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Exact Analytical Solution for Nonuniform Multiconductor Transmission Lines with the Aid of the Solution of a Corresponding Matrix Riccati Equation

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Abstract

Nonuniform multiconductor transmission lines (NMTLs) are formulated from the telegrapher equations into a first-order supervector/supermatrix differential equation combining the voltage and current vectors. This equation is converted to one for the waves propagating in the two directions of the transmission line. Both equations can be represented in a common general supermatrix form. The supermatrizant describing the solutions of this general equation is evaluated via a transformation with an elementary relative matrix and the solution of a matrix Riccati equation simplifying the resulting telegrapher equations to a structure to which the usual sum rules for product integrals can easily be applied. The block elements of the supermatrizant are (mainly) composed by matrizants of *NxN* matrices. For a single nonuniform line and for circulant NMTLs the supermatrizants are represented by exponentials.

1
Introduction

There are several techniques to solve the telegrapher equations for NMTLs [1-3]. The present paper continues this discussion, utilizing the product integral [4] and relative matrices [5] for extending these results.
The telegrapher equations for an NMTL for N conductors (plus reference) read

$$\begin{aligned}\frac{\partial}{\partial z}\mathbf{v}(z,s) &= -\mathbf{Z}'(z,s)\,\mathbf{i}(z,s)+\mathbf{v}'^{(s)}(z,s)\\ \frac{\partial}{\partial z}\mathbf{i}(z,s) &= -\mathbf{Y}'(z,s)\,\mathbf{v}(z,s)+\mathbf{i}'^{(s)}(z,s)\end{aligned} \tag{1.1}$$

The vectors (little bold face letters) have N components and the matrices (capital bold face letters) are NxN. Here the treatment is in complex-frequency domain as indicated by the two-sided Laplace-transform variable $s = \Omega + j\omega$, and z is the position along the line. For brevity, we sometimes will suppress the dependence on s and z. As usual the vectors **v** and **i** denote the voltage and the current at z, respectively. The per-unit-length source vectors are indicated by an upper dash and s in brackets. The per-unit-length impedance (**Z'**) and admittance (**Y'**) matrices can be comined to give

$$\mathbf{P}(z,s) := \left(\mathbf{Z}'(z,s)\,\mathbf{Y}'(z,s)\right)^{1/2} \qquad \text{(positive real (p.r.) square root)} \tag{1.2}$$

$\equiv$ propagation matrix

$$\mathbf{Z}_c(z,s) := \mathbf{P}(z,s)\,\mathbf{Y}'^{-1}(z,s) = \mathbf{P}^{-1}(z,s)\,\mathbf{Z}'(z,s) = \left(\mathbf{Z}_c(z,s)\right)^{T} \tag{1.3}$$

$\equiv$ characteristic impedance matrix

where reciprocity has been assumed.

2
The supermatrizants for voltage and current vectors and for the wave variables

As is shown in [3] and [6] the voltage and current vectors can be combined with the aid of the characteristic impedance matrix to a supervector of $2N$ components, and the combined telegrapher equations can be represented as a first order differential equation in supermatrix form

$$\frac{\partial}{\partial z}\begin{pmatrix}\mathbf{v}_1(z,s)\\ \mathbf{v}_2(z,s)\end{pmatrix} = \begin{pmatrix}\mathbf{P}_{11}(z,s) & \mathbf{P}_{12}(z,s)\\ \mathbf{P}_{21}(z,s) & \mathbf{P}_{22}(z,s)\end{pmatrix}\begin{pmatrix}\mathbf{v}_1(z,s)\\ \mathbf{v}_2(z,s)\end{pmatrix} + \begin{pmatrix}\mathbf{v}'^{(s)}_1(z,s)\\ \mathbf{v}'^{(s)}_2(z,s)\end{pmatrix} \tag{2.1}$$

where the supermatrices are $2N\times 2N$, and where the block matrices $\mathbf{P}_{ij}$ depend on the chosen representation. In the representation of voltage and current vectors we identify

$$\begin{aligned} &\mathbf{v}_1(z,s)=\mathbf{v}(z,s); \quad \mathbf{v}_2(z,s)=\mathbf{Z}_c(z,s)\,\mathbf{i}(z,s) \\ &\mathbf{v}'^{(s)}_1(z,s)=\mathbf{v}'^{(s)}(z,s); \quad \mathbf{v}'^{(s)}_2(z,s)=\mathbf{Z}_c(z,s)\,\mathbf{i}'^{(s)}(z,s) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\mathbf{P}_{11}(z,s)=\mathbf{0}; \quad \mathbf{P}_{12}(z,s)=-\mathbf{P}(z,s)=\mathbf{P}_{21}(z,s) \\ &\mathbf{P}_{22}(z,s)=\left(\frac{\partial}{\partial z}\mathbf{Z}_c(z,s)\right)\mathbf{Z}_c^{-1}(z,s)=:\mathrm{D}_z\mathbf{Z}_c(z,s) \end{aligned} \tag{2.3}$$

The product derivative D_z is discussed (e.g.) in [6].

In order to obtain the representation in wave variables with (+) giving the direction of increasing z (right), and (-) giving the direction of decreasing z (left), we have to observe the following relations:

$$\begin{pmatrix} \mathbf{v}_{(+)}(z,s) \\ \mathbf{v}_{(-)}(z,s) \end{pmatrix} = \left[\overline{\mathbf{R}}+\overline{\mathbf{P}}_{er}\right]\begin{pmatrix} \mathbf{v}(z,s) \\ \mathbf{Z}_c(z,s)\,\mathbf{i}(z,s) \end{pmatrix}$$

and (2.4)

$$\begin{pmatrix} \mathbf{v}'^{(s)}_{(+)}(z,s) \\ \mathbf{v}'^{(s)}_{(-)}(z,s) \end{pmatrix} = \left[\overline{\mathbf{R}}+\overline{\mathbf{P}}_{er}\right]\begin{pmatrix} \mathbf{v}'^{(s)}(z,s) \\ \mathbf{Z}_c(z,s)\mathbf{i}(z,s) \end{pmatrix}$$

Here the supermatrices $\overline{\mathbf{R}}$ and $\overline{\mathbf{P}}_{er}$ describe reflections

$$\overline{\mathbf{R}}:=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}=\overline{\mathbf{R}}^{-1} \tag{2.5}$$

and permutations

$$\overline{\mathbf{P}}_{er}:=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}=\overline{\mathbf{P}}_{er}^{-1} \tag{2.6}$$

respectively. Then we find

$$\begin{aligned} &\mathbf{v}_1(z,s)=\mathbf{v}_{(+)}(z,s); \quad \mathbf{v}_2(z,s)=\mathbf{v}_{(-)}(z,s) \\ &\mathbf{v}'^{(s)}_1(z,s)=\mathbf{v}'^{(s)}_{(+)}(z,s); \quad \mathbf{v}'^{(s)}_2(z,s)=\mathbf{v}'^{(s)}_{(-)}(z,s) \end{aligned} \tag{2.7}$$

and

$$\mathbf{P}_{11}(z,s) = -\mathbf{P}(z,s) + \frac{1}{2}\mathrm{D}_z\mathbf{Z}_c(z,s); \quad \mathbf{P}_{12}(z,s) = \mathbf{P}_{21}(z,s) = -\frac{1}{2}\mathrm{D}_z\mathbf{Z}_c(z,s)$$

$$\mathbf{P}_{22}(z,s) = \mathbf{P}(z,s) + \frac{1}{2}\mathrm{D}_z\mathbf{Z}_c(z,s) \tag{2.8}$$

It is convenient to rewrite equation (2.1) in the supermatrix form

$$\frac{\partial}{\partial z}\,\overline{\mathbf{v}}\,(z,s) = \overline{\mathbf{P}}\,\overline{\mathbf{v}}(z,s) + \overline{\mathbf{v}}'^{(s)}(z,s) \tag{2.1'}$$

Its solution can be derived using the methods given in [4].
The solution reads

$$\overline{\mathbf{v}}(z,s) = \overline{\mathbf{M}}_{z_0}^{z}\left(\overline{\mathbf{P}}\right)\overline{\mathbf{v}}(z_0,s) + \int_{z_0}^{z}\overline{\mathbf{K}}(z,\zeta,s)\,\overline{\mathbf{v}}'^{(s)}(\zeta,s)\,d\zeta \tag{2.9}$$

with the supermatrizant [4]

$$\overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{P}}) = \overline{\mathbf{1}} + \int_{z_0}^{z}\overline{\mathbf{P}}(\zeta,s)d\zeta + \int_{z_0}^{z}\overline{\mathbf{P}}(\zeta,s)\int_{z_0}^{\zeta}\overline{\mathbf{P}}(\eta,s)d\eta d\zeta + \ldots \tag{2.10}$$

and the Green's function

$$\overline{\mathbf{K}}(z,\zeta;s) = \overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{P}})\overline{\mathbf{M}}_{\zeta}^{z_0}(\overline{\mathbf{P}}) \tag{2.11}$$

Thus it becomes quite obvious that our further investigation has to focus on the calculation of the supermatrizant $\overline{\mathbf{M}}_{\cdot}^{\cdot}(\overline{\mathbf{P}})$.

3
Determination of the supermatrizant via the solution of a matrix Riccati equation

For the determination of the supermatrizant it is sufficient to solve equation (2.1') without sources [4]. First we generalize equation (2.1') to

$$\frac{\partial}{\partial z}\overline{\mathbf{V}}(z,s) = \overline{\mathbf{P}}\,\overline{\mathbf{V}}(z,s) \quad \text{or} \quad \mathrm{D}_z\,\overline{\mathbf{V}}(z,s) = \overline{\mathbf{P}}(z,s) \tag{3.1}$$

where $\overline{\mathbf{V}} = (\overline{\mathrm{v}}_{ik})$ denotes the *2 N x 2 N* integralsupermatrix which columns are composed by *2N* linear independent solutions of (2.1'). Next the generalized telegrapher equation (3.1) is transformed with the aid of the relative supermatrix

$$\overline{\mathbf{T}} = \begin{pmatrix} 1 & \mathbf{T}_{12} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \overline{\mathbf{T}}^{-1} = \begin{pmatrix} 1 & -\mathbf{T}_{12} \\ 0 & 1 \end{pmatrix} \tag{3.2}$$

to a corresponding equation for $\overline{\mathbf{W}}$, with

$$\overline{\mathbf{V}} = \overline{\mathbf{T}}\,\overline{\mathbf{W}} \tag{3.3}$$

giving

$$D_z \overline{\mathbf{W}} = \overline{\mathbf{T}}^{-1} \overline{\mathbf{P}}\,\overline{\mathbf{T}} - \overline{\mathbf{T}}^{-1} \frac{\partial \overline{\mathbf{T}}}{\partial z} =: \overline{\mathbf{Q}} \tag{3.4}$$

The block elements of $\overline{\mathbf{Q}}$ turn out to be

$$\begin{aligned} &\mathbf{Q}_{11} = \mathbf{P}_{11} - \mathbf{T}_{12}\mathbf{P}_{21}; \quad && \mathbf{Q}_{12} = \mathbf{P}_{11}\mathbf{T}_{12} - \mathbf{T}_{12}\mathbf{P}_{21}\mathbf{T}_{12} - \mathbf{T}_{12}\mathbf{P}_{22} + \mathbf{P}_{12} - \frac{\partial \mathbf{T}_{12}}{\partial z} \\ &\mathbf{Q}_{21} = \mathbf{P}_{21}; && \mathbf{Q}_{22} = \mathbf{P}_{21}\mathbf{T}_{12} + \mathbf{P}_{22} \end{aligned} \tag{3.5}$$

As will be shown later the requirement that the upper block matrix $\mathbf{Q}_{12}$ should vanish simplifies the construction for the solution of $\overline{\mathbf{M}}(\overline{\mathbf{P}})$. In this case we obtain a relative half-reduced supermatrix [5], and $\mathbf{T}_{12}$ has to fulfill the matrix Riccati equation

$$\frac{\partial \mathbf{T}_{12}}{\partial z} = \mathbf{P}_{11}\mathbf{T}_{12} - \mathbf{T}_{12}\mathbf{P}_{21}\mathbf{T}_{12} - \mathbf{T}_{12}\mathbf{P}_{22} + \mathbf{P}_{12} \tag{3.6}$$

In what follows it is assumed that equation (3.6) is solved for $\mathbf{T}_{12}$. Then we continue our calculation decomposing $\overline{\mathbf{Q}}$ into two summands

$$\overline{\mathbf{Q}} = \overline{\mathbf{Q}}_1 + \overline{\mathbf{Q}}_2 := \begin{pmatrix} \mathbf{Q}_{11} & 0 \\ 0 & \mathbf{Q}_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \mathbf{Q}_{21} & 0 \end{pmatrix} \tag{3.7}$$

This has the advantage that we can apply a sum rule for the matrizant [4] giving

$$\overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{Q}}) = \overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{Q}})\overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{S}}) \tag{3.8}$$

with

$$\overline{\mathbf{S}} = \left(\overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{Q}}_1)\right)^{-1} \overline{\mathbf{Q}}_2 \overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{Q}}_1) \tag{3.9}$$

Using equation (2.10) we easily recognize that the matrizant for $\overline{\mathbf{Q}}_1$ can be written as

$$\overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{Q}}_1) = \begin{pmatrix} \mathbf{M}_{z_0}^{z}(\mathbf{Q}_{11}) & 0 \\ 0 & \mathbf{M}_{z_0}^{z}(\mathbf{Q}_{22}) \end{pmatrix} \tag{3.10}$$

With this expression and its inverse

$$\left(\overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{Q}}_1)\right)^{-1}=\begin{pmatrix}\left(\mathbf{M}_{z_0}^{z}(\mathbf{Q}_{11})\right)^{-1}=\mathbf{M}_{z}^{z_0}(\mathbf{Q}_{11}) & \mathbf{0}\\ \mathbf{0} & \left(\mathbf{M}_{z_0}^{z}(\mathbf{Q}_{22})\right)^{-1}=\mathbf{M}_{z}^{z_0}(\mathbf{Q}_{22})\end{pmatrix} \tag{3.11}$$

$\overline{\mathbf{S}}$ is obtained as

$$\overline{\mathbf{S}}=\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{M}_{z}^{z_0}(\mathbf{Q}_{22})\,\mathbf{Q}_{21}(z)\mathbf{M}_{z_0}^{z}(\mathbf{Q}_{11}) & \mathbf{0}\end{pmatrix} \tag{3.12}$$

The matrizant for $\overline{\mathbf{S}}$ only contains two terms (compare with eq. (2.10))

$$\begin{aligned}\overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{S}})&=\overline{\mathbf{I}}+\int_{z_0}^{z}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{M}_{\zeta}^{z_0}(\mathbf{Q}_{22})\,\mathbf{Q}_{21}(\zeta)\mathbf{M}_{z_0}^{\zeta}(\mathbf{Q}_{11}) & \mathbf{0}\end{pmatrix}d\zeta\\ &=\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ \int_{z_0}^{z}\mathbf{M}_{\zeta}^{z_0}(\mathbf{Q}_{22})\,\mathbf{Q}_{21}(\zeta)\mathbf{M}_{z_0}^{\zeta}(\mathbf{Q}_{11})d\zeta & \mathbf{1}\end{pmatrix}\end{aligned} \tag{3.13}$$

Eventually we find the preliminary final result (see eq. (3.8))

$$\overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{Q}})=\begin{pmatrix}\mathbf{M}_{z_0}^{z}(\mathbf{Q}_{11}) & \mathbf{0}\\ \mathbf{M}_{z_0}^{z}(\mathbf{Q}_{22})\int_{z_0}^{z}\mathbf{M}_{\zeta}^{z_0}(\mathbf{Q}_{22})\,\mathbf{Q}_{21}(\zeta)\mathbf{M}_{z_0}^{\zeta}(\mathbf{Q}_{11})d\zeta & \mathbf{M}_{z_0}^{z}(\mathbf{Q}_{22})\end{pmatrix} \tag{3.14}$$

which in a last step has to be retransformed to the matrizant for $\overline{\mathbf{P}}$. We have

$$\begin{aligned}\overline{\mathbf{V}}(z)&=\overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{P}})\,\overline{\mathbf{V}}(z_0)=\overline{\mathbf{T}}(z)\overline{\mathbf{W}}(z)=\overline{\mathbf{T}}(z)\overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{Q}})\,\overline{\mathbf{W}}(z_0)\\ &=\overline{\mathbf{T}}(z)\overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{Q}})\,\overline{\mathbf{T}}^{-1}(z_0)\,\overline{\mathbf{V}}(z_0)\end{aligned} \tag{3.15}$$

and therefore the relation

$$\overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{P}})=\overline{\mathbf{T}}(z)\overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{Q}})\overline{\mathbf{T}}^{-1}(z_0) \tag{3.16}$$

This solution has to be inserted in equation (2.9) where also sources are taken into account.

4
Applications

More explicit expressions for the supermatrizant $\overline{\mathbf{M}}\ (\overline{\mathbf{Q}})$ can be derived in the cases $N = 1$ (i.e. if there is only one line and the reference conductor) and if the matrices $\mathbf{Q}_{11}(z)$, $\mathbf{Q}_{11}(z')$ and $\mathbf{Q}_{22}(z)$, $\mathbf{Q}_{22}(z')$ commute for all $z, z' \in (a,b)$ along the line. Then one gets

$$\mathbf{M}_{z_0}^{z}(\mathbf{Q}_{11}) = e^{\int_{z_0}^{z} Q_{11}(z')dz'} \quad ; \quad \mathbf{M}_{z_0}^{z}(\mathbf{Q}_{22}) = e^{\int_{z_0}^{z} Q_{22}(z')dz'}$$
$$\mathbf{M}_{\zeta}^{z_0}(\mathbf{Q}_{ii}) = e^{-\int_{z_0}^{\zeta} Q_{ii}(z')dz'} \qquad (i = 1,2) \tag{4.1}$$

An interesting group of matrices which are also most relevant for practical applications are circulant matrices [1]. They form an abelian group (they even have a richer structure) with respect to matrix multiplication. Thus all elements of this group can be simultaneously diagonalized with the aid of one and the same unitary matrix, called the Fourier matrix [7].
This diagonalization procedure (simularity transformations) results in diagonal expressions for the quantities in equation (4.1). In effect, the calculation with circulant matrices leads to equations which formally resemble those for the one dimensional ($N = 1$) case [1].

A further simplification (compared to (4.1)) can be achieved for $N = 1$. Then we have to deal with *2 x 2* matrices which elements are scalar complex functions. For the matrizant we obtain

$$\overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{Q}}) = \begin{pmatrix} e^{\int_{z_0}^{z} Q_{11}(z')dz'} & 0 \\ e^{\int_{z_0}^{z} Q_{22}(z')dz'} \int_{z_0}^{z} Q_{21}(\zeta) e^{\int_{z_0}^{z}(Q_{11}(z')-Q_{22}(z'))dz'} d\zeta & e^{\int_{z_0}^{z} Q_{22}(z')dz'} \end{pmatrix} \tag{4.2}$$

$$=: \begin{pmatrix} f_1(z,z_0) & 0 \\ f_{21}(z,z_0) & f_2(z,z_0) \end{pmatrix}$$

The scalar Riccati equation (in the representation for the voltage and current vectors) reads

$$\frac{\partial T_{12}}{\partial z} = P(z,s)\,T_{12}^2(z) - T_{12}(z)\frac{\partial}{\partial z}\ell n\left(Z_c(z,s)\right) - P(z,s) \tag{4.3}$$

For the lossless exponential line ,e. q., T_{12} becomes constant (w.r.t.z), and equation (4.3) can easily be solved. The line parameters, here the propagation function P(z,s) and the characteristic impedance function $Z_c(z,s)$ determine T_{12} via equation (4.3). Knowing $T_{12}(z,s)$ one finally calculates $\overline{\mathbf{M}}$ $(\overline{\mathbf{P}})$ as

$$\overline{\mathbf{M}}_{z_0}^{z}(\overline{\mathbf{P}}) = \begin{pmatrix} f_1(z,z_0)+T_{12}(z)f_{21}(z,z_0) & -f_1(z,z_0)T_{12}(z_0)-T_{12}(z)T_{12}(z_0)f_{21}(z,z_0)+T_{12}(z)f_2(z,z_0) \\ f_{21}(z,z_0) & -f_{21}(z,z_0)T_{12}(z_0)+f_2(z,z_0) \end{pmatrix} \tag{4.4}$$

With the solutions (4.2) and (4.4) we present the general structure of the solution for *all* one-dimensional ($N = 1$) nonuniform transmission lines.

5
Concluding remarks

We have shown that the use of relative matrices together with the product integral gives some interesting ways to represent and compute the supermatrizants representing waves on NMTLs. In deriving the supermatrizants an intermediate step requires the solution of a matrix Riccati equation.
This very closely resembles the equivalence of the telegrapher equations and the Riccati equation for the positon dependent reflection function [1]. Moreover, it may be useful to observe the Lie group properties of the matrix Riccati equations [8] to study solutions in terms of Lie series.

Another advantageous aspect of the matrizant solution is the connection to the scattering supermatrix $\overline{\mathbf{S}}_c$ for purposes of network simulation. This connection is established by the comparison of the two matrix equations ($z_0 = 0$ near end of the line, $z = \ell$ far end of the line)

$$\begin{pmatrix} \mathbf{v}_{(+)}(\ell) \\ \mathbf{v}_{(-)}(\ell) \end{pmatrix} = \overline{\mathbf{M}}_{o}^{\ell}(\overline{\mathbf{P}}) \begin{pmatrix} \mathbf{v}_{(+)}(0) \\ \mathbf{v}_{(-)}(0) \end{pmatrix} \tag{5.1}$$

and

$$\begin{pmatrix} \mathbf{v}_{(-)}(0) \\ \mathbf{v}_{(+)}(\ell) \end{pmatrix} = \overline{\mathbf{S}}_c \begin{pmatrix} \mathbf{v}_{(+)}(0) \\ \mathbf{v}_{(-)}(\ell) \end{pmatrix} \tag{5.2}$$

which results in the relation

$$\mathbf{S}_{c11} = -\mathbf{M}_{22}^{-1}\mathbf{M}_{21}; \quad \mathbf{S}_{c12} = \mathbf{M}_{22}^{-1} \tag{5.3}$$

$$\mathbf{S}_{c21} = \mathbf{M}_{11} - \mathbf{M}_{12}\mathbf{M}_{22}^{-1}\mathbf{M}_{21}; \quad \mathbf{S}_{c22} = \mathbf{M}_{12}\mathbf{M}_{22}^{-1}$$

between the block matrix elements of $\overline{\mathbf{M}}(\overline{\mathbf{P}})$ *and* $\bar{\mathbf{S}}_c$.

Thus also nonuniform MTLs can be well treated in a network simulation [9].

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