\ \overleftrightarrow{\sigma}(\vec{r}) &\equiv \text{conductivity dyadic} \end{aligned} \tag{1.6}$$

These appear in the electric and magnetic (polarization) current densities in the target as

$$\begin{aligned} \tilde{\vec{J}}_e(\vec{r},s) &= \left[\overleftrightarrow{\sigma}(\vec{r}) + s\left[\overleftrightarrow{\varepsilon}(\vec{r}) - \varepsilon_0\overleftrightarrow{1}\right]\right]\cdot\tilde{\vec{E}}(\vec{r},s) \\ \tilde{\vec{J}}_h(\vec{r},s) &= s\left[\overleftrightarrow{\mu}(\vec{r}) - \mu_0\overleftrightarrow{1}\right]\cdot\tilde{\vec{H}}(\vec{r},s) \end{aligned} \tag{1.7}$$

and fit in the Maxwell equations as

$$\begin{aligned} \nabla\times\tilde{\vec{E}}(\vec{r},s) &= -s\,\tilde{\vec{B}}(\vec{r},s) = -s\,\overleftrightarrow{\mu}(\vec{r})\cdot\tilde{\vec{H}}(\vec{r},s) \\ &= -\tilde{\vec{J}}_h(\vec{r},s) = s\,\mu_0\,\tilde{\vec{H}}(\vec{r},s) \end{aligned}$$

$$\nabla \times \tilde{\vec{H}}(\vec{r},s) = \overleftrightarrow{\sigma}(\vec{r}) \cdot \tilde{\vec{E}}(\vec{r},s) + s\,\tilde{\vec{D}}(\vec{r},s) = \left[\overleftrightarrow{\sigma}(\vec{r}) + s\,\overleftrightarrow{\varepsilon}(\vec{r})\right] \cdot \tilde{\vec{E}}(\vec{r},s)$$
$$= \tilde{\vec{J}}_e(\vec{r},s) + s\,\varepsilon_0\,\tilde{\vec{E}}(\vec{r},s) \tag{1.8}$$

One should note that the above fields are total fields, including both incident and scattered parts. For present purposes we can neglect permittivity (an MSI approximation) to give

$$\begin{aligned}
\tilde{\vec{J}}_e(\vec{r},s) &= \overleftrightarrow{\sigma}(\vec{r}) \cdot \tilde{\vec{E}}(\vec{r},s) \\
\nabla \cdot \tilde{\vec{J}}_e(\vec{r},s) &= 0 \\
\nabla \cdot \tilde{\vec{J}}_h(\vec{r},s) &= -s\,\mu_0\,\nabla \cdot \tilde{\vec{H}}(\vec{r},s)
\end{aligned} \tag{1.9}$$

This will simplify some of the computations.

For the special case that the target is uniform and isotropic the constitutive parameters reduce to μ, ε, and σ with $s\varepsilon$ negligible compared to σ. Then we have a Green's function appropriate to the target medium as

$$\begin{aligned}
\tilde{G}(\vec{r},\vec{r}\,';s) &= \frac{e^{-\tilde{\gamma}(s)|\vec{r}-\vec{r}\,'|}}{4\pi\,|\vec{r}-\vec{r}\,'|} \\
\nabla'\tilde{G}(\vec{r},\vec{r}\,';s) &= -\nabla\tilde{G}(\vec{r},\vec{r}\,';s) \\
&= [1 + \tilde{\gamma}(s)|\vec{r}-\vec{r}\,'|]\,e^{-\gamma|\vec{r}-\vec{r}\,'|}\,\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}
\end{aligned} \tag{1.10}$$

with wave parameters for the target medium as

$$\begin{aligned}
\tilde{\gamma}(s) &= \left[s\,\mu[\sigma + s\varepsilon]\right]^{1/2} \equiv \text{propagation constant} \\
\tilde{Z}(s) &= \left[\frac{s\mu}{\sigma + s\varepsilon}\right]^{1/2} \equiv \text{wave impedance}
\end{aligned} \tag{1.11}$$

which simplify under our MSI approximations to

$$\tilde{\gamma}(s) = [s\mu\sigma]^{1/2}$$
$$\tilde{Z}(s) = \left[\frac{s\mu}{\sigma}\right]^{1/2} \tag{1.12}$$

As discussed in [3, 18] the natural frequencies (poles in the $s = \Omega + j\omega$ plane) of such targets under the MSI approximations lie on the negative real s axis. On the real s axis (Ω axis) the fields, currents, etc., are all real, corresponding to the two-sided Laplace transform (symbol ~ above) of the real-valued temporal quantities. The exterior Green's function causes no problem in this regard as $G_0(\vec{r},\vec{r}\,')$ is purely real. However, the interior Green's function $\tilde{G}(\vec{r},\vec{r}\,';s)$ has a branch cut on the negative real s axis. Formulations which utilize this (as in Section 4) can separate real and imaginary parts as

$$[s\,\mu\,\sigma]^{1/2} = \pm j\,[-s\,\mu\,\sigma]^{1/2}$$
$$\mathrm{Re}\left[\tilde{G}(\vec{r},\vec{r}\,';s)\right] = \frac{\cos\left([-s\,\mu\,\sigma]^{1/2}\,|\vec{r}-\vec{r}\,'|\right)}{4\pi\,|\vec{r}-\vec{r}\,'|}$$
$$\mathrm{Im}\left[\tilde{G}(\vec{r},\vec{r}\,';s)\right] = \mp\frac{\sin\left([-s\,\mu\,\sigma]^{1/2}\,|\vec{r}-\vec{r}\,'|\right)}{4\pi\,|\vec{r}-\vec{r}\,'|}$$
$$\mathrm{Re}\left[\nabla'\tilde{G}(\vec{r},\vec{r}\,';s)\right] = \left[\cos\left([-s\,\mu\,\sigma]^{1/2}\,|\vec{r}-\vec{r}\,'|\right) + [-s\,\mu\,\sigma]^{1/2}\,|\vec{r}-\vec{r}\,'|\,\sin\left([-s\,\mu\,\sigma]^{1/2}\,|\vec{r}-\vec{r}\,'|\right)\right]\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$
$$\mathrm{Im}\left[\nabla'\tilde{G}(\vec{r},\vec{r}\,';s)\right] = \pm\left[[-s\,\mu\,\sigma]^{1/2}\,|\vec{r}-\vec{r}\,'|\cos[-s\,\mu\,\sigma]^{1/2}\,|\vec{r}-\vec{r}\,'| + \sin\left([-s\,\mu\,\sigma]^{1/2}\,|\vec{r}-\vec{r}\,'|\right)\right]\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \tag{1.13}$$

2. Incident Fields for Use with Quasi-Magnetostatic Scattering

The fields (total fields) are divided, as usual, into incident and scattered fields as

$$\begin{aligned}\tilde{\vec{E}}(\vec{r},s) &= \tilde{\vec{E}}^{(inc)}(\vec{r},s) + \tilde{\vec{E}}^{(sc)}(\vec{r},s)\\ \tilde{\vec{H}}(\vec{r},s) &= \tilde{\vec{H}}^{(inc)}(\vec{r},s) + \tilde{\vec{H}}^{(sc)}(\vec{r},s)\end{aligned} \tag{2.1}$$

The incident fields arise from some distant sources and are defined as those fields which would exist in the absence of the target, i.e., with the target replace by the exterior medium. The scattered fields are the change in the fields associated with the introduction of the target.

One often considers the incident field as a plane wave, but that is not appropriate in the present case. Here we think of an incident magnetic field which is approximately uniform in the vicinity of the target. So think of a uniform z-directed magnetic field as

$$\tilde{\vec{H}}^{(inc)}(\vec{r},s) = H_0 \vec{1}_z \tag{2.2}$$

This is associated with a ϕ-independent, ϕ-directed vector potential (usual (Ψ,ϕ,z) cylindrical coordinates)

$$\begin{aligned}H_0 \vec{1}_z &= \frac{1}{\mu_0}\nabla\times\vec{A}(\vec{r}) = \frac{1}{\mu_0\Psi}\frac{\partial}{\partial\Psi}\left(\Psi A_\phi\right)\vec{1}_Z\\ A_\phi &= \frac{\mu_0 H_0}{\Psi}\int_0^\Psi \Psi'\,d\Psi' = \mu_0\frac{\Psi}{2}H_0\end{aligned} \tag{2.3}$$

The integral is taken from zero to avoid an insignificant constant term.

The incident electric field is in general derived from scalar and vector potentials as

$$\tilde{\vec{E}}^{(inc)}(\vec{r},s) = -s\vec{A}(\vec{r}) - \nabla\Phi(\vec{r}) \tag{2.4}$$

Setting the scalar potential to zero we have

$$\vec{\tilde{E}}^{(inc)}(\vec{r},s) = -s\mu_0 \frac{\Psi}{2} H_0 \vec{1}_\phi \tag{2.5}$$

This is a kind of "minimal" electric field associated with the magnetic field in (2.2). Of course the incident magnetic field cannot be perfectly uniform for $s \neq 0$ because this would have zero curl and hence zero electric field. The above represents appropriate leading terms, another MSI approximation. Note also that both fields above have zero divergence.

The scalar potential is associated with quasi-electrostatic scattering which exists, but which we ignore, since we are looking at near scattered magnetic fields. Another way to look at this is to think of not one, but many, incident plane waves, all with $\vec{H}$ polarized parallel to $\vec{1}_z$. The electric field associated with each plane wave is perpendicular to $\vec{1}_z$, but has x and y components of various magnitudes depending on the direction of incidence (perpendicular to $\vec{1}_z$). Summing over these electric fields gives a small electric field on the z axis for an appropriate set of such plane waves.

The choice of the z axis being arbitrary we can relabel $\vec{1}_z$ as $\vec{1}_h$ and have

$$\begin{aligned} \vec{\tilde{H}}^{(inc)}(\vec{r},s) &= H_0 \vec{1}_h \\ \vec{A}(\vec{r}) &= \frac{\mu_0 H_0}{2} \vec{1}_h \times \vec{r} \\ \vec{\tilde{E}}^{(inc)}(\vec{r},s) &= -\frac{s\mu_0 H_0}{2} \vec{1}_h \times \vec{r} \end{aligned} \tag{2.6}$$

This is suitable for use in the usual Cartesian (x, y, z) and spherical (r, θ, ϕ) coordinate systems. By successive choices of $\vec{1}_h$ in three orthogonal directions, one can construct a set of incident fields to give (by linear combination) an arbitrary incident-magnetic-field polarization, and can construct the magnetic polarizability dyadic (to be discussed later).

3. Volume Integral Equations

Integrating over the electric and magnetic currents in the targets as in (1.7) we have the pair of integral equations [3, 18]

$$
\begin{aligned}
\tilde{\vec{E}}^{(sc)}(\vec{r},s) &= \frac{1}{\sigma_1 + s\varepsilon_1}\left\langle \nabla \tilde{G}_1(\vec{r},\vec{r}\,';s)\ ,\ \nabla' \cdot \tilde{\vec{J}}_e(\vec{r}\,',s)\right\rangle \\
&\quad - s\mu_0 \left\langle \tilde{G}_1(\vec{r},\vec{r}\,';s)\ ,\ \tilde{\vec{J}}_e(\vec{r}\,',s)\right\rangle \\
&\quad - \left\langle \nabla \tilde{G}_1(\vec{r},\vec{r}\,';s) \underset{\prime}{\times} \tilde{\vec{J}}_h(\vec{r}\,',s)\right\rangle \\
\tilde{\vec{H}}^{(sc)}(\vec{r},s) &= \frac{1}{s\mu_0}\left\langle \nabla \tilde{G}_1(\vec{r},\vec{r}\,';s)\ ,\ \nabla' \cdot \tilde{\vec{J}}_h(\vec{r}\,',s)\right\rangle \\
&\quad - (\sigma_1 + s\varepsilon_1)\left\langle \tilde{G}_1(\vec{r},\vec{r}\,';s)\ ,\ \tilde{\vec{J}}_h(\vec{r}\,',s)\right\rangle \\
&\quad + \left\langle \nabla \tilde{G}_1(\vec{r},\vec{r}\,';s) \underset{\prime}{\times} \tilde{\vec{J}}_e(\vec{r}\,',s)\right\rangle
\end{aligned}
\tag{3.1}
$$

where the symmetric products are integrals over the target volume with respect to the common $(\vec{r}\,')$ coordinates.

Under our MSI approximations these integral equations reduce to

$$
\begin{aligned}
\tilde{\vec{E}}^{(sc)}(\vec{r},s) &= -s\mu_0 \left\langle G_0(\vec{r},\vec{r}\,')\ ,\ \tilde{\vec{J}}_e(\vec{r}\,',s)\right\rangle \\
&\quad - \left\langle \nabla G_0(\vec{r},\vec{r}\,') \underset{\prime}{\times} \tilde{\vec{J}}_h(\vec{r}\,',s)\right\rangle \\
\tilde{\vec{H}}^{(sc)}(\vec{r},s) &= \frac{1}{s\mu_0}\left\langle \nabla G_0(\vec{r},\vec{r}\,')\ ,\ \nabla' \cdot \tilde{\vec{J}}_h(\vec{r}\,',s)\right\rangle \\
&\quad + \left\langle \nabla G_0(\vec{r},\vec{r}\,') \underset{\prime}{\times} \tilde{\vec{J}}_e(\vec{r}\,',s)\right\rangle
\end{aligned}
\tag{3.2}
$$

Substituting from (1.7) and (1.9) we have

$$\vec{\tilde{E}}^{(sc)}(\vec{r},s) = -s\mu_0 \left\langle G_0(\vec{r},\vec{r}\,') \; , \; \overset{\leftrightarrow}{\sigma}(\vec{r}\,') \cdot \vec{\tilde{E}}(\vec{r}\,',s) \right\rangle$$

$$- s \left\langle \nabla G_0(\vec{r},\vec{r}\,') \underset{,}{\times} \left[\overset{\leftrightarrow}{\mu}(\vec{r}\,') - \mu_0 \overset{\leftrightarrow}{1} \right] \cdot \vec{\tilde{H}}(\vec{r}\,',s) \right\rangle$$

$$\vec{\tilde{H}}^{(sc)}(\vec{r},s) = -\left\langle \nabla G_0(\vec{r},\vec{r}\,') \; , \; \nabla' \cdot \vec{\tilde{H}}(\vec{r}\,',s) \right\rangle \tag{3.3}$$

$$+ \left\langle \nabla G_0(\vec{r},\vec{r}\,') \underset{,}{\times} \overset{\leftrightarrow}{\sigma}(\vec{r}\,') \cdot \vec{\tilde{E}}(\vec{r}\,',s) \right\rangle$$

Note that by substituting $-\nabla'$ for ∇ we have an alternate form for the integrals.

At this point, let us consider these volume integrals and what happens at the surface. Let V denote the volume not including the surface, with S denoting the surface in the sense of a thin layer of vanishing thickness enclosing the surface. Then we have

$$\left\langle \nabla G_0(\vec{r},\vec{r}\,') \; , \; \nabla' \cdot \vec{\tilde{H}}(\vec{r}\,',s) \right\rangle$$

$$= \left\langle \nabla G_0(\vec{r},\vec{r}\,') \; , \; \nabla' \cdot \vec{\tilde{H}}(\vec{r}\,',s) \right\rangle_V \tag{3.4}$$

$$+ \left\langle \nabla G_0(\vec{r},\vec{r}\,') \; , \; \vec{1}_S(\vec{r}_S\,') \cdot \left[\vec{\tilde{H}}_+(\vec{r}\,',s) - \vec{\tilde{H}}_-(\vec{r}\,',s) \right] \right\rangle_S$$

$$\vec{r}_S \, , \vec{r}_S\,' \in S \quad , \quad \vec{1}_S(\vec{r}_S\,') \equiv \text{outward pointing unit normal to S at } \vec{r}_S$$

The integral over S is a surface integral (two dimensions), and the δ function in $\nabla' \cdot \vec{H}$ at S due to the discontinuity in normal $\vec{H}$ there is integrated out in the normal direction. Subscripts + and – are for parameters just outside and just inside S respectively.

If the target is uniform and isotropic, then we also have in V

$$\nabla' \cdot \vec{\tilde{H}}(\vec{r}\,',s) = \frac{1}{\mu} \nabla' \cdot \vec{\tilde{B}}(\vec{r}\,',s) = 0 \tag{3.5}$$

In this case (3.3) reduces to

$$\vec{\tilde{E}}^{(sc)}(\vec{r},s) = -s\mu_0\sigma\left\langle G_0(\vec{r},\vec{r}')\ ,\ \vec{\tilde{E}}(\vec{r}',s)\right\rangle_V - s[\mu-\mu_0]\left\langle \nabla G_0(\vec{r},\vec{r}') \underset{,}{\times} \vec{\tilde{H}}(\vec{r}',s)\right\rangle_V$$

$$\vec{\tilde{H}}^{(sc)}(\vec{r},s) = -\left\langle \nabla G_0(\vec{r},\vec{r}')\ ,\ \vec{1}_S(\vec{r}_s')\cdot\left[\vec{\tilde{H}}_+(\vec{r}',s) - \vec{\tilde{H}}_-(\vec{r}',s)\right]\right\rangle_S + \sigma\left\langle \nabla G_0(\vec{r},\vec{r}') \underset{,}{\times} \vec{\tilde{E}}(\vec{r}',s)\right\rangle_V \tag{3.6}$$

Noting that normal $\vec{B}$ is continuous through S we have

$$\vec{1}_S(\vec{r}_s')\cdot\vec{\tilde{B}}(\vec{r}_s',s) = \mu_0\,\vec{1}_S(\vec{r}_s')\cdot\vec{\tilde{H}}_+(\vec{r}_s',s) = \mu\,\vec{1}_S(\vec{r}_s')\cdot\vec{\tilde{H}}_-(\vec{r}_s',s)$$

$$\vec{1}_S(\vec{r}_s')\cdot\left[\vec{\tilde{H}}_+(\vec{r}_s',s) - \vec{\tilde{H}}_-(\vec{r}_s',s)\right] = \left[\mu_r - 1\right]\vec{1}_S(\vec{r}_s')\cdot\vec{\tilde{H}}_-(\vec{r}_s',s) = \left[1-\mu_r^{-1}\right]\vec{1}_S(\vec{r}_s')\cdot\vec{\tilde{H}}_+(\vec{r}_s',s) \tag{3.7}$$

$$\mu_r \equiv \frac{\mu}{\mu_0}$$

Furthermore, utilizing [13]

$$\nabla'\times\left[G_0(\vec{r},\vec{r}')\vec{\tilde{H}}(\vec{r},\vec{r}')\right] = \left[\nabla' G_0(\vec{r},\vec{r}')\right]\times\vec{\tilde{H}}(\vec{r}',s) + G_0(\vec{r},\vec{r}')\nabla'\times\vec{\tilde{H}}(\vec{r}',s) \tag{3.8}$$

with (A.6) we have

$$\begin{aligned}\left\langle \nabla G_0(\vec{r},\vec{r}') \underset{,}{\times} \vec{\tilde{H}}(\vec{r}',s)\right\rangle_V &= -\left\langle \nabla' G_0(\vec{r},\vec{r}') \underset{,}{\times} \vec{\tilde{H}}(\vec{r}',s)\right\rangle_V \\ &= -\int_V \nabla'\times\left[G_0(\vec{r},\vec{r}')\,\vec{\tilde{H}}(\vec{r}',s)\right]dV' + \left\langle G_0(\vec{r},\vec{r}'),\ \nabla'\times\vec{\tilde{H}}(\vec{r}',s)\right\rangle_V \\ &= -\left\langle G_0(\vec{r},\vec{r}'),\ \vec{1}_S(\vec{r}_s')\times\vec{\tilde{H}}(\vec{r}_s',s)\right\rangle_S + \sigma\left\langle G_0(\vec{r},\vec{r}'),\ \vec{\tilde{E}}(\vec{r}',s)\right\rangle_V\end{aligned} \tag{3.9}$$

Similarly we have

$$\left\langle \nabla G_0(\vec{r},\vec{r}') \underset{\prime}{\times} \vec{\tilde{E}}(\vec{r}',s)\right\rangle_V = -\left\langle \nabla' G_0(\vec{r},\vec{r}') \underset{\prime}{\times} \vec{\tilde{E}}(\vec{r}',s)\right\rangle_V$$

$$= -\int_V \nabla' \times \left[G_0(\vec{r},\vec{r}')\, \vec{\tilde{E}}(\vec{r}',s)\right] dV' + \left\langle G_0(\vec{r},\vec{r}'), \nabla' \times \vec{\tilde{E}}(\vec{r}',s)\right\rangle_V \tag{3.10}$$

$$= -\left\langle G_0(\vec{r},\vec{r}'), \vec{1}_S(\vec{r}_s') \times \vec{\tilde{E}}(\vec{r}_s',s)\right\rangle_S - s\mu \left\langle G_0(\vec{r},\vec{r}'), \vec{\tilde{H}}(\vec{r}',s)\right\rangle_V$$

Substituting in (3.6) gives

$$\vec{\tilde{E}}^{(sc)}(\vec{r},s) = -s\mu\sigma\left\langle G_0(\vec{r},\vec{r}'), \vec{\tilde{E}}(\vec{r}',s)\right\rangle_V$$

$$+ s[\mu - \mu_0]\left\langle G_0(\vec{r},\vec{r}_s'), \vec{1}_S(\vec{r}_s') \times \vec{\tilde{H}}(\vec{r}_s,s)\right\rangle_S$$

$$\vec{\tilde{H}}^{(sc)}(\vec{r},s) = [\mu_r - 1]\left\langle \nabla' G_0(\vec{r},\vec{r}'), \vec{1}_S(\vec{r}_s) \cdot \vec{\tilde{H}}_-(\vec{r}_s,s)\right\rangle_S \tag{3.11}$$

$$- s\mu\sigma\left\langle G_0(\vec{r},\vec{r}'), \vec{\tilde{H}}(\vec{r}',s)\right\rangle_V - \sigma\left\langle G_0(\vec{r},\vec{r}_s'), \vec{1}_S(\vec{r}_s') \times \vec{\tilde{E}}(\vec{r}_s',s)\right\rangle_S$$

as an alternate form. Note that for s real ($s_\alpha < 0$ for natural frequencies) all terms in the above are real, thereby simplifying the computations.

4. Surface Integral Equations for Uniform Isotropic Target

Following [16] we have the external fields (subscript +) in terms of equivalent surface current densities (electric and magnetic) on S (just outside S) as

$$
\begin{aligned}
\tilde{\vec{E}}_+(\vec{r},s) = T\,\tilde{\vec{E}}^{(inc)}(\vec{r},s) - T\int_S \Big\{ s\mu_0 \left[\vec{1}_S(\vec{r}_s\,') \times \tilde{\vec{H}}_+(\vec{r}_s\,',s)\right] \tilde{G}_1(\vec{r},\vec{r}_s\,';s) \\
- \left[\vec{1}_S(\vec{r}_s\,') \times \tilde{\vec{E}}_+(\vec{r}_s\,',s)\right] \times \nabla' \tilde{G}_1(\vec{r},\vec{r}_s\,';s) - \left[\vec{1}_S(\vec{r}_s\,') \cdot \tilde{\vec{E}}_+(\vec{r}_s\,',s)\right] \nabla' \tilde{G}_1(\vec{r},\vec{r}_s\,';s) \Big\} dS' \\
\tilde{\vec{H}}_+(\vec{r},s) = T\,\tilde{\vec{H}}^{(inc)}(\vec{r},s) + T\int_S \Big\{ [\sigma_1 + s\varepsilon_1] \left[\vec{1}_S(\vec{r}_s\,') \times \tilde{\vec{E}}_+(\vec{r}_s\,',s)\right] \tilde{G}_1(\vec{r},\vec{r}_s\,';s) \\
+ \left[\vec{1}_S(\vec{r}_s\,') \times \tilde{\vec{H}}_+(\vec{r}_s\,',s)\right] \times \nabla' \tilde{G}_1(\vec{r},\vec{r}_s\,';s) + \left[\vec{1}_S(\vec{r}_s\,') \cdot \tilde{\vec{H}}_+(\vec{r}_s\,',s)\right] \nabla' \tilde{G}_1(\vec{r},\vec{r}_s\,';s) \Big\} dS'
\end{aligned}
\tag{4.1}
$$

$$
T = \begin{cases} 1 \text{ for } \vec{r} \text{ outside (away from } S) \\ 2 \text{ for } \vec{r} \text{ on } S\ (\vec{r} = \vec{r}_s\,') \text{ for smooth } S \end{cases}
$$

For $\vec{r}$ away from S the scattered field is then

$$
\begin{aligned}
\tilde{\vec{E}}_+^{(sc)}(\vec{r},s) &= -\int_S \Big\{ s\mu_0 \left[\vec{1}_S(\vec{r}_s\,') \times \tilde{\vec{H}}_+(\vec{r}_s\,',s)\right] \tilde{G}_1(\vec{r},\vec{r}_s\,';s) \\
&\quad - \left[\vec{1}_S(\vec{r}_s\,') \times \tilde{\vec{E}}_+(\vec{r}_s\,',s)\right] \times \nabla' \tilde{G}_1(\vec{r},\vec{r}_s\,';s) - \left[\vec{1}_S(\vec{r}_s\,') \cdot \tilde{\vec{E}}_+(\vec{r}_s\,',s)\right] \nabla' \tilde{G}_1(\vec{r},\vec{r}_s\,';s) \Big\} dS \\
\tilde{\vec{H}}_+^{(sc)}(\vec{r},s) &= \int_S \Big\{ [\sigma_1 + s\varepsilon_1] \left[\vec{1}_S(\vec{r}_s\,') \times \tilde{\vec{E}}_+(\vec{r}_s\,',s)\right] \tilde{G}_1(\vec{r},\vec{r}_s\,';s) \\
&\quad + \left[\vec{1}_S(\vec{r}_s\,') \times \tilde{\vec{H}}_+(\vec{r}_s\,',s)\right] \times \nabla' \tilde{G}_1(\vec{r},\vec{r}_s\,';s) + \left[\vec{1}_S(\vec{r}_s\,') \cdot \tilde{\vec{H}}_+(\vec{r}_s\,',s)\right] \nabla' \tilde{G}_1(\vec{r},\vec{r}_s\,';s) \Big\} dS
\end{aligned}
\tag{4.2}
$$

Equations similar to (4.1) can be written for the interior fields (subscript –) as

$$\tilde{\vec{E}}_-(\vec{r},s)=T\int_S\Big\{s\mu\Big[\vec{1}_S(\vec{r}_s')\times\tilde{\vec{H}}_-(\vec{r}_s',s)\Big]\tilde{G}(\vec{r},\vec{r}_s';s)$$
$$-\Big[\vec{1}_S(\vec{r}_s')\times\tilde{\vec{E}}_-(\vec{r}_s',s)\Big]\times\nabla'\tilde{G}(\vec{r},\vec{r}_s';s)-\Big[\vec{1}_S(\vec{r}_s')\cdot\tilde{\vec{E}}_-(\vec{r}_s',s)\Big]\nabla'\tilde{G}(\vec{r},\vec{r}_s';s)\Big\}dS'$$
$$\tilde{\vec{H}}_-(\vec{r},s)=-T\int_S\Big\{[\sigma+s\varepsilon]\Big[\vec{1}_S(\vec{r}_s')\times\tilde{\vec{E}}_-(\vec{r}_s',s)\Big]\tilde{G}(\vec{r},\vec{r}_s';s)$$
$$+\Big[\vec{1}_S(\vec{r}_s')\times\tilde{\vec{H}}_-(\vec{r}_s',s)\Big]\times\nabla'\tilde{G}(\vec{r},\vec{r}_s';s)+\Big[\vec{1}_S(\vec{r}_s')\cdot\tilde{\vec{H}}_-(\vec{r}_s',s)\Big]\nabla'\tilde{G}(\vec{r},\vec{r}_s';s)\Big\}dS' \tag{4.3}$$
$$T=\begin{cases}1 \text{ for } \vec{r} \text{ inside (away from } S)\\ 2 \text{ for } \vec{r} \text{ on } S\ (\vec{r}=\vec{r}_s) \text{ for smooth } S\end{cases}$$

Taking the limits from both sides of S, take the tangential components by operating by $\vec{1}_S(\vec{r}_s)\times$ and noting the continuity of the tangential components as

$$\begin{aligned}\vec{1}_S(\vec{r}_s)\times\tilde{\vec{E}}_+(\vec{r}_s,s) &= \vec{1}_S(\vec{r}_s)\times\tilde{\vec{E}}_-(\vec{r}_s,s)\\ \vec{1}_S(\vec{r}_s)\times\tilde{\vec{H}}_+(\vec{r}_s,s) &= \vec{1}_S(\vec{r}_s)\times\tilde{\vec{H}}_-(\vec{r}_s,s)\end{aligned} \tag{4.4}$$

and discontinuity of the normal components as

$$\begin{aligned}[\sigma_1+s\varepsilon_1]\vec{1}_S(\vec{r}_s)\cdot\tilde{\vec{E}}_+(\vec{r}_s,s) &= [\sigma+s\varepsilon]\vec{1}_S(\vec{r}_s)\cdot\tilde{\vec{E}}_-(\vec{r}_s,s)\\ \mu_0\vec{1}_S(\vec{r}_s)\cdot\tilde{\vec{H}}_+(\vec{r}_s',s) &= \mu\vec{1}_S(\vec{r}_s)\cdot\tilde{\vec{H}}_-(\vec{r}_s,s)\end{aligned} \tag{4.5}$$

giving

$$\vec{1}_S(\vec{r}_s)\times\tilde{\vec{E}}^{(inc)}(\vec{r}_s,s) = \vec{1}_S(\vec{r}_s)\times\int_S\Big\{\Big[\vec{1}_S(\vec{r}_s')\times\tilde{\vec{H}}_+(\vec{r}_s',s)\Big]\Big[s\mu_0\tilde{G}_1(\vec{r}_s,\vec{r}_s';s)+s\mu\tilde{G}(\vec{r}_s,\vec{r}_s';s)\Big]$$
$$-\Big[\vec{1}_S(\vec{r}_s')\times\tilde{\vec{E}}_+(\vec{r}_s',s)\Big]\times\Big[\nabla'\tilde{G}_1(\vec{r}_s,\vec{r}_s';s)+\nabla'\tilde{G}(\vec{r}_s,\vec{r}_s';s)\Big] \tag{4.6}$$
$$-\Big[\vec{1}_S(\vec{r}_s')\cdot\tilde{\vec{E}}_+(\vec{r}_s',s)\Big]\Big[\nabla'\tilde{G}_1(\vec{r}_s,\vec{r}_s';s)+\frac{\sigma_1+s\varepsilon_1}{\sigma+s\varepsilon}\nabla'\tilde{G}(\vec{r}_s,\vec{r}_s';s)\Big]\Big\}dS'$$

$$\vec{1}_S(\vec{r}_S)\times\vec{\tilde{H}}^{(inc)}(\vec{r}_S,s)=-\vec{1}_S(\vec{r}_S)\times\int_S\left\{\left[\vec{1}_S(\vec{r}\,'_S)\times\vec{\tilde{E}}_+(\vec{r}\,'_S,s)\right]\left[[\sigma_1+s\varepsilon_1]\tilde{G}_1(\vec{r}_S,\vec{r}\,'_S;s)+[\sigma+s\varepsilon]\tilde{G}(\vec{r}_S,\vec{r}\,'_S;s)\right]\right.$$
$$+\left[\vec{1}_S(\vec{r}\,'_S)\times\vec{\tilde{H}}_+(\vec{r}\,'_S,s)\right]\times\left[\nabla'\tilde{G}_1(\vec{r}_S,\vec{r}\,'_S;s)+\nabla'\tilde{G}(\vec{r}_S,\vec{r}\,'_S;s)\right]$$
$$\left.+\left[\vec{1}_S(\vec{r}\,'_S)\cdot\vec{\tilde{H}}_+(\vec{r}\,'_S,s)\right]\left[\nabla'\tilde{G}_1(\vec{r}_S,\vec{r}\,'_S;s)+\frac{\mu_0}{\mu}\nabla'\tilde{G}(\vec{r}_S,\vec{r}\,'_S;s)\right]\right\}dS'$$

Noting that the normal components on S satisfy [13, 16]

$$\begin{aligned}
\vec{1}_S(\vec{r}_S)\cdot\vec{\tilde{E}}_+(\vec{r}_S) &= -\frac{1}{\sigma_1+s\varepsilon_1}\nabla_S\cdot\left[\vec{1}_S(\vec{r}_S)\times\vec{\tilde{H}}_+(\vec{r}_S)\right]\\
&= \frac{1}{\sigma_1+s\varepsilon_1}\nabla_S\cdot\left[\vec{1}_S(\vec{r}_S)\times\vec{\tilde{H}}_+(\vec{r}_S)\right]\\
\vec{1}_S(\vec{r}_S)\cdot\vec{\tilde{H}}_+(\vec{r}_S) &= \frac{1}{s\mu_0}\nabla_S\cdot\left[\vec{1}_S(\vec{r}_S)\times\vec{\tilde{E}}_+(\vec{r}_S)\right]\\
&= -\frac{1}{s\mu_0}\vec{1}_S(\vec{r}_S)\cdot\left[\nabla_S\times\vec{\tilde{E}}_+(\vec{r}_S)\right]
\end{aligned}\tag{4.7}$$

then (4.6) involves only the tangential components of electric and magnetic fields just outside S, appropriately giving four scalar equations in four unknowns. While (4.6) uses the fields just outside S, one can easily convert this to fields inside S, if desired, by using (4.4) and (4.5).

Under our MSI approximations the scattered magnetic field in (4.2) reduces to

$$\begin{aligned}
\vec{\tilde{H}}_+^{(sc)}(\vec{r},s) = \int_S&\left\{\left[\vec{1}_S(\vec{r}\,'_S)\times\vec{\tilde{H}}_+(\vec{r}\,'_S,s)\right]\times\nabla' G_0(\vec{r},\vec{r}\,'_S)\right.\\
&\left.+\left[\vec{1}_S(\vec{r}\,'_S)\cdot\vec{\tilde{H}}_+(\vec{r}\,'_S,s)\right]\nabla' G_0(\vec{r},\vec{r}\,'_S)\right\}dS'
\end{aligned}\tag{4.8}$$

Since we are interested only in the low-frequency scattered magnetic field, this equation can be used to find this field from surface values on S , whether computed from (4.6) or from the surface fields computed via volume integrals as in Section 3. We shall return to this point later when considering the magnetic polarizability of the target.

Also under our MSI approximations the integral equations (4.6) simplify as

$$
\begin{aligned}
\vec{1}_S(\vec{r}_S)\times\tilde{\vec{E}}^{(inc)}(\vec{r}_S,s) = \vec{1}_S(\vec{r}_S)\times\int_S\Bigg\{&\left[\vec{1}_S(\vec{r}\,'_S)\times\tilde{\vec{H}}_+(\vec{r}\,'_S,s)\right]\left[s\mu_0 G_0(\vec{r}_S,\vec{r}\,'_S)+s\mu\tilde{G}(\vec{r}_S,\vec{r}\,'_S;s)\right]\\
&-\left[\vec{1}_S(\vec{r}\,'_S)\times\tilde{\vec{E}}_+(\vec{r}\,'_S,s)\right]\times\left[\nabla' G_0(\vec{r}_S,\vec{r}\,'_S)+\nabla'\tilde{G}(\vec{r}_S,\vec{r}\,'_S;s)\right]\\
&-\left[\vec{1}_S(\vec{r}\,'_S)\cdot\tilde{\vec{E}}_+(\vec{r}\,'_S,s)\right]\nabla' G_0(\vec{r}_S,\vec{r}\,'_S)\Bigg\}\,dS'
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
\vec{1}_S(\vec{r}_S)\times\tilde{\vec{H}}^{(inc)}(\vec{r}_S,s) = -\vec{1}_S(\vec{r}_S)\times\int_S\Bigg\{&\left[\vec{1}_S(\vec{r}\,'_S)\times\tilde{\vec{E}}_+(\vec{r}\,'_S,s)\right]\sigma\,\tilde{G}(\vec{r}_S,\vec{r}\,'_S;s)\\
&+\left[\vec{1}_S(\vec{r}\,'_S)\times\tilde{\vec{H}}_+(\vec{r}\,'_S,s)\right]\times\left[\nabla' G_0(\vec{r}_S,\vec{r}\,'_S)+\nabla'\tilde{G}(\vec{r}_S,\vec{r}\,'_S;s)\right]\\
&+\left[\vec{1}_S(\vec{r}\,'_S)\cdot\tilde{\vec{H}}_+(\vec{r}\,'_S,s)\right]\left[\nabla' G_0(\vec{r}_S,\vec{r}\,'_S)+\mu_r^{-1}\nabla'\tilde{G}(\vec{r}_S,\vec{r}\,'_S;s)\right]\Bigg\}\,dS'
\end{aligned}
$$

Here terms involving $\sigma_1 + s\varepsilon_1$ and $s\varepsilon$ have been eliminated by comparison to σ. The exterior Green's function has no delay, but the interior Green's function has its frequency dependence retained since for natural frequencies of interest the target thickness is related to the "skin depth". Noting that the natural frequencies of interest are found on the negative real s axis where the fields are all real, the above equations can be split into real and imaginary parts using (1.13). The imaginary part must give zero for all negative real s. The real part can then be used to find natural frequencies by setting the incident field to zero.

5. Magnetic Dipole Moment from Volume Integrals

Dividing the magnetic dipole moment into the two parts associated with electric and magnetic currents [1, 3, 18] we have

$$\begin{aligned}\tilde{\vec{m}}(s) &= \tilde{\vec{m}}_e(s) + \tilde{\vec{m}}_h(s)\\ \tilde{\vec{m}}_e(s) &= \frac{1}{2}\int_V \vec{r}\,' \times \tilde{\vec{J}}_e(\vec{r}\,',s)\,dV'\\ \tilde{\vec{m}}_h(s) &= \frac{1}{s\mu_0}\int_V \tilde{\vec{J}}_h(\vec{r}\,',s)\,dV' = \int_V \left[\frac{\overleftrightarrow{\mu}(\vec{r}\,')}{\mu_0} - \overleftrightarrow{1}\right]\cdot \tilde{\vec{H}}_-(\vec{r}\,',s)\,dV'\end{aligned} \tag{5.1}$$

From the Maxwell equation and (C.3) we have the electric part as

$$\begin{aligned}\tilde{\vec{m}}_e(s) &= \frac{1}{2}\int_V \vec{r}\,' \times \tilde{\vec{J}}_e(\vec{r}\,',s)\,dV' = \frac{1}{2}\int_V \vec{r}\,' \times \left[\nabla' \times \tilde{\vec{H}}_-(\vec{r}\,',s)\right]dV'\\ &= \frac{1}{2}\int_S \vec{r}\,' \times \left[\vec{1}_S(\vec{r}\,'_s) \times \tilde{\vec{H}}_+(\vec{r}\,',s)\right] + \int_V \tilde{\vec{H}}_-(\vec{r}\,',s)\,dV'\end{aligned} \tag{5.2}$$

noting the continuity of tangential $\vec{H}$ through S. Using (C.1) we find the magnetic part as

$$\begin{aligned}\tilde{\vec{m}}_h(s) &= \frac{1}{\mu_0}\int_V \tilde{\vec{B}}_-(\vec{r}\,',s)\,dV' - \int_V \tilde{\vec{H}}_-(\vec{r}\,',s)\,dV'\\ &= \frac{1}{\mu_0}\int_S \left[\vec{1}_S(\vec{r}\,'_s)\cdot \tilde{\vec{B}}_-(\vec{r}\,'_s,s)\right]\vec{r}\,'_s - \int_V \tilde{\vec{H}}_-(\vec{r}\,',s)\,dV'\\ &= \int_S \left[\vec{1}_S(\vec{r}\,'_s)\cdot \tilde{\vec{H}}_+(\vec{r}\,'_s,s)\right]\vec{r}\,'_s\,dS' - \int_V \tilde{\vec{H}}_-(\vec{r}\,',s)\,dV'\end{aligned} \tag{5.3}$$

noting that $\vec{B}$ has zero divergence and that its normal component is continuous through S. Combining these results we have

$$\tilde{\vec{m}}(s) = \frac{1}{2}\int_S \vec{r}\,'_s \times \left[\vec{1}_S(\vec{r}\,'_s) \times \tilde{\vec{H}}_+(\vec{r}\,'_s,s)\right]dS' + \int_S \left[\vec{1}_S(\vec{r}\,'_s)\cdot \tilde{\vec{H}}_+(\vec{r}\,'_s,s)\right]\vec{r}\,'_s\,dS' \tag{5.4}$$

So the magnetic dipole moment can also be evaluated by integrals of the magnetic field just outside S.

The leading term in the scattered magnetic field is the magnetic-dipole term given by

$$\begin{aligned}\tilde{\vec{H}}_d^{(sc)}(\vec{r},s) &= \frac{1}{4\pi r^3}\left[3\,\vec{1}_r\,\vec{1}_r - \overleftrightarrow{1}\right]\cdot\tilde{\vec{m}}(s) \\ &= \frac{1}{4\pi r^3}\left[3\,\vec{1}_r\,\vec{1}_r - \overleftrightarrow{1}\right]\cdot\tilde{\overleftrightarrow{M}}(s)\cdot\tilde{\vec{H}}^{(inc)}(\vec{0},s)\end{aligned} \tag{5.5}$$

This is valid for distances r from the target (say the approximate center of the target) large compared to a which is defined as some characteristic dimension of the target.

6. Magnetic Dipole Moment Evaluated from Scattered Magnetic Field Given by Surface Integral

The scattered magnetic field is given by surface integrals as in (4.8) which looks somewhat different from (5.3) and (5.4). So let us expand the scattered magnetic field for large r ($r >> a$) for which we need

$$
\begin{aligned}
\nabla' G_0(\vec{r},\vec{r}') &= \frac{1}{4\pi r^2} \frac{\vec{1}_r - \frac{\vec{r}'}{r}}{|\vec{1}_r - \frac{\vec{r}'}{r}|^3} \\
&= \frac{1}{4\pi r^2}\left[\vec{1}_r - \frac{\vec{r}'}{r}\right]\left[1 - 2\frac{\vec{1}_r \cdot \vec{r}'}{r} + \left(\frac{r'}{r}\right)^2\right]^{-3/2} \\
&= \frac{1}{4\pi r^2}\left[\vec{1}_r - \frac{\vec{r}'}{r}\right]\left[1 + 3\frac{\vec{1}_r \cdot \vec{r}'}{r} + O\left(r^{-2}\right)\right] \\
&= \frac{1}{4\pi r^2}\left[\vec{1}_r - \frac{\vec{r}'}{r} + \frac{3}{r}\vec{1}_r, \vec{1}_r \cdot \vec{r}' + O\left(r^{-2}\right)\right] \text{ as } r \to \infty
\end{aligned}
\tag{6.1}
$$

So consider (4.8) expanded in inverse powers of r.

The r^{-2} term is

$$
\tilde{\vec{H}}_{-2}^{(sc)}(\vec{r},s) = \frac{1}{4\pi r^2}\int_S \left\{\left[\vec{1}_S(\vec{r}_s\,') \times \tilde{\vec{H}}_+(\vec{r}_s\,',s)\right] \times \vec{1}_r + \left[\vec{1}_S(\vec{r}_s\,') \cdot \tilde{\vec{H}}_+(\vec{r}_s\,',s)\right]\vec{1}_r\right\} dS' \tag{6.2}
$$

Considering the second term we have

$$
\begin{aligned}
\int_S \vec{1}_S(\vec{r}_s\,') \cdot \tilde{\vec{H}}_+(\vec{r}_s\,',s)\, dS' &= \frac{1}{\mu_0}\int_S \vec{1}_S(\vec{r}_s\,') \cdot \tilde{\vec{B}}_+(\vec{r}_s\,',s)\, dS' \\
&= \frac{1}{\mu_0}\int_V \nabla' \cdot \tilde{\vec{B}}_-(\vec{r}_s\,',s)\, dS' = 0
\end{aligned}
\tag{6.3}
$$

noting the continuity of the normal component of $\vec{B}$ through S and the zero divergence of $\vec{B}$. This is just a statement of the well-known fact that the magnetic monopole moment is zero. Considering the first term we have

$$\int_S \vec{1}_S(\vec{r}'_s) \times \tilde{\vec{H}}_+(\vec{r}'_s, s)\, dS' = \int_V \nabla' \times \tilde{\vec{H}}_-(\vec{r}_s, s)\, dS'$$

$$= \int_V \tilde{\vec{J}}_e(\vec{r}'_s, s)\, dV' = s\,\tilde{\vec{p}}(s) \tag{6.4}$$

$$\tilde{\vec{p}}(s) \equiv \text{electric dipole moment}$$

noting the continuity of tangential $\vec{H}$ through S. Thus we have

$$\tilde{\vec{H}}_{-2}^{(sc)}(\vec{r}, s) = \frac{s}{4\pi r^2}\,\tilde{\vec{p}}(s) \times \vec{1}_r \to \vec{0} \quad \text{for } s \to 0 \tag{6.5}$$

This term is neglected since it is negligible compared to the near quasi-static magnetic field due to the induced magnetic dipole moment at our low frequencies of interest. Note that other contributions of order r^{-2} and r^{-1} have also been neglected in approximating $\tilde{\vec{G}}_1(\vec{r}, \vec{r}'; s)$ by $G_0(\vec{r}, \vec{r}')$, consistent with our MSI approximations.

The r^{-3} term is

$$\tilde{\vec{H}}_{-3}^{(sc)}(\vec{r}, s) = \frac{1}{4\pi r^3} \int_S \left\{ \left[\vec{1}_S(\vec{r}'_s) \times \tilde{\vec{H}}_+(\vec{r}'_s, s) \right] \left[3\,\vec{1}_r \vec{1}_r - \overleftrightarrow{1} \right] \cdot \vec{r}'_s \right.$$
$$\left. + \left[\vec{1}_S(\vec{r}'_s) \cdot \tilde{\vec{H}}_+(\vec{r}'_s, s) \right] \left[3\,\vec{1}_r \vec{1}_r - \overleftrightarrow{1} \right] \cdot \vec{r}'_s \right\} dS' \tag{6.6}$$

The second term can be readily evaluated by bringing out the r-dependent dyadic to agree with the second term in (5.4) which combined with (5.5) gives that part of the magnetic-dipole field. However, in the first term the dyadic is buried between terms depending on $\vec{r}'_s$. So let us consider another approach by writing

$$\tilde{\vec{H}}_{-3}^{(sc)}(\vec{r}, s) = \frac{1}{4\pi r^3} \left[3\,\vec{1}_r \vec{1}_r - \overleftrightarrow{1} \right] \cdot \tilde{\vec{m}}(s) + \text{possible higher order terms} \tag{6.7}$$

In this case the magnetic-dipole term is the leading term in a multipole expansion. As is well known, the multipole terms are mutually orthogonal on a shperical surface of radius r (outside the target) [2, 11]. This orthogonality is in terms of the tangential (θ, ϕ) components of the fields which are expanded in vector spherical harmonics ($\vec{Q}$ and $\vec{R}$ functions), in terms of which the dipole terms are the lowest order for electromagnetic fields away from sources. The resulting integrals then effectively occur on the unit sphere.

The tangential part of the field of a magnetic dipole on a sphere of radius r is found from (5.5) as

$$\overleftrightarrow{1}_r \cdot \tilde{\vec{H}}_d^{(sc)}(\vec{r},s) = -\frac{1}{4\pi r^3}\overleftrightarrow{1}_r \cdot \tilde{\vec{m}}(s) \tag{6.8}$$

The transverse dyad (to $\vec{r}$) and various identities involving integrals on the unit sphere that we will use here are discussed in Appendix B. As an example, if we think of a z directed magnetic dipole we have

$$\begin{aligned} \tilde{\vec{m}}(s) &= \tilde{m}(s)\vec{1}_z \\ \overleftrightarrow{1}_r \cdot \tilde{\vec{H}}_d^{(sc)}(\vec{r},s) &= \frac{1}{4\pi r^3}\vec{1}_\theta \sin(\theta)\tilde{\vec{m}}(s) \end{aligned} \tag{6.9}$$

Using three orthogonal axes (say $\vec{1}_x, \vec{1}_y, \vec{1}_z$) the three magnetic-dipole terms are readily constructed. We can evaluate these all at once using the form in (6.8)

Noting that any higher order terms are orthogonal on the unit sphere to such magnetic-dipole terms we multiply both sides of (6.7) by $\overleftrightarrow{1}_r$ and integrate over the unit sphere S_1 to obtain

$$\begin{aligned} \frac{1}{4\pi r^3}\left\{-\int_{S_1}\overleftrightarrow{1}_r\, dS_1\right\} \cdot \tilde{\vec{m}}(s) &= -\frac{1}{4\pi r^3}\frac{8\pi}{3}\tilde{\vec{m}}(s) \\ &= \frac{1}{4\pi r^3}\int_{S_1}\overleftrightarrow{1}_r \cdot \left\{\int_S\left\{\left[\vec{1}_S(\vec{r}_s')\times\tilde{\vec{H}}_+(\vec{r}_s',s)\right]\times\left[3\,\vec{1}_r\vec{1}_r-\overleftrightarrow{1}\right]\cdot\vec{r}_s' \right.\right. \\ &\qquad \left.\left. +\left[\vec{1}_S(\vec{r}_s')\cdot\tilde{\vec{H}}_+(\vec{r}_s',s)\right]\left[3\,\vec{1}_r\vec{1}_r-\overleftrightarrow{1}\right]\cdot\vec{r}_s'\right\}dS'\right\}dS_1 \\ &= \frac{1}{4\pi r^3}\int_S\int_{S_1}\overleftrightarrow{1}_r\left\{\left[\vec{1}_S(\vec{r}_s')\times\tilde{\vec{H}}_+(\vec{r}_s',s)\right]\times\left[3\,\vec{1}_r\vec{1}_r-\overleftrightarrow{1}\right]\cdot\vec{r}_s'\right\}dS_1\,dS' \\ &\quad -\frac{1}{4\pi r^3}\int_S\int_{S_1}\left[\vec{1}_S(\vec{r}_s')\cdot\tilde{\vec{H}}_+(\vec{r}_s',s)\right]\overleftrightarrow{1}_r\cdot\vec{r}_s'\,dS_1\,dS' \end{aligned} \tag{6.10}$$

Having reversed the order of integration in (6.10) we can evaluate the integrals over S_1 first. The first integral uses (A.3) to give

$$\begin{aligned}
&\int_{S_1} \overleftrightarrow{1}_r \cdot \left\{ \left[\vec{1}_S(\vec{r}'_s) \times \tilde{\vec{H}}_+(\vec{r}'_s, s) \right] \times \left[3\,\vec{1}_r\,\vec{1}_r - \overleftrightarrow{1} \right] \cdot \vec{r}'_s \right\} dS_1 \\
&= \left\{ \int_{S_1} -\overleftrightarrow{1}_r \times \left[3\,\vec{1}_r\,\vec{1}_r - \overleftrightarrow{1} \right] \cdot \vec{r}'_s \, dS_1 \right\} \cdot \left[\vec{1}_S(\vec{r}'_s) \times \tilde{\vec{H}}_+(\vec{r}'_s, s) \right] dS_1 \\
&= \left\{ \int_{S_1} \left\{ \left[-3\,\overleftrightarrow{1}_r \times \vec{1}_r \right] \vec{1}_r \cdot \vec{r}'_s + \overleftrightarrow{1}_r \times \vec{r}'_s \right\} dS_1 \right\} \cdot \left[\vec{1}_S(\vec{r}'_s) \times \tilde{\vec{H}}_+(\vec{r}'_s, s) \right] \\
&= \left\{ -3\,\overleftrightarrow{1} \times \left\{ \int_{S_1} \vec{1}_r\,\vec{1}_r \, dS_1 \right\} \cdot \vec{r}'_s + \left\{ \int_{S_1} \overleftrightarrow{1}_r \, dS_1 \right\} \times \vec{r}'_s \right\} \cdot \left[\vec{1}_S(\vec{r}'_s) \times \tilde{\vec{H}}_+(\vec{r}'_s, s) \right] \\
&= \left\{ -3\,\frac{4\pi}{3}\,\overleftrightarrow{1} \times \vec{r}'_s + \frac{8}{3}\pi\,\overleftrightarrow{1} \times \vec{r}'_s \right\} \cdot \left[\vec{1}_S(\vec{r}'_s) \times \tilde{\vec{H}}_+(\vec{r}'_s, s) \right] \\
&= \frac{8\pi}{3} \left\{ -\frac{1}{2}\,\vec{r}'_s \times \left[\vec{1}_S(\vec{r}'_s) \times \tilde{\vec{H}}_+(\vec{r}'_s, s) \right] \right\}
\end{aligned} \tag{6.11}$$

The second integral over S_1 is more straight forward giving

$$\begin{aligned}
&\int_{S_1} \left[\vec{1}_S(\vec{r}'_s) \cdot \tilde{\vec{H}}_+(\vec{r}'_s, s) \right] \overleftrightarrow{1}_r \cdot \vec{r}'_s \, dS_1 \\
&= \left[\vec{1}_S(\vec{r}'_s) \cdot \tilde{\vec{H}}_+(\vec{r}'_s, s) \right] \left\{ \int_{S_1} \overleftrightarrow{1}_r \, dS_1 \right\} \cdot \vec{r}'_s \\
&= \frac{8\pi}{3} \left[\vec{1}_S(\vec{r}'_s) \cdot \tilde{\vec{H}}_+(\vec{r}'_s, s) \right] \vec{r}'_s
\end{aligned} \tag{6.12}$$

Substituting in (6.10) and clearing common factors gives

$$\tilde{\vec{m}}(s) = \frac{1}{2} \int_S \vec{r}'_s \times \left[\vec{1}_S(\vec{r}'_s) \times \tilde{\vec{H}}_+(\vec{r}'_s, s) \right] dS' + \int_S \left[\vec{1}_S(\vec{r}'_s) \cdot \tilde{\vec{H}}_+(\vec{r}'_s, s) \right] \vec{r}'_s \, dS' \tag{6.13}$$

in agreement with (5.4).

So, now by two different routes we have found the induced magnetic dipole moment as an integral over the magnetic field just outside S. These can, in turn, be found from the volume integral equations (Section 3, applicable to an isotropic and inhomogeneous targets), and the surface integral equations (Section 4, applicable only to uniform isotropic targets.

7. Magnetic Polarizability

Having found the magnetic dipole moment for an arbitrarily oriented incident magnetic field (described in Section 2), we are now in a position to find the magnetic polarizability dyadic. For this purpose, consider three orthogonal incident magnetic fields, each producing an induced magnetic field as

$$H_0 \vec{1}_h \rightarrow \tilde{\vec{m}}^{(h)}(s) \quad \text{for} \quad h = x, y, z \tag{7.1}$$

where Cartesian (x, y, z) coordinates are used for convenience here. The coordinate origin ($\vec{r} = \vec{0}$) is taken at some central location which, for symmetric targets, lies on an axis or point of symmetry.

Next form a dyadic incident field as

$$\overleftrightarrow{H}_0 \equiv H_0 \overleftrightarrow{1} = H_0 \left[\vec{1}_x \vec{1}_x + \vec{1}_y \vec{1}_y + \vec{1}_z \vec{1}_z \right] \tag{7.2}$$

and a dyadic from the dipole moments as

$$\begin{aligned} \tilde{\overleftrightarrow{m}}(s) &\equiv \left(\tilde{\vec{m}}^{(x)}(s) \,,\, \tilde{\vec{m}}^{(y)}(s) \,,\, \tilde{\vec{m}}^{(z)}(s) \right) && \text{(vectors as matrix columns)} \\ &= \begin{pmatrix} \tilde{\vec{m}}^{(x)}(s) \\ \tilde{\vec{m}}^{(y)}(s) \\ \tilde{\vec{m}}^{(z)}(s) \end{pmatrix} && \text{(vectors as matrix rows)} \\ &= \tilde{\overleftrightarrow{m}}(s)^T && \text{(symmetric)} \end{aligned} \tag{7.3}$$

From the equation

$$\begin{aligned} \tilde{\vec{m}}^{(h)}(s) &= \tilde{\overleftrightarrow{M}}(s) \cdot \left[H_0 \vec{1}_h \right] \\ \tilde{\overleftrightarrow{M}}(s) &= \tilde{\overleftrightarrow{M}}^T(s) \cdot \quad \text{(symmetric)} \end{aligned} \tag{7.4}$$

we have

$$\overleftrightarrow{\tilde{m}}(s) = \overleftrightarrow{\tilde{M}}(s) \cdot \left[H_0 \overleftrightarrow{1}\right] = \left[H_0 \overleftrightarrow{1}\right] \cdot \overleftrightarrow{\tilde{M}}(s)$$
$$\overleftrightarrow{\tilde{M}}(s) = \frac{1}{H_0} \overleftrightarrow{\tilde{m}}(s) \tag{7.5}$$

justifying the construction of $\overleftrightarrow{\tilde{m}}(s)$ from either rows or columns in (7.3).

In some cases, one can reduce the complexity by judicious choice of coordinate directions since the Cartesian coordinates can be translated to any origin of convenience and rotated about this origin. If the target has point symmetry (rotations and/or reflections) the situation is significantly simplified [4, 6, 17, 18]. If the target has two (or more) symmetry planes, then choosing two of the $\overrightarrow{1}_h$ as perpendicular to these planes, and the third as perpendicular to these makes the $\overrightarrow{\tilde{m}}^{(h)}(s)$ parallel to the respective $\overrightarrow{1}_h$. With this choice of axes, then $\overleftrightarrow{\tilde{M}}(s)$ is diagonal. Note that in such a case, one need not compute the eigenvectors of $\overleftrightarrow{\tilde{M}}(s)$; they are known a priori. A similar situation applies to bodies of revolution whether discrete as in C_N for $N \geq 3$, or continuous as in $C_{\infty a}$. In addition, in such cases two of the eigenvalues are equal (double degeneracy, requiring only two solutions of the integral equations).

8. Scaling of Target and Frequency

For a general case of $\overleftrightarrow{\mu}(\vec{r})$ and $\overleftrightarrow{\sigma}(\vec{r})$ varying throughout the target volume we have the integral equations (3.3) to describe the scattering. Setting the incident fields to zero gives equations for natural frequencies and modes. From a more general point of view we can find scaling relationships for frequencies in terms of the other parameters. Consider two cases: the original case (subscript 1) and the scaled case (subscript 2). The original case has

$$
\begin{aligned}
\tilde{\vec{E}}_1^{(sc)}(\vec{r}_1,s) &= -s_1\mu_0 \left\langle G_0(\vec{r}_1,\vec{r}\,'_1),\ \overleftrightarrow{\sigma}_1(\vec{r}\,'_1)\cdot\tilde{\vec{E}}_1(\vec{r}\,'_1,s)\right\rangle \\
&\quad - s_1\left\langle \nabla_1 G_0(\vec{r}_1,\vec{r}\,'_1) \underset{,}{\times} \left[\overleftrightarrow{\mu}_1(\vec{r}\,'_1)-\mu_0\overleftrightarrow{1}\right]\cdot\tilde{\vec{H}}_1(\vec{r}\,'_1,s)\right\rangle \\
\tilde{\vec{H}}_1^{(sc)}(\vec{r}_1,s) &= -\left\langle \nabla_1 G_0(\vec{r}_1,\vec{r}\,'_1),\ \nabla'_1\cdot\tilde{\vec{H}}_1(\vec{r}\,'_1,s)\right\rangle \\
&\quad + \left\langle \nabla_1 G_0(\vec{r}_1,\vec{r}\,'_1) \underset{,}{\times} \overleftrightarrow{\sigma}_1(\vec{r}\,'_1)\cdot\tilde{\vec{E}}_1(\vec{r}\,'_1,s)\right\rangle
\end{aligned}
\tag{8.1}
$$

with the integrals above as volume integrals. The scaled case is like the above with the subscript 1 replaced by 2.

In the scaling start with coordinate dilation

$$
\vec{r}\,'_2 = \chi_1\,\vec{r}_1 \tag{8.2}
$$

which immediately scales other parameters as

$$
\begin{aligned}
dV'_2 &= \chi_1^3\ dV'_1\ ,\quad \nabla_2 = \chi_1^{-1}\nabla_1 \\
G_0(\vec{r}_2,\vec{r}\,'_2) &= \chi_1^{-1} G_0(\vec{r}_1,\vec{r}\,'_1)
\end{aligned}
\tag{8.3}
$$

Scale the constitutive parameters as

$$
\overleftrightarrow{\sigma}_2(\vec{r}_2) = \chi_2\,\overleftrightarrow{\sigma}_1(\vec{r}_1)\ ,\quad \overleftrightarrow{\mu}_2(\vec{r}_2) = \chi_3(\vec{r}_3) \tag{8.4}
$$

With the magnetic field kept the same magnitude we have

$$
\tilde{\vec{H}}_2(\vec{r}_2,s_2) = \tilde{\vec{H}}_1(\vec{r}_1,s_1)\ ,\quad \tilde{\vec{E}}_2(\vec{r}_2,s_2) = \chi_4\,\tilde{\vec{E}}_1(\vec{r}_1,s_1) \tag{8.5}
$$

applying to both incident and scattered fields. The frequency finally scales as

$$s_2 = \chi_5 \, s_1 \tag{8.6}$$

All the above χ_n are taken as positive (and hence real).

Rewriting (8.1) for the scaled condition (subscript 2) and then substituting the scaling relations (8.2) through (8.6) gives

$$\begin{aligned}
\chi_4 \vec{\tilde{E}}_1^{(sc)}(\vec{r}_1, s_1) &= -\chi_1^2 \chi_2 \chi_4 \chi_5 \, s_1 \, \mu_0 \left\langle G_0(\vec{r}_1, \vec{r}\,'_1), \; \overleftrightarrow{\sigma}(\vec{r}\,'_1) \cdot \vec{\tilde{E}}_1(\vec{r}\,'_1, s) \right\rangle \\
&\quad - \chi_1 \chi_5 \, s_1 \left\langle \nabla_1 G_0(\vec{r}_1, \vec{r}\,'_1) \times \left[\chi_3 \overleftrightarrow{\mu}_2(\vec{r}\,'_1) - \mu_0 \overleftrightarrow{1} \right] \cdot \vec{\tilde{H}}_1(\vec{r}\,'_1, s) \right\rangle \\
\vec{\tilde{H}}_1^{(sc)}(\vec{r}_1, s_1) &= -\left\langle \nabla_1 G_0(\vec{r}_1, \vec{r}\,'_1), \; \nabla' \cdot \vec{\tilde{H}}_1(\vec{r}\,'_1, s) \right\rangle \\
&\quad - \chi_1 \chi_2 \chi_4 \left\langle \nabla_1 G_0(\vec{r}_1, \vec{r}\,'_1) \times \overleftrightarrow{\sigma}_1(\vec{r}\,'_1) \cdot \vec{\tilde{E}}_1(\vec{r}\,'_1, s) \right\rangle
\end{aligned} \tag{8.7}$$

Requiring that these equations be the same as (8.1) implies

$$\chi_1^2 \chi_2 \chi_5 = 1 \;, \quad \frac{\chi_1 \chi_5}{\chi_4} = 1 \;, \quad \chi_3 = 1 \;, \quad \chi_1 \chi_2 \chi_4 = 1 \tag{8.8}$$

These are satisfied provided

$$\chi_3 = 1 \;, \quad \chi_5 = \frac{1}{\chi_2 \, \chi_1{}^2} \;, \quad \chi_4 = \frac{1}{\chi_2 \, \chi_1} (= \chi_5 \, \chi_1) \tag{8.9}$$

so that χ_1 (coordinate scaling) and χ_2 (conductivity scaling) can be independently specified, giving χ_4 (electric field scaling relative to magnetic field) and χ_5 (frequency scaling). Note that the permeability magnitude is retained constant in the scaling, i.e., $\overleftrightarrow{\mu}(\vec{r})$ is kept the same "relative permeability" since μ_0 (relating to the external medium) is assumed constant. Viewed another way, the χ_5 result implies that $s \, \sigma \, a^2$ is invariant to the scaling where a is some characteristic linear dimension of the target. Similarly, the χ_4 result implies that $\vec{\tilde{E}} \, a \sigma$ or $\vec{\tilde{E}}/(s\,a)$ is invariant to the scaling. The magnetic-dipole moment and magnetic polarizability dyadic scale in the well-known way proportional to a^3.

This scaling can be carried further in an approximate sense if we assume that the permeability of the target is large compared to μ_0. Then the scattered magnetic energy outside the target can be neglected compared to that inside. Setting $\mu_0 = 0$ for this case in (8.7) gives

$$\frac{\chi_1 \chi_5 \chi_3}{\chi_4} = 1 \quad , \quad \chi_1 \chi_2 \chi_4 = 1 \tag{8.10}$$

in place of (8.8). These are satisfied provided

$$\chi_5 = \frac{1}{\chi_3 \chi_2 \chi_1^2} \quad , \quad \chi_4 = \frac{1}{\chi_1 \chi_2} (= \chi_5 \chi_2 \chi_1) \tag{8.11}$$

Now $s \mu \sigma a^2$ is invariant to the scaling. This suggests that one define for the target

$$\begin{aligned} \tau \equiv \mu \sigma a^2 &\equiv \text{characteristic time} \\ \mu &\equiv \text{characteristic permeability} \\ \sigma &\equiv \text{characteristic conductivity} \\ a &\equiv \text{characteristic distance (size)} \end{aligned} \tag{8.12}$$

so that $s\tau$ is the invariant way to describe frequencies, including natural frequencies.

More generally, we can scale this way keeping

$$\mu_r = \frac{\mu}{\mu_0} \equiv \text{characteristic relative permeability} \tag{8.13}$$

as a parameter which is fixed for a particular calculation. By choosing various values of μ_r one can calculate $s_\alpha \tau$ as a function of μ_r where s_α is any given natural frequency. Also note from the form of $\tilde{G}$ (the internal-medium Green's function) in (1.10), (1.12), and (4.9), that $\tilde{\gamma} \mid \vec{r} - \vec{r}' \mid$ scales as $[s\tau]^{1/2}$, again giving the $s\tau$ combination in the surface integral equations for a homogeneous isotropic target. One can also consider the solution for the canonical problem of the perfectly conducting sphere to observe the same behavior. In this case the lowest order $s_\alpha \tau$ (negative) is of the order of 1, but with various higher order $s_\alpha \tau$ with $\mid s_\alpha \tau \mid$ extending upward from this. For targets continuously deformed from a sphere we can expect similar behavior, but different target topologies (e.g., a toroid) can give somewhat different results.

9. Scaling of Thin-Shell Target and Frequency

A special case of interest is a target with a thin metal shell as illustrated in fig. 9.1. In this case the shell has a high conductivity σ and may have a high permeability μ. However, the shell thickness d is small, i.e.,

$$d << a \tag{9.1}$$

Consistent with the MSI approximations, all permittivity is neglected as well as the conductivities of all media except the shell. The media external and internal to the shell have permeability μ_0.

By a thin-shell target we mean one with shell thickness small compared to skin depth so that

$$s\mu\sigma d^2 << 1 \tag{9.2}$$

In this case, we can think of the shell as located on the surface S with equivalent sheet parameters to represent σ and μ. In this case, the natural frequencies correspond to L/R time constants where the inductance depends on the loop area for currents circulating around the interior volume, but resistance depending on the shell conductivity.

The surface electric current density as indicated in fig. 9.2A is given by

$$\vec{1}_S \times \left[\tilde{\vec{H}}^{(out)}(\vec{r}_S, s) - \tilde{\vec{H}}^{(in)}(\vec{r}_S, s) \right] = \tilde{\vec{J}}_{S_e}(\vec{r}_S, s) = \tilde{\vec{J}}_{S_e}(\vec{r}_S, s)$$

$$= \overleftrightarrow{G}_S(\vec{r}_S) \cdot \tilde{\vec{E}}^{(avg)}(\vec{r}_S, s) \simeq h\, \tilde{\vec{E}}_{S_e}(\vec{r}_S, s)$$

$$\tilde{\vec{J}}_{S_e}(\vec{r}_S, s) \equiv \text{surface electric current density on } S$$

$$\overleftrightarrow{G}_S(\vec{r}_S) = d\, \overleftrightarrow{1}_S(\vec{r}_S) \cdot \overleftrightarrow{\sigma}(\vec{r}_S) \cdot \overleftrightarrow{1}_S(\vec{r}_S) \equiv \text{sheet conductance} \tag{9.3}$$

$$\tilde{\vec{E}}^{(avg)}(\vec{r}_S, s) \simeq \frac{1}{2}\left[\tilde{\vec{E}}^{(out)}(\vec{r}_S, s) + \tilde{\vec{E}}^{(in)}(\vec{r}_S, s) \right] \text{(only components parallel to } S \text{ significant)}$$

$$\overleftrightarrow{1}_S(\vec{r}_S) \equiv \overleftrightarrow{1} - \vec{1}_S(\vec{r}_S)\, \vec{1}_S(\vec{r}_S)$$

$$\equiv \text{transverse dyadic on } S$$

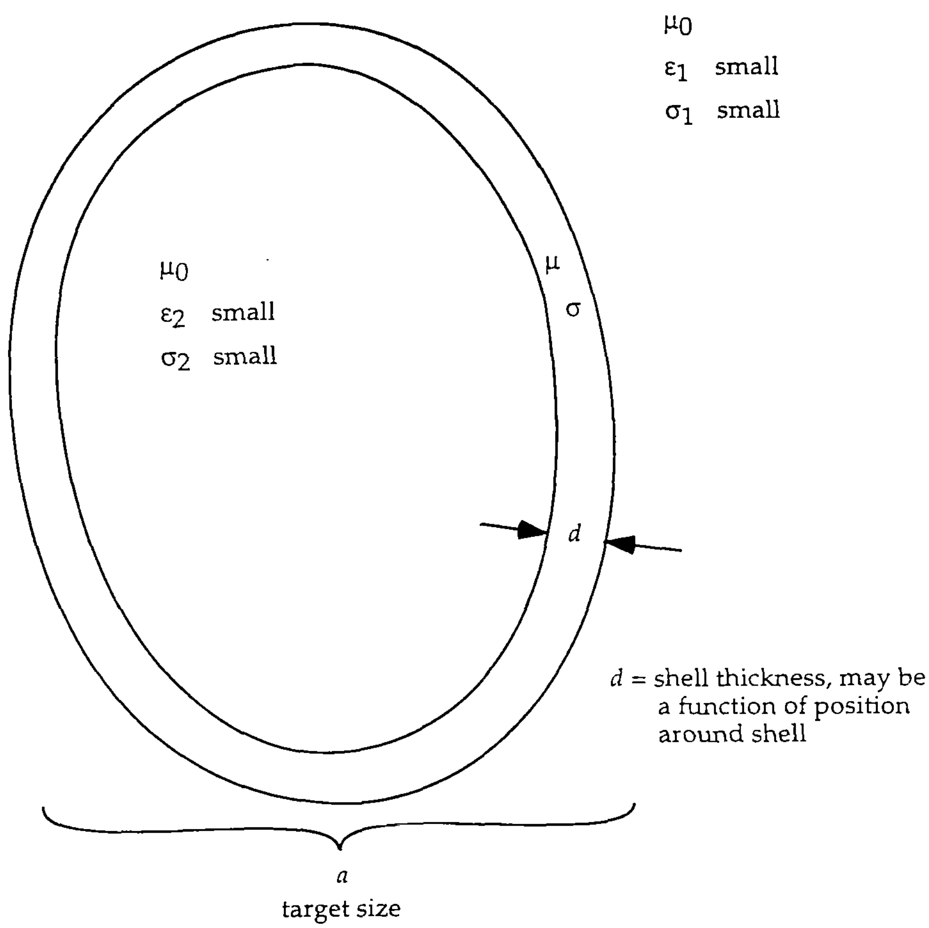
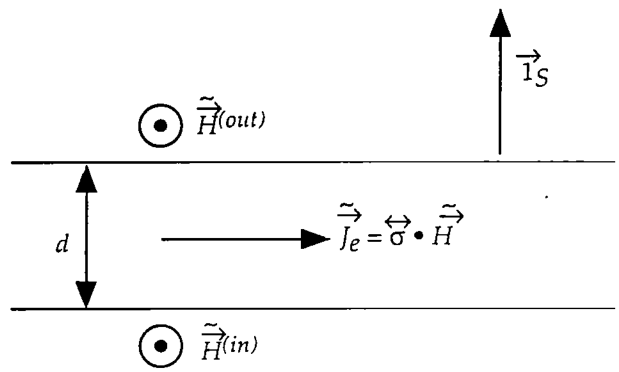
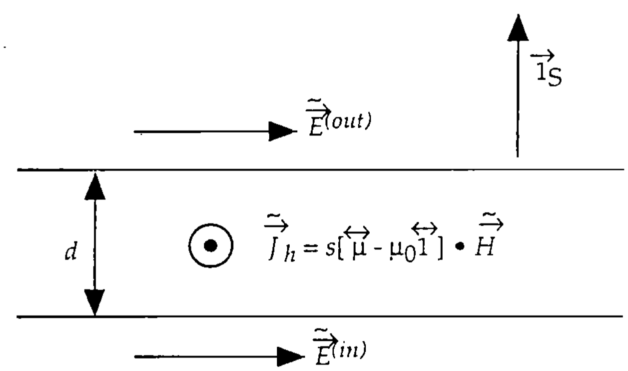


Fig. 9.1. Thin-Shell Target



A. Electric current



B. Magnetic current

Fig. 9.2. Equivalent Boundary Conditions for Thin Shell

Note the use of the average electric field since the inside and outside fields (transverse components) may be discontinuous by the surface magnetic current density. Similarly, the surface magnetic current density as indicated in fig. 9.2B is given by

$$
\begin{aligned}
\overrightarrow{1}_S(\vec{r}_s) \times \left[\tilde{\vec{E}}^{(out)}(\vec{r}_s, s) - \tilde{\vec{E}}^{(in)}(\vec{r}_s, s) \right] &= -\tilde{\vec{J}}_{s_h}(\vec{r}_s, s) \\
&= -\overleftrightarrow{G}_{s_h}(\vec{r}_s) \cdot \tilde{\vec{H}}^{(avg)}(\vec{r}_s, s) \simeq -h\,\tilde{\vec{J}}_h(\vec{r}_s, s)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\vec{J}}_{s_h}(\vec{r}_s, s) &\equiv \text{surface magnetic current density on } S \\
\tilde{\overleftrightarrow{G}}_{s_h}(\vec{r}_s) &= s\,L_s(\vec{r}_s) \equiv \text{ sheet magnetic admittance} \\
\overleftrightarrow{L}_s(\vec{r}_s) &= d\,\overleftrightarrow{1}_S(\vec{r}_s) \cdot \left[\overleftrightarrow{\mu}(\vec{r}_s) - \mu_0 \overleftrightarrow{1} \right] \cdot \overleftrightarrow{1}_S(\vec{r}_s) \equiv \text{sheet inductance} \\
\tilde{\vec{H}}^{(avg)}(\vec{r}_s, s) &\simeq \frac{1}{2} \left[\tilde{\vec{H}}^{(out)}(\vec{r}_s, s) + \tilde{\vec{H}}^{(in)}(\vec{r}_s, s) \right] \text{(only components parallel to S significant)}
\end{aligned}
\tag{9.4}
$$

Note here that we are concerned with the limit of small d , and the above are only approximations.

For the simpler case that the shell is non-permeable the volume integrals in (8.1) reduce to

$$
\begin{aligned}
\tilde{\vec{E}}_1(\vec{r}_1, s) &= -s_1 \mu_0 \left\langle G_0(\vec{r}_1, \vec{r}'_{s_1}),\ \overleftrightarrow{G}_{s_1}(\vec{r}'_{s_1}) \cdot \tilde{\vec{E}}_1(\vec{r}'_{s_1}) \right\rangle_S \\
\tilde{\vec{H}}_1(\vec{r}_1, s) &= \left\langle \nabla_1 G_0(\vec{r}_1, \vec{r}'_1) \underset{,}{\times} \overleftrightarrow{G}_{s_1}(\vec{r}'_{s_1}) \cdot \tilde{\vec{E}}_1(\vec{r}'_{s_1}) \right\rangle_S
\end{aligned}
\tag{9.5}
$$

Tangential $\vec{E}$ now being continuous through the shell, we need not distinguish which side of S for the integration. For our scaling as in the previous section we now have

$$
dS'_2 = \chi_1^2\ dS'_1 \quad , \quad \overleftrightarrow{G}_{s_2}(\vec{r}_{s_2}) = \chi_6\, \overleftrightarrow{G}_{s_1}(\vec{r}_{s_1})
\tag{9.6}
$$

Rewriting (9.5) for the scaled fields (subscript 2) and applying the scaling relations gives

$$
\begin{aligned}
\chi_4\, \tilde{\vec{E}}_1(\vec{r}_1, s) &= -\chi_1\, \chi_6\, \chi_4\, \chi_5\, s_1\, \mu_0 \left\langle G_0(\vec{r}_1, \vec{r}'_{s_1}) \cdot \overleftrightarrow{G}_{s_1}(\vec{r}'_{s_1}) \cdot \tilde{\vec{E}}_1(\vec{r}'_{s_1}) \right\rangle_S \\
\tilde{\vec{H}}_1(\vec{r}_1, s) &= \chi_6\, \chi_4 \left\langle \nabla_1 G_0(\vec{r}_1, \vec{r}'_{s_1}) \underset{,}{\times} \overleftrightarrow{G}_{s_1}(\vec{r}'_{s_1}) \cdot \tilde{\vec{E}}_1(\vec{r}'_{s_1}) \right\rangle_S .
\end{aligned}
\tag{9.7}
$$

This then gives the constraints among the scaling parameters

$$\chi_1 \chi_6 \chi_5 = 1 \quad , \quad \chi_6 \chi_4 = 1 \tag{9.8}$$

which are rearranged as

$$\chi_5 = \frac{1}{\chi_6 \chi_1} \quad , \quad \chi_4 = \frac{1}{\chi_6} \left(= \chi_5 \chi_1\right) \tag{9.10}$$

So now $\vec{r}$ scaling and $\overleftrightarrow{G}_s$ scaling can be independently specified to scale the frequency. Thus $s\, G_s\, a$, or better $s \mu_0\, G_s\, a$, is now invariant to the scaling, as is $\vec{E}\, G_s$, where G_s is some characteristic sheet conductance. Now we have the characteristic time as

$$\tau \equiv \mu_0\, G_s\, a \equiv \text{ characteristic time} \tag{9.10}$$

Here we can identify $\mu_0 a$ as like an inductance and G_s^{-1} as a resistance, giving an L/R time constant. This is related to the properties of this structure as a shield, for which it is not skin depth in the shield, but such an L/R time which dominates the low-frequency penetration of magnetic fields into the shield [14]. As before, the lowest natural frequency s_α is related to this and $s_\alpha \tau$ is of order 1. For non-spherical shells one will need to use the various dimensions of the structure (different choices for a) to obtain these lower natural frequencies.

As discussed in the previous section, the permeability magnitude is unchanged in the scaling. Essentially, it is μ / μ_0 or μ_r that is unchanged in magnitude (just moved $\vec{r}_1 \to \vec{r}_2$). In the present context of a thin-shell target, our attention shifts from $\overleftrightarrow{\mu}(\vec{r})$ to $\overleftrightarrow{L}_s(\vec{r}_s)$. Scaling via the volume formulas we have

$$d_2 = \chi_1\, d_1 \tag{9.11}$$

implying that $\overleftrightarrow{L}_s$, being proportional to d, scales the same way, i.e.,

$$\overleftrightarrow{L}_{s_2}(\vec{r}_{s_2}) = \chi_1\, \overleftrightarrow{L}_{s_1}(\vec{r}_{s_1}) \tag{9.12}$$

In terms of the characteristic size a, then we have $(\mu_0 a)^{-1} L_s$ invariant to the scaling, L_s being some characteristic value of the sheet inductance.

10. Pole Representation

As discussed in [3, 18] the magnetic polarizability dyadic has the SEM representation

$$\tilde{\overleftrightarrow{M}}(s) = \tilde{\overleftrightarrow{M}}(\infty) + \sum_{\alpha} M_\alpha \vec{M}_\alpha \vec{M}_\alpha [s - s_\alpha]^{-1}$$

$$\vec{M}_\alpha \cdot \vec{M}_\alpha = 1 \;, \quad \vec{M}_\alpha = \text{real vector}$$

$$M_\alpha = \text{real scalar}$$

$$s_\alpha < 0 \text{ (all negative real natural frequencies)}$$

$$\tilde{\overleftrightarrow{M}}(\infty) \equiv \text{magnetic polarizability of perfectly conducting target (negligible field penetration at sufficiently high frequencies)} \tag{10.1}$$

$$\frac{1}{S}\tilde{\overleftrightarrow{M}}(s) = \frac{1}{S}\tilde{\overleftrightarrow{M}}(0) + \sum_{\alpha} \frac{M_\alpha}{s_\alpha} \vec{M}_\alpha \vec{M}_\alpha [s - s_\alpha]^{-1}$$

$$\tilde{\overleftrightarrow{M}}(0) = \overleftrightarrow{0} \text{ for non-permeable target}$$

These various terms can be computed from the appropriate integral equations discussed previously.

In the context of the volume integral equations (3.3) we can write a supervector field (applying to incident, scattered, and total fields)

$$\left(\tilde{\vec{F}}_\ell(\vec{r},s)\right) = \left(\tilde{\vec{H}}(\vec{r},s)\right) \;, \quad N_e \tilde{\vec{E}}(\vec{r},s)$$

$$\ell = \begin{cases} 1 \Rightarrow \text{magnetic field} \\ 2 \Rightarrow \text{normalized electric field} \end{cases} \tag{10.2}$$

$$N_e = \text{normalized scalar}$$

The function of N_e is to make the electric part comparable to the magnetic part, if desired. For example, looking at (2.5) for the incident field one might choose

$$N_e = [s\mu a]^{-1}$$
$$a \equiv \text{characteristic size of target} \tag{10.3}$$

which also gives the two parts in (10.2) the same dimensions.

The volume integral equations take the general form

$$(\vec{\tilde{F}}_\ell^{(sc)}(\vec{r},s)) = \left\langle (\overleftrightarrow{\tilde{X}}_{\ell,\ell}^{(V)}(\cdot)) \;\odot\!\!, \; (\vec{\tilde{F}}_\ell(\vec{r},s)) \right\rangle_V \tag{10.4}$$

by appropriate manipulation of the terms in (3.3). Here we indicate the operator as a dyadic with multiplication in generalized dot-product sense (for supervectors and supermatrices) for dimensional consistency. Of course, the operator acts in the sense of integrals and derivatives over the volume domain V. Separating out the incident and scattered fields gives

$$\left\langle (1_{\ell,\ell'})\;\delta(\vec{r}-\vec{r}') - (\overleftrightarrow{\tilde{X}}_{\ell,\ell'}^{(V)}(\cdot)) \;\odot\!\!, \; (\vec{\tilde{F}}_\ell^{(sc)}(\vec{r}',s)) \right\rangle_V = \left\langle (\overleftrightarrow{\tilde{X}}_{\ell,\ell'}^{(V)}(\cdot)) \;\odot\!\!, \; (\vec{\tilde{F}}_\ell^{(inc)}\vec{r},s) \right\rangle \tag{10.5}$$

Matricizing this in the usual moment-method form by appropriately zoning the volume gives

$$\begin{gathered} (\tilde{F}_n^{(sc)}(s)) = (\tilde{X}_{n,m}^{(V)}(s)) \odot (\tilde{F}_n(s)) \\ \left[\left(1_{n,m}\right) - \left(\tilde{X}_{n,m}^{(V)}(s)\right) \right] \cdot (\tilde{F}_n^{(sc)}(s)) = (\tilde{F}_n^{(inc)}(s)) \end{gathered} \tag{10.6}$$

noting the six field components for each spatial location. The integration and differentiation has been subsumed in the form of the matrix elements which are only functions of s in this form.

The form of the solution is

$$\begin{aligned} \left(\vec{\tilde{F}}_\ell^{(sc)}(\vec{r},s) \right) = \sum_\alpha \eta_\alpha(\vec{1}_h) \left(\tilde{F}_\ell^{(sc)}(\vec{r}) \right)_\alpha [s-s_\alpha]^{-1} \\ + \text{ entire function} \end{aligned} \tag{10.7}$$

with the corresponding numerical form

$$\begin{aligned} \left(\tilde{F}_n^{(sc)}(s)\right) = \sum_\alpha \eta_\alpha(\vec{1}_h) \left(\tilde{F}_n^{(sc)}(s)\right)_\alpha [s-s_\alpha]^{-1} \\ + \text{ entire function} \end{aligned} \tag{10.8}$$

Following the basic development for SEM poles [12] we have

$$\left\langle (\overleftrightarrow{1}_{\ell,\ell'})\delta(\vec{r}-\vec{r}\,') \cdot (\tilde{\overleftrightarrow{X}}^{(V)}_{\ell,\ell'})(\cdot)\Big|_{s=s_\alpha} \odot_{,} (\vec{F}^{(sc)}_{\ell}(\vec{r}\,'))_\alpha \right\rangle_V = (\vec{0}_\ell)$$

$$\left[(1_{n,m}) - (\tilde{X}^{(V)}_{n,m}(s_\alpha))\right] \cdot (F^{(sc)}_n)_\alpha = (0_n)$$

natural modes

$$\det\left(\left[\left(1_{n,m}\right) - \left(\tilde{X}^{(V)}_{n,m}(s_\alpha)\right)\right]\right) = 0 \quad \text{natural frequencies}$$

$$\left\langle (\vec{L}_\ell(\vec{r}\,))_\alpha \odot_{,} (\overleftrightarrow{1}_{\ell,\ell'})\delta(\vec{r}-\vec{r}\,') - (\tilde{\overleftrightarrow{X}}^{(V)}_{\ell,\ell'})(\cdot)\Big|_{s=s_\alpha} \right\rangle_V = (\vec{0}_\ell)$$

$$(L_n)_\alpha \cdot \left[(1_{n,m}) - (\tilde{X}^{(V)}_{n,m}(s_\alpha))\right] = (0_n)$$

left modes

$$\eta_\alpha(\vec{1}_h) = -\frac{\left\langle (\vec{L}_\ell(\vec{r}\,))_\alpha \odot_{,} (\tilde{\overleftrightarrow{X}}_{\ell,\ell'}(\cdot))\Big|_{s=s_\alpha} \odot_{,} (\tilde{\vec{F}}^{(inc)}_\ell(\vec{r}\,', s_\alpha))\right\rangle_V}{\left\langle (\vec{L}_\ell(\vec{r}\,))_\alpha \odot_{,} \frac{\partial}{\partial s}(\tilde{\overleftrightarrow{X}}_{\ell,\ell'}(\cdot))\Big|_{s=s_\alpha} \odot_{,} (\tilde{\vec{F}}_\ell(\vec{r}\,'))_\alpha \right\rangle_V} \tag{10.9}$$

$$= -\frac{(L_n)_\alpha \cdot (\tilde{X}_{n,m}(s_\alpha)) \cdot (\tilde{F}^{(inc)}_n(s_\alpha))}{(L_n)_\alpha \cdot \frac{d}{ds}(\tilde{X}_{n,m}(s))\Big|_{s=s_\alpha} \cdot (F^{(sc)}_n)_\alpha}$$

$\equiv$ coupling coefficients

Here we have used the class-1 form of the coupling coefficients (no frequency dependence). This is the appropriate form for computing the terms in (10.1).

From (5.1) we have in volume integral form the magnetic-dipole residues

$$\vec{m}^{(h)}_\alpha = \eta_\alpha(\vec{1}_h)\left\{\frac{1}{2}\int_V \vec{r}\,' \times \left[\tilde{\overleftrightarrow{\sigma}}(\vec{r}\,') \cdot \left[\tilde{N}_e^{-1}(s_\alpha)\,\vec{F}^{(sc)}_{2_\alpha}(\vec{r}\,')\right]\right] dV' + \int_V \left[\frac{\overleftrightarrow{\mu}(\vec{r}\,')}{\mu_0} - \overleftrightarrow{1}\right] \cdot \vec{F}^{(sc)}_{1_\alpha}(\vec{r}\,')\, dV' \right\} \tag{10.10}$$

Forming the dyadic as in (7.3) we then find

$$\overleftrightarrow{M}_\alpha = \frac{1}{H_0}\overleftrightarrow{m}_\alpha = M_\alpha\,\vec{M}_\alpha\,\vec{M}_\alpha \tag{10.11}$$

Diagonalization of $\overleftrightarrow{M}_\alpha$ gives the orientation of $\vec{M}_\alpha$, so we can now choose $\vec{1}_h$ as

$$\vec{1}_h \equiv \vec{M}_\alpha \tag{10.12}$$

giving

$$\begin{aligned} M_\alpha = \frac{\eta_\alpha(\vec{1}_h)}{H_0}\,\vec{1}_h \cdot \Bigg\{ \frac{1}{2}\int_V \vec{r}' \times \left[\overleftrightarrow{\sigma}(\vec{r}') \cdot \left[N_e^{-1}(s_\alpha)\,\vec{F}^{(sc)}_{2_\alpha}(\vec{r}') \right]\right] dV' \\ + \int_V \left[\frac{\overleftrightarrow{\mu}(\vec{r}')}{\mu_0} - \overleftrightarrow{1} \right] \cdot \vec{F}^{(sc)}_{1_\alpha}(\vec{r}')\, dV' \Bigg\} \end{aligned} \tag{10.13}$$

As discussed in Section 7, target symmetry can be used to find the orientations of the $\vec{1}_h$ a priori.

An alternate approach uses the surface integral equations (4.9) (together with (4.7) to put things in terms of tangential fields just outside S). Then repeat the steps beginning with (10.2), noting that there are only four instead of six field components to consider. In the integral equations, this can be handed by use of $\overleftrightarrow{1}_S(\vec{r}_s)\cdot$ instead of $\vec{1}_S(\vec{r}_s)\times$. The volume integrals then become surface integrals with kernel $(\tilde{\overleftrightarrow{X}}^{(S)}_{\ell,\ell'}(\cdot))$. For the matrix (numerical) form in (10.6) we have a matrix $(\tilde{X}^{(S)}_{n,m}(s))$ resulting from zoning S instead of V. This is followed through (10.9). The magnetic dipole moment is now expressed by surface integrals of the magnetic field just outside S, giving residues

$$\begin{aligned} \vec{m}^{(h)}_\alpha = \eta_\alpha(\vec{1}_h) \Bigg\{ \frac{1}{2}\int_S \vec{r}'_s \times \left[\vec{1}_S(\vec{r}'_s) \times \vec{F}^{(sc)}_{1_\alpha}(\vec{r}'_s) \right] dS' \\ + \int_S \left[\vec{1}_S(\vec{r}'_s) \cdot \vec{F}^{(sc)}_{1_\alpha}(\vec{r}'_s) \right] \vec{r}'_s \, dS' \Bigg\} \end{aligned} \tag{10.14}$$

The normal component can be found from (4.7) as

$$\begin{aligned} \vec{1}_S(\vec{r}'_s) \cdot \vec{F}^{(sc)}_{1_\alpha}(\vec{r}'_s) &= \frac{1}{s\mu_0} \nabla_s \cdot \left[\vec{1}_S(\vec{r}'_s) \times \left[\tilde{N}_e^{-1}(s_\alpha)\,\vec{F}^{(sc)}_{2_\alpha}(\vec{r}'_s) \right]\right] \\ &= \frac{1}{s\mu_0 \tilde{N}_e(s_\alpha)} \nabla_s \cdot \left[\vec{1}_S(\vec{r}'_s) \times \vec{F}^{(sc)}_{2_\alpha}(\vec{r}'_s) \right] \end{aligned} \tag{10.15}$$

Following steps (10.11) and (10.12) we then have

$$M_\alpha = \frac{\eta_\alpha(\vec{1}_h)}{H_0}\,\vec{1}_h \cdot \left\{ \frac{1}{2}\int_S \vec{r}'_s \times \left[\vec{1}_S(\vec{r}'_s) \times \vec{F}_{1_\alpha}^{(sc)}(\vec{r}'_s) \right] dS' + \int_S \left[\vec{1}_S(\vec{r}'_s) \cdot \vec{F}_{1_\alpha}^{(sc)}(\vec{r}'_s) \right] \vec{r}'_s \, dS' \right\} \tag{10.16}$$

In general, these integrals will have to be cast in numerical form for computation of typical-target parameters.

11. Concluding Remarks

As we have seen, construction of the magnetic polarizability and associated pole terms for MSI can be accomplished in more than one way, depending on the use of volume or surface integral equations. For the case of uniform, isotropic targets the two approaches involving, on the one hand volume integrals of electric and magnetic currents, and on the other hand integration over the magnetic field just exterior to the target, have been shown to be equivalent.

The reader should note the approximations involved when using these results to model experimental data. Besides the basic MSI approximations for a quasi-magnetostatic analysis, other phenomena may be present. Ferromagnetic targets may possibly be nonlinear (depending on field strengths) and may have some initial magnetization. Furthermore, we have assumed for this analysis that $\overleftrightarrow{\mu}(\vec{r})$ is independent of frequency, whereas we may need a more general $\tilde{\overleftrightarrow{\mu}}(\vec{r},s)$ to better account for the magnetic properties. If one has such a better model of the permeability, this can be used in the foregoing integral equations to obtain the various pole parameters.

As the reader may suspect, the present paper is leading in the direction of numerical computation of the MSI parameters for various targets. These may be canonical targets (such as analytically calculable shapes for accuracy comparison), or shapes approximating real targets of interest. This will require matricizing the integral equations using standard moment-method techniques [7-10, 15]. Our present plans are to first consider bodies of revolution due to the numerical simplifications associated with the symmetry, while still being able to compute some shapes of interest.

Appendix A. Some Useful Identities

Here we summarize some identities that are useful in understanding this paper. More extensive lists are found in [13]. First we have the usual Cartesian (x, y, z) and spherical (r, θ, ϕ) coordinates, with

$$
\begin{aligned}
\vec{r} &= r\,\vec{1}_r = x\,\vec{1}_x + y\,\vec{1}_y + z\,\vec{1}_z \\
\overleftrightarrow{1} &= \vec{1}_x\vec{1}_x + \vec{1}_y\vec{1}_y + \vec{1}_z\vec{1}_z = \vec{1}_r\vec{1}_r + \vec{1}_\theta\vec{1}_\theta + \vec{1}_\phi\vec{1}_\phi \\
&\equiv \text{identity dyadic} \\
\overleftrightarrow{1}_r &= \overleftrightarrow{1} - \vec{1}_r\vec{1}_r = \vec{1}_\theta\vec{1}_\theta + \vec{1}_\phi\vec{1}_\phi \\
&\equiv \text{transverse identity dyadic with respect to } \vec{1}_r
\end{aligned}
\tag{A.1}
$$

Considering the coordinate vector $\vec{r}$ we have

$$
\begin{aligned}
&\nabla\vec{r} = \overleftrightarrow{1}\ ,\quad \nabla\cdot\vec{r} = 3 \text{ (space dimension)}\ ,\quad \nabla\times\vec{r} = \vec{0} \\
&\nabla(\vec{r}\cdot\vec{r}) = \nabla(r^2) = 2\,\vec{r}\ ,\quad \nabla r = \vec{1}_r\ (r \neq 0) \\
&\nabla\cdot\vec{1}_r = \frac{2}{r}
\end{aligned}
\tag{A.2}
$$

Some vector/dyadic relationships include

$$
\begin{aligned}
&\overleftrightarrow{1}\times\vec{u} = \vec{u}\times\overleftrightarrow{1} = -\left[\overleftrightarrow{1}\times\vec{u}\right]^T \quad \text{(general antisymmetric dyadic)} \\
&\vec{u}\times\left[\vec{v}\times\vec{w}\right] = \vec{u}\cdot\left[\vec{w}\,\vec{v} - \vec{v}\,\vec{w}\right] = -\left[\vec{w}\,\vec{v} - \vec{v}\,\vec{w}\right]\cdot\vec{u} \\
&\left[\overleftrightarrow{A}\times\vec{u}\right]^T = -\vec{u}\times\overleftrightarrow{A}^T \\
&\overleftrightarrow{A}\cdot\left[\vec{u}\times\vec{v}\right] = \left[\overleftrightarrow{A}\times\vec{u}\right]\cdot\vec{v} = -\left[\overleftrightarrow{A}\times\vec{v}\right]\cdot\vec{u}
\end{aligned}
\tag{A.3}
$$

Involving the del operator we also have

$$\nabla\left[\vec{u}(\vec{r})\cdot\vec{v}(\vec{r})\right] = \vec{u}(\vec{r})\times\left[\nabla\times\vec{v}(\vec{r})\right] + \vec{v}(\vec{r})\times\left[\nabla\times\vec{u}(\vec{r})\right]$$
$$+\ \vec{u}(\vec{r})\cdot\left[\nabla\,\vec{v}(\vec{r})\right] + \vec{v}(\vec{r})\cdot\left[\nabla\,\vec{u}(\vec{r})\right]$$
$$\nabla\cdot\left[\vec{u}(\vec{r})\,\vec{v}(\vec{r})\right] = \left[\nabla\cdot\vec{u}(\vec{r})\right]\vec{v}(\vec{r}) + \vec{u}(\vec{r})\cdot\left[\nabla\,\vec{v}(\vec{r})\right] \tag{A.4}$$
$$\nabla\times\left[\vec{u}(\vec{r})\,\vec{v}(\vec{r})\right] = \vec{u}(\vec{r})\left[\nabla\cdot\vec{v}(\vec{r})\right] - \vec{v}\left[\nabla\cdot\vec{u}(\vec{r})\right]$$
$$+\ \vec{v}(\vec{r})\cdot\left[\nabla\,\vec{u}(\vec{r})\right] - \vec{u}(\vec{r})\cdot\left[\nabla\,\vec{v}(\vec{r})\right]$$

Letting certain of these vector functions be $\vec{r}$ we have the special cases

$$\nabla\left[\vec{r}\cdot\vec{v}(\vec{r})\right] = \vec{r}\times\left[\nabla\times\vec{v}(\vec{r})\right] + \vec{r}\cdot\left[\nabla\,\vec{v}(\vec{r})\right] + \vec{v}(\vec{r})$$
$$\nabla\cdot\left[\vec{v}(\vec{r})\,\vec{r}\right] = \left[\nabla\cdot\vec{v}(\vec{r})\right]\vec{r} + \vec{v}(\vec{r})$$
$$\nabla\times\left[\vec{r}\times\vec{v}(\vec{r})\right] = \left[\nabla\cdot\vec{v}(\vec{r})\right]\vec{r} - 2\,\vec{v}(\vec{r}) - \vec{r}\cdot\left[\nabla\,\vec{v}(\vec{r})\right] \tag{A.5}$$
$$= \nabla\cdot\left[\vec{v}(\vec{r})\,\vec{r}\right] - 3\,\vec{v}(\vec{r}) - \vec{r}\cdot\left[\nabla\,\vec{v}(\vec{r})\right]$$

These further simplify if $\vec{v}$ takes on special forms (e.g., zero divergence).

There are the "Gauss' theorems"

$$\int_V \nabla f(\vec{r})\,dV = \int_S \vec{1}_S(\vec{r}_S)\; f(\vec{r}_S)\,dS$$
$$\int_V \nabla\cdot\vec{v}(\vec{r})\,dV = \int_S \vec{1}_S(\vec{r}_S)\cdot\vec{v}(\vec{r}_S)\,dS \tag{A.6}$$
$$\int_V \nabla\times\vec{v}(\vec{r})\,dV = \int_S \vec{1}_S(\vec{r}_S)\times\vec{v}(\vec{r})\,dS$$

where S is a closed surface bounding a volume V and suitable conditions are imposed on the derivatives and where

$$\vec{r}_S \in S \quad (\vec{r} \text{ on } S)$$
$$\vec{1}_S(\vec{r}_S) \equiv \text{outward pointing unit vector at } \vec{r}_S \tag{A.7}$$

These formulas are also extended to dyadics as

$$\int_V \nabla \vec{v}(\vec{r})\, dV = \int_S \vec{1}_S(\vec{r}_S)\ \vec{v}(\vec{r}_S)\, dS$$
$$\int_V \nabla \cdot \overleftrightarrow{A}(\vec{r})\, dV = \int_S \vec{1}_S(\vec{r}_S) \cdot \overleftrightarrow{A}(\vec{r}_S)\, dS \tag{A.8}$$
$$\int_V \nabla \times \overleftrightarrow{A}(\vec{r})\, dV = \int_S \vec{1}_S(\vec{r}_S) \times \overleftrightarrow{A}(\vec{r}_S)\, dS$$

Appendix B. Integrals over the Unit Sphere

Consider a closed surface S bounding a volume V, and a unit sphere with surface S_1 bounding a volume V_1 for which we have

$$\begin{aligned} S &\equiv \text{area of } S \\ S_1 &\equiv \text{area of } S_1 = 4\pi \\ V &\equiv \text{volume of } V \\ V_1 &\equiv \text{volume of } V_1 = @\,\pi \end{aligned} \tag{B.1}$$

Then we have

$$\begin{aligned} &\int_S dS = S \quad , \quad \int_V dV = V \\ &\int_{S_1} dS = S_1 = 4\pi \quad , \quad \int_{V_1} dV = V_1 = \frac{4}{3}\pi \end{aligned} \tag{B.2}$$

Including any constant scalar, vector, or dyadic in the integrand of such integrals merely multiplies such a term by the above areas and volumes as appropriate.

Using the divergence theorem we have

$$\int_S \vec{v}_1(\vec{r}) \cdot \vec{1}_S(\vec{r})\, dS = \int_V \nabla \cdot \vec{v}_1(\vec{r})\, dV \tag{B.3}$$

Taking the choice of constant $\vec{v}$ we have

$$\begin{aligned} \vec{v}_1 &\equiv \text{constant vector} \\ \nabla \cdot \vec{v}_1 &= 0 \\ \vec{v}_1 \cdot \int_S \vec{1}_S(\vec{r})\, dS &= 0 \end{aligned} \tag{B.4}$$

Letting $\vec{v}_1$ be any vector, say successively choosing it as unit vectors $\vec{1}_x, \vec{1}_y, \vec{1}_z$ gives

$$\int_S \vec{1}_S(\vec{r})\, dS = \vec{0} \tag{B.5}$$

On S_1 (of radius 1 centered on $\vec{r} = \vec{0}$) we have

$$\begin{aligned} \vec{1}_S(\vec{r}) &= \vec{1}_r \\ \int_{S_1} \vec{1}_r(\vec{r})\, dS &= \vec{0} \end{aligned} \tag{B.6}$$

A related dyadic formula is

$$\int_V \nabla \vec{v}_2(\vec{r})\, dV = \int_S \vec{1}_S(\vec{r})\, \vec{v}_2(\vec{r})\, dS \tag{B.7}$$

Taking the choice of $\vec{v}_2$ as $\vec{r}$ we have

$$\begin{aligned} \vec{v}_2 &\equiv \vec{r} \\ \nabla \vec{r} &= \overleftrightarrow{1} \\ \int_V \overleftrightarrow{1}\, dV &= \int_S \vec{1}_S(\vec{r})\, \vec{r}\, dS \end{aligned} \tag{B.8}$$

giving

$$\int_S \vec{1}_S(\vec{r})\, \vec{r}\, dS = V\, \overleftrightarrow{1} \tag{B.9}$$

On S_1 we have

$$\begin{aligned} \vec{1}_S(\vec{r}) &= \vec{1}_r = \vec{r} \\ \int_{S_1} \vec{1}_r\, \vec{1}_r\, dS &= \frac{4}{3}\pi\, \overleftrightarrow{1} \end{aligned} \tag{B.10}$$

Combining (B.10) with (B.2) gives

$$\int_{S_1} \overleftrightarrow{1}_r \, dS = \int_{S_1} \left[\overleftrightarrow{1} - \vec{1}_r \, \vec{1}_r \right] dS = \int_{S_1} \left[\vec{1}_\theta \, \vec{1}_\theta + \vec{1}_\phi \, \vec{1}_\phi \right] dS$$
$$= 4\pi \overleftrightarrow{1} - \frac{4}{3}\pi \overleftrightarrow{1} = \frac{8}{3}\pi \overleftrightarrow{1} \tag{B.11}$$

Using similar manipulations substituting $(\vec{r} \cdot \vec{r})^n$ for scalar functions and $(\vec{r} \cdot \vec{r})^n \vec{r}$ for vector functions, various other results can be obtained.

From the various identities in (A.5) one can apply the "Gauss theorems" in (A.6) to obtain some useful integrals. From the second of (A.5) we have

$$\begin{aligned}\int_V \vec{v}(\vec{r})\,dV &= \int_V \nabla\cdot\left[\vec{v}(\vec{r})\,\vec{r}\right]dV - \int_V \left[\nabla\cdot\vec{v}(\vec{r})\right]\vec{r}\,dV \\ &= \int_S \left[\vec{1}_S(\vec{r}_s)\cdot\vec{v}(\vec{r}_s)\right]\vec{r}_s\,dS - \int_V \left[\nabla\cdot\vec{v}(\vec{r})\right]\vec{r}\,dV \\ &= \int_S \left[\vec{1}_S(\vec{r}_s)\cdot\vec{v}(\vec{r}_s)\right]\vec{r}_s\,dS \quad \text{if} \quad \nabla\cdot\vec{v}(\vec{r}) = 0\end{aligned} \tag{C.1}$$

From the third of (A.5) followed by the second of (A.3) we have

$$\begin{aligned}\int_V \vec{r}\cdot\left[\nabla\vec{v}(\vec{r})\right]dV &= -\int_V \nabla\times\left[\vec{r}\times\vec{v}(\vec{r})\right]dV + \int_V \nabla\cdot\left[\vec{v}(\vec{r})\,\vec{r}\right]dV \\ &\qquad -3\int_V \vec{v}(\vec{r})\,dV \\ &= -\int_S \vec{1}_S(\vec{r}_s)\times\left[\vec{r}_s\times\vec{v}(\vec{r}_s)\right]dS + \int_S \left[\vec{1}_S(\vec{r}_s)\cdot\vec{v}(\vec{r}_s)\right]\vec{r}_s\,dV \\ &\qquad -3\int_V \vec{v}(\vec{r})\,dV \\ &= \int_S \left[\vec{1}_S(\vec{r}_s)\cdot\vec{r}_s\right]\vec{v}(\vec{r}_s)\,dS - 3\int_V \vec{v}(\vec{r})\,dV\end{aligned} \tag{C.2}$$

From the first of (A.5) and (C.2) we have

$$\begin{aligned}\int_V \vec{r}\times\left[\nabla\times\vec{v}(\vec{r})\right]dV &= \int_V \nabla\left[\vec{r}\cdot\vec{v}(\vec{r})\right]dV - \int_V \vec{r}\cdot\left[\nabla\vec{v}(\vec{r})\right]dV - \int_V \vec{v}(\vec{r})\,dV \\ &= \int_S \left[\vec{r}_s\cdot\vec{v}(\vec{r}_s)\right]\vec{1}_S(\vec{r}_s)\,dS - \int_S \left[\vec{1}_S(\vec{r}_s)\cdot(\vec{r}_s)\right]\vec{v}(\vec{r}_s)\,dS + 2\int_V \vec{v}(\vec{r})\,dr \\ &= \int_S \vec{r}_s\times\left[\vec{1}_S(\vec{r}_s)\times\vec{v}(\vec{r}_s)\right]dS + 2\int_V \vec{v}(\vec{r})\,dV\end{aligned} \tag{C.3}$$

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