Interaction Notes

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Decomposition of the Backscattering Dyadic

Carl E. Baum
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Abstract

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1. Introduction

In remote sensing via scattered electromagnetic waves one attempts to find out as much about the target as he can. For given directions of incidence and scattering this process is described by a scattering dyadic or matrix [1], which in the right coordinates is basically a 2 x 2 matrix. The information we seek is in this scattering dyadic and we would like to know how much information we can get out of this.

Consider an incident wave as indicated in fig. 1.1 with

$$\tilde{\vec{E}}^{(inc)}(\vec{r},s) = E_0 \tilde{f}(s) \vec{1}_p e^{-\tilde{\gamma}\vec{1}_i \cdot \vec{r}}$$

$$\tilde{\vec{H}}^{(inc)}(\vec{r},s) = \frac{E_0}{Z_0} \tilde{f}(s) \vec{1}_i \times \vec{1}_p e^{-\tilde{\gamma}\vec{1}_i \cdot \vec{r}}$$

$\vec{1}_i \equiv$ direction of incidence, $\vec{1}_p \equiv$ polarization, $\vec{1}_i \cdot \vec{1}_p = 0$

$\sim \equiv$ Laplace transform (two-sided) over time t

$s \equiv \Omega + j\omega \equiv$ Laplace-transform variable or complex frequency

$\gamma \equiv \frac{s}{c} = s[\mu_0 \varepsilon_0]^{\frac{1}{2}} \equiv$ propagation constant (free space)

$$Z_0 \equiv \left[\frac{\mu_0}{\varepsilon_0}\right]^{\frac{1}{2}} \equiv \text{wave impedance (free space)} \tag{1.1}$$

$\mu_0 \equiv$ permeability (free space)

$\varepsilon_0 \equiv$ permittivity (free space)

$f(t) \equiv$ incident waveform

In time domain the incident wave takes the form

$$\vec{E}^{(inc)}(\vec{r},t) = E_0 f\left(t - \frac{\vec{1}_i \cdot \vec{r}}{c}\right) \vec{1}_p$$

$$\vec{H}^{(inc)}(\vec{r},t) = \frac{E_0}{Z_0} f\left(t - \frac{\vec{1}_i \cdot \vec{r}}{c}\right) \vec{1}_i \times \vec{1}_p \tag{1.2}$$

where $\vec{1}_i$ and $\vec{1}_p$ are taken real and time invariant.

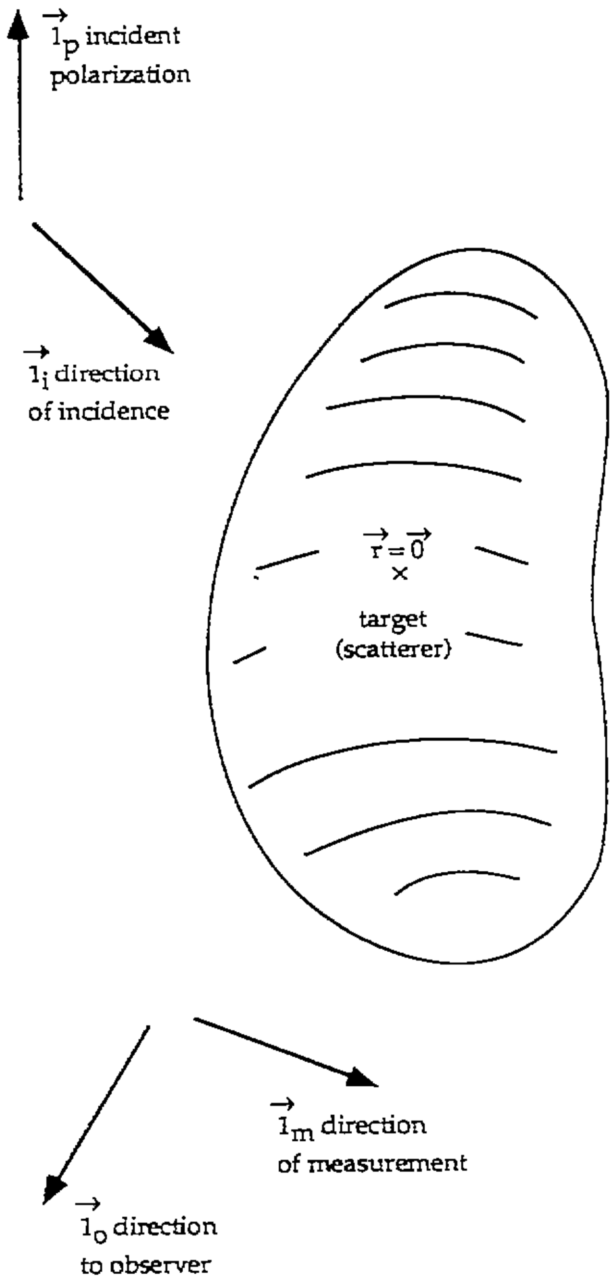


Fig. 1.1 Target with Incident and Scattered Waves

Let $\vec{r} = \vec{0}$ (coordinate center) be taken as some convenient position near the target (say, the center of the minimum circumscribing sphere). The scattered field in the far field takes the form

$$
\begin{aligned}
\tilde{\vec{E}}_f(\vec{r},s) &= \frac{e^{-\gamma r}}{4\pi r}\,\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o,\vec{1}_i;s)\cdot\tilde{\vec{E}}^{(inc)}(\vec{0},s)\,,\ \tilde{\vec{E}}_f(\vec{r},s)\cdot\vec{1}_o = 0 \\
\tilde{\vec{H}}_f(\vec{r},s) &= \frac{1}{Z_0}\vec{1}_o\,\tilde{\vec{E}}_f(\vec{r},s) \\
\vec{1}_o &= \text{direction to observer},\ \vec{1}_m = \text{direction of measurement} \\
\vec{1}_o\cdot\vec{1}_m &= 0 \text{ (only transverse fields to measure)} \\
r &\equiv |\vec{r}| \\
\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o,\vec{1}_i;s) &= \tilde{\overleftrightarrow{\Lambda}}^T(-\vec{1}_i,-\vec{1}_o;s) \equiv \text{scattering dyadic}
\end{aligned}
\tag{1.3}
$$

In time domain this takes the form

$$
\begin{aligned}
\vec{E}_f(\vec{r},t) &= \frac{1}{4\pi r}\overleftrightarrow{\Lambda}(\vec{1}_o,\vec{1}_i;s)\circ\vec{E}^{(inc)}\left(\vec{0},t-\frac{\vec{1}_o\cdot\vec{r}}{c}\right) \\
\vec{H}_f(\vec{r},t) &= \frac{1}{Z_0}\vec{1}_o\times\vec{E}_f(\vec{r},s) \\
&\circ \equiv \text{convolution with respect to time}
\end{aligned}
\tag{1.4}
$$

where $\vec{1}_o$ is taken real and time invariant.

In time domain the scattering dyadic becomes a convolution operator, but one can consider $\overleftrightarrow{\Lambda}$ (without convolution) as a dyadic delta-function scattering response. Note in time-domain form that

$$
\overleftrightarrow{\Lambda}(\vec{1}_o,\vec{1}_i;t) = \overleftrightarrow{\Lambda}^T(-\vec{1}_i,-\vec{1}_o;t) \tag{1.5}
$$

is real valued, scattering real incident electric field into real scattered far electric field.

Our interest in this paper is primarily in backscattering, for which

$$\vec{1}_o = -\vec{1}_i$$

$$\tilde{\overleftrightarrow{\Lambda}}_b(\vec{1}_i, s) \equiv \tilde{\overleftrightarrow{\Lambda}}(-\vec{1}_i, \vec{1}_i; s) = \tilde{\overleftrightarrow{\Lambda}}_b^T(\vec{1}_i, s) \tag{1.6}$$
$$= \text{complex symmetric dyadic (not Hermitian)}$$

In time domain, this is

$$\overleftrightarrow{\Lambda}_b(\vec{1}_i, t) = \overleftrightarrow{\Lambda}_b^T(\vec{1}_i, t) \tag{1.7}$$
$$= \text{real symmetric dyadic (special case of Hermitian)}$$

For the case of backscattering, it is convenient to consider these dyadics (matrices) as 2 x 2, since there are no components involving the direction $\vec{1}_i$. In the usual h, v radar coordinates we have

$$\vec{1}_i \times \vec{1}_v = \vec{1}_h \quad , \quad \vec{1}_v \cdot \vec{1}_i = 0 \tag{1.8}$$

so that $(\vec{1}_h, \vec{1}_v, -\vec{1}_i)$ forms a right-handed set of coordinate directions. Here, h stands for "horizontal" (as in horizon) and v for "vertical". So, in 2 x 2 form we can write (suppressing $\vec{1}_i$)

$$\tilde{\overleftrightarrow{\Lambda}}_b(s) = \begin{pmatrix} \tilde{\Lambda}_{b_{h,h}}(s) & \tilde{\Lambda}_{b_{h,v}}(s) \\ \tilde{\Lambda}_{b_{v,h}}(s) & \tilde{\Lambda}_{b_{v,v}}(s) \end{pmatrix} \tag{1.9}$$
$$\tilde{\Lambda}_{b_{h,v}}(s) = \tilde{\Lambda}_{b_{v,h}}(s)$$

Thus, there are three complex numbers (for any given s) which characterize this matrix. In time domain, we have

$$\overleftrightarrow{\Lambda}_b(t) = \begin{pmatrix} \tilde{\Lambda}_{b_{h,h}}(t) & \tilde{\Lambda}_{b_{h,v}}(t) \\ \tilde{\Lambda}_{b_{v,h}}(t) & \tilde{\Lambda}_{b_{v,v}}(t) \end{pmatrix} \tag{1.10}$$
$$\Lambda_{b_{h,v}}(t) = \tilde{\Lambda}_{b_{v,h}}(t)$$

which is characterized by three real numbers (for any given t).

2. Diagonalization of the Backscattering Dyadic

So now diagonalize the frequency-domain form of the backscattering dyadic as

$$\tilde{\overleftrightarrow{\Lambda}}_b(s) = \sum_{\beta=1}^{2} \tilde{\lambda}_\beta(s)\, \tilde{\overrightarrow{1}}_\beta(s)\, \tilde{\overrightarrow{1}}_\beta(s)$$

$$\tilde{\overleftrightarrow{\Lambda}}_b(s) \cdot \tilde{\overrightarrow{1}}_\beta(s) = \tilde{\overrightarrow{1}}_\beta(s) \cdot \tilde{\overleftrightarrow{\Lambda}}_b(s) = \tilde{\lambda}_\beta(s)\, \tilde{\overrightarrow{1}}_\beta(s) \tag{2.1}$$

$$\tilde{\overrightarrow{1}}_{\beta_1}(s) \cdot \tilde{\overrightarrow{1}}_{\beta_1}(s) = 1_{\beta_1,\beta_2} = \begin{cases} 1 \text{ for } \beta_1 = \beta_2 \\ 0 \text{ for } \beta_1 \neq \beta_2 \end{cases} \quad \text{(orthonormal)}$$

Note that the left and right eigenvectors are the same for a symmetric dyadic (matrix). These vectors are in general two-component complex. As discussed in Appendix A the above is a valid representation except in the special case that

$$\tilde{\Lambda}_{b_{h,h}}(s) - \tilde{\Lambda}_{b_{v,v}}(s) = \pm j\, 2\, \tilde{\Lambda}_{b_{h,v}}(s) \neq 0 \tag{2.2}$$

Dot products can be used to define direction cosines in the two-dimensional h, v space as

$$\tilde{\overrightarrow{1}}_\beta = \cos\left(\tilde{\psi}'_\beta\right) \tilde{\overrightarrow{1}}_h + \cos\left(\tilde{\psi}'_\beta\right) \overrightarrow{1}_v$$

$$\cos\left(\tilde{\psi}_\beta\right) \equiv \tilde{\overrightarrow{1}}_\beta \cdot \overrightarrow{1}_h \ , \ \cos\left(\tilde{\psi}'_\beta\right) \equiv \tilde{\overrightarrow{1}}_\beta \cdot \overrightarrow{1}_v \tag{2.3}$$

From this we have

$$\overrightarrow{1}_\beta \cdot \overrightarrow{1}_\beta = 1 = \cos^2\left(\tilde{\psi}_\beta\right) + \cos^2\left(\tilde{\psi}'_\beta\right)$$

$$\tilde{\psi}'_\beta = \tilde{\psi}_\beta \pm \frac{\pi}{2}$$

$$\overrightarrow{1}_1 \cdot \overrightarrow{1}_2 = 0 = \cos\left(\tilde{\psi}_1\right)\cos\left(\tilde{\psi}_2\right) + \sin\left(\tilde{\psi}_1\right)\sin\left(\tilde{\psi}_2\right) \tag{2.4}$$

$$= \cos\left(\tilde{\psi}_1 - \tilde{\psi}_2\right)$$

$$\tilde{\psi}_2 - \tilde{\psi}_1 = \pm\frac{\pi}{2} \ , \ \pm\frac{3\pi}{2}$$

Taking $\tilde{\psi}_\beta$ (in general, complex) as the angle with respect to the positive h axis we have

$$\tilde{\overrightarrow{1}}_\beta = \cos\left(\tilde{\psi}_\beta\right) \overrightarrow{1}_h + \sin\left(\tilde{\psi}_\beta\right) \overrightarrow{1}_v \tag{2.5}$$

Furthermore take the convention

$$\tilde{\psi}_2(s) = \tilde{\psi}_1(s) + \frac{\pi}{2} \tag{2.6}$$

However, there is still a sign ambiguity on $\vec{1}_\beta$ since it can be replaced by $-\vec{1}_\beta$ in (2.1) with no change. One can adopt a convention that

$$0 \le \tilde{\psi}_1(s) < \pi \quad , \quad \frac{\pi}{2} \le \tilde{\psi}_2(s) < \frac{3\pi}{2} \tag{2.7}$$

This still leaves the problem as to which is "1" and which is "2". One can let this be defined by

$$\left|\tilde{\lambda}_1(s)\right| > \left|\tilde{\lambda}_2(s)\right| \ge 0 \tag{2.8}$$

An exception to this is the case when the magnitudes are equal, making it difficult to decide the ordering. Furthermore, if the two eigenvalues are equal, except in the case of (2.2), then we have

$$\tilde{\overleftrightarrow{\Lambda}}_b(s) = \tilde{\lambda}(s)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \ , \quad \tilde{\lambda}(s) = \tilde{\lambda}_1(s) = \tilde{\lambda}_2(s) \tag{2.9}$$

and the angle $\tilde{\psi}_1$ (and hence $\tilde{\psi}_2$) is undetermined. Also since the $\tilde{\lambda}_\beta(s)$ are, in general, functions of frequency, then as one varies frequency the definitions of "1" and "2" in (2.8) may interchange. So one may adopt a different convention (in broadband) to keep track of the eigenvalue variation.

For distinct eigenvalues $\tilde{\psi}_1$ (and hence $\tilde{\psi}_2$) is determined and can be considered as some characteristic of the target. The backscattering dyadic as in (1.9) is characterized by three complex numbers. In the diagonal form in (2.1) there are still three distinct complex numbers $\tilde{\lambda}_1(s)$, $\tilde{\lambda}_2(s)$, and $\tilde{\Psi}_1(s)$. What can we tell about the target from these? This has both narrowband and broadband aspects.

The reader can note that the above diagonalization is not the only way to decompose the scattering matrices. Various other bases, such as the Pauli spin matrices are also used [6, 10].

3. Form of the Backscattering Dyadic for Canonical Scattering Models

So as to better understand the relation of the backscattering dyadic to the target, let us consider the form this dyadic takes for three kinds of targets. For the moment we are considering that measurements are being taken at some fixed frequency $s = j\omega$.

3.1 Line scatterer

Our first example is defined by

$$\left|\tilde{\lambda}_1(s)\right| >> \left|\tilde{\lambda}_2(s)\right| \simeq 0 \tag{3.1}$$

so that, to a good approximation, the scattering dyadic has only one non-zero eigenvalue. This is easily tested via [8]

$$\det(\tilde{\overleftrightarrow{\Lambda}}_b(s)) \simeq 0 \quad , \quad \mathrm{tr}(\tilde{\overleftrightarrow{\Lambda}}_b(s)) \simeq \tilde{\lambda}_1(s) \tag{3.2}$$

or, more realistically, small in magnitude compared to one or more matrix elements.

In this case we can write

$$\tilde{\overleftrightarrow{\Lambda}}_b(s) = \tilde{\lambda}_1(s)\, \vec{\tilde{1}}_1(s)\, \vec{\tilde{1}}_1(s) = \tilde{\lambda}_1(s) \begin{pmatrix} \cos^2(\tilde{\psi}_1(s)) & \cos(\tilde{\psi}_1(s))\sin(\tilde{\psi}_1(s)) \\ \cos(\tilde{\psi}_1(s))\sin(\tilde{\psi}_1(s)) & \sin^2(\tilde{\psi}_1(s)) \end{pmatrix} \tag{3.3}$$

Noting that $\vec{\tilde{1}}_1$ is characterized by the angle $\tilde{\psi}_1$, then the scattering dyadic is characterized by two complex numbers, $\tilde{\lambda}_1$ giving the amplitude and $\tilde{\psi}_1$ giving the direction.

As illustrated in fig. 3.1, if $\tilde{\psi}_1$ is a real angle it defines a direction in h, v space. Physically this could correspond to a long, thin target aligned in this direction. Note that the substitution

$$\vec{\tilde{1}}_1(s) \to -\vec{\tilde{1}}_1(s) \quad , \quad \tilde{\psi}_1(s) \to \tilde{\psi}_1(s) \pm \pi \tag{3.4}$$

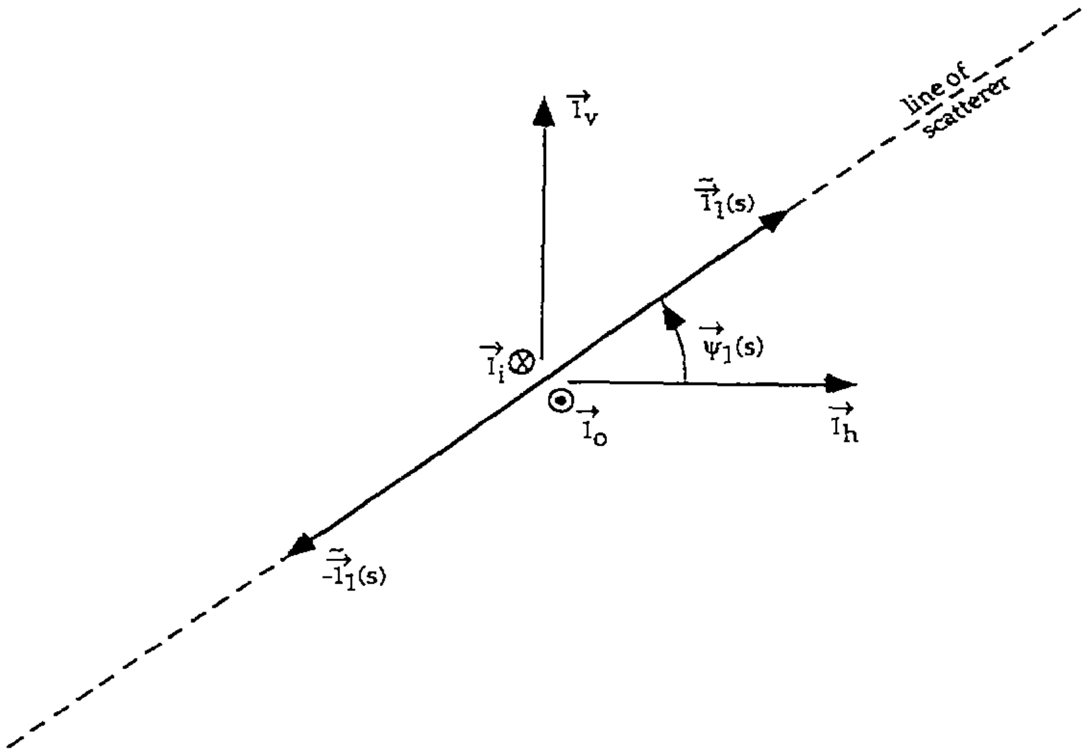


Fig. 3.1. Line Scatterer

leaves (3.3) invariant, so that the alignment of the target is equally well characterized by both directions. For synthetic-aperture-radar (SAR) images one might consider characterizing the pixels, paths, and/or areas by double headed arrows in cases that (3.3) applies. Thereby one may identify some geometric properties of the scatterers, both individually and collectively.

Of course, $\tilde{\psi}_1$ may be complex. If there is a significant imaginary part, one may have to interpret the physical properties differently. One may also consider how to add this imaginary part to a SAR image.

3.2 Rotational scatterer

Our second example is defined by

$$\tilde{\lambda}_1(s) = \tilde{\lambda}_2(s) = \tilde{\lambda}(s) \tag{3.5}$$

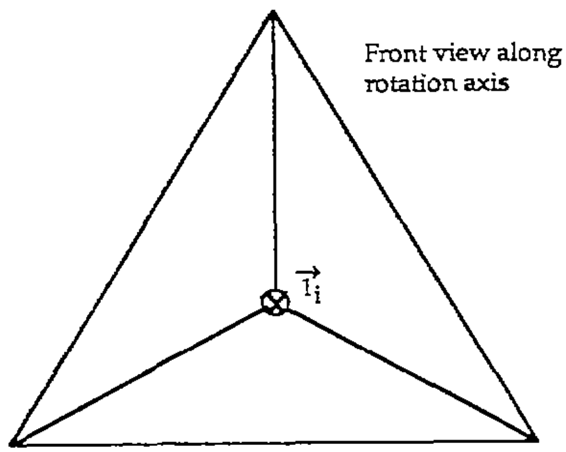
at least to a good approximation. Then the backscattering dyadic becomes

$$\begin{aligned} &\tilde{\overleftrightarrow{\Lambda}}_b(s) = \tilde{\lambda}(s)\,\overleftrightarrow{1}_i \\ &\overleftrightarrow{1}_i = \overrightarrow{1}_h\overrightarrow{1}_h + \overrightarrow{1}_v\overrightarrow{1}_v = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \equiv \text{identity (transverse to } \overrightarrow{1}_i \text{ in two dimentional form)} \\ &\det(\tilde{\overleftrightarrow{\Lambda}}_b(s)) = \tilde{\lambda}^2(s) \quad , \quad \mathrm{tr}(\tilde{\overleftrightarrow{\Lambda}}_b(s)) = 2\tilde{\lambda}(s) \end{aligned} \tag{3.6}$$

which is characterized by a single complex number. Physically, this means no depolarization. If the incident wave has some polarization $\overrightarrow{1}_{inc}$ in the h, v system the received scattered wave will be polarized in the same direction (noting the generally complex coefficient $\tilde{\lambda}$).

Various types of targets have this signature. Quite generally any scatterer with C_N symmetry (N-fold rotation axis along $\overrightarrow{1}_i$) for $N \geq 3$ has this property [8, 9]. This belongs to much more than spheres (although for spheres this applies to all $\overrightarrow{1}_i$), or in general form to a target with O_3 symmetry (orthogonal group in three dimensions). It applies as well to more general structures such as propellers (with at least three blades). So, for a label, let us refer to this class of scatterers as "rotational".

Figure 3.2 shows an example of a trihedral corner reflector with C_{3a} symmetry (containing, in addition, three axial symmetry planes). Note that this condition applies for $\overrightarrow{1}_i$ aligned along the



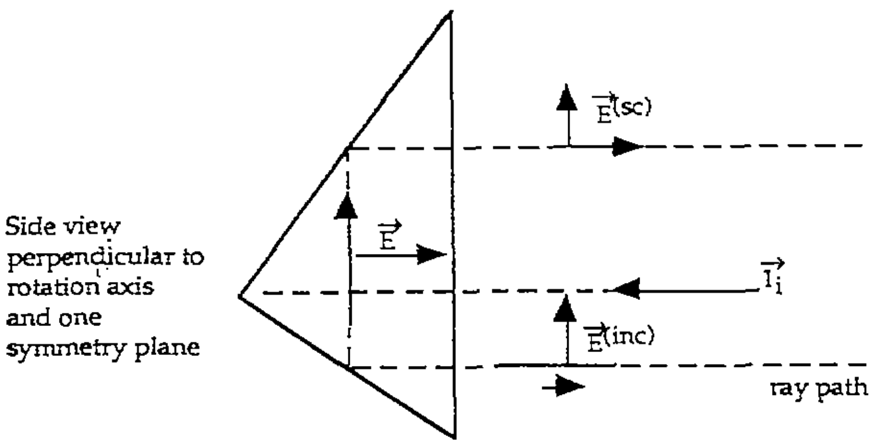


Fig. 3.2. Trihedral Corner Reflector

symmetry axis. In fig. 3.2, one can follow a ray (vertical polarization) to observe the scattering, say first from the "bottom" plate and second, from the dihdral "upper" corner to give a scattering sense of positive (for assumed perfectly conducting sheets and including only specular diffraction), opposite to that of a perfectly conducting plate perpendicular to $\vec{1}_i$.

3.3 Dihedral corner reflector

A popular type of scattering model is that of a dihedral corner reflector (also called diplane) [2, 3]. As illustrated in fig. 3.3, look along the dihedral edge which we label as the $\vec{1}_1$ direction (frequency independent in this case). If the corner reflector is made of perfectly conducting sheets, an incident electric field polarized in the $\vec{1}_2$ direction is reflected with a positive sign as can be seen by following the wave along a ray path. However, a wave polarized in the $\vec{1}_2$ direction ($\perp \vec{1}_1$ and $\vec{1}_i$) is reflected back toward the observer with a negative sign. As such, the canonical dihedral model takes the form

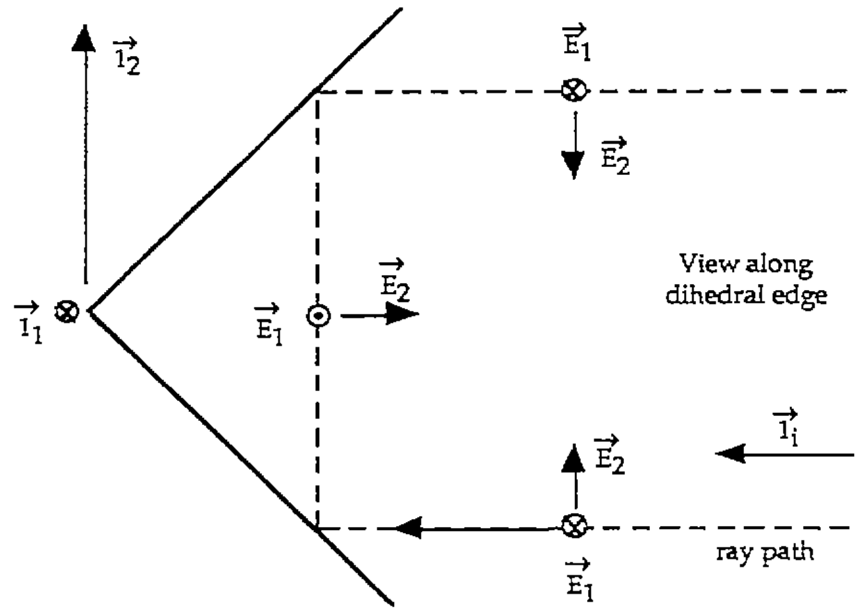
$$\overleftrightarrow{\tilde{\Lambda}}_b(s) = \tilde{\lambda}(s)\left[\vec{1}_1\vec{1}_1 - \vec{1}_2\vec{1}_2\right] \quad , \quad \tilde{\lambda}_1(s) = -\tilde{\lambda}_2(s) = \tilde{\lambda}(s) \tag{3.7}$$

Here, the eigenvectors have been taken as real and frequency independent, corresponding to a physical corner reflector. As in fig. 3.3, we can then look along the $\vec{1}_i$ direction and see that the $\vec{1}_\beta$ and associated angles ψ_β corresponding to the corner reflector are precisely the eigenvectors and angles discussed in Section 2. One might consider using the two double-headed arrows (or a single one with another symbol) on SAR images for identifying some of the scattering geometries.

Note that, in contrast to the line scatterer, both angles ψ_1 and ψ_2 appear in the scattering response. Which angle is which depends on the signs on the two eigenvalues. If one has no phase information (for a single frequency) then there is a sign ambiguity for the two orthogonal polarizations. Furthermore, other structures modelled as dihedral corner reflectors (such as tree trunks in combination with ground surface) are constructed of lossy dielectrics, thereby changing the reflection coefficients for the two polarizations (making $\tilde{\lambda}_1 \neq -\tilde{\lambda}_2$). Such a physical dihedral then does not exactly follow the dihedral form in (3.7).

More generally, one can have a model of the form

$$\overleftrightarrow{\tilde{\Lambda}}_b(s) = \tilde{\lambda}(s)\left[\vec{\tilde{1}}_1(s)\vec{\tilde{1}}_1(s) - \vec{\tilde{1}}_2(s)\vec{\tilde{1}}_2(s)\right] \tag{3.8}$$



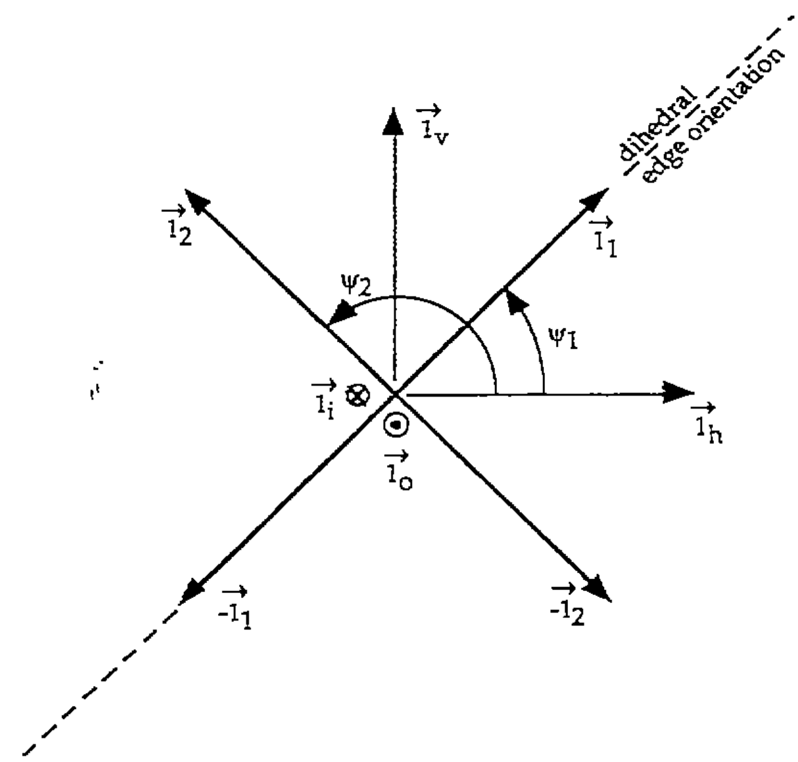


Fig. 3.3. Dihedral Corner Reflector

as a generalized dihedral model. The eigenvectors are allowed to be frequency dependent and perhaps complex. In processing scattering data, one may encounter such a case which is not a physical dihedral, but a mathematical one nonetheless. Note that (3.8) is characterized by two complex numbers, $\tilde{\lambda}$ and $\tilde{\psi}_1$ ($\tilde{\psi}_2$ being determined by $\tilde{\psi}_1$). In matrix form this is

$$\begin{aligned}
\tilde{\overleftrightarrow{\Lambda}}_b(s) &= \tilde{\lambda}(s)\begin{pmatrix} \cos^2(\tilde{\psi}_1(s)) - \sin^2(\tilde{\psi}_1(s)) & 2\cos(\tilde{\psi}_1(s))\sin(\tilde{\psi}_1(s)) \\ 2\cos(\tilde{\psi}_1(s))\sin(\tilde{\psi}_1(s)) & \sin^2(\tilde{\psi}_1(s)) - \cos^2(\tilde{\psi}_1(s)) \end{pmatrix} \\
&= \tilde{\lambda}(s)\begin{pmatrix} \cos(2\tilde{\psi}_1(s)) & \sin(2\tilde{\psi}_1(s)) \\ \sin(2\tilde{\psi}_1(s)) & -\cos(2\tilde{\psi}_1(s)) \end{pmatrix} \qquad (3.9) \\
\det(\tilde{\overleftrightarrow{\Lambda}}_b(s)) &= \tilde{\lambda}^2(s) \quad , \quad \mathrm{tr}(\tilde{\overleftrightarrow{\Lambda}}_b(s)) = 0
\end{aligned}$$

so this model is characterized by zero trace.

3.4 Combinations of the above

For more general $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$, there is an ambiguity of which model to choose. Even other models, such as based on helix scattering (related to circular polarization) [2], can be used in this decomposition of the scattering dyadic. A linear combination of any two of the three (or more) models will suffice. The angle $\tilde{\psi}_1$ (noting ambiguity with $-\tilde{\psi}_1$ and $\pm\tilde{\psi}_2$ giving $\tilde{\psi}_1 + \pi$ and $\tilde{\psi}_1 \pm \pi/2$) is a common feature except in the special case that only the rotational scattering model is required. Basically, for a single frequency we only have the three complex numbers given in various forms (based on various representations of the symmetrical backscattering dyadic) with which to work. For greater resolution of the scattering we need more information.

4. Backscattering Temporal Operator

There is a phase ambiguity on $\overleftrightarrow{\tilde{\Lambda}}_b(j\omega)$. Relative phases between the $\tilde{\Lambda}_{b_{n,m}}(j\omega)$ can be readily obtained, but one does not know generally the distance to the target with sufficient accuracy, giving an overall phase ambiguity. For CW measurements then one knows $\left|\tilde{\lambda}_1(j\omega)\right|$ and $\tilde{\lambda}_2(j\omega)/\tilde{\lambda}_1(j\omega)$, as well as $\vec{\tilde{1}}_1(j\omega)$ and $\vec{\tilde{1}}_2(j\omega)$ modulo sign (180° ambiguity not appearing in the dyadic products of the eigenvectors).

As we increase the bandwidth we would like the $\tilde{\lambda}_\beta(s)$ (or $\tilde{\lambda}_\beta(j\omega)$) to be continuous functions of frequency. In such a context we need to choose which eigenvalue is $\tilde{\lambda}_1$ by some dominant characteristic over the frequency band (instead of the largest of the two eigenvalues at a single frequency).

For a pulse (actual or synthesized from multiple single-frequency measurements) we have the sign of the scattered pulse (in each polarization) compared to the incident pulse. (We also have the time delay to the target.) This gives us more physical information.

In any event, we have some approximation of

$$\overleftrightarrow{\Lambda}_b(t) = \begin{pmatrix} \Lambda_{b_{h,h}}(t) & \Lambda_{h,v}(t) \\ \Lambda_{b_{v,h}}(t) & \Lambda_{h_{v,v}}(t) \end{pmatrix} = \overleftrightarrow{\Lambda}_b^T(t) \tag{4.1}$$

$$= \text{real symmetric dyadic}$$

This can be diagonalized in time domain with *real* eigenvalues and *real* eigenvectors as

$$\overleftrightarrow{\Lambda}_b(t) = \sum_{\beta=1}^{2} \lambda_\beta^{(t)}(t)\, \vec{1}_\beta^{(t)}(t)\, \vec{1}_\beta^{(t)}(t) \tag{4.2}$$

The superscripts t distinguish these from the frequency domain form. They are in general, not the same, i.e., $\tilde{\lambda}_\beta(t)$ is in general *not* the Laplace transform of $\lambda_\beta^{(t)}(t)$, and similarly for the eigenvectors. If the eigenvectors are time independent then the eigenvalues have a transform relationship, so one needs to be careful in going between frequency and time for individual parts of the diagonal form of the scattering dyadic.

5. Temporal Windows

Suppose now that we have some measurements of the three distinct scattering-matrix elements as functions of time. Furthermore, suppose that we have what appear to be two or more distinct scattering events in time which we label successively $1, \ldots, N$ as indicated in fig. 5.1. Then define a time window T_n encompassing the nth scattering event defined by $\Lambda^{(n)}_{b_{h,h}}(t)$, $\Lambda^{(n)}_{b_{h,v}}(t)$, and $\Lambda^{(n)}_{b_{v,v}}(t)$ (all contained within the same time window). Then we can attempt to analyze the scattering in each as separate events, perhaps each with its own scattering model.

One can consider each event in time and/or frequency domains. Related to this separation into windows is the window Laplace/Fourier transform [12] in which one defines some kind of window function, say $F_n(t)$, which multiplies the three waveforms of interest so as to make these effectively zero outside the window of interest. This window function might be just one inside the window and zero outside, or have some smoothing properties imposed near the window beginning and end in time. The important thing is the result that one can analyze the temporal and frequency-spectral properties in each window appropriate to the particular scattering event. (If some scattering events overlap in time then one can model them together with appropriate models in the same window.)

With the scattering events separated we can revisit the question of scattering models. In Section 3, we discussed a few models based on single-frequency data. We can now consider the same on a wideband basis. If the temporal eigenvectors (real) as in (4.2) are time invariant, then they apply in frequency domain and represent some characteristic direction for the target, e.g., in the line or dihedral sense. However, much more information is available in the temporal and spectral properties.

Different types of scatterers have different temporal, or equivalently, spectral dependences. Various types of scattering models exist for various types of targets [5, 12]. Examples include the generalized cone with form $\overleftrightarrow{K} s^{-1}$ or $\overleftrightarrow{K} u(t)$, a temporal integrator with $\overleftrightarrow{K}$ symmetric and real [11]. An important model is the singularity expansion method (SEM), using poles at the natural frequencies s_α (in left half plane) and residues using vectors $\vec{c}_\alpha$ describing directions much like the eigenvectors in Section 3 [4]. Such resonant scatterers can be discriminated based on the pattern of the s_α set using various types of procedures such as Prony method, matrix pencil, and E/K pulse filters. The reader can consult the references for yet more models.

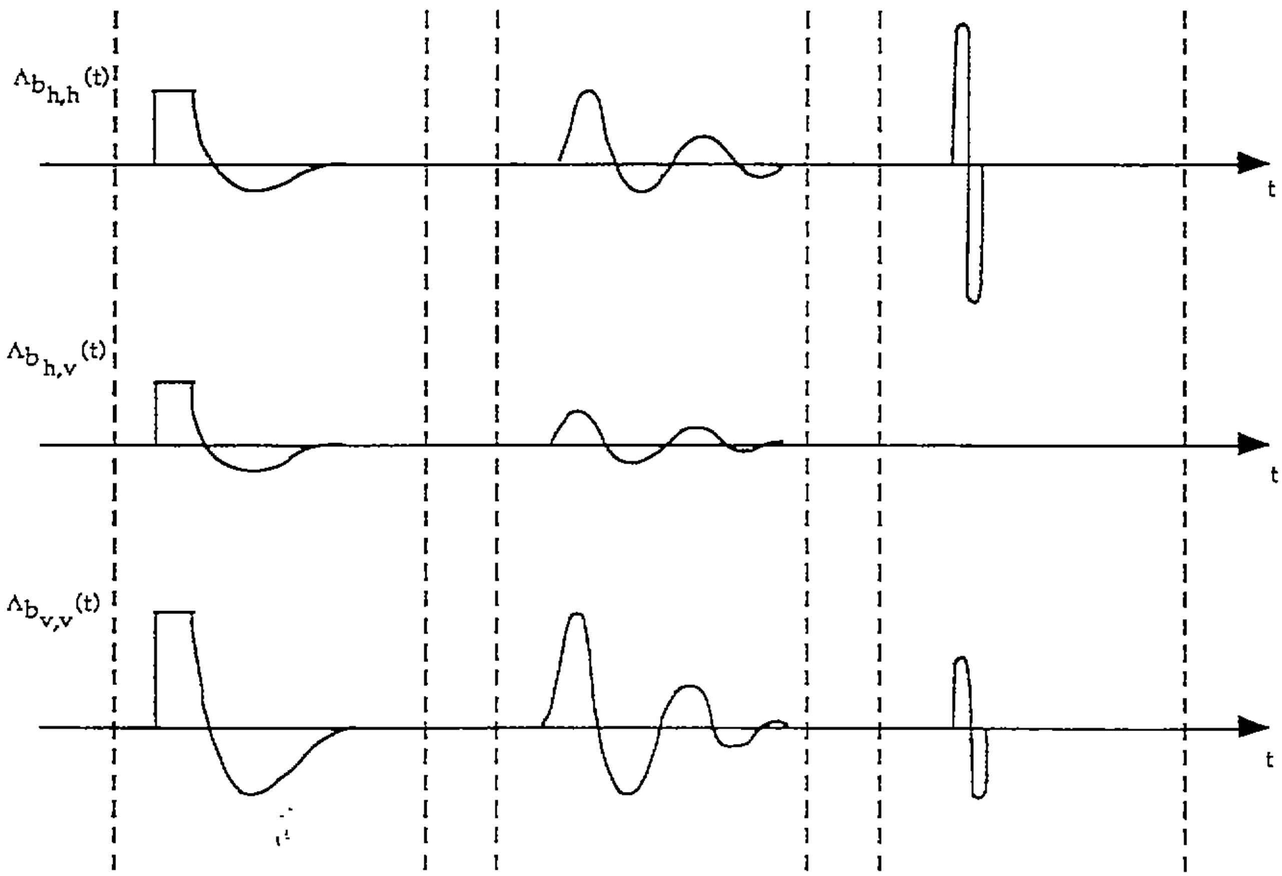


Fig. 5.1. Scattering Events Separated into Temporal Windows.

6. Concluding Remarks

While polarization allows one to obtain more information about the target via the scattering dyadic (single frequency), the amount of information is still limited. Fitting the scattering data by simple models (line, rotational, and dihedral) reveals an ambiguity in that any two of these will suffice.

Extending the bandwidth of our radar to multiple frequencies retaining phase, or even a pulse, much more information becomes available. In this context, one can use more and better models for more accurate target identification.

Appendix A. Diagonalization of 2 x 2 Complex Symmetric Matrix

Consider the matrix

$$\left(A_{n,m}\right) \equiv \begin{pmatrix} a & b \\ b & d \end{pmatrix} \equiv 2 \times 2 \text{ complex symmetric matrix (elements complex numbers)} \tag{A.1}$$

The eigenvalues λ_β are found from

$$\begin{aligned}
&\det\left(\begin{pmatrix} a-\lambda_\beta & b \\ b & d-\lambda_\beta \end{pmatrix}\right) = 0 \\
&\left[a-\lambda_\beta\right]\left[d-\lambda_\beta\right] - b^2 = 0 \\
&\lambda_\beta^2 - [a+d]\lambda_\beta + ad - b^2 = 0 \\
&\lambda_\beta = \frac{1}{2}\left[a+d \pm \left[[a+d]^2 - 4ad + 4b^2\right]^{\frac{1}{2}}\right] \\
&\beta = 1, 2 \text{ (for upper and lower signs respectively)}
\end{aligned} \tag{A.2}$$

As is well known, if these two eigenvalues are distinct, the matrix is diagonalizable with a complete set of eigenvectors spanning the space (two dimensional in this case).

So consider the case that the two eigenvalues are equal. This occurs if

$$\begin{aligned}
[a+d]^2 - 4ad + 4b^2 &= 0 \\
[a-d]^2 + 4b^2 &= 0 \\
a-d &= \pm\, j2b
\end{aligned} \tag{A.3}$$

Note that this constraint does not apply to a real symmetric matrix unless $b = 0$. A real symmetric matrix (special case of Hermitian) is always diagonalizable. So there is a class of matrices satisfying (A.3) which is diagonalizable.

Now letting $b \neq 0$ with the constraint of (A.3) we have the two equal eigenvalues

$$\lambda_\beta = \frac{a+d}{2} \tag{A.4}$$

The corresponding eigenvector equation is

$$
\begin{pmatrix} a & b \\ b & d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_\beta = \lambda_\beta \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_\beta
$$

$$
\begin{pmatrix} \frac{a-d}{2} & b \\ b & \frac{d-a}{2} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_\beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{A.5}
$$

$$
b \begin{pmatrix} \pm j & 1 \\ 1 & \mp j \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_\beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

Choosing the upper signs the only solution has

$$
x_1 = jx_2 \tag{A.6}
$$

meaning only one eigenvector. For lower signs

$$
x_1 = -jx_2 \tag{A.7}
$$

giving also only one eigenvector. So for $b \neq 0$ a 2 x 2 complex symmetric matrix with equal eigenvalues is not diagonalizable.

Conversely, a complex symmetric matrix is diagonalizable if the two eigenvalues are distinct, *or* $b = 0$ (diagonal matrix).

Now assume that there are two linearly independent eigenvectors. These then span the two dimensional space. If $\lambda_1 \neq \lambda_2$, then

$$
\begin{aligned}
(x_n)_2 \cdot (A_{n,m}) \cdot (x_n)_1 = \lambda_2 (x_n)_1 \cdot (x_n)_2 &= \lambda_1 (x_n)_1 \cdot (x_n)_2 \\
(x_n)_1 \cdot (x_n)_2 &= 0
\end{aligned} \tag{A.8}
$$

so that the eigenvectors are orthogonal in the dot-product (symmetric-product) sense. Write a general two-component vector as

$$
(y_n) = c_1 (x_n)_1 + c_2 (x_n)_2 \tag{A.9}
$$

due to the linear independence of the eigenvectors, with generally both c_1 and c_2 non zero. Then dot multiply by $(x_n)_1$ to obtain

$$(x_n)_1 \cdot (y_n) = c_1 (x_n)_1 \cdot (x_n)_1 \tag{A.10}$$

Choose (y_n) (e.g., $(x_n)_1^*$) such that this is non zero. With c_1 bounded we have

$$(x_n)_1 \cdot (x_n)_1 \neq 0 \tag{A.11}$$

and similarly for $(x_n)_2$, allowing us to scale the vectors to normalize them. Summarizing we have

$$(x_n)_{\beta_1} \cdot (x_n)_{\beta_2} = 1_{\beta_1 \beta_2} = \begin{cases} 1 \text{ for } \beta_1 = \beta_2 \\ 0 \text{ for } \beta_1 = \beta_2 \end{cases} \text{ (orthonormal)} \tag{A.12}$$

For $\lambda_1 = \lambda_2$, any linear combination of two linearly independent eigenvectors can be used as a basis (not necessarily orthogonal (defined by (A.8))). By a procedure known as Gram-Schmidt orthogonalization a set of orthogonal eigenvectors can be constructed [7]. Since we are here concerned with the two-dimensional space we can use two real unit vectors such as

$$(x_n)_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad (x_n)_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{A.13}$$

as our orthonormal eigenvectors (thereby avoiding the distinction between symmetric and inner products). Since for this case we must have $b = 0$, then we can see that from (A.5), the above choices work. Furthermore, in this case the complex-symmetric matrix has $a = d$ and the matrix is just

$$(A_{n,m}) = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.14}$$

i.e., a scalar times the identity.

The reader can note that the foregoing results can be generalized to general complex matrices of the form

$$(A_{n,m}) = \begin{pmatrix} a & b_1 \\ b_2 & d \end{pmatrix} \equiv 2 \times 2 \text{ complex matrix} \tag{A.15}$$

By replacing

$$b^2 = b_1 b_2 \tag{A.16}$$

the criterion in (A.3) for equal eigenvalues is replaced by

$$a - d = \pm j2 \left[b_1 b_2 \right]^{\frac{1}{2}} \tag{A.17}$$

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