IN
S37

DEPARTMENT OF THE AIR FORCE

AIR FORCE RESEARCH LABORATORY
DIRECTED ENERGY DIRECTORATE/PA
3550 ABERDEEN AVE, SE
KIRTLAND AFB, NEW MEXICO 87117-5776

Date: 31 MAR 98

MEMORANDUM FOR AFRL/DEHP

ATTENTION: Dr. Carl E. Baum

FROM: AFRL/DEOB-PA (6-6246)

SUBJECT: Security Review Case Number(s): DE 98-234

Action on the case(s) mentioned above has been completed with the following determination, as marked:

X **CLEARED FOR OPEN PUBLICATION/PRESENTATION (NO CHANGES).**

___ **CLEARED FOR OPEN PUBLICATION/PRESENTATION AS AMENDED.** The amendments (changes, deletions, etc.) denoted by the words "as amended" -- are mandatory. Information to be deleted will be inside brackets.

___ **CLEARED FOR OPEN PUBLICATION/PRESENTATION WITH RECOMMENDED CHANGES.** Recommended changes are not mandatory but highly recommended. This information will be marked "recommended changes."

___ **DISAPPROVED FOR PUBLIC RELEASE** for the following reason(s):

___ **CONTAINED CLASSIFIED INFORMATION**

___ **CONTAINED TECHNICAL SECURITY INFORMATION NOT RELEASABLE TO THE PUBLIC**

___ **RELEASE DENIED FOR OTHER AF/DoD/GOVERNMENT REASONS**

___ The project officer must notify the author/contractor of the clearance determination and provide them with a copy of this memorandum and stamped document.

___ The final document(s) (paper, speech, etc.) may require higher headquarters review. Therefore, please submit 12 copies to AFRL/DEOB-PA with the PA form. Allow 45 WORKING DAYS for completion of review.

___ **COMMENT:** ______________________________

Judy L. Johnston
JUDY L JOHNSTON
Public Affairs Specialist

Attachment(s):
1. DE 98-234
2. ___________

Interaction Notes

Note 537

18 January 1998

Symmetric Renormalization of the Nonuniform Multiconductor-Transmission-Line Equations with a Single Modal Speed for Analytically Solvable Sections

Carl E. Baum

Air Force Research Laboratory
Directed Energy Directorate

Abstract

A nonuniform multiconductor transmission line (NMTL) can have its propagation represented (and computed numerically) by a product integral corresponding to dot multiplication of the product integrals for a set of uniform sections. For a smoother transition from one section to the next (continuous characteristic-impedance matrix) one can interpolate the characteristic-impedance matrix for each section from its end-point values. The procedure discussed in this paper concerns the case of all modal speeds the same (such as the case of perfect conductors with a uniform dielectric medium). The interpolation is accomplished in a way that preserves the symmetry of the characteristic-impedance matrix (and hence reciprocity) throughout each section.

Contents

1. Introduction

A nonuniform multiconductor transmission line (NMTL) has the general telegrapher equations as

$$\begin{aligned} \frac{\partial}{\partial z}\left(\tilde{V}_n(z,s)\right) &= -\left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{I}_n(s)\right) + \left(\tilde{V}_n^{(s)'}(z,s)\right) \\ \frac{\partial}{\partial z}\left(\tilde{I}_n(z,s)\right) &= -\left(\tilde{Y}'_{n,m}(z,s)\right) \cdot \left(\tilde{V}_n(s)\right) + \left(\tilde{I}_n^{(s)'}(z,s)\right) \end{aligned} \tag{1.1}$$

for N conductors plus reference (zero voltage). The vectors have N components and the matrices are $N \times N$. These equations can be combined in various ways to give a single supermatrix equation ($2N \times 2N$) [5, 6, 13]. From a numerical point of view these can be solved via the product integral [6]; this is basically a staircase approximation with the per-unit-length impedance and admittance matrices assumed as constants within each short section of the transmission line.

If the modal speeds are all the same the matrix equations simplify considerably [4, 5]. This corresponds to the case that perfect conductors (transmission-line conductors) are embedded in a uniform medium, but the conductor positions and sizes can vary as one proceeds along the transmission line (increasing z coordinate). This allows one to concentrate on the characteristic-impedance matrix (a function of z) for diagonalizing the system of equations. In [5 (Section 6)] a method for common diagonalization over a section of NMTL is developed by normalizing this impedance matrix by postmultiplication by the matrix inverse at one value of z (the beginning of the section), diagonalizing the product at another value of z (the end of the section), and interpolating the eigenvalues between 1 (at the beginning) and the ending values. This gives an approximate way to define each section in such a way that analytic techniques can be used to define the propagation in terms of tabulated functions (exponentials, Bessel functions, etc.). Furthermore, the characteristic-impedance matrix is made to smoothly transition from one section to the next (only slope discontinuities). Unfortunately, however, the interpolation between section end points produces an approximate characteristic-impedance matrix which is, in general, to some degree nonsymmetric. Ideally a symmetric interpolation matrix is desirable, thereby satisfying reciprocity. In this paper such a symmetric interpolation is developed.

2. Propagation Matrix as a Scalar Function Times the Identity Matrix

Assume a uniform medium with permittivity ε and permeability μ giving

$$
\begin{aligned}
v &= [\mu\,\varepsilon]^{-\frac{1}{2}} \equiv \text{wave speed} \\
Z &= \left[\frac{\mu}{\varepsilon}\right]^{\frac{1}{2}} = Y^{-1} \equiv \text{wave impedance}
\end{aligned}
\tag{2.1}
$$

There can also be a conductivity σ, but this can be combined with ε to give a complex, frequency-dependent permittivity. If μ and ε are positive real constants this will give special convenient properties for waves in time domain. Assuming perfectly conducting transmission-line conductors we have

$$
\begin{aligned}
\left(\tilde{Z}'_{n,m}(z,s)\right) &= s\,\mu\left(f_{g_{n,m}}(2)\right) \equiv \text{per-unit-length impedance matrix} \\
\left(\tilde{Y}'_{n,m}(z,s)\right) &= s\,\varepsilon\left(f_{g_{n,m}}(2)\right)^{-1} \equiv \text{per-unit-length admittance matrix} \\
\left(\tilde{\gamma}_{c_{n,m}}(s)\right) &= \left[\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right)\right]^{\frac{1}{2}} \\
&= \tilde{\gamma}(s)\left(1_{n,m}\right) \equiv \text{propagation matrix} \\
\tilde{\gamma}(s) &= \frac{s}{v} \equiv \text{propagation constant} \\
\sim\ &\equiv \text{two-sided Laplace transform (over time } t) \\
s &= \Omega + j\omega \equiv \text{Laplace-transform variable or complex frequency} \\
\left(Z_{c_{n,m}}(z)\right) &= Z\left(f_{g_{n,m}}(z)\right) = \left(Y_{c_{n,m}}(z)\right)^{-1} \equiv \text{characteristic-impedance matrix} \\
\left(f_{g_{n,m}}(z)\right) &= \left(f_{g_{n,m}}(z)\right)^{T} \equiv \text{geometric-factor matrix (dimensionless)}
\end{aligned}
\tag{2.2}
$$

The geometric-factor matrix is not only symmetric (reciprocity); it is real-valued. Furthermore, due to its impedance/admittance properties in (2.2), the passive nature makes it positive semi-definite, and we will generally assume it to be positive definite except in special limiting cases. Then we have

$$
\begin{aligned}
\left(f_{g_{n,m}}(z)\right) &= \sum_{\beta=1}^{N} f_{\beta}(z)\left(f_n(z)\right)_{\beta}\left(f_n(z)\right)_{\beta} \\
f_{\beta}(z) &> 0 \quad \text{(positive eigenvalues)} \\
\left(f_n(z)\right)_{\beta} &= \text{real eigenvectors} \\
\left(f_n(z)\right)_{\beta_1}\cdot\left(f_n(z)\right)_{\beta_2} &= 1_{\beta_1,\beta_2} \quad \text{(orthonormal)}
\end{aligned}
\tag{2.3}
$$

As a real symmetric matrix (special case of Hermitian matrix) this is always possible. This allows us to write

$$\begin{aligned} \left(f_{g_{n,m}}(z)\right)^{\alpha} &= \sum_{\beta=1}^{N} f_{\beta}^{\alpha}(z)\,(f_n(z))_{\beta}(f_n(z))_{\beta} \\ f_{\beta}^{\alpha}(z) &> 0 \text{ for } \alpha \text{ real} \end{aligned} \tag{2.4}$$

This gives us expressions for and ways to compute the inverse matrix as well as the square-root matrix and its inverse. Note that for real α, the matrix as above is also positive definite, real, and symmetric.

3. Symmetric Renormalization for Each Section

Starting from some z_0 (which can be zero meters if desired) let us label a set of values of z as

$$z_0 < z_1 < z_2 < \cdots < z_\ell < z_{\ell+1} < \cdots \tag{3.1}$$

Then some section (ℓth) of the transmission line is defined by

$$z_\ell \leq z \leq z_{\ell+1} \tag{3.2}$$

For this section, let us choose $(f_{g_{n,m}}(z_\ell))$ for purposes of renormalizing the telegrapher equations via

$$\begin{aligned}
\left(\tilde{v}_n^{(\ell)}(z,s)\right) &= \left(f_{g_{n,m}}(z_\ell)\right)^{-\frac{1}{2}} \cdot \left(\tilde{V}_n(z,s)\right) = \text{normalized voltage vector} \\
\left(\tilde{i}_n^{(\ell)}(z,s)\right) &= Z\left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} \cdot \left(\tilde{I}_n(z,s)\right) = \text{normalized current vector} \\
\left(\tilde{v}_n^{(s,\ell)'}(z,s)\right) &= \left(f_{g_{n,m}}(z_\ell)\right)^{-\frac{1}{2}} \cdot \left(\tilde{V}_n^{(s)'}(z,s)\right) = \text{normalized per-unit-length voltage-source vector} \\
\left(\tilde{i}_n^{(s,\ell)'}(z,s)\right) &= Z\left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} \cdot \left(\tilde{I}_n^{(s)'}(z,s)\right) = \text{normalized per-unit-length current-source vector}
\end{aligned} \tag{3.3}$$

Note that the normalized voltages and currents have the same units by appropriate inclusion of Z.

The telegrapher equations (1.1) then become

$$\begin{aligned}
\frac{\partial}{\partial z}\left(\tilde{v}_n^{(\ell)}(z,s)\right) &= -\tilde{\gamma}(s)\left(X_{n,m}^{(\ell)}(z)\right) \cdot \left(\tilde{i}_n^{(\ell)}(z,s)\right) + \left(\tilde{v}_n^{(s,\ell)'}(z,s)\right) \\
\frac{\partial}{\partial z}\left(\tilde{i}_n^{(\ell)}(z,s)\right) &= -\tilde{\gamma}(s)\left(X_{n,m}^{(\ell)}(z)\right)^{-1} \cdot \left(\tilde{v}_n^{(\ell)}(z,s)\right) + \left(\tilde{i}_n^{(s,\ell)'}(z,s)\right) \\
\left(X_{n,m}^{(\ell)}(z)\right) &= \tilde{\gamma}_{(s)}^{-1} \left(f_{g_{n,m}}(z_\ell)\right)^{-\frac{1}{2}} \cdot \left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(f_{g_{n,m}}(z_\ell)\right)^{-\frac{1}{2}} Y \\
&= \left(f_{g_{n,m}}(z_\ell)\right)^{-\frac{1}{2}} \cdot \left(f_{g_{n,m}}(z)\right) \cdot \left(f_{g_{n,m}}(z_\ell)\right)^{-\frac{1}{2}} \\
&= \left(X_{n,m}^{(\ell)}(z)\right)^T \\
\left(X_{n,m}^{(\ell)}(z)\right)^{-1} &= \tilde{\gamma}_{(s)}^{-1} Z \left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} \cdot \left(\tilde{Y}_{c_{n,m}}(z,s)\right) \cdot \left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} \\
&= \left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} \cdot \left(f_{g_{n,m}}(z)\right)^{-1} \cdot \left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}}
\end{aligned} \tag{3.4}$$

Now $\left(X_{n,m}^{(\ell)}(z)\right)$ (as well as its inverse) is symmetric and positive definite (as a product of positive definite matrices). As such it can be diagonalized with positive eigenvalues and real eigenvectors.

Instead of diagonalizing $\left(X_{n,m}^{(\ell)}(z)\right)$ for each z in the interval (and thereby have eigenvectors undesirably as a function of z), let us perform the diagonalization at $z = z_{\ell+1}$ giving

$$
\begin{aligned}
&\left(X_{n,m}^{(\ell)}(z_{\ell+1})\right) = \sum_{\beta=1}^{N} X_{\beta}^{(\ell)}(z_{\ell+1}) \left(x_n^{(\ell)}\right)_\beta \left(x_n^{(\ell)}\right)_\beta \\
&X_{\beta}^{(\ell)}(z_{\ell+1}) > 0 \\
&\det\left(\left(\tilde{X}_{n,m}^{(\ell)}(z_{\ell+1})\right) - X_{\beta}^{(\ell)}(z_{\ell+1}) \left(1_{n,m}\right)\right) = 0 \\
&\left(\left(X_{n,m}^{(\ell)}(z_{\ell+1})\right) \cdot \left(x_n^{(\ell)}\right)_\beta = \left(x_n^{(\ell)}\right)_\beta \cdot \left(X_{n,m}^{(\ell)}(z_{\ell+1})\right) = X_{\beta}^{(\ell)}(z_{\ell+1}) \left(x_n^{(\ell)}\right)_\beta\right) \\
&\left(x_n^{(\ell)}\right)_{\beta_1} \cdot \left(x_n^{(\ell)}\right)_{\beta_2} = 1_{\beta_1\beta_2} \quad \text{(orthonormal)} \\
&\left(x_n^{(1)}\right)_\beta \quad \text{real}
\end{aligned}
\tag{3.5}
$$

Note that the eigenvectors are not indicated as functions of z (deliberately). At $z = z_\ell$ (beginning of the interval) we have

$$
\begin{aligned}
&\left(X_{n,m}^{(\ell)}(z_\ell)\right) = \left(1_{n,m}\right) = \sum_{\beta=1}^{N} \left(x_n^{(\ell)}\right)_\beta \left(x_n^{(\ell)}\right)_\beta \\
&X_{\beta}^{(\ell)}(z_\ell) = 1 \quad \text{for} \quad \beta = 1, 2, \cdots, N
\end{aligned}
\tag{3.6}
$$

The same eigenvectors can be used in this case since any complete set can be used for the identity.

So let us define an approximation to $\left(X_{n,m}^{(\ell)}(z)\right)$ as

$$
\begin{aligned}
&\left(X_{n,m}^{(\ell,a)}(z)\right) \equiv \sum_{\beta=1}^{N} X_{\beta}^{(\ell,a)}(z) \left(x_n^{(\ell)}\right)_\beta \left(x_n^{(\ell)}\right)_\beta \\
&\left(X_{n,m}^{(\ell,a)}(z_\ell)\right) = \left(1_{n,m}\right) = \left(X_{n,m}^{(\ell)}(z_\ell)\right) \\
&\left(X_{n,m}^{(\ell,a)}(z_{\ell+1})\right) = \left(X_{n,m}^{(\ell)}(z_{\ell+1})\right)
\end{aligned}
\tag{3.7}
$$

(with superscript a identifying an approximation). The eigenvalues of the approximating matrix are chosen to match those of the original matrix at the interval end points as

$$X_\beta^{(\ell,a)}(z_\ell) = X_\beta^{(\ell)}(z_\ell) = 1 \quad , \quad \tilde{X}_\beta^{(\ell,a)}(z_{\ell+1}) = \tilde{X}_\beta^{(\ell)}(z_{\ell+1}) \tag{3.8}$$

and assume a convenient smooth functional form (polynomial, exponential, etc. in z) in the interval. Of course, let us require

$$X_\beta^{(\ell,a)}(z) > 0 \quad \text{for} \quad z_\ell \le z \le z_{\ell+1} \tag{3.9}$$

so that the approximating matrix is real, symmetric, and positive definite in the interval.

From (3.4), using the approximating matrix we have an approximation to the geometric-factor matrix as

$$\begin{aligned}
\left(f_{g_{n,m}}^{(\ell,a)}(z)\right) &= \left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} \cdot \left(X_{n,m}^{(\ell,a)}(z)\right) \cdot \left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} \\
&= \sum_{\beta=1}^{N} X_\beta^{(\ell,a)}(z) \left[\left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} \cdot (x_n)_\beta\right]\left[\left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} \cdot (x_n)_\beta\right] \\
&= \text{real, symmetric, positive-definite matrix} \\
\left(f_{g_{n,m}}^{(\ell,a)}(z_\ell)\right) &= \left(f_{g_{n,m}}(z_\ell)\right) \quad , \quad \left(f_{g_{n,m}}^{(\ell,a)}(z_{\ell+1})\right) = \left(f_{g_{n,m}}(z_{\ell+1})\right)
\end{aligned} \tag{3.10}$$

By this interpolation scheme then $\left(f_{g_{n,m}}^{(\ell,a)}(z)\right)$ is kept continuous as one passes from one interval (ℓ) to the next $(\ell+1)$, crossing each boundary between adjacent intervals. Furthermore, as compared to the scheme in [5], the geometric-factor matrix (and hence characteristic impedance matrix) is symmetric for all z, exactly satisfying reciprocity.

4. Solution for Each Section in Normalized Form

In supermatrix form the telegrapher equations as in (3.4) become

$$\frac{\partial}{\partial z}\begin{pmatrix}(\tilde{v}_n^{(\ell)}(z,s))\\(\tilde{i}_n^{(\ell)}(z,s))\end{pmatrix} = \left(\left(\tilde{\xi}_{n,m}^{(\ell)}(z,s)\right)_{p,p'}\right) \odot \begin{pmatrix}(\tilde{v}_n^{(\ell)}(z,s))\\(\tilde{i}_n^{(\ell)}(z,s))\end{pmatrix} + \begin{pmatrix}(\tilde{v}_n^{(s,\ell)'}(z,s))\\(\tilde{i}_n^{(s,\ell)'}(z,s))\end{pmatrix}$$

$$\left(\left(\tilde{\xi}_{n,m}^{(\ell)}(z,s)\right)_{p,p'}\right) = -\tilde{\gamma}(s)\begin{pmatrix}(0_{n,m}) & \left(X_{n,m}^{(\ell)}(z)\right)\\ \left(X_{n,m}^{(\ell)}(z)\right)^{-1} & (0_{n,m})\end{pmatrix} \tag{4.1}$$

The related supermatrizant equation is

$$\frac{\partial}{\partial z}\left(\left(\tilde{\Xi}_{n,m}^{(\ell)}(z,z_\ell;s)\right)_{p,p'}\right) = \left(\left(\tilde{\xi}_{n,m}^{(\ell)}(z,s)\right)\right)_{p,p'} \odot \left(\left(\tilde{\Xi}_{n,m}^{(\ell)}(z,z_\ell;s)\right)_{p,p'}\right)$$

$$\left(\left(\tilde{\Xi}_{n,m}^{(\ell)}(z_\ell,z_\ell;s)\right)_{p,p'}\right) = \left(\left(1_{n,m}\right)_{p,p'}\right) = \begin{pmatrix}(1_{n,m}) & (0_{n,m})\\(0_{n,m}) & (1_{n,m})\end{pmatrix} \tag{4.2}$$

(boundary condition)

with the solution in product-integral form [6] as

$$\left(\left(\tilde{\Xi}_{n,m}^{(\ell)}(z_\ell,z_\ell;s)\right)_{p,p'}\right) = \prod_{z_\ell}^{z} e^{\left(\left(\tilde{\xi}_{n,m}^{(\ell)}(z',s)\right)_{p,p'}\right)dz'} \tag{4.3}$$

In terms of this, the solution to (4.1) is

$$\begin{pmatrix}\left(\tilde{v}_n^{(\ell)}(z,s)\right)\\ \left(\tilde{i}_n^{(\ell)}(z,s)\right)\end{pmatrix} = \left(\left(\tilde{\Xi}_{n,m}^{(\ell)}(z,z_\ell;s)\right)_{p,p'}\right) \odot \begin{pmatrix}\left(\tilde{v}_n^{(\ell)}(z,s)\right)\\ \left(\tilde{i}_n^{(\ell)}(z,s)\right)\end{pmatrix}$$

$$+\int_{z_\ell}^{z}\left(\left(\tilde{\Xi}_{n,m}^{(\ell)}(z,z';s)\right)_{p,p'}\right) \odot \begin{pmatrix}\left(\tilde{v}_n^{(s,\ell)'}(z',s)\right)\\ \left(\tilde{i}_n^{(s,\ell)'}(z',s)\right)\end{pmatrix}dz' \tag{4.4}$$

So all we need is the solution (product integral) for the matrizant. For convenience we also have

$$\left(\left(\tilde{\Xi}_{n,m}^{(\ell)}(z,z';s)\right)_{p,p'}\right) = \left(\left(\tilde{\Xi}^{(\ell)}(z,z_\ell;s)\right)_{p,p'}\right) \odot \left(\left(\tilde{\Xi}^{(\ell)}(z_\ell,z';s)\right)_{p,p'}\right)$$
$$= \left(\left(\tilde{\Xi}^{(\ell)}(z,z_\ell;s)\right)_{p,p'}\right) \odot \left(\left(\tilde{\Xi}^{(\ell)}(z',z_\ell;s)\right)_{p,p'}\right)^{-1} \tag{4.5}$$

so that one can have all supermatrizants referenced to z_ℓ if desired.

In Section 3 an approximation is developed for $\left(X_{n,m}^{(\ell)}(z)\right)$. This can be used to give an approximation for the matrizant via

$$\left(\left(\tilde{\xi}_{n,m}^{(\ell,a)}(z,s)\right)_{p,p'}\right) = -\tilde{\gamma}(s)\begin{pmatrix} (0_{n,m}) & \left(X^{(\ell,a)}(z)\right) \\ -\left(X^{(\ell,a)}(z)\right)^{-1} & (0_{n,m}) \end{pmatrix}$$
$$= -\tilde{\gamma}(s)\begin{pmatrix} (0_{n,m}) & \sum_{\beta=1}^{N} X_\beta^{(\ell,a)}(z)\left(x_n^{(\ell)}\right)_\beta \left(x_n^{(\ell)}\right)_\beta \\ \sum_{\beta=1}^{N} X_\beta^{(\ell,a)^{-1}}(z)\left(x_n^{(\ell)}\right)_\beta \left(x_n^{(\ell)}\right)_\beta & (0_{n,m}) \end{pmatrix}$$
$$= -\tilde{\gamma}(s)\sum_{\beta=1}^{N}\begin{pmatrix} (0_{n,m}) & X_\beta^{(\ell,a)}(z)\left(x_n^{(\ell)}\right)_\beta \left(x_n^{(\ell)}\right)_\beta \\ X_\beta^{(\ell,a)^{-1}}(z)\left(x_n^{(\ell)}\right)_\beta \left(x_n^{(\ell)}\right)_\beta & (0_{n,m}) \end{pmatrix}$$
$$= -\tilde{\gamma}(s)\sum_{\beta=1}^{N}\left[\left(x_n^{(\ell)}\right)_\beta \left(x_n^{(\ell)}\right)_\beta\right] \otimes \begin{pmatrix} 0 & X_\beta^{(\ell,a)}(z) \\ X_\beta^{(\ell,a)^{-1}}(z) & 0 \end{pmatrix} \tag{4.6}$$
$$\left(\left(\tilde{\Xi}_{n,m}^{(\ell,a)}(z,z_\ell;s)\right)_{p,p'}\right) = \prod_{z_\ell}^{z} e^{\left(\left(\tilde{\xi}_{n,m}^{(\ell,a)}(z',s)\right)_{p,p'}\right) dz'}$$

This can then be applied to the normalized voltage and current variables in (4.1) to give approximations $\left(\tilde{v}_n^{(\ell,a)}(z,s)\right)$ and $\left(\tilde{i}_n^{(\ell,a)}(z,s)\right)$. (See Appendix A for the use of the direct product $\otimes$ in forming supermatrices.)

Notice in (4.6) that there is a partial decomposition of the matrizant since

$$\left[\left[\left(x_n^{(\ell)}\right)_{\beta_1}\left(x_n^{(\ell)}\right)_{\beta_1}\right] \otimes \begin{pmatrix} 0 & X_{\beta_1}^{(\ell,a)}(z') \\ X_{\beta_1}^{(\ell,a)^{-1}}(z') & 0 \end{pmatrix}\right]$$

$$\odot \left[\left[\left(x_n^{(\ell)}\right)_{\beta_2}\left(x_n^{(\ell)}\right)_{\beta_2}\right] \otimes \begin{pmatrix} 0 & X_{\beta_2}^{(\ell,a)}(z'') \\ X_{\beta_2}^{(\ell,a)^{-1}}(z'') & 0 \end{pmatrix}\right]$$

$$= \left[\left[\left(x_n^{(\ell)}\right)_{\beta_1}\left(x_n^{(\ell)}\right)_{\beta_1}\right] \cdot \left[\left(x_n^{(\ell)}\right)_{\beta_2}\left(x_n^{(\ell)}\right)_{\beta_2}\right]\right]$$

$$\otimes \left[\begin{pmatrix} 0 & X_{\beta_1}^{(\ell,a)}(z') \\ X_{\beta_1}^{(\ell,a)^{-1}}(z') & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & X_{\beta}^{(\ell,a)}(z'') \\ X_{\beta}^{(\ell,a)^{-1}}(z'') & 0 \end{pmatrix}\right] \tag{4.7}$$

$$= \left(0_{n,m}\right) \otimes \left(0_{p,p'}\right) \quad \text{for} \quad \beta_1 \neq \beta_2 \text{ and all } (z', z'') \text{ pairs in the interval}$$

With the terms in the exponential thereby commuting we have (using (B.8) and (B.20))

$$e^{\left[\left(\tilde{\xi}_{n,m}^{(\ell,a)}(z,s)\right)_{p,p'}\right]}$$

$$= \bigodot_{\beta=1}^{N} \exp\left(-\tilde{\gamma}(s)\left[\left(x_n^{(\ell)}\right)_{\beta}\left(x_n^{(\ell)}\right)_{\beta}\right] \otimes \begin{pmatrix} 0 & X_{\beta}^{(\ell,a)}(z) \\ X_{\beta}^{(\ell,a)^{-1}}(z) & 0 \end{pmatrix}\right)$$

$$= \bigodot_{\beta=1}^{N} \left[\left[\left(1_{n,m}\right) - \left(x_n^{(\ell)}\right)_{\beta}\left(x_n^{(\ell)}\right)_{\beta}\right] \otimes \left(1_{p,p'}\right)\right.$$

$$\left. + \left[\left(x_n^{(\ell)}\right)_{\beta}\left(x_n^{(\ell)}\right)_{\beta}\right] \otimes \exp\left(-\tilde{\gamma}(s)\begin{pmatrix} 0 & X_{\beta}^{(\ell,a)}(z) \\ X_{\beta}^{(\ell,a)^{-1}}(z) & 0 \end{pmatrix}\right)\right] \tag{4.8}$$

where the terms in the N-fold products over the eigenindex β can be taken in any order due to the commutativity of the terms. This result can be further simplified by successively dot multiplying the terms and using the orthogonality of the eigendyads as

$$
\begin{aligned}
&e^{\left(\left(\tilde{\xi}_{n,m}(z,s)\right)_{p,p'}\right)} \\
&= \left[\left[\left(1_{n,m}\right) - \sum_{\beta=1}^{N}\left(x_n^{(\ell)}\right)_\beta\left(x_n^{(\ell)}\right)_\beta\right] \otimes \left(1_{p,p'}\right)\right. \\
&\qquad \left. + \sum_{\beta=1}^{N}\left[\left(x_n^{(\ell)}\right)_\beta\left(x_n^{(\ell)}\right)_\beta\right] \otimes \exp\left(-\tilde{\gamma}(s)\begin{pmatrix} 0 & X_\beta^{(\ell,a)}(z) \\ X_\beta^{(\ell,a)^{-1}}(z) & 0 \end{pmatrix}\right)\right] \qquad (4.9) \\
&= \sum_{\beta=1}^{N}\left[\left(x_n^{(\ell)}\right)_\beta\left(x_n^{(\ell)}\right)_\beta\right] \otimes \exp\left(-\tilde{\gamma}(s)\begin{pmatrix} 0 & X_\beta^{(\ell,a)}(z) \\ X_\beta^{(\ell,a)^{-1}}(z) & 0 \end{pmatrix}\right)
\end{aligned}
$$

where the expansion of the identity as a sum of eigendyads in (3.6) is used. This is extended to the product integral (using (B.14) and (B.27) as

$$
\begin{aligned}
&\left(\left(\tilde{\Xi}_{n,m}^{(\ell,a)}(z,z_\ell;s)\right)_{p,p'}\right) \\
&= \bigodot_{\beta=1}^{N}\left[\prod_{z_\ell}^{z}\exp\left(-\tilde{\gamma}(s)\left[(x_n)_\beta\,(x_n)_\beta\right] \otimes \begin{pmatrix} 0 & X_\beta^{(\ell,a)}(z) \\ X_\beta^{(\ell,a)^{-1}}(z) & 0 \end{pmatrix}dz'\right)\right] \\
&= \bigodot_{\beta=1}^{N}\left[\left[\left(1_{n,m}\right) - \left(x_n^{(\ell)}\right)_\beta\left(x_n^{(\ell)}\right)_\beta\right] \otimes \left(1_{p,p'}\right)\right. \qquad (4.10) \\
&\qquad \left. + \left[\left(x_n^{(\ell)}\right)_\beta\left(x_n^{(\ell)}\right)_\beta\right] \otimes \left[\prod_{z_\ell}^{z}\exp\left(-\tilde{\gamma}(s)\begin{pmatrix} 0 & X_\beta^{(\ell,a)}(z') \\ X_\beta^{(\ell,a)^{-1}}(z') & 0 \end{pmatrix}dz'\right)\right]\right]
\end{aligned}
$$

Again the N-fold products can be taken in any order due to the commutativity of the terms. (See Appendix B for various formulae involving the product integral.) This can be further simplified by multiplying out the terms and using the orthogonality of the eigendyads as

$$
\begin{aligned}
&\left(\left(\Xi_{n,m}^{(\ell,a)}(z,z_\ell;s)\right)_{p,p'}\right)\\
&=\left[\left(1_{n,m}\right)-\sum_{\beta=1}^{N}\left(x_n^{(\ell)}\right)_\beta\left(x_n^{(\ell)}\right)_\beta\right]\otimes\left(1_{p,p'}\right)\\
&\quad+\sum_{\beta=1}^{N}\left[\left(x_n^{(\ell)}\right)_\beta\left(x_n^{(\ell)}\right)_\beta\right]\otimes\left[\prod_{z_\ell}^{z}\exp\left(-\tilde{\gamma}(s)\begin{pmatrix}0 & X_\beta^{(\ell,a)}(z')\\ X_\beta^{(\ell,a)^{-1}}(z') & 0\end{pmatrix}dz'\right)\right]\\
&=\sum_{\beta=1}^{N}\left[\left(x_n^{(\ell)}\right)_\beta\left(x_n^{(\ell)}\right)_\beta\right]\otimes\left[\prod_{z_\ell}^{z}\exp\left(-\tilde{\gamma}(s)\begin{pmatrix}0 & X_\beta^{(\ell,a)}(z')\\ X_\beta^{(\ell,a)^{-1}}(z') & 0\end{pmatrix}dz'\right)\right]
\end{aligned}
\tag{4.11}
$$

The matrizant is now decomposed into a sum of product integrals, with each product integral over only a 2 x 2 matrix function. This constituent product integral is the next step.

5. Solution for Normalized Supermatrizant via Second-Order Differential Equation

In order to solve for the matrizant, it is instructive to return to the telegrapher equations (3.4) which are now stated in terms of our approximation for $\left(X_{n,m}(z)\right)$. Writing out the telegrapher equations in terms of our approximation we have

$$
\begin{aligned}
\frac{\partial}{\partial z}\left(\tilde{v}_n^{(\ell,a)}(z,s)\right) &= -\tilde{\gamma}(s)\left(X_{n,m}^{(\ell,a)}(z)\right)\cdot\left(\tilde{i}_n^{(\ell,a)}(z,s)\right)+\left(\tilde{v}_n^{(s,\ell)'}(z,s)\right)\\
\frac{\partial}{\partial z}\left(\tilde{i}_n^{(\ell,a)}(z,s)\right) &= -\tilde{\gamma}(s)\left(X_{n,m}^{(\ell,a)}(z)\right)^{-1}\cdot\left(\tilde{v}_n^{(\ell,a)}(z,s)\right)+\left(\tilde{i}_n^{(s,\ell)'}(z,s)\right)\\
\frac{\partial}{\partial z}\begin{pmatrix}\left(\tilde{v}_n^{(\ell,a)}(z,s)\right)\\ \left(\tilde{i}_n^{(\ell,a)}(z,s)\right)\end{pmatrix} &= \left(\left(\xi_{n,m}^{(\ell,a)}(z,s)\right)_{p,p'}\right)\odot\begin{pmatrix}\left(\tilde{v}_n^{(\ell,a)}(z,s)\right)\\ \left(\tilde{i}_n^{(\ell,a)}(z,s)\right)\end{pmatrix}+\begin{pmatrix}\left(\tilde{v}_n^{(s,a)'}(z,s)\right)\\ \left(\tilde{i}_n^{(s,a)'}(z,s)\right)\end{pmatrix}
\end{aligned}
\tag{5.1}
$$

For purposes of the matrizant we need only consider the homogeneous equations (no sources). From the diagonal representation of the coefficient matrix in (3.7) we have

$$
\left(X_{n,m}^{(\ell,a)}(z)\right)^{\pm 1} = \sum_{\beta=1}^{N} X_{\beta}^{(\ell,a)^{\pm 1}}(z)\left(x_n^{(\ell)}\right)_{\beta}\left(x_n^{(\ell)}\right)_{\beta}
\tag{5.2}
$$

Defining

$$
\begin{aligned}
\tilde{v}_{\beta}^{(\ell)}(z,s) &\equiv \left(x_n^{(\ell)}\right)_{\beta}\cdot\left(\tilde{v}_n^{(\ell,a)}(z,s)\right)\\
\tilde{i}_{\beta}^{(\ell)}(z,s) &\equiv \left(x_n^{(\ell)}\right)_{\beta}\cdot\left(\tilde{v}_n^{(\ell,a)}(z,s)\right)
\end{aligned}
\tag{5.3}
$$

we have

$$
\begin{aligned}
\left(\tilde{v}_n^{(\ell,a)}(z,s)\right) &= \sum_{\beta=1}^{N}\tilde{v}_{\beta}^{(\ell)}(z,s)\left(x_n\right)_{\beta}\\
\left(\tilde{i}_n^{(\ell,a)}(z,s)\right) &= \sum_{\beta=1}^{N}\tilde{i}_{\beta}^{(\ell)}(z,s)\left(x_n\right)_{\beta}
\end{aligned}
\tag{5.4}
$$

scalarizing the equations (homogeneous) as

$$\frac{\partial}{\partial z}\tilde{v}_{\beta}^{(\ell)}(z,s) = -\tilde{\gamma}(s)\; X_{\beta}^{(\ell,a)}(z)\; \tilde{i}_{\beta}^{(\ell)}(z,s)$$
$$\frac{\partial}{\partial z}\tilde{i}_{\beta}^{(\ell)}(z,s) = -\tilde{\gamma}(s)\; X_{\beta}^{(\ell,a)^{-1}}(z)\; \tilde{v}_{\beta}^{(\ell)}(z,s) \tag{5.5}$$
$$\frac{\partial}{\partial z}\begin{pmatrix}\tilde{v}_{\beta}^{(\ell)}(z,s)\\ \tilde{i}_{\beta}^{(\ell)}(z,s)\end{pmatrix} = -\tilde{\gamma}(s)\begin{pmatrix}0 & X_{\beta}^{(\ell,a)}(z)\\ X_{\beta}^{(\ell,a)^{-1}}(z) & 0\end{pmatrix}\cdot\begin{pmatrix}\tilde{v}_{\beta}^{(\ell)}(z,s)\\ \tilde{i}_{\beta}^{(\ell)}(z,s)\end{pmatrix}$$

The product integral of the coefficient matrix is exactly that appearing in (4.11).

Following the procedure in [5] we form the second-order differential equations

$$\left[\frac{\partial^2}{\partial z^2} - \left[\frac{\partial}{\partial z}\ell n\left(X_{\beta}^{(\ell,a)}(z)\right)\right]\frac{\partial}{\partial z} - \tilde{\gamma}^2(s)\right]\tilde{v}_{\beta}^{(\ell)}(z,s) = 0$$
$$\left[\frac{\partial^2}{\partial z^2} + \left[\frac{\partial}{\partial z}\ell n\left(X_{\beta}^{(\ell,a)}(z)\right)\right]\frac{\partial}{\partial z} - \tilde{\gamma}^2(s)\right]\tilde{i}_{\beta}^{(\ell)}(z,s) = 0 \tag{5.6}$$

These have two linearly independent solutions (superscript $\delta = 1, 2$) related by

$$\frac{\partial}{\partial z}\tilde{v}_{\beta}^{(\ell,\delta)}(z,s) = -\tilde{\gamma}(s)\; X_{\beta}^{(\ell,a)}(z)\; \tilde{i}_{\beta}^{(\ell,a)}(z,s)$$
$$\frac{\partial}{\partial z}\tilde{i}_{\beta}^{(\ell,\delta)}(z,s) = -\tilde{\gamma}(s)\; X_{\beta}^{(\ell,a)^{-1}}(z)\; \tilde{v}_{\beta}^{(\ell,a)}(z,s) \tag{5.7}$$
$$\delta = 1, 2$$

Examples of such solution (linear and exponential variation of eigenvalues) are treated in [5 (Appendices)].

This can be restated for the matrizant

$$\left(\tilde{\psi}_{p,p'}^{(\ell,a)}(z,z_\ell;s)\right)_{\beta} = \prod_{z_\ell}^{z}\exp\left(-\tilde{\gamma}(s)\begin{pmatrix}0 & X_{\beta}^{(\ell,a)}(z')\\ X_{\beta}^{(\ell,a)^{-1}}(z') & 0\end{pmatrix}dz'\right)$$
$$\frac{d}{dz}\left(\tilde{\psi}_{p,p'}^{(\ell,a)}(z,z_\ell;s)\right)_{\beta} = \exp\left(-\tilde{\gamma}(s)\begin{pmatrix}0 & X_{\beta}^{(\ell,a)}(z)\\ X_{\beta}^{(\ell,a)^{-1}}(z) & 0\end{pmatrix}\right)\cdot\left(\tilde{\psi}_{p,p'}(z,z_\ell;s)\right)_{\beta} \tag{5.8}$$
$$\left(\tilde{\psi}_{p,p'}(z_\ell,z_\ell;s)\right)_{\beta} = \left(1_{p,p'}\right) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

such as appears in (4.11). Writing out the equations for the four components we have the first-order differential equations

$$\begin{aligned}
\frac{\partial}{\partial z}\,\tilde{\Psi}^{(\ell,\alpha)}_{1,p';\beta}(z,z_\ell;s) &= -\tilde{\gamma}(s)\,X^{(\ell,\alpha)}_{\beta}(z)\;\tilde{\Psi}^{(\ell,\alpha)}_{2,p';\beta}(z,z_\ell;s)\\
\frac{\partial}{\partial z}\,\tilde{\Psi}^{(\ell,\alpha)}_{2,p';\beta}(z,z_\ell;s) &= -\tilde{\gamma}(s)\,X^{(\ell,\alpha)^{-1}}_{\beta}(z)\;\tilde{\Psi}^{(\ell,\alpha)}_{1,p';\beta}(z,z_\ell;s)\\
p' &= 1,2
\end{aligned} \tag{5.9}$$

and second-order differential equations

$$\begin{aligned}
&\left[\frac{\partial^2}{\partial z^2} - \left[\frac{\partial}{\partial z}\,\ell n\left(X^{(\ell,\alpha)}_{\beta}(z)\right)\right]\frac{\partial}{\partial z} - \tilde{\gamma}^2(s)\right]\tilde{\Psi}^{(\ell,\alpha)}_{1,p';\beta}(z,z_\ell;s) = 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(voltage like)}\\
&\left[\frac{\partial^2}{\partial z^2} + \left[\frac{\partial}{\partial z}\,\ell n\left(X^{(\ell,\alpha)}_{\beta}(z)\right)\right]\frac{\partial}{\partial z} - \tilde{\gamma}^2(s)\right]\tilde{\Psi}^{(\ell,\alpha)}_{2,p';\beta}(z,z_\ell;s) = 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(current like)}\\
&\qquad p' = 1,2
\end{aligned} \tag{5.10}$$

subject to the boundary conditions at $z = z_\ell$ in (5.8). These second-order equations are the same as (5.6) and have the solutions in (5.7). Therefore we write

$$\begin{aligned}
\tilde{\Psi}^{(\ell,\alpha)}_{1,p';\beta}(z,z_\ell;s) &= \sum_{\delta=1}^{2} d^{(\beta,\delta)}_{1,p'}\;\tilde{v}^{(\ell,\delta)}_{\beta}(z,s)\\
\tilde{\Psi}^{(\ell,\alpha)}_{2,p';\beta}(z,z_\ell;s) &= \sum_{\delta=1}^{2} d^{(\beta,\delta)}_{2,p'}\;\tilde{i}^{(\ell,\delta)}_{\beta}(z,s)\\
p' &= 1,2
\end{aligned} \tag{5.11}$$

There are then eight coefficients $d^{(\beta,\delta)}_{p,p'}$ for each β to determine.

From (5.7) and (5.9) we find

$$
\begin{aligned}
\frac{\partial}{\partial z}\tilde{\psi}^{(\ell,a)}_{1,p';\beta}(z,z_\ell;s) &= -\tilde{\gamma}(s)\,X^{(\ell,a)}_{\beta}(z)\sum_{\delta=1}^{2} d^{(\beta,\delta)}_{1,p'}\;\tilde{i}^{(\ell,\delta)}_{\beta}(z,s)\\
&= -\tilde{\gamma}(s)\,X^{(\ell,a)}_{\beta}(z)\sum_{\delta=1}^{2} d^{(\beta,\delta)}_{2,p'}\;\tilde{i}^{(\ell,\delta)}_{\beta}(z,s)\\
\frac{\partial}{\partial z}\tilde{\psi}^{(\ell,a)}_{2,p';\beta}(z,z_\ell;s) &= -\tilde{\gamma}(s)\,X^{(\ell,a)^{-1}}_{\beta}(z)\sum_{\delta=1}^{2} d^{(\beta,\delta)}_{2,p'}\;\tilde{v}^{(\ell,\delta)}_{\beta}(z,s)\\
&= -\tilde{\gamma}(s)\,X^{(\ell,a)^{-1}}_{\beta}(z)\sum_{\delta=1}^{2} d^{(\beta,\delta)}_{1,p'}\;\tilde{v}^{(\ell,\delta)}_{\beta}(z,s)\\
p' &= 1,2
\end{aligned}
\tag{5.12}
$$

from which we conclude

$$
d^{(\beta,\delta)}_{1,p'} = d^{(\beta,\delta)}_{2,p'} \quad \text{for} \quad p' = 1,2 \text{ and } \delta = 1,2
\tag{5.13}
$$

leaving only four unknown coefficients. Apply boundary conditions at $z = z_\ell$ from (5.8) as

$$
\begin{aligned}
\tilde{\psi}^{(\ell,a)}_{1,1;\beta}(z_\ell,z_\ell;s) &= 1 = \sum_{\delta=1}^{2} d^{(\beta,\delta)}_{1,1}\;\tilde{v}^{(\ell,\delta)}_{\beta}(z_\ell,s)\\
\tilde{\psi}^{(\ell,a)}_{1,2;\beta}(z_\ell,z_\ell;s) &= 0 = \sum_{\delta=1}^{2} d^{(\beta,\delta)}_{1,2}\;\tilde{v}^{(\ell,\delta)}_{\beta}(z_\ell,s)\\
\tilde{\psi}^{(\ell,a)}_{2,1;\beta}(z_\ell,z_\ell;s) &= 0 = \sum_{\delta=1}^{2} d^{(\beta,\delta)}_{1,1}\;\tilde{i}^{(\ell,\delta)}_{\beta}(z_\ell,s)\\
\tilde{\psi}^{(\ell,a)}_{2,2;\beta}(z_\ell,z_\ell;s) &= 1 = \sum_{\delta=1}^{2} d^{(\beta,\delta)}_{1,2}\;\tilde{i}^{(\ell,\delta)}_{\beta}(z_\ell,s)
\end{aligned}
\tag{5.14}
$$

giving

$$
\begin{aligned}
\begin{pmatrix}1\\0\end{pmatrix} &= \begin{pmatrix}\tilde{v}^{(\ell,1)}_{\beta}(z_\ell,s) & \tilde{v}^{(\ell,2)}_{\beta}(z_\ell,s)\\ \tilde{i}^{(\ell,1)}_{\beta}(z_\ell,s) & \tilde{i}^{(\ell,2)}_{\beta}(z_\ell,s)\end{pmatrix}\cdot\begin{pmatrix}d^{(\beta,1)}_{1,1}\\ d^{(\beta,2)}_{1,1}\end{pmatrix}\\
\begin{pmatrix}1\\0\end{pmatrix} &= \begin{pmatrix}\tilde{i}^{(\ell,1)}_{\beta}(z_\ell,s) & \tilde{i}^{(\ell,2)}_{\beta}(z_\ell,s)\\ \tilde{v}^{(\ell,1)}_{\beta}(z_\ell,s) & \tilde{v}^{(\ell,2)}_{\beta}(z_\ell,s)\end{pmatrix}\cdot\begin{pmatrix}d^{(\beta,1)}_{1,2}\\ d^{(\beta,2)}_{1,2}\end{pmatrix}
\end{aligned}
\tag{5.15}
$$

This is inverted as

$$
\begin{pmatrix} d_{1,1}^{(\beta,1)} \\ d_{1,1}^{(\beta,2)} \end{pmatrix} = \tilde{\Delta}_{\beta}^{(\ell)}(s) \begin{pmatrix} \tilde{i}_{\beta}^{(\ell,2)}(z_\ell,s) & -\tilde{v}_{\beta}^{(\ell,2)}(z_\ell,s) \\ -\tilde{i}_{\beta}^{(\ell,1)}(z_\ell,s) & \tilde{v}_{\beta}^{(\ell,1)}(z_\ell,s) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tilde{\Delta}_{\beta}^{(\ell)}(s) \begin{pmatrix} \tilde{i}_{\beta}^{(\ell,2)}(z_\ell,s) \\ -\tilde{i}_{\beta}^{(\ell,1)}(z_\ell,s) \end{pmatrix}
$$

$$
\begin{pmatrix} d_{1,2}^{(\beta,1)} \\ d_{1,2}^{(\beta,2)} \end{pmatrix} = -\tilde{\Delta}_{\beta}^{(\ell)}(s) \begin{pmatrix} \tilde{v}_{\beta}^{(\ell,2)}(z_\ell,s) & -\tilde{i}_{\beta}^{(\ell,2)}(z_\ell,s) \\ -\tilde{v}_{\beta}^{(\ell,1)}(z_\ell,s) & \tilde{i}_{\beta}^{(\ell,1)}(z_\ell,s) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\tilde{\Delta}_{\beta}^{(\ell)}(s) \begin{pmatrix} \tilde{v}_{\beta}^{(\ell,2)}(z_\ell,s) \\ -\tilde{v}_{\beta}^{(\ell,1)}(z_\ell,s) \end{pmatrix}
$$

$$
\begin{aligned}
\tilde{\Delta}_{\beta}^{(\ell)^{-1}}(s) &= \tilde{v}_{\beta}^{(\ell,1)}(z_\ell,s)\, \tilde{i}_{\beta}^{(\ell,2)}(z_\ell,s) - \tilde{v}_{\beta}^{(\ell,2)}(z_\ell,s)\, \tilde{i}_{\beta}^{(\ell,1)}(z_\ell,s) \\
&= -\tilde{\gamma}^{-1}(s)\left[\tilde{v}_{\beta}^{(\ell,1)}(z_\ell,s) \frac{\partial}{\partial z} \tilde{v}_{\beta}^{(\ell,2)}(z,s)\Big|_{z=z_\ell} - \tilde{v}_{\beta}^{(\ell,2)}(z_\ell,s) \frac{\partial}{\partial z} \tilde{v}_{\beta}^{(\ell,1)}(z,s)\Big|_{z=z_\ell} \right] \qquad (5.16) \\
&= \tilde{\gamma}^{-1}(s)\left[\tilde{i}_{\beta}^{(\ell,1)}(z_\ell,s) \frac{\partial}{\partial z} \tilde{i}_{\beta}^{(\ell,2)}(z,s)\Big|_{z=z_\ell} - \tilde{i}_{\beta}^{(\ell,2)}(z_\ell,s) \frac{\partial}{\partial z} \tilde{i}_{\beta}^{(\ell,1)}(z,s)\Big|_{z=z_\ell} \right]
\end{aligned}
$$

thereby solving for all the coefficients. Note that the determinant above has the form of a Wronskian of the two independent solutions of the second-order differential equation. This will be useful when specific problems are considered and special functions (e.g., Bessel functions) are encountered. Note that $\tilde{\gamma}(s)$ can be included with ∂z as $\partial(\tilde{\gamma}(s)z)$ for this purpose.

So now we have the matrizant

$$
\begin{aligned}
\left(\tilde{\psi}_{p,p'}^{(\ell,a)}(z,z_\ell;s)\right)_{\beta} &= \prod_{z_\ell}^{z} \exp\left(-\tilde{\gamma}(s) \begin{pmatrix} 0 & X_{\beta}^{(\ell,a)}(z) \\ X_{\beta}^{(\ell,a)^{-1}}(z) & 0 \end{pmatrix} dz' \right) \\
&= \tilde{\Delta}_{\beta}^{(\ell)}(s) \begin{pmatrix} \left[\tilde{i}_{\beta}^{(\ell,2)}(z_\ell,s)\, v_{\beta}^{(\ell,1)}(z,s) - \tilde{i}_{\beta}^{(\ell,1)}(z_\ell,s) \tilde{v}_{\beta}^{(\ell,2)}(z,s) \right] & \left[-\tilde{v}_{\beta}^{(\ell,2)}(z_\ell,s)\, \tilde{v}_{\beta}^{(\ell,1)}(z,s) + \tilde{v}_{\beta}^{(\ell,1)}(z_\ell,s) \tilde{v}_{\beta}^{(\ell,2)}(z,s) \right] \\ \left[\tilde{i}_{\beta}^{(\ell,2)}(z_\ell,s)\, \tilde{i}_{\beta}^{(\ell,1)}(z,s) - \tilde{i}_{\beta}^{(\ell,1)}(z_\ell,s) \tilde{i}_{\beta}^{(\ell,2)}(z,s) \right] & \left[-\tilde{v}_{\beta}^{(\ell,2)}(z_\ell,s)\, \tilde{i}_{\beta}^{(\ell,1)}(z,s) + \tilde{v}_{\beta}^{(\ell,1)}(z_\ell,s) \tilde{i}_{\beta}^{(\ell,2)}(z,s) \right] \end{pmatrix} \qquad (5.17)
\end{aligned}
$$

The reader can verify that this is the 2 x 2 identity for $z = z_\ell$. Inserting this in (4.11) as

$$
\left(\left(\tilde{\Xi}_{n,m}^{(\ell,a)}(z,z_\ell;s) \right)_{p,p'} \right) = \sum_{\beta=1}^{N} \left[\left(x_n^{(\ell)} \right)_{\beta} \left(x_n^{(\ell)} \right)_{\beta} \right] \otimes \left(\tilde{\psi}_{p,p'}^{(\ell,a)}(z,z_\ell;s) \right)_{\beta} \qquad (5.18)
$$

we have the supermatrizant for the ℓ^{th} section. This, in turn, can be substituted in (4.4) to solve for the voltages and currents in the ℓ^{th} section, noting that

$$
\begin{aligned}
\left(\left(\Xi^{(\ell)}_{n,m}(z,z';s)\right)_{p,p'}\right) &= \left(\left(\tilde{\Xi}^{(\ell)}(z,z_\ell;s)\right)_{p,p'}\right) \odot \left(\left(\tilde{\Xi}^{(\ell)}(z_\ell,z';s)\right)_{p,p'}\right) \\
&= \left[\sum_{\beta=1}^{N}\left[\left(x_n^{(\ell)}\right)_\beta\left(x_n^{(\ell)}\right)_\beta\right] \otimes \left(\tilde{\psi}_{p,p'}(z,z_\ell;s)\right)_\beta\right] \\
&\qquad \odot \left[\sum_{\beta=1}^{N}\left[\left(x_n^{(\ell)}\right)_\beta\left(x_n^{(\ell)}\right)_\beta\right] \otimes \left(\tilde{\psi}_{p,p'}(z_\ell,z')\right)_\beta\right] \qquad (5.19) \\
&= \sum_{\beta=1}^{N}\left[\left(x_n^{(\ell)}\right)_\beta\left(x_n^{(\ell)}\right)_\beta\right] \otimes \left[\left(\tilde{\psi}_{p,p'}(z,z_\ell;s)\right)_\beta \cdot \left(\tilde{\psi}_{p,p'}(z_\ell,z';s)\right)_\beta\right] \\
&= \sum_{\beta=1}^{N}\left[\left(x_n^{(\ell)}\right)_\beta\left(x_n^{(\ell)}\right)_\beta\right] \otimes \left[\left(\tilde{\psi}_{p,p'}(z,z_\ell;s)\right)_\beta \cdot \left(\tilde{\psi}_{p,p'}(z',z_\ell;s)\right)_\beta^{-1}\right]
\end{aligned}
$$

which is needed in case there are sources to be integrated.

6. Full Product Integral Decomposed by Sections

Returning to (1.1) the NMTL equations can be written in supermatrix form as

$$\frac{\partial}{\partial z}\begin{pmatrix}\left(\tilde{V}_n(z,s)\right)\\ Z\left(\tilde{I}_n(z,s)\right)\end{pmatrix} = \begin{pmatrix}(0_{n,m}) & -Y\left(\tilde{Z}'_{n,m}(z,s)\right)\\ -Z\left(\tilde{Y}'_{n,m}(z,s)\right) & (0_{n,m})\end{pmatrix} \odot \begin{pmatrix}\left(\tilde{V}_n(z,s)\right)\\ Z\left(\tilde{I}_n(z,s)\right)\end{pmatrix} + \begin{pmatrix}\left(\tilde{V}_n^{(s)'}(z,s)\right)\\ Z\left(\tilde{I}_n^{(s)'}(z,s)\right)\end{pmatrix}$$

$$= -\tilde{\gamma}(s)\begin{pmatrix}(0_{n,m}) & \left(f_{g_{n,m}}(z)\right)\\ \left(f_{g_{n,m}}(z)\right)^{-1} & (0_{n,m})\end{pmatrix} \odot \begin{pmatrix}\left(\tilde{V}_n(z,s)\right)\\ Z\left(\tilde{I}_n(z,s)\right)\end{pmatrix} + \begin{pmatrix}\left(\tilde{V}_n^{(s)'}(z,s)\right)\\ Z\left(\tilde{I}_n^{(s)'}(z,s)\right)\end{pmatrix} \tag{6.1}$$

where the wave impedance/admittance has been included to make the voltage and current variables have the same units. To solve this equation we have the supermatrizant equation

$$\begin{aligned}
\frac{\partial}{\partial z}\left(\left(\tilde{\Phi}_{n,m}(z,z_0;s)\right)_{p,p'}\right) &= \left(\left(\tilde{\Phi}_{n,m}(z,s)\right)_{p,p'}\right) \odot \left(\left(\tilde{\Phi}_{n,m}(z,z_0;s)\right)_{p,p'}\right)\\
\left(\left(\tilde{\Phi}_{n,m}(z,s)\right)_{p,p'}\right) &= -\tilde{\gamma}(s)\begin{pmatrix}(0_{n,m}) & \left(f_{g_{n,m}}(z)\right)\\ \left(f_{g_{n,m}}(z)\right)^{-1} & (0_{n,m})\end{pmatrix}\\
\left(\left(\tilde{\Phi}_{n,m}(z_0,z_0;s)\right)_{p,p'}\right) &= \left((1_{n,m})_{p,p'}\right) = (1_{n,m}) \otimes (1_{p,p'})\\
\left(\left(\tilde{\Phi}_{n,m}(z,z_0;s)\right)_{p,p'}\right)^{-1} &= \left(\left(\tilde{\Phi}_{n,m}(z_0,z;s)\right)_{p,p'}\right)
\end{aligned} \tag{6.2}$$

For the ℓth section of the NMTL we have

$$\begin{pmatrix}\left(\tilde{V}_n(z,s)\right)\\ Z\left(\tilde{I}_n(z,s)\right)\end{pmatrix} = \left(\left(\tilde{\Phi}_{n,m}(z,z_\ell)\right)_{p,p'}\right) \odot \begin{pmatrix}\left(\tilde{V}_n(z_\ell,s)\right)\\ Z\left(\tilde{I}_n(z_\ell,s)\right)\end{pmatrix}$$

$$+\int_{z_\ell}^{z}\left(\left(\tilde{\Phi}_{n,m}(z_{\ell+1},z';s)\right)_{p,p'}\right)\odot\begin{pmatrix}\left(\tilde{V}_n^{(s)'}(z',s)\right)\\ Z\left(\tilde{I}_n^{(s)'}(z',s)\right)\end{pmatrix}dz' \tag{6.3}$$

$$\left(\left(\tilde{\Phi}_{n,m}(z_{\ell+1},z';s)\right)_{p,p'}\right)=\left(\left(\tilde{\Phi}_{n,m}(z_{\ell+1},z_\ell;s)\right)_{p,p'}\right)\odot\left(\left(\tilde{\Phi}_{n,m}(z',z_\ell;s)\right)_{p,p'}\right)^{-1}$$

Applying the normalization as in Section 3 we have

$$\begin{pmatrix}\left(\tilde{V}_n(z,s)\right)\\ Z\left(\tilde{I}_n(z,s)\right)\end{pmatrix}=\begin{pmatrix}\left(f_{g_{n,m}}(z_\ell)\right)^{1/2} & (0_{n,m})\\ (0_{n,m}) & \left(f_{g_{n,m}}(z_\ell)\right)^{-1/2}\end{pmatrix}\odot\begin{pmatrix}\left(\tilde{v}_n^{(\ell)}(z,s)\right)\\ \left(\tilde{i}_n^{(\ell)}(z,s)\right)\end{pmatrix}$$

$$\begin{pmatrix}\left(\tilde{V}_n^{(s)'}(z,s)\right)\\ Z\left(\tilde{I}_n(z,s)\right)\end{pmatrix}=\begin{pmatrix}\left(f_{g_{n,m}}(z_\ell)\right)^{1/2} & (0_{n,m})\\ (0_{n,m}) & \left(f_{g_{n,m}}(z_\ell)\right)^{-1/2}\end{pmatrix}\odot\begin{pmatrix}\left(\tilde{v}_n^{(s)'}(z,s)\right)\\ \left(\tilde{i}_n^{(s)'}(z,s)\right)\end{pmatrix} \tag{6.4}$$

Then following the approximation procedure in the previous sections we have approximate voltages and currents given by

$$\frac{\partial}{\partial z}\begin{pmatrix}\left(\tilde{V}_n^{(\ell,a)}(z,s)\right)\\ Z\left(\tilde{I}_n^{(\ell,a)}(z,s)\right)\end{pmatrix}=\left(\left(\tilde{\Phi}_{n,m}^{(\ell,a)}(z,s)\right)_{p,p'}\right)\odot\begin{pmatrix}\left(\tilde{V}_n^{(\ell,a)}(z,s)\right)\\ Z\left(\tilde{I}_n^{(\ell,a)}(z,s)\right)\end{pmatrix}+\begin{pmatrix}\left(\tilde{V}_n^{(s)'}(z,s)\right)\\ Z\left(\tilde{I}_n^{(s)'}(z,s)\right)\end{pmatrix}$$

$$\begin{pmatrix}\left(\tilde{V}_n^{(\ell,a)}(z,s)\right)\\ Z\left(\tilde{I}_n^{(\ell,a)}(z,s)\right)\end{pmatrix}=\left(\left(\tilde{\Phi}_{n,m}^{(\ell,a)}(z,z_\ell;s)\right)_{p,p'}\right)\odot\begin{pmatrix}\left(\tilde{V}_n^{(\ell,a)}(z,s)\right)\\ Z\left(\tilde{I}_n^{(\ell,a)}(z,s)\right)\end{pmatrix}$$

$$+\int_{z_\ell}^{z}\left(\left(\tilde{\Phi}_{n,m}^{(\ell,a)}(z,z';s)\right)_{p,p'}\right)\odot\begin{pmatrix}\left(\tilde{V}_n^{(s)'}(z',s)\right)\\ Z\left(\tilde{I}_n^{(s)'}(z',s)\right)\end{pmatrix}dz'$$

$$\left(\left(\tilde{\Phi}_{n,m}^{(\ell,a)}(z,s)\right)_{p,p'}\right)$$

$$= \begin{pmatrix} \left(f_{g_{n,m}}(z_\ell)\right)^{1/2} & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(f_{g_{n,m}}(z_\ell)\right)^{-1/2} \end{pmatrix} \odot \left(\left(\xi_{n,m}^{(\ell,a)}(z,s)\right)_{p,p'}\right) \odot \begin{pmatrix} \left(f_{g_{n,m}}(z_\ell)\right)^{-\frac{1}{2}} & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} \end{pmatrix}$$

$$= -\tilde{\gamma}(s) \begin{pmatrix} \left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(f_{g_{n,m}}(z_\ell)\right)^{-\frac{1}{2}} \end{pmatrix} \odot \begin{pmatrix} \left(0_{n,m}\right) & \left(X_{n,m}^{(\ell,a)}(z)\right) \\ \left(X_{n,m}^{(\ell)}(z)\right)^{-1} & \left(0_{n,m}\right) \end{pmatrix} \odot \begin{pmatrix} \left(f_{g_{n,m}}(z_\ell)\right)^{-\frac{1}{2}} & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} \end{pmatrix}$$

$$\left(\left(\tilde{\Phi}_{n,m}^{(\ell,a)}(z,z_\ell;s)\right)_{p,p'}\right)$$

$$= \begin{pmatrix} \left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(f_{g_{n,m}}(z_\ell)\right)^{-\frac{1}{2}} \end{pmatrix} \odot \left(\left(\Xi_{n,m}^{(\ell,a)}(z,z_\ell;s)\right)_{p,p'}\right) \odot \begin{pmatrix} \left(f_{g_{n,m}}(z_\ell)\right)^{-\frac{1}{2}} & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(f_{g_{n,m}}(z_\ell)\right)^{-\frac{1}{2}} \end{pmatrix} \odot \left[\sum_{\beta=1}^{N}\left[\left(x_n^{(\ell)}\right)_\beta\left(x_n^{(\ell)}\right)_\beta\right] \otimes \left(\tilde{\psi}_{p,p'}^{(\ell,a)}(z,z_\ell;s)\right)_\beta\right]$$

$$\odot \begin{pmatrix} \left(f_{g_{n,m}}(z_\ell)\right)^{-\frac{1}{2}} & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} \end{pmatrix} \tag{6.5}$$

$$\left(\left(\tilde{\Phi}_{n,m}^{(\ell,a)}(z,z';s)\right)_{p,p'}\right)$$

$$= \begin{pmatrix} \left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(f_{g_{n,m}}(z_\ell)\right)^{-\frac{1}{2}} \end{pmatrix} \odot \left[\sum_{\beta=1}^{N}\left[\left(x_n^{(\ell)}\right)_\beta\left(x_n^{(\ell)}\right)_\beta\right] \otimes \left(\tilde{\psi}_{p,p'}^{(\ell,a)}(z,z_\ell;s)\right)_\beta \cdot \left(\tilde{\psi}_{p,p'}^{(\ell,a)}(z',z_\ell;s)\right)_\beta^{-1}\right]$$

$$\odot \begin{pmatrix} \left(f_{g_{n,m}}(z_\ell)\right)^{-\frac{1}{2}} & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} \end{pmatrix}$$

With these results we are in a position to write the solution (approximate) for the complete ℓth section of the line ($z_\ell \le z \le z_{\ell+1}$) as

$$\begin{pmatrix} \left(\tilde{V}_n^{(\ell,a)}(z_{\ell+1},s)\right) \\ Z\left(\tilde{I}_n^{(\ell,a)}(z_{\ell+1},s)\right) \end{pmatrix} = \left(\left(\tilde{\Phi}_{n,m}^{(\ell,a)}(z_{\ell+1},z_\ell;s)\right)_{p,p'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n^{(\ell,a)}(z_\ell,s)\right) \\ Z\left(\tilde{I}_n^{(\ell,a)}(z_\ell,s)\right) \end{pmatrix}$$

$$+\int_{z_\ell}^{z_{\ell+1}}\left(\left(\tilde{\Phi}_{n,m}^{(\ell,a)}(z_{\ell+1},z';s)\right)_{p,p'}\right)\odot\begin{pmatrix}\left(\tilde{V}_n^{(s)'}(z',s)\right)\\ Z\left(\tilde{I}_n^{(s)'}(z',s)\right)\end{pmatrix} \tag{6.6}$$

Now to go to the next section ($\ell+1$) we have the continuity of voltage and current as

$$\begin{pmatrix}\left(\tilde{V}_n^{(\ell+1,a)}(z_{\ell+1},s)\right)\\ Z\left(\tilde{I}_n^{(\ell+1,a)}(z_{\ell+1},s)\right)\end{pmatrix}=\begin{pmatrix}\left(\tilde{V}_n^{(\ell,a)}(z_\ell,s)\right)\\ Z\left(\tilde{I}_n^{(\ell,a)}(z_\ell,s)\right)\end{pmatrix} \tag{6.7}$$

By this procedure, one can go from the voltage at $z=z_0$ (0th section) up through the L-1st section to z_L by successively computing the voltage and current at $z_1, z_2, \ldots, z_L$ via (6.6).

For the case that the sources are zero along the entire NMTL, we have a matrizant for the entire line as

$$\begin{gathered}\begin{pmatrix}\left(\tilde{V}_n^{(L,a)}(z_L,s)\right)\\ Z\left(\tilde{I}_n^{(L,a)}(z_L,s)\right)\end{pmatrix}=\left(\left(\tilde{\Phi}_{n,m}^{(a)}(z_L,z_0;s)\right)_{p,p'}\right)\odot\begin{pmatrix}\left(\tilde{V}_n^{(\ell,a)}(z_0,s)\right)\\ Z\left(\tilde{I}_n^{(\ell,a)}(z_0,s)\right)\end{pmatrix}\\ \left(\left(\tilde{\Phi}_{n,m}^{(a)}(z_L,z_0;s)\right)_{p,p'}\right)=\mathop{\bigodot}_{\ell=0}^{L-1}\left(\left(\tilde{\Phi}_{n,m}^{(\ell,a)}(z_{\ell+1},z_\ell;s)\right)_{p,p'}\right)\end{gathered} \tag{6.8}$$

with multiplication as ℓ ascends always taken on the left (as in Appendix B). With the section matrizants as in (6.5) the terms involving the geometric-factor matrices can be combined to give

$$\left(\left(\tilde{\Phi}_{n,m}^{(a)}(z_L,z_0;s)\right)_{p,p'}\right)=\begin{pmatrix}\left(f_{g_{n,m}}(z_{L-1})\right)^{\frac{1}{2}} & (0_{n,m})\\ (0_{n,m}) & \left(f_{g_{n,m}}(z_{L-1})\right)^{-\frac{1}{2}}\end{pmatrix}$$

$$\odot\left[\mathop{\bigodot}_{\ell=1}^{L-1}\left[\left[\sum_{\beta=1}^{N}\left[\left(x_n^{(\ell)}\right)_\beta\left(x_n^{(\ell)}\right)_\beta\right]\otimes\left(\tilde{\psi}_{p,p'}^{(\ell,a)}(z_{\ell+1},z_\ell;s)\right)_\beta\right]\right.$$

$$\odot \begin{pmatrix} \left(f_{g_{n,m}}(z_\ell)\right)^{-\frac{1}{2}} \cdot \left(f_{g_{n,m}}(z_{\ell-1})\right)^{\frac{1}{2}} & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(f_{g_{n,m}}(z_\ell)\right)^{\frac{1}{2}} \cdot \left(f_{g_{n,m}}(z_{\ell-1})\right) \end{pmatrix} \Bigg] \Bigg] \tag{6.9}$$

$$\odot \left[\sum_{\beta=1}^{N} \left[\left(x_n^{(\ell)} \right)_\beta \left(x_n^{(\ell)} \right)_\beta \right] \odot \left(\tilde{\psi}_{p,p'}^{(0,a)}(z_1, z_0; s) \right)_\beta \right] \odot \begin{pmatrix} \left(f_{g_{n,m}}(z_0)\right)^{-\frac{1}{2}} & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(f_{g_{n,m}}(z_0)\right)^{\frac{1}{2}} \end{pmatrix}$$

7. Concluding Remarks

For the case that all the modal speeds on an NMTL are the same we now have a way to approximately calculate its response based on an interpolation scheme for the geometric-factor matrix (or equivalently, characteristic-impedance matrix) in individual sections of the line. This makes a smoother transition in going from one section to the next, the discontinuity being in the slope (first derivative) at the section boundaries. Compared to the scheme in [6], this also maintains a symmetric characteristic-impedance matrix in the interpolation throughout each section. A previous paper [4] has considered the high-frequency, or early-time propagation on such NMTLs. The present paper extends this to all frequencies within the transmission-line approximation.

While the present development has been in the context of an approximation, this need not be the only case of interest. Since the characteristic-impedance matrix is now symmetric everywhere along the NMTL, thereby satisfying reciprocity, we can use this to *define* NMTL examples for which the solution is *exact* (within the transmission-line assumption). These can in turn be constructed, thereby giving a *synthesis* procedure for NMTLs for special applications.

Appendix A. Operations with Supermatrices

In [1-3] supervectors and supermatrices are formed to arbitrary levels of partition, i.e., dimatrices (two sets of indices), trimatrices (three sets of indices), etc. For present purposes we will be considering dimatrices and divectors (two sets of indices), corresponding to the cases in this paper.

Summarizing, we have addition

$$\begin{aligned} \left(\left(a_{n,m}\right)_{u,v}\right) + \left(\left(b_{n,m}\right)_{u,v}\right) &= \left(\left(a_{n,m}\right)_{u,v} + \left(b_{n,m}\right)_{u,v}\right) \\ &= \left(\left(c_{n,m}\right)_{u,v}\right) \\ c_{n,m;u,v} &= a_{n,m;u,v} + b_{n,m;u,v} \end{aligned} \tag{A.1}$$

and generalized dot product (contractive multiplication)

$$\begin{aligned} \left(\left(a_{n,m}\right)_{u,v}\right) \odot \left(\left(b_{n,m}\right)_{u,v}\right) &= \left(\sum_{v'=1}^{M}\left(a_{n,m}\right)_{u,v'} \cdot \left(b_{n,m}\right)_{v',v}\right) \\ &= \left(\left(d_{n,m}\right)_{u,v}\right) \\ d_{n,m;u,v} &= \sum_{v'=1}^{M} \sum_{m'=1}^{N} a_{n,m';u,v'}\, b_{m',m;v',v} \end{aligned} \tag{A.2}$$

Here the matrices are taken as square with

$$\begin{aligned} n, m &= 1, 2, \ldots, N \quad (\text{for } N \times N \text{ blocks}) \\ u, v &= 1, 2, \ldots, M \quad (\text{for } M \times M \text{ matrix of blocks or submatrices}) \end{aligned} \tag{A.3}$$

Similar to supermatrices, supervectors follow the same rules as above except that each index set has only one element (e.g., $((e_n)_v)$. Here the index sets are set apart by semicolons when writing out various elements (or even higher-order aggregates, each semicolons basically replacing a set of parentheses). Each index pair of indices in an index set for a supermatrix takes the form of (row, column). This differs from the convention in [7, 8, 11] where the two row indices are first, followed by the two column indices. We find our convention more convenient, especially when multiple levels of partition are involved. For supervectors, the index set is a set of single indices which are all separated by semicolons when exhibiting an individual element. Note the vectors are not distinguished as rows or columns; for contractive multiplication (dot product) when placed on the right summation is over the last of the index pair of a preceding matrix, and when placed on the left the summation is over the first of the index pair of a following matrix.

Note that for these operations the supermatrices and supermatrices are partitioned in compatible ways such that the operations make sense. This can be extended to rectangular supermatrices, but for present purposes the supermatrices are square and of *symmetric compatible order* so that both addition and dot multiplication are defined in any order [1–3]. This also allows one to form an inverse supermatrix in terms of its blocks [3].

One can construct supermatrices and supervectors from elementary blocks in various ways. The direct sum $\oplus$ is applied to supervectors as

$$\begin{aligned} \left((e_n)_v\right) &= (e_n)_1 \oplus (e_n)_2 \oplus \ldots \oplus (e_n)_M \\ &\equiv \bigoplus_{v=1}^{M} (e_n)_v \end{aligned} \tag{A.4}$$

and to supermatrices (block-diagonal) as

$$\begin{aligned} \left(\left(A_{n,m}\right)_{u,v}\right) &= \left(A_{n,m}\right)_{1,1} \oplus \left(A_{n,m}\right)_{2,2} \oplus \ldots \oplus \left(A_{n,m}\right)_{M,M} \\ &\equiv \bigoplus_{u'=1}^{M} \left(A_{n,m}\right)_{u',u'} \\ &= \begin{pmatrix} \left(A_{n,m}\right)_{1,1} & & & O \\ & \left(A_{n,m}\right)_{M,M} & & \\ & & \ddots & \\ O & & & \left(A_{n,m}\right)_{M,M} \end{pmatrix} \\ \left(A_{n,m}\right)_{u,v} &= \left(0_{n,m}\right)_{u,v} \quad \text{for } u \neq v \end{aligned} \tag{A.5}$$

Note that the direct sum in non-commutative, but that sums and dot products of supermatrices/ supervectors formed as direct sums are themselves direct sums. Furthermore the inverse of such a supermatrix is merely

$$\left(\left(A_{n,m}\right)_{u,v}\right)^{-1} = \bigoplus_{u'=1}^{M} \left(A_{n,m}\right)_{u',u'}^{-1} \tag{A.6}$$

and the eigenvalues and eigenvectors are found by diagonalizing the individual (square) blocks or submatrices. From this we have the relationships

$$
\begin{aligned}
tr\left(\left((A_{n,m})_{u,v}\right)\right) &= \sum_{u'=1}^{M} tr\left((A_{n,m})_{u',u'}\right) \\
&= \sum_{u'=1}^{M} \sum_{m'=1}^{N} A_{m',m';u',u'} \\
&= \sum_{u'=1}^{M} \sum_{m'=1}^{N} \left[\text{eigenvalues of } (A_{n,m})_{u',u'}\right] \\
\det\left(\left((A_{n,m})_{u,v}\right)\right) &= \prod_{u'=1}^{M} \det\left((A_{n,m})_{u',u'}\right) \\
&= \prod_{u'=1}^{M} \prod_{m'=1}^{N} \left[\text{eigenvalues of } (A_{n,m})_{u',u'}\right]
\end{aligned}
\tag{A.7}
$$

Now we come to the direct product $\otimes$, also known as the Kronecker product [7]. This is used in group theory to form the direct product of groups [12]. Here, however, we are concerned with products of matrices to form supermatrices as

$$
\begin{aligned}
&\left((B_{n,m})_{u,v}\right) = (B'_{n,m}) \otimes (B''_{u,v}) \\
&= \begin{pmatrix}
(B'_{n,m})B''_{1,1} & (B'_{n,m})B''_{1,2} & \cdots & (B'_{n,m})B''_{1,M} \\
(B'_{n,m})B''_{2,1} & (B'_{n,m})B''_{2,2} & & \vdots \\
\vdots & & \ddots & \vdots \\
(B'_{n,m})B''_{M,1} & \cdots & \cdots & (B'_{n,m})B''_{M,M}
\end{pmatrix} \\
&B_{n,m;u,v} = B'_{n,m}\, B''_{u,v}
\end{aligned}
\tag{A.8}
$$

Here we can see that order is important, so the direct product is *non-commutative*. In this form we see the simple interpretation that the blocks are just the first matrix multiplied by an element of the second matrix. (Here the blocks and the supermatrix are square, but these can be rectangular in a more general case.) By ordering the indices as we have, the above can be extended to higher-order supermatrices by successive direct-product multiplication on the right by $(B''_{q,q'})$ etc. The direct product can be readily seen to have the properties

$$
\left((B_{n,m})_{u,v}\right)^T = (B'_{n,m})^T \otimes (B''_{n,m})^T
$$

$$\left(\left(B_{n,m}\right)_{u,\mathrm{v}}\right)^{\dagger} = \left(B'_{n,m}\right)^{\dagger} \otimes \left(B''_{n,m}\right)^{T} \tag{A.9}$$

from which we find that $\left(\left(B_{n,m}\right)_{u,\mathrm{v}}\right)$ is orthogonal (unitary) if both $\left(B'_{n,m}\right)$ and $\left(B'_{n,m}\right)$ are orthogonal (unitary) [11]. Furthermore, we also easily find

$$\begin{aligned} \left(B'_{n,m}\right) \otimes \left[\left(B''_{u,\mathrm{v}}\right) + \left(C''_{u,\mathrm{v}}\right)\right] &= \left(B'_{n,m}\right) \otimes \left(B''_{u,\mathrm{v}}\right) + \left(B'_{n,m}\right) \otimes \left(B''_{u,\mathrm{v}}\right) \\ \left[\left(B'_{n,m}\right) + \left(C''_{n,m}\right)\right] \otimes \left(B''_{u,\mathrm{v}}\right) &= \left(B'_{n,m}\right) \otimes \left(B''_{u,\mathrm{v}}\right) + \left(C'_{n,m}\right) \otimes \left(B''_{u,\mathrm{v}}\right) \end{aligned} \tag{A.10}$$

As special important cases we have

$$\begin{aligned} \left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right) &= \left(1_{n,m}\right) \otimes \left(1_{u,\mathrm{v}}\right) \text{ (identity supermatrix)} \\ 1_{n,m;u,\mathrm{v}} &= \begin{cases} 1 & \text{for } n = m \text{ and } u = \mathrm{v} \\ 0 & \text{otherwise} \end{cases} \\ \left(\left(0_{n,m}\right)_{u,\mathrm{v}}\right) &= \left(0_{n,m}\right) \otimes \left(0_{u,\mathrm{v}}\right) \text{ (zero supermatrix)} \\ 0_{n,m;u,\mathrm{v}} &= 0 \\ \left(0_{n,m}\right) \otimes \left(B''_{u,\mathrm{v}}\right) &= \left(0_{n,m}\right) \otimes \left(0_{u,\mathrm{v}}\right) = \left(B'_{n,m}\right) \otimes \left(0_{u,\mathrm{v}}\right) \end{aligned} \tag{A.11}$$

An important general property of such supermatrices is [7, 8, 11]

$$\begin{aligned} &\left[\left(B'_{n,m}\right) \otimes \left(B''_{u,\mathrm{v}}\right)\right] \odot \left[\left(C'_{n,m}\right) \otimes \left(C''_{u,\mathrm{v}}\right)\right] \\ &\quad = \left[\left(B'_{n,m}\right) \cdot \left(C'_{n,m}\right)\right] \odot \left[\left(B''_{u,\mathrm{v}}\right) \cdot \left(C''_{u,\mathrm{v}}\right)\right] \end{aligned} \tag{A.12}$$

This can be applied to supervectors of the form

$$\begin{aligned} \left(\left(b_n\right)_u\right) &= \left(b'_n\right) \otimes \left(b''_u\right) \\ &= \begin{pmatrix} (b'_n) b''_1 \\ (b'_n) b''_2 \\ \vdots \\ (b'_n) b''_M \end{pmatrix} \end{aligned} \tag{A.13}$$

by regarding one of the supermatrices in (A.12), say $\left(\left(C_{n,m}\right)_{u,\mathrm{v}}\right)$ as having a single column for dot multiplication on the right, or a single row for dot multiplication on the left as

$$\left[\left(B'_{n,m}\right) \otimes \left(B''_{u,\mathrm{v}}\right)\right] \odot \left(\left(b_n\right)_u\right) = \left[\left(B'_{n,m}\right) \cdot \left(b'_n\right)\right] \otimes \left[\left(B''_{u,\mathrm{v}}\right) \cdot \left(b''_u\right)\right]$$

$$\left((b_n)_u\right) \odot \left[\left(B'_{n,m}\right) \otimes \left(B''_{u,v}\right)\right] = \left[\left(b'_n\right) \cdot \left(B'_{n,m}\right)\right] \otimes \left[\left(b''_u\right) \cdot \left(B''_{u,v}\right)\right] \tag{A.14}$$

The eigenvalues and eigenvectors of such a direct product matrix can be found from those of the constituent matrices which have the form

$$\begin{aligned} \left(B'_{n,m}\right) \cdot \left(b_n'^{(n')}\right) &= \lambda'_{n'}\left(b_n'^{(n')}\right) \text{ for } n' = 1, 2, \ldots, N \\ \left(B''_{u,v}\right) \cdot \left(b_u'^{(u')}\right) &= \lambda''_{u'}\left(b_u'^{(u')}\right) \text{ for } u' = 1, 2, \ldots, M \end{aligned} \tag{A.15}$$

The eigenvalues need not be distinct, nor the eigenvectors all linearly independent in the general case for each of the above matrices. Considering arbitrary values of n' and u' we have

$$\begin{aligned} &\left[\left(B'_{n,m}\right) \otimes \left(B''_{u,v}\right)\right] \odot \left[\left(b_n'^{(n')}\right) \otimes \left(b_u''^{(u')}\right)\right] = \left[\left(B'_{n,m}\right) \cdot \left(b_n'^{(n')}\right)\right] \otimes \left[\left(B''_{n,m}\right) \cdot \left(b_u''^{(u')}\right)\right] \\ &= \left[\lambda'_{n'} \left(b_n'^{(n')}\right)\right] \otimes \left[\lambda''_{u''} \left(b_n''^{u'}\right)\right] = \lambda'_{n'} \lambda''_{u'}\left(b_n'^{(n')}\right) \otimes \left(b_u''^{(u')}\right) \end{aligned} \tag{A.16}$$

which says that

$$\begin{aligned} \left(b_n'^{(n')}\right) \otimes \left(b_u''^{(u')}\right) &= \text{eigenvectors of } \left(B'_{n,m}\right) \otimes \left(B''_{u,v}\right) \\ \lambda_{n';u'} = \lambda'_{n'}\lambda''_{u''} &= \text{eigenvalues of } \left(B'_{n,m}\right) \otimes \left(B''_{u,v}\right) \end{aligned} \tag{A.17}$$

there being NM of each of these (not necessarily all distinct). One can similarly construct left eigenvectors as well. With these results we also have

$$\begin{aligned} tr\left(\left(B_{n,m}\right)_{u,v}\right) &= \sum_{n=1}^{N} \sum_{u=1}^{M} B'_{n,n} B''_{u,u} \\ &= \left[\sum_{n=1}^{N} B'_{n,n}\right] \left[\sum_{u=1}^{M} B''_{u,u}\right] \\ &= tr\left(\left(B'_{n,m}\right)\right)\, tr\left(\left(B''_{u,v}\right)\right) \\ &= \left[\sum_{n'=1}^{N} \lambda'_{n'}\right] \left[\sum_{u'=1}^{M} \lambda''_{u'}\right] \\ \det\left(\left(B_{n,m}\right)_{u,v}\right) &= \prod_{n'}^{N} \prod_{u'=1}^{M} \lambda''_{n'}\, \lambda''_{u'} \end{aligned}$$

$$= \left[\prod_{n'=1}^{N} \lambda_{n}'^{M}\right]\left[\prod_{u'=1}^{M} \lambda_{u'}''^{N}\right] \tag{A.18}$$

$$= \det^{M}\left(\left(B'_{n,m}\right)\right) \det^{N}\left(\left(B''_{u,\mathrm{v}}\right)\right)$$

Extending from (A.12) we have if either of

$$\left(B'_{n,m}\right) \cdot \left(C'_{n,m}\right) = \left(0_{n,m}\right) \ , \ \left(B''_{u,\mathrm{v}}\right) \cdot \left(C''_{u,\mathrm{v}}\right) = \left(0_{n,m}\right) \tag{A.19}$$

holds, then

$$\left[\left(B'_{n,m}\right) \otimes \left(B''_{n,m}\right)\right] \odot \left[\left(C'_{n,m}\right) \otimes \left(C''_{u,\mathrm{v}}\right)\right] = \left(0_{n,m}\right) \otimes \left(0_{u,\mathrm{v}}\right) \tag{A.20}$$

of course, the eigenvalues are then all zero. Other special products (such as the identity) appearing in (A.19) also simplify matters.

One can also have functions of supermatrices. As usual these can be defined in terms of power (Taylor) series describing the functions as

$$f(\zeta) = \sum_{\ell=0}^{\infty} \alpha_{\ell} \ \zeta^{\ell}$$

$$\alpha_{\ell} = \frac{1}{\ell!} \frac{d^{\ell}}{d\zeta^{\ell}} f(\zeta)\bigg|_{\zeta=0} \tag{A.21}$$

$$f\left(\left(a_{n,m}\right)_{u,\mathrm{v}}\right) = \sum_{\ell=0}^{\infty} \alpha_{\ell}\left(\left(a_{n,m}\right)_{u,\mathrm{v}}\right)^{\ell}$$

where due attention is paid to the radius of convergence of the series. For a direct-sum supermatrix as in (A.5) we have

$$f\left(\left(A_{n,m}\right)_{u,\mathrm{v}}\right) = f\left(\left(A_{n,m}\right)_{u',u'}\right)$$

$$= \begin{pmatrix} f\left(\left(A_{n,m}\right)_{1,1}\right) & & O \\ & f\left(\left(A_{n,m}\right)_{2,2}\right) & \\ O & & f\left(\left(A_{n,m}\right)_{M,M}\right) \end{pmatrix} \tag{A.22}$$

For the direct-product supermatrix we have from (A.12) for positive integer powers

$$\left[(B'_{n,m})^{\ell} \otimes (B''_{u,v})^{\ell}\right] \odot \left[(B'_{n,m}) \otimes (B''_{u,v})\right] = (B'_{n,m})^{\ell+1} \otimes (B''_{u,v})^{\ell+1}$$
$$\left[(B'_{n,m}) \otimes (B''_{u,v})\right]^{\ell} = (B'_{n,m})^{\ell} \otimes (B''_{u,v})^{\ell} \tag{A.23}$$

For $\ell = 0$ we have

$$\left[(B'_{n,m}) \otimes (B''_{u,v})\right] = (B'_{n,m})^{0} \otimes (B'_{u,v})^{0} = (1_{n,m}) \otimes (1_{u,v})$$
$$= \left((1_{n,m})_{u,v}\right) \tag{A.24}$$

provided the constituent matrices are both non singular (no zero eigenvalues since 0^0 is undefined) and have complete sets of eigenvectors spanning their respective spaces. There are cases, however, for which (A.24) does not hold. For the series expansion in (A.21), the $\ell = 0$ term is defined to be the supermatrix identity as

$$f\left((B'_{n,m}) \otimes (B''_{u,v})\right) = \alpha_0 (1_{n,m}) \otimes (1_{u,v}) + \sum_{\ell=1}^{\infty} \alpha_\ell \left[(B'_{n,m}) \otimes (B''_{u,v})\right]^{\ell}$$
$$= \alpha_0 (1_{n,m}) \otimes (1_{u,v}) + \sum_{\ell=1}^{\infty} \alpha_\ell (B'_{n,m})^{\ell} \otimes (B''_{u,v})^{\ell} \tag{A.25}$$

If either of the constituent matrices is the identity we readily have

$$f\left((1_{n,m}) \otimes (B''_{u,v})\right) = (1_{n,m}) \otimes f\left((B''_{u,v})\right)$$
$$f\left((B'_{n,m}) \otimes (1_{u,v})\right) = f\left((1_{n,m})\right) \otimes (1_{u,v}) \tag{A.26}$$

If, however, we have

$$(B'_{n,m})^{\ell} = (B'_{n,m}) \quad \text{for } \ell = 1, 2, \ldots \tag{A.27}$$

but not being the identity (as in the case of a dyadic product of a pair of biorthonormal right and left eigenvectors), then we have the more general expression

$$f\left(\left(B'_{n,m}\right) \otimes \left(B''_{u,\mathrm{v}}\right)\right) = \alpha_0 \left[\left(1_{n,m}\right) - \left(B'_{n,m}\right)\right] \otimes \left(1_{u,\mathrm{v}}\right) + \left(B'_{n,m}\right) \otimes f\left(\left(B''_{u,\mathrm{v}}\right)\right) \tag{A.28}$$

and similarly for $\left(B''_{u,\mathrm{v}}\right)^{\ell}$ as in (A.25) we have

$$f\left(\left(B'_{n,m}\right) \otimes \left(B''_{u,\mathrm{v}}\right)\right) = \alpha_0 \left(1_{n,m}\right) \otimes \left[\left(1_{u,\mathrm{v}}\right) - \left(B''_{u,\mathrm{v}}\right)\right] + f\left(\left(B'_{n,m}\right) \otimes \left(B''_{u,\mathrm{v}}\right)\right) \tag{A.29}$$

As one can see, these reduce to (A.24) for the special case of identity matrices.

A general property of a supermatrix as in (A.8) is its inverse given by

$$\left[\left(B'_{n,m}\right) \otimes \left(B''_{u,\mathrm{v}}\right)\right]^{-1} = \left(B'_{n,m}\right)^{-1} \otimes \left(B''_{n,m}\right)^{-1} \tag{A.30}$$

which can be verified by dot multiplying the two sides and applying (A.12) to give the identity supermatrix in (A.11). Of course *both* constituent matrices must be nonsingular for this to apply.

B. Product Integrals

As in Section 4, the product integral is a general way to give the solution of the first-order linear matrix (or supermatrix) differential equation [10]. This has been applied to the solution of NMTLs in [6] with the appendices tabulating many of the properties of product integrals. Here we develop some additional properties.

First consider functions of square matrices as in (A.21). Beginning with scalar functions suppose that we have

$$f(a+b) \; = \; g(a,b) \equiv \text{some combination of functions of } a \text{ and } b \text{ separately} \tag{B.1}$$

From (A.21) we have [9]

$$\begin{aligned} f(a+b) &= \sum_{\ell=0}^{\infty} \alpha_\ell [a+b]^\ell = \sum_{\ell=0}^{\infty} \alpha_\ell \left[\sum_{\ell'=0}^{\ell} \binom{\ell}{\ell'} a^{\ell-\ell'} b^{\ell'} \right] \\ \binom{\ell}{\ell'} &= \frac{\ell!}{[\ell-\ell']!\,\ell!} = \binom{\ell}{\ell-\ell'} \equiv \text{binomial coefficient} \end{aligned} \tag{B.2}$$

If $N \times N$ matrices $(a_{n,m})$ and $(b_{n,m})$ commute then we also have (since the matrices move through each other just like scalars)

$$\begin{aligned} f\big((a_{n,m}) + (b_{n,m})\big) &= \sum_{\ell=0}^{\infty} \alpha_\ell \big[(a_{n,m}) + (b_{n,m})\big]^\ell \\ &= \sum_{\ell'=0}^{\infty} \alpha_\ell \left[\sum_{\ell'=0}^{\ell} \binom{\ell}{\ell'} (a_{n,m})^{\ell-\ell'} \cdot \; (b_{n,m})^{\ell'} \right] \end{aligned} \tag{B.3}$$

with appropriate attention to the radius of convergence of the series in terms of the matrix eigenvalues. The point is that if the matrices commute the manipulations of scalars and matrices in the various functions in f and g are exactly the same so that

$$f\big((a_{n,m}) + (b_{n,m})\big) \; = \; g\big((a_{n,m}),\, (b_{n,m})\big) \tag{B.4}$$

Note that all multiplications of matrices are dot products in these series. As examples one can consider the well-known expansions of trigonometric, hyperbolic, exponential, etc., functions of a sum in terms of functions of a and b separately, and hence of $(a_{n,m})$ and $(b_{n,m})$ separately. Note the use of integer powers (ℓ) in the expansions! Fractional powers can be a problem since one needs to define sheets for the roots of the various eigenvalues. A simple, but important, example is

$$e^{(a_{n,m})+(b_{n,m})} = e^{(a_{n,m})} \cdot e^{(b_{n,m})} = e^{(b_{n,m})} \cdot e^{(a_{n,m})} \tag{B.5}$$

Note also that commuting matrices have

$$f_1\big((a_{n,m})\big) \cdot f_2\big((b_{n,m})\big) = f_2\big((b_{n,m})\big) \cdot f_1\big((a_{n,m})\big) \tag{B.6}$$

where both f_1 and f_2 are assumed to have power-series representations as in (A.19).

Now extend the properties of commuting matrices to the product integral. This solves the matrizant differential equation as

$$\begin{aligned}
&\frac{d}{dz}\big(A_{n,m}(z,z_0)\big) = \big(a_{n,m}(z)\big) \cdot \big(A_{n,m}(z,z_0)\big) \\
&\big(A_{n,m}(z_0,z_0)\big) = \big(1_{n,m}\big) \\
&\big(A_{n,m}(z)\big) = \prod_{z_0}^{z} e^{(a_{n,m}(z'))\,dz'} \qquad \text{(product integral)}
\end{aligned} \tag{B.7}$$

The product integral can be considered as the limit of a product as

$$\begin{aligned}
&\Delta z = \frac{z-z_0}{L} \quad , \quad z_p = z_0 + p\Delta z \\
&\prod_{z_0}^{z} e^{(a_{n,m}(z'))\,dz'} \\
&\quad = \lim_{\substack{L\to\infty \\ (\Delta z\to 0)}} e^{(a_{n,m}(z_L))\,\Delta z} \cdot \ldots \cdot e^{(a_{n,m}(z_2))\,\Delta z} \cdot e^{(a_{n,m}(z_1))\,\Delta z} \\
&\quad = \lim_{L\to\infty} \bigodot_{p=1}^{L} e^{(a_{n,m}(z_p))\Delta z}
\end{aligned} \tag{B.8}$$

with continued dot multiplication taken to the *left*.

Summarizing some important properties of the product integral we have

$$\det\left(\prod_{z_0}^{z} e^{(a_{n,m}(z'))dz'}\right) = \exp\left(\int_{z_0}^{z} tr((a_{n,m}(z')))dz'\right)$$

$$\prod_{z_0}^{z} e^{(a_{n,m}(z'))dz'} = \left[\prod_{z_0'}^{z} e^{(a_{n,m}(z'))dz'}\right] \cdot \left[\prod_{z_0}^{z_0'} e^{(a_{n,m}(z'))dz'}\right] \tag{B.9}$$

$$\left[\prod_{z_0}^{z} e^{(a_{n,m}(z'))dz'}\right]^{-1} = \prod_{z}^{z_0} e^{(a_{n,m}(z'))dz'}$$

The inverse operation is the product derivative D_z given by

$$D_z\left(\prod_{z_0}^{z} e^{(a_{n,m}(z'))dz'}\right) = D_z((A_{n,m}(z,z_0)))$$

$$= \left[\frac{\partial}{\partial z}(A_{n,m}(z,z_0))\right] \cdot (A_{n,m}(z,z_0))^{-1} \tag{B.10}$$

$$= (a_{n,m}(z))$$

which for scalars is just the logarithmic derivative. The sum rule is

$$\prod_{z_0}^{z} e^{[(a_{n,m}(z')) + (b_{n,m}(z'))]dz'}$$

$$= (A_{n,m}(z,z_0)) \cdot \prod_{z_0}^{z} e^{(A_{n,m}(z,z_0))^{-1} \cdot (b_{n,m}(z')) \cdot (A_{n,m}(z,z_0))} \tag{B.11}$$

The similarity rule is

$$(P_{n,m}(z)) \cdot \left[\prod_{z_0}^{z} e^{(b_{n,m}(z'))\,dz'}\right] \cdot (P_{n,m}(z_0))^{-1}$$

$$= \prod_{z_0}^{z} e^{\left[D_{z'}((P_{n,m}(z'))) + (P_{n,m}(z')) + (b_{n,m}(z')) \cdot (P_{n,m}(z'))^{-1}\right]dz'} \tag{B.12}$$

Now if $(a_{n,m}(z'))$ and $(a_{n,m}(z''))$ commute pairwise for every pair (z', z'') on the interval we have

$$
\begin{aligned}
&\prod_{z_0}^{z} e^{(a_{n,m}(z'))dz'} \\
&= \lim_{L\to\infty} \exp\left(\sum_{p=1}^{\infty}\left(a_{n,m}(z_p)\right)\Delta z\right) \\
&= \exp\left(\int_{z_0}^{z}\left(a_{n,m}(z')\right) dz'\right)
\end{aligned}
\tag{B.13}
$$

In this case the product integral can be written in terms of the usual sum integral.

Now consider two matrices $(a_{n,m}(z))$ and $(b_{n,m}(z))$ such that $(a_{n,m}(z'))$ and $(b_{n,m}(z''))$ commute for every pair (z',z'') on the interval from z_0 to z. Then we have

$$
\begin{aligned}
&\prod_{z_0}^{z} e^{[(a_{n,m}(z')) + (b_{n,m}(z'))]dz'} \\
&= \lim_{L\to\infty} \bigodot_{p=1}^{L} e^{[(a_{n,m}(z_p)) + (b_{n,m}(z_p))]\Delta z} \\
&= \lim_{L\to\infty} \bigodot_{p=1}^{L} e^{(a_{n,m}(z_p))\Delta z} \cdot e^{(b_{n,m}(z_p))\Delta z} \\
&= \left[\lim_{L\to\infty} \bigodot_{p=1}^{L} e^{(a_{n,m}(z_p))\Delta z}\right] \cdot \left[\lim_{L\to\infty} \bigodot_{p=1}^{L} e^{(b_{n,m}(z_p))\Delta z}\right] \\
&= \left[\prod_{z_0}^{z} e^{(a_{n,m}(z'))dz'}\right] \cdot \left[\prod_{z_0}^{z} e^{(b_{n,m}(z'))dz'}\right] \\
&= \left[\prod_{z_0}^{z} e^{(b_{n,m}(z'))dz'}\right] \cdot \left[\prod_{z_0}^{z} e^{(a_{n,m}(z'))dz'}\right]
\end{aligned}
\tag{B.14}
$$

which is a commuting product of product integrals.

For supermatrices constructed by the direct product (Appendix A) there are special results. First the series representation in (A.23) gives for $[N \times N] \otimes [M \times M]$ supermatrices for the exponential function

$$e^{\left(B'_{n,m}\right)\otimes\left(B''_{u,v}\right)} = \left(1_{n,m}\right)\otimes\left(1_{u,v}\right)+\sum_{\ell=1}^{\infty}\frac{1}{\ell!}\left[\left(B'_{n,m}\right)\otimes\left(B''_{u,v}\right)\right]^{\ell}$$
$$= \left(1_{n,m}\right)\otimes\left(1_{u,v}\right)+\sum_{\ell=1}^{\infty}\frac{1}{\ell!}\left(B'_{n,m}\right)^{\ell}\otimes\left(B''_{u,v}\right)^{\ell} \qquad \text{(B.15)}$$

which can simplify the computation of this function. For either of the constituent matrices as an identity we then have

$$e^{\left(1_{n,m}\right)\otimes\left(B''_{u,v}\right)} = \left(1_{n,m}\right)\otimes e^{\left(B''_{u,v}\right)}$$
$$e^{\left(B'_{n,m}\right)\otimes\left(1_{u,v}\right)} = e^{\left(B'_{n,m}\right)}\otimes\left(1_{u,v}\right) \qquad \text{(B.16)}$$

For such terms appearing in product integrals we have

$$\prod_{z_0}^{z} e^{\left(1_{n,m}\right)\otimes\left(B''_{u,v}(z')\right)dz'} = \left(1_{n,m}\right)\otimes\left[\prod_{z_0}^{z} e^{\left(B''_{u,v}\right)dz'}\right]$$
$$\prod_{z_0}^{z} e^{\left(B'_{n,m}(z')\right)\otimes\left(1_{u,v}\right)dz'} = \left[\prod_{z_0}^{z} e^{\left(B'_{n,m}(z')\right)dz'}\right]\otimes\left(1_{u,v}\right) \qquad \text{(B.17)}$$

Another special case of interest has

$$\left(B'_{n,m}\right)^{\ell} = \left(B'_{n,m}\right) \text{ for } \ell = 1,2,\cdots,\infty \qquad \text{(B.18)}$$

where $\left(B'_{n,m}\right)$ may be singular with one or more zero eigenvalues. This means that

$$\left(B'_{n,m}\right)^{0} \neq \left(1_{n,m}\right) \qquad \text{(undefined)} \qquad \text{(B.19)}$$

Then we have

$$e^{\left(B'_{n,m}\right)\otimes\left(B''_{u,v}\right)} = \left(1_{n,m}\right)\otimes\left(1_{u,v}\right)+\left(B'_{n,m}\right)\otimes\left[\sum_{\ell=1}^{\infty}\frac{1}{\ell!}\left(B''_{u,v}\right)^{\ell}\right]$$
$$= \left[\left(1_{n,m}\right)-\left(B'_{n,m}\right)\right]\otimes\left(1_{u,v}\right)+\left(B'_{n,m}\right)\otimes e^{\left(B''_{u,v}\right)} \qquad \text{(B.20)}$$

Noting that

$$\left[\left[(1_{n,m})-(B'_{n,m})\right]\otimes(1_{u,v})\right]^{\ell} = \left[(1_{n,m})-(B'_{n,m})\right]\otimes(1_{u,v}) \text{ for } \ell=1,2,\cdots,\infty$$

$$\left[\left[(1_{n,m})-(B'_{n,m})\right]\otimes(1_{u,v})\right] \odot \left[\left[(1_{n,m})-(B'_{n,m})\right]\otimes(1_{u,v})+(B'_{n,m})\otimes e^{\left(B''_{n,m}(z_p)\right)\Delta z}\right] = \left[(1_{n,m})-(B'_{n,m})\right]\otimes(1_{u,v}) \tag{B.21}$$

we have the product integral formula

$$\begin{aligned}
&\left[\left[(1_{n,m})-(B'_{n,m})\right]\otimes(1_{u,v})\right] \odot \left[\prod_{z_0}^{z} e^{(B'_{n,m})\otimes(B''_{u,v}(z'))dz'}\right] \\
&\quad = \left[\left[(1_{n,m})-(B'_{n,m})\right]\otimes(1_{u,v})\right] \\
&\qquad \odot \left[\lim_{L\to\infty} \bigodot_{p=1}^{L} \left[\left[(1_{n,m})-(B'_{n,m})\right]\otimes(1_{u,v})+(B'_{n,m})\otimes e^{\left(B''_{u,v}(z_p)\right)\Delta z}\right]\right] \\
&\quad = \lim_{L\to\infty} \bigodot_{p=1}^{L} \left[\left[(1_{n,m})-(B'_{n,m})\right]\otimes(1_{u,v})\right] \\
&\quad = \left[(1_{n,m})-(B'_{n,m})\right]\otimes(1_{u,v})
\end{aligned} \tag{B.22}$$

Noting that

$$\left[(B'_{n,m})\otimes(1_{u,v})\right]^{\ell} = (B'_{n,m})\otimes(1_{u,v}) \text{ for } \ell=1,2,\cdots,\infty$$

$$\left[(B'_{n,m})\otimes(1_{u,v})\right] \odot \left[\left[(1_{n,m})-(B'_{n,m})\right]\otimes(1_{u,v}) + (B'_{n,m})\otimes e^{\left(B''_{n,m}(z_p)\right)\Delta z}\right] = (B'_{n,m})\otimes e^{\left(B''_{n,m}(z_p)\right)\Delta z} \tag{B.23}$$

we similarly have

$$\left[(B'_{n,m})\otimes(1_{u,v})\right] \odot \left[\prod_{z_0}^{z} e^{(B'_{n,m})\otimes(B''_{u,v}(z'))dz'}\right]$$

$$= \left[\left(B'_{n,m}\right) \otimes \left(1_{u,v}\right)\right] \odot \left[\lim_{L\to\infty} \bigodot_{p=1}^{L} \left[\left(1_{n,m}\right) - \left(B'_{n,m}\right)\right] \otimes \left(1_{u,v}\right) + \left(B'_{n,m}\right) \otimes e^{\left(B''_{u,v}\left(z_p\right)\right)\Delta z}\right]$$

$$= \lim_{L\to\infty} \bigodot_{p=1}^{L} \left[\left(B'_{n,m}\right) \otimes e^{\left(B''_{u,v}\left(z_p\right)\right)\Delta z}\right] \tag{B.24}$$

$$= \left(B'_{n,m}\right) \otimes \left[\lim_{L\to\infty} \bigodot_{p=1}^{L} e^{\left(B''_{u,v}\left(z'\right)\right)dz}\right]$$

$$= \left(B'_{n,m}\right) \otimes \left[\prod_{z_0}^{z} e^{\left(B''_{u,v}\left(z'\right)\right)dz'}\right]$$

Adding (B.22) to (B.24) and noting that the coefficient on the left-hand side is the identity we have

$$\prod_{z_0}^{z} e^{\left(B'_{n,m}\right) \otimes \left(B''_{u,v}\left(z'\right)\right)dz'}$$

$$= \left[\left(1_{n,m}\right) - \left(B'_{n,m}\right)\right] \otimes \left(1_{u,v}\right) + \left(B'_{n,m}\right) \otimes \left[\prod_{z_0}^{z} e^{\left(B''_{u,v}\left(z'\right)\right)dz'}\right] \tag{B.25}$$

The reader can observe that $\left(B'_{n,m}\right)$ can be the zero or identity matrix in the above formula, replicating known results.

A similar case has

$$\left(B''_{u,v}\right)^{\ell} = \left(B''_{u,v}\right) \text{ for } \ell = 1, 2, \cdots, \infty$$

$$\left(B''_{u,v}\right)^{0} \neq \left(1_{n,m}\right) \quad \text{(undefined)} \tag{B.26}$$

$$e^{\left(B'_{n,m}\right) \otimes \left(B''_{u,v}\right)} = \left(1_{n,m}\right) \otimes \left[\left(1_{u,v}\right) - \left(B''_{u,v}\right)\right] + e^{\left(B'_{n,m}\right)} \otimes \left(B''_{u,v}\right)$$

Following the previous steps we obtain

$$\left[\left(1_{n,m}\right) \otimes \left[\left(1_{u,v}\right) - \left(B''_{u,v}\right)\right]\right] \odot \left[\prod_{z_0}^{z} e^{\left(B'_{n,m}\left(z'\right)\right) \otimes \left(B''_{u,v}\right)dz'}\right]$$

$$= \left(1_{n,m}\right) \otimes \left[\left(1_{u,v}\right) - \left(B''_{u,v}\right)\right]$$

$$\left[\left(1_{n,m}\right) \otimes \left(B''_{u,v}\right)\right] \odot \left[\prod_{z_0}^{z} e^{\left(B'_{n,m}(z')\right) \otimes \left(B''_{u,v}\right) dz'}\right]$$

$$= \left[\prod_{z_0}^{z} e^{\left(B'_{n,m}(z')\right) dz'}\right] \otimes \left(B''_{u,v}\right) \tag{B.27}$$

$$\prod_{z_0}^{z} e^{\left(B'_{n,m}(z')\right) \otimes \left(B''_{u,v}\right) dz'}$$

$$= \left(1_{n,m}\right) \otimes \left[\left(1_{u,v}\right) - \left(B''_{u,v}\right)\right] + \left[\prod_{z_0}^{z} e^{\left(B'_{n,m}(z')\right) dz'}\right] \otimes \left(B''_{u,v}\right)$$

References

1. C. E. Baum, T. K. Liu, and F. M. Tesche, On the Analysis of General Multiconductor Transmission-Line Networks, Interaction Note 350, November 1978, also contained in C. E. Baum, Electromagnetic Topology for the Analysis and Design of Complex Electromagnetic Systems, pp. 467—547, in J. E. Thompson and L. H. Luessen (eds.), *Fast Electrical and Optical Measurements*, Martinus Nijhoff, Dordrecht, 1986.

2. C. E. Baum, Electromagnetic Topology: A Formal Approach to the Analysis and Design of Complex Electronic Systems, Interaction Note 400, and Proc. EMC Symposium, Zurich, March 1981, pp. 209–214.

3. C. E. Baum, On the Use of Electromagnetic Topology for the Decomposition of Scattering Matrices for Complex Physical Structures, Interaction Note 454, July 1985.

4. C. E. Baum, High-Frequency Propagation on Nonuniform Multiconductor Transmission Lines in Uniform Media, Interaction Note 463, March 1988, and Int'l. J. Numerical Modelling: Electronic Networks, Devices, and Fields, 1988, pp. 175–188.

5. C. E. Baum, Approximation of Nonuniform Multiconductor Transmission Lines by Analytically Solvable Section, Interaction Note 490, October 1992.

6. C. E. Baum, J. B. Nitsch, and R. J. Sturm, Nonuniform Multiconductor Transmission Lines, Interaction Note 516, February 1996.

7. J. S. Lomont, *Applications of Finite Groups*, Academic Press, 1959.

8. E. P. Wigner, Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra, Academic Press, 1959.

9. M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions*, AMS 55, U. S. Gov't Printing Office, 1964.

10. J. D. Dollard and C. N. Friedman, *Product Integration*, Addison-Wesley, 1979.

11. J. F. Cornwell, *Group Theory in Physics*, Academic Press, 1984.

12. C. E. Baum and H. N. Kritikos, Symmetry in Electromagnetics, ch. 1, pp. 1–90, in C. E. Baum and H. N. Kritikos (eds.), *Electromagnetic Symmetry*, Taylor & Francis, 1995.

13. C. E. Baum, J. B. Nitsch, and R. J. Sturm, Analytical Solution for Uniform and Nonuniform Multiconductor Transmission Lines with Sources, ch. 18, pp. 433–464, in W. Ross Stone (ed.), *Review of Radio Science 1993–1996*, Oxford U. Press, 1996.