Interaction Notes

Note 541

24 June 1998

On the Theory of Nonuniform Multiconductor Transmission Lines: an Approach to Irregular Wire Configurations.

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Abstract

The theory of nonuniform transmission lines is extended to lines which strongly deviate from uniformity. Based on differential geometry the wires are described as curves in threedimensional space. The electric field is derived from a very general mixed potential ansatz. Only minor approximations are introduced in the course of the derivation. Terms, which describe the curvature of the wires. are explicitly taken into account. The result is a system of linear first order differential equations with position dependent coefficients. It is presented in supermatrix notation.

Introduction

Nonuniform or irregular wire configurations are very common in modern cable networks and transmission line systems. The majority consists of several coupled wires and they are designed to transmit high frequency signals. This situation differs significantly from the early applications of nonuniform transmission lines, where adapters and matching circuits in simple transmission lines were of main interest. The early papers, [1] through [5], reflect these facts. In more modern applications complex structures dominate. One of the first thorough concepts to analyse multiconductor transmission line networks has become known under BLT equations, [6]. It has been developed on the basis of topology and is able to describe networks in terms of transmission-line sections, named tubes, and in terms of junctions. In its original form the BLT concept was limited to uniform transmission lines, but it became kind of a framework for the later developed general theory of nonuniform multiconductor transmission lines (NMTL). In [7] circulant NMTLs were studied. Stress was laid on symmetry and other foundamental aspects involved with symmetry. When applicable, the method developed there significantly reduces the numerical effort, since it allows to decouple the equations. Also limited is the class of NMTLs, which was studied in [8]. The idea behind that paper was to assemble lines of analytically solvable sections. The limiting factor of this model is the fact, that only a small number of solvable NMTLs exist. A first step towards general NMTL was done in [9]. Starting from a general nonuniform ansatz an approximation was introduced in the course of the paper. Roughly speaking, the position dependence of the characteristic impedance matrix (and with it the reflections) were first neglected within certain sections, and reestablished later in a corrective step at the junctions. This limitation was overcome in [10]. In this paper the position dependence of the characteristic impedance matrix was fully considered and a new system of NMTL-equations was presented as a set of coupled linear first order differential equations, the coefficients of which are position dependent. This system was solved and an exact solution was presented in [10]. These results were then integrated in the BLT concept, [11]. In this way a very versatile concept for analysing networks of nonuniform multiconductor transmission lines was obtained.

In the cource of the development, just described, the major part of the work was devoted to the mathematical and numerical problems. The main reason is the fact that, compared to the uniform case, the solution of the nonuniform one is much more complex. Moreover, it is not straight forward to apply the solution presented in [10] to real cases and therefore further adaptation to established standard functions and methods seems to be desirable. In a currently published paper [12] a single wire structure (one conductor plus current return) has been studied. Compared to [10], the complexity of the solution was considerably reduced.

In contrast to the intense endeavour to solve the mathematical problems which are involved with the modified transmission line equations, the physical aspects did not gain the same degree of attention. The classical transmission line model represents an idealized case based on far-reaching assumptions. The wires are straight and parallel. The propagation is therefore position independent and independent of direction. The only mode which is considered is the TEM mode. The interaction between two wires reduces to the interaction between two adjacent differential length elements of the wires. Consequently, capacitance and inductance per unit length are the only characteristics of the line. This is no longer true when strong departures from uniformity are considered. Then the elements of the matrices characterizing the transmission line can no longer be interpreted in terms of capacitance and inductance per unit length, only. A theory which would allow an alternative interpretation is still missing. For slight departures from uniformity one might rely upon perturbation theory. It says that the results of the uniform model are conserved in an approximate sense.

There is still another unsolved question. At high frequences, particularly when strong nonuniformities are involved longitudinal modes may significantly contribute to the total field. The integral over the electric field will then strongly depend on the integration path. This means that the variable "voltage" becomes an illdefined quantity.

Résumé: when developing a theory of nonuniform tranmission lines with strong deviations from uniformity in mind the uniform transmission line theory seems no longer to be a good basis to start with. More appropriate seems to be a general ansatz which allows to describe very irregular wire configurations from the beginning. No further limiting assumptions, in particular what the involved modes are concerned, should be part of the ansatz. Since the method of solving linear first order differential equation systems with position dependent coefficients is well established, one should try to derive a similar system of equations.

This paper is aimed at deriving a system of equations which describe the propagation of signals on very general wire structures. To avoid the difficulties involved with the variable "voltage" we shall use the variable "charge density", instead of voltage. This choice has the additional advantage that all conductors are equivalent i.e. the reference conductor needs not to be defined until a real system is considered.

Description of the Wire Geometry

Consider a multiconductor system which consists of N thin, cylindrical wires of good conductivity. Each wire is indicated by an index i, $1 \le i \le N$. The system is assumed to be in free space. There shall be no electric contact between the wires. We describe the axis of cylinder of wire i by the local vector $\mathbf{r}_i\,(l_i)$, (Fig. 1). l_i is the length parameter of wire i. The tangential unit vector $\mathbf{s}_i(l_i)$ at point $\mathbf{r}_i$ is then defined by $\mathbf{s}_i\,(l_i) = \mathrm{d}\,\mathbf{r}_i/\mathrm{d}l_i$. (note: if a curve parameter other than l_i is chosen, then $\mathbf{s}_i$ is no longer a unit vector)

The curve parameter l_i is a good choice when considering a single wire or a system of uncoupled wires. Then the signals propagate independently on the individual wires and individual length parameters are just the right variable to discribe the propagation. In case of coupled wires we are interested in a collective phenomenon i.e. in a wave which is guided by an ensemble of wires. The variable z, which is common to all wires, is more appropriate in this case, provided that an important restriction is acceptable, i.e. $\mathbf{r}_i(z)$ must be a unique function of z . That means that backbending wires (Fig.1) cannot be treated with the equations we are going to develop. In the first step we will confine ourselfes to thin wires. In this approximation the current vector always points in the direction of the wire axis.

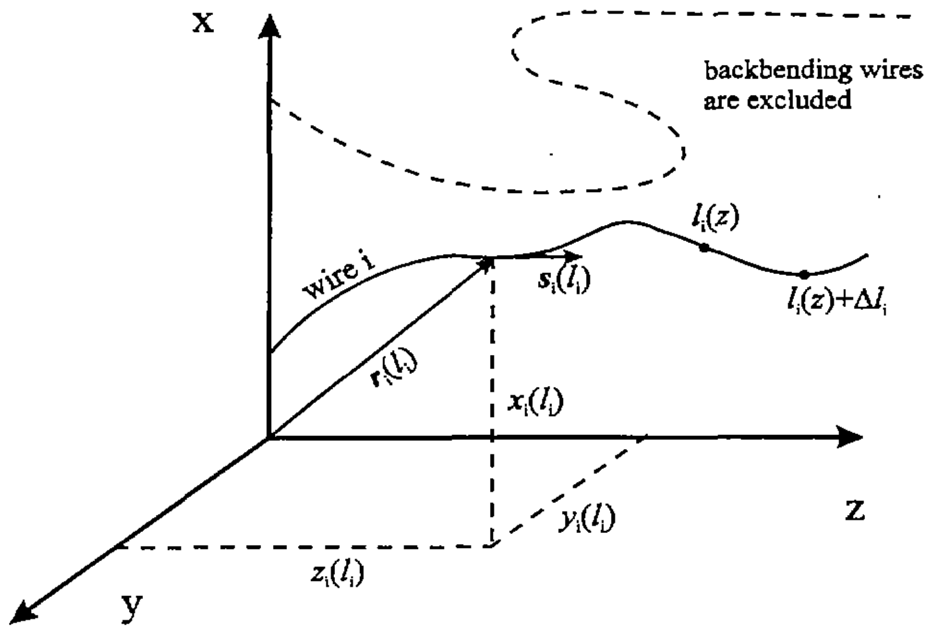

Fig. 1: The wires are described by the local vector $r_i(l_i) = u_x x_i(l_i) + u_y y_i(l_i) + u_z z(l_i)$. s_i is the tangential unit vector at point $r_i(l_i)$.

Derivation of the Generalized Transmission Line Equations

Consider the surface S_i of the cylindrical conductor i with the current density $J_{i,surf}(r_{s,i}')$ and the charge density $\rho_{i,surf}(r_{s,i}')$, ($r_{s,i}' \in S_i$). We want to calculate the scattered electric field $E_i^s(r)$ at point r which shall be either exterior of wire i ($r \notin S_i$) or on its surface ($r \in S_i$). To determine the electric field in free space we start from the mixed potential representation of the electric field. For time-harmonic fields we get:

$$E_i^s(r) = -j\omega A_i(r) - \nabla\Phi_i(r)$$

Φ - *scalar potential*; A - *vector potential, with*

$$A_i(r) = \frac{1}{\varepsilon_0 c^2}\int_{S_i} G(r,r_{s,i}')J_{i,surf}(r_{s,i}')dS_i', \quad \Phi_i(r) = \frac{1}{\varepsilon_0}\int_{S_i} G(r,r_{s,i}')\rho_{i,surf}(r_{s,i}')dS_i' \qquad (1)$$

$$G(r,r_{s,i}') = \frac{1}{4\pi|r-r_{s,i}'|}e^{-jk|r-r_{s,i}'|} \qquad \textit{Green's function of free space}$$

Eq.(1) is a frequently used starting point for theoretical considerations in electrodynamics. For our purposes it is preferable to start from a somewhat different form. This is shown in (2). The transition from (1) to (2) is described in appendix 1. Note that (2) is an approximation. The contributions from both ends of the wire have been dropped. Since the theory shall first be developed for regions which are far apart from the ends the contribution of the endterms is taken to be negligible.

$$E_i^s(r) \approx -\frac{j\omega}{\varepsilon_0 c^2}\int_{S_i} G(r,r_{s,i}')J_{surf}(r_{s,i}')dS_i' - \frac{1}{\varepsilon_0}\int_{S_i} G(r,r_{s,i}')\nabla'\rho_{surf}(r_{s,i}')dS_i' \qquad (2)$$

Next we adapt the integrals in (2) to the geometry of cylinders and to the thin wire approximations. We proceed in the following way:

1. Introduce cylinder coordinates. If the radius a_i of cylinder i is fixed, the differential area on the surface is given by $dS_i' = a_i\, d\varphi_i'\, dl_i'$.

2. The thin wire approximation suggests that $J_{i,surf}(r_{s,i}')$ and $\rho_{i,surf}(r_{s,i}')$ depend on l_i' only, and do not depend on φ_i'. In other words, $J_{i,surf}(r_{s,i}')$ and $\rho_{i,surf}(r_{s,i}')$ are assumed to be equally distributed over the circumference of any randomly selected cross-section.

3. According to 2. the centers of the charge and current distributions lie on the axis of the cylinder. For small radii a_i (small compared to the distance from the axis of wire i to the surface of other wires) one can replace the charge and current density on the surface by a total charge and current concentrated on the axis of the wires. This means, that $J_{i,surf}(r_{s,i}')$ and $\rho_{i,surf}(r_{s,i}')$ are replaced by the current $J_{i,tot}(r_i')$ and the line charge $\rho_{i,tot}(r_i')$ without significant loss of accuracy. Consequently the vectors $\nabla\rho_{i,tot}$ und $J_{i,tot}$ have the direction of s_i.

4. After having concentrated charge and current on the axis, $G(r,r_{s,i}')$ has to be adjusted accordingly. The vector $r_{s,i}'$ which indicates the location of the sources, is replaced by r_i'. Thus $|r-r_{s,i}'|$ changes into $|r-r_i'|$. That means that all distances to the field point r are measured now from the axis of wire i. Moreover $G(r,r_i')$ is now independent of φ_i' and consequently can be put in front of the integral over φ_i'.

From 4. follows

$$E_i^s(r) \approx -\frac{j\omega}{\varepsilon_0\, c^2}\cdot\int_L G(r,r_i')\left(\int_0^{2\pi} J_{i,surf}(r_{s,i}')\; a_i\; d\varphi_i'\right)dl_i' - \frac{1}{\varepsilon_0}\cdot\int_L G(r,r_i')\left(\int_0^{2\pi}\nabla'\rho_{i,surf}(r_{s,i}')\; a_i\; d\varphi_i'\right)dl_i' \qquad (3)$$

Since the vector $\nabla'\rho_{i,surf}$ points in the direction of $s_i(r_i')$ it can be written as

$$\nabla'\varrho(r_i') = s_i(r_i') \cdot \frac{\partial}{\partial s_i}\varrho(r_i') \tag{4}$$

The derivative with respect to s_i is sometimes refered to as the directional derivative. According to 3. we write

$$\begin{aligned} \rho_{i,surf}(r_{s,i}) &\approx \frac{\rho_{i,tot}(r_i)}{2\pi a_i} \\ J_{i,surf}(r_{s,i}) &\approx \frac{J_{i,tot}(r_i)}{2\pi a_i} \end{aligned} \tag{5}$$

The integrals over φ_i' can now be replaced by simple expressions.

$$\begin{aligned} \int_0^{2\pi} J_{i,surf}(r_{s,i}')\, a_i\, d\varphi_i' &= s_i(r_i') \cdot J_{i,tot}(r_i') \\ \int_0^{2\pi} \nabla'\rho_{i,surf}(r_{s,i}')\, a_i\, d\varphi_i' &= s_i(r_i') \cdot \frac{\partial}{\partial s_i}\, q_{i,tot}(r_i') \end{aligned} \tag{6}$$

Boundary Conditions

In a configuration as shown in Fig. 1 there is a mutual interaction between all individual wires. On the surface of a randomly selected wire j ($j=i$ is allowed) the tangential component of the electric field must vanish (for ideal conductors). In case of N wires the electric field at r_j ,which is on wire j, is a superposition of all $E_i^s(r_j)$, $1 \le i \le N$. (Index i in E_i indicates that it is due to $\nabla\rho_{i,tot}$ und $J_{i,tot}$ on wire i). In the thin wire approximation the boundary condition reduces to the component along the axis of cylinder.

$$s_j(r_j)\cdot E^{inc}(r_j) + \sum_{i=1}^{N} s_j(r_j)\cdot E_i^s(r_j) = \begin{cases} 0 & \textit{for ideal conductors} \\ Z_{j,s}(r_j)J_{j,tot}(r_j) & \textit{for conductors with finite but still good conductivity} \end{cases} \tag{7}$$

$E^{inc}(r_j)$ denotes the external E-field incident on wire j, and $Z_{j,s}(r_j)$ is the surface impedance of wire j at r_j . Eq.(7) represents N equations ($1 \le j \le N$) for $2N$ variables $J_{i,tot}(r_j)$ and $\rho_{i,tot}(r_j)$, ($1 \le i \le N$).

The system of equations is completed by adding N equations ($1 \le i \le N$) which represent the continuity condition for each individual wire.

$$\frac{d}{dl_i} J_{i,tot}(\boldsymbol{r}_i) = -j\omega \rho_{i,tot}(\boldsymbol{r}_i) \tag{8}$$

Adaption of the Equation System

Eqs .(7) and (8) form an integro differential equation system. There are many ways to solve it, and among them are quite a number of numerical methods. The latter are well suited to study special cases. Since we are more interested in general properties of nonuniform transmission lines we prefer an analytical solution. A successful way could be to convert (7) and (8) to a first order differential equation system. We will follow this idea.

Let $\boldsymbol{r}_j'$, $\boldsymbol{r}_i'$ and l_i be unique functions of z. Then we are allowed to substitute in (3) z' for the integration variable l_i'. The requirement of uniqueness somewhat reduces the class of problems, to which this theory is applicable. However, in most cases of practical importance this condition seems to be fullfilled. Starting from (3) this substitution yields the following expression for $\boldsymbol{E}_i^s(\boldsymbol{r})$

$$\begin{aligned} \boldsymbol{E}_i^s(\boldsymbol{r}(z)) \approx & -\frac{j\omega}{\varepsilon_0 c^2} \int_{z=0}^{z_L} G(\boldsymbol{r}(z),\, \boldsymbol{r}_i(z'))\; \boldsymbol{s}_i(z')\; J_{i,tot}(z')\; \frac{dl_i'}{dz'}\; dz' \\ & -\frac{1}{\varepsilon_0} \int_{z=0}^{z_L} G(\boldsymbol{r}(z),\, \boldsymbol{r}_i(z')) \cdot \boldsymbol{s}_i(z') \cdot \frac{d}{dz'}\, \rho_{i,tot}(z')\; \frac{dl_i'}{ds_i}\; dz' \end{aligned} \tag{9}$$

A chance to convert (9) to a differential equation is to put $s_i(z')\cdot d\rho_{i,tot}/dz'$ and $s_i(z')\cdot J_{i,tot}$ as factors in front of the integrals. This is feasible when making a first order Taylor approximation of either vector function at position z.

$$\begin{aligned} \boldsymbol{s}_i(\boldsymbol{r}_i(z')) \cdot \frac{d}{dz'}\rho_{i,tot}(\boldsymbol{r}_i(z')) \approx & \; \boldsymbol{s}_i(\boldsymbol{r}_i(z)) \cdot \frac{d}{dz}\rho_{i,tot}(\boldsymbol{r}_i(z)) \\ & + \left[\frac{d}{dz}\boldsymbol{s}_i(\boldsymbol{r}_i(z))\right] \cdot \frac{d}{dz}\rho_{i,tot}(\boldsymbol{r}_i(z))\cdot(z'-z) + \boldsymbol{s}_i(\boldsymbol{r}_i(z)) \cdot \left[\frac{d^2}{dz^2}\rho_{i,tot}(\boldsymbol{r}_i(z))\right](z'-z) \end{aligned} \tag{10}$$

$$\begin{aligned} \boldsymbol{s}_i(\boldsymbol{r}_i(z')) \cdot J_{i,tot}(\boldsymbol{r}_i(z')) \approx & \; \boldsymbol{s}_i(\boldsymbol{r}_i(z)) \cdot J_{i,tot}(\boldsymbol{r}_i(z)) \\ & + \left[\frac{d}{dz}\boldsymbol{s}_i(\boldsymbol{r}_i(z))\right] \cdot J_{i,tot}(\boldsymbol{r}_i(z))(z'-z) + \boldsymbol{s}_i(\boldsymbol{r}_i(z)) \cdot \left[\frac{d}{dz}J_{i,tot}(\boldsymbol{r}_i(z))\right](z'-z) \end{aligned} \tag{11}$$

At this point it's worth the effort to reflect about the approximations (10) and (11). Since all terms of second order and higher have been dropped, (10) and (11) are valid only in a limited interval around z. This is tolerable as long as the range of suffiently exact approximation covers completely the range where the Green's function $G(\boldsymbol{r}(z),\boldsymbol{r}_{s,i}(z'))$ differs significantly from zero. This argument is based on the idea that $G(\boldsymbol{r}(z),\boldsymbol{r}_{s,i}(z'))$ can be interpreted in (9) as the weighting function of the vector functions $\boldsymbol{s}_i(z')\cdot d\rho_{i,tot}/dz'$ and $\boldsymbol{s}_i(z')\cdot J_{i,surf}(z')$. It seems to be evident that the decision whether or not (10) and (11) are good approximations depends strongly on the individual case. So we postpone this discussion till real cases are considered.

To make (13) easier to manage we introduce in (12) a number of abbreviations for the remaining integrals.

$$
\begin{aligned}
I_{j,i}^{J,(0)}(z) &= \int G(\boldsymbol{r}_j(z),\boldsymbol{r}_i(z'))\frac{dl_i'}{dz'}dz' \;;\quad I_{j,i}^{J,(1)}(z) = \int G(\boldsymbol{r}_j(z),\boldsymbol{r}_i(z'))\frac{dl_i'}{dz'}(z'-z)dz' \\
I_{j,i}^{\rho,(0)}(z) &= \int G(\boldsymbol{r}_j(z),\boldsymbol{r}_i(z'))\frac{dl_i'}{ds_i'}dz' \;;\quad I_{j,i}^{\rho,(1)}(z) = \int G(\boldsymbol{r}_j(z),\boldsymbol{r}_i(z'))\frac{dl_i'}{ds_i'}(z'-z)dz'
\end{aligned}
\tag{12}
$$

We then substitute (10) and (11) into (9), rename the integrals by applying (12) and finally substitute the resulting $\boldsymbol{E}_i^s(\boldsymbol{r}_j)$ into (7). Eq.(7) then reduces to

$$
\begin{aligned}
\boldsymbol{s}_j(z)\cdot\boldsymbol{E}^{inc}(\boldsymbol{r}_j) = \sum_{i=1}^{N} \Bigg(&\left[\boldsymbol{s}_j(z)\boldsymbol{s}_i(z)\frac{j\omega I_{j,i}^{J,(0)}(z)}{\varepsilon_0 c^2}\right] J_{i,tot}(z) \\
&+\left[\boldsymbol{s}_j(z)\frac{d}{dz}\boldsymbol{s}_i(z)\frac{j\omega I_{j,i}^{J,(1)}(z)}{\varepsilon_0 c^2}\right] J_{i,tot}(z) \\
&+\left[\boldsymbol{s}_j(z)\boldsymbol{s}_i(z)\frac{j\omega I_{j,i}^{J,(1)}(z)}{\varepsilon_0 c^2}\right] \frac{d}{dz}J_{i,tot}(z) \\
&+\left[\boldsymbol{s}_j(z)\boldsymbol{s}_i(z)\;\frac{I_{j,i}^{\rho,(0)}(z)}{\varepsilon_0}\right] \frac{d}{dz}\rho_{i,tot}(z) \\
&+\left[\boldsymbol{s}_j(z)\frac{d}{dz}\boldsymbol{s}_i(z)\;\frac{I_{j,i}^{\rho,(1)}(z)}{\varepsilon_0}\right] \frac{d}{dz}\rho_{i,tot}(z) \\
&+\left[\boldsymbol{s}_j(z)\;\boldsymbol{s}_i(z)\;\frac{I_{j,i}^{\rho,(1)}(z)}{\varepsilon_0}\right] \frac{d^2}{dz^2}\rho_{i,tot}(z)\Bigg)
\end{aligned}
\tag{13}
$$

Eq.(13) shows a clear structure. The terms set in square brackets are characteric of the transmission line. They depend only on the geometry of the conductors and consist of scalar products of the tangent vectors or the derivatives thereof and of the integrals shown in (12). We temporarily rename the terms in square brackets in the order as they are listed in (13) and call them $A^{(1)}_{j,i}(z)$ through $A^{(6)}_{j,i}(z)$.

It is our goal to derive a first order differential equation sytem. The sixth term of (13) does not fit in this scheme because it contains a second order derivative. We drop this term without further discussion but we keep in mind that some justification has to be given later on.

Since the homogeneous differential equation system shall be written in matrix notation the terms in (13) are rearranged.

$$\sum_{i=1}^{N} \left[A^{(4)}{}_{j,i} + A^{(5)}{}_{j,i} \right] \cdot \frac{d}{dz}\rho_{i,tot}(z) + A^{(3)}{}_{j,i} \cdot \frac{d}{dz} J_{i,tot}(z) = - \sum_{i=1}^{N} \left[A^{(1)}{}_{j,i} + A^{(2)}{}_{j,i} \right] \cdot J_{i,tot}(z) \tag{14}$$

With the continuity conditions, now written as derivative with respect to z, the system of equations is completed.

$$\frac{d}{dz} J_{i,tot}(z) = -j\omega \frac{dl_i}{dz} \rho_{i,tot}(z) \tag{15}$$

After having combined the $\rho_{i,tot}$ and the $J_{i,tot}$ in a $2N$-dimensional vector (14) und (15) can now be written in matrix form:

$$\begin{bmatrix} \left[\kappa_{mn}^{((1))}(z) \right] & \left[\kappa_{mn}^{((2))}(z) \right] \\ 0 & \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \end{bmatrix} \frac{d}{dz} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_N \\ J_1 \\ J_2 \\ \vdots \\ J_N \end{pmatrix} = \begin{bmatrix} \left[\quad 0 \quad \right] & \left[\kappa_{mn}^{((3))}(z) \right] \\ -j\omega \begin{bmatrix} \ddots & & 0 \\ & \frac{dl_n}{dz} & \\ 0 & & \ddots \end{bmatrix} & 0 \end{bmatrix} \cdot \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_N \\ J_1 \\ J_2 \\ \vdots \\ J_N \end{pmatrix} \tag{16}$$

The following abbreviations have been used

$$\begin{aligned} \kappa_{mn}^{((1))} &= A^{(4)}{}_{j,i} + A^{(5)}{}_{j,i} && \text{with } m=j \\ \kappa_{mn}^{((2))} &= A^{(3)}{}_{j,i} && \text{and } n=i \\ \kappa_{mn}^{((3))}(z) &= A^{(1)}{}_{j,i} + A^{(2)}{}_{j,i} \end{aligned} \tag{17}$$

Since (16) is not the final form which we are looking for, we determine the inverse of the matrix which is on the left side of (16) and multiply (16) from the left. In this way we get (18). Here we have taken it for granted that the inverse exists. When applying the theory to a real multiconductor this has to be proven case by case.

$$\frac{d}{dz}\begin{pmatrix}\rho_1\\ \rho_2\\ \vdots\\ \rho_N\\ J_1\\ J_2\\ \vdots\\ J_N\end{pmatrix} = \begin{bmatrix}\left[\kappa_{mn}^{((1))}(z)\right] & \left[\kappa_{mn}^{((2))}(z)\right]\\ 0 & \begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}\end{bmatrix}^{-1} \cdot \begin{bmatrix}\left[\quad 0 \quad\right] & \left[\kappa_{mn}^{((3))}(z)\right]\\ -j\omega\begin{bmatrix}\ddots & & 0\\ & \frac{dl_n}{dz} & \\ 0 & & \ddots\end{bmatrix} & 0\end{bmatrix} \cdot \begin{pmatrix}\rho_1\\ \rho_2\\ \vdots\\ \rho_N\\ J_1\\ J_2\\ \vdots\\ J_N\end{pmatrix} \tag{18}$$

To reach the compact form, as shown in (20), the inverse matrix is further developed.

$$\begin{bmatrix}\left[\kappa_{mn}^{((1))}(z)\right] & \left[\kappa_{mn}^{((2))}(z)\right]\\ 0 & \begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}\end{bmatrix}^{-1} = \begin{bmatrix}\left[\kappa_{mn}^{((1))}(z)\right]^{-1} & \left[\kappa_{mn}^{((1))}(z)\right]^{-1}\cdot\left[\kappa_{mn}^{((2))}(z)\right]\\ 0 & \begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}\end{bmatrix} \tag{19}$$

Based on (19) we are now able to rewrite (18) in the wanted form

$$\frac{d}{dz}\begin{pmatrix}\boldsymbol{\rho}\\ \boldsymbol{\varsigma}\end{pmatrix} = \begin{pmatrix}\boldsymbol{C}_{11} & \boldsymbol{C}_{12}\\ \boldsymbol{C}_{21} & \boldsymbol{O}\end{pmatrix} \odot \begin{pmatrix}\boldsymbol{\rho}\\ \boldsymbol{\varsigma}\end{pmatrix} \tag{20}$$

Eq. (20) is written in supermatrix/supervector notation. The elements $C_{v,w}$ of the supermatrix are matrices. They are defined by

$$C_{11} = +j\omega \left(\kappa_{mn}^{((1))}(z) \right)^{-1} \cdot \left(\kappa_{mn}^{((2))}(z) \right) \cdot \left(\frac{dl_m}{dz} \delta_{mn} \right)$$

$$C_{12} = \left(\kappa_{mn}^{((1))}(z) \right)^{-1} \cdot \left(\kappa_{mn}^{((3))}(z) \right) \quad ; \quad C_{21} = -j\omega \left(\frac{dl_m}{dz} \delta_{mn} \right) \tag{21}$$

$$\rho = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_N \end{pmatrix} \quad ; \quad \varsigma = \begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_N \end{pmatrix}$$

With (20) we obtained an equation system the form of which is similar to that of the nonuniform transmission line equations investigated earlier. Therefore the mathematical methods developed there to solve the equations are also applicable to the new system. There is, however, a significant difference to the former situation. The coefficients, the physical meaning of which was rather rather nebulous, are now easily interpreted by the interaction-terms, as given in (21).

Conclusion

A new theory of nonuniform transmission lines has been presented. It was derived from the mixed potential representation of the electric field combined with a very general geometrical description which covers even strongly bent wires. In the course of the development some approximations had to be made. These will limit the class of wire configurations to which the new theory is applicable. Fortunately it seems that most cases of practical relevance can be treated by the theory presented here, and mainly the extrem and to some degree hypothetical cases are excluded. Further work is necessary to develop precise criteria on the basis of which one can decide whether the theory is applicable. A general statement which might not be decisive enough but which points in the right direction would be that the thin wire assumptions have to be met and that Eq. (10) and (11) must represent reasonable approximations.

Appendix

This is to show that the gradient of the scalar potential Φ_i, as used in (1), can be expressed by an integral over the gradient of the charge density ρ_i and a function $\boldsymbol{F}_i(\boldsymbol{r}, \boldsymbol{r}_i)$, which takes into account the sources located at the ends of conductor i. This suggests the ansatz

$$\nabla\varphi_i = \frac{1}{\varepsilon_0}\int_{S_i} G(\boldsymbol{r},\boldsymbol{r}_{s,i})\,\nabla'\rho_{i,surf}(\boldsymbol{r}_{s,i}')\,dS_i' + \boldsymbol{F}_i(\boldsymbol{r},\boldsymbol{r}_i'(z{=}0)) - \boldsymbol{F}_i(\boldsymbol{r},\boldsymbol{r}_i'(z{=}z_L)) \qquad \text{(A1.1)}$$

An explicit expression of $\boldsymbol{F}_i(\boldsymbol{r}, \boldsymbol{r}_i)$ will be given at the end of this appendix in (A1.6).

We start from (1) and apply the following general relations

* $\nabla'(G\cdot\rho) = G\cdot\nabla'\rho + \rho\cdot\nabla' G$, which is a generalisation of the product rule of differentiation
* $\nabla G(\boldsymbol{r},\boldsymbol{r}') = -\nabla' G(\boldsymbol{r},\boldsymbol{r}')$, which describes a special symmetry of Green's function G

Substituting these into (1) yields

$$\begin{aligned}\nabla\varphi_i &= -\frac{1}{\varepsilon_0}\int_{S_i}\nabla' G(\boldsymbol{r},\boldsymbol{r}_{s,i}')\rho_{i,surf}(\boldsymbol{r}_{s,i}')\,dS_i' \\ &= -\frac{1}{\varepsilon_0}\int_{S_i}\nabla'\left(G(\boldsymbol{r},\boldsymbol{r}_{s,i}')\rho_{i,surf}(\boldsymbol{r}_{s,i}')\right)dS_i' + \frac{1}{\varepsilon_0}\int_{S_i} G(\boldsymbol{r},\boldsymbol{r}_{s,i}')\,\nabla'\rho_{i,surf}(\boldsymbol{r}_{s,i}')\,dS_i'\end{aligned} \qquad \text{(A1.2)}$$

The second term on the right-hand side of (A1.2) is identical with the leading term of (A1.1). In thin wire approximation, the first one takes the form

$$\begin{aligned}-\frac{1}{\varepsilon_0}\int_{z=0}^{z_L}\nabla'\left(G(\boldsymbol{r},\boldsymbol{r}_i')\rho_{i,tot}(\boldsymbol{r}_i')\right)dl_i' &= -\boldsymbol{u}_x\int_{z=0}^{z_L}\frac{\partial}{\partial x'}\left(G(\boldsymbol{r},\boldsymbol{r}_i')\cdot\rho_{i,tot}(\boldsymbol{r}_i')\right)dl_i' \\ -\boldsymbol{u}_y\int_{z=0}^{z_L}\frac{\partial}{\partial y'}\left(G(\boldsymbol{r},\boldsymbol{r}_i')\cdot\rho_{i,tot}(\boldsymbol{r}_i')\right)dl_i' &- \boldsymbol{u}_z\int_{z=0}^{z_L}\frac{\partial}{\partial z'}\left(G(\boldsymbol{r},\boldsymbol{r}_i')\cdot\rho_{i,tot}(\boldsymbol{r}_i')\right)dl_i'\end{aligned} \qquad \text{(A1.3)}$$

where $\boldsymbol{u}_x$, $\mathbf{u}_y$, and $\boldsymbol{u}_z$ are the unit vectors with respect to the x-, y-, and z-axis. The expressions on the right-hand side of (A1.3) are all of similar structure. That's why it is sufficient to evaluate the z-component, only. The others are then determined by analogy. The differential length dl_i' is given by

$$dl_i' = \sqrt{\left(\frac{dx_i'}{dt}\right)^2 + \left(\frac{dy_i'}{dt}\right)^2 + \left(\frac{dz_i'}{dt}\right)^2}\cdot dt \qquad \text{(A1.4)}$$

t is an arbitrary curve parameter. When evaluating the z-component we choose $t = z$ and have

$$
\begin{aligned}
& - \boldsymbol{u}_z \int_{z=0}^{z_L} \frac{\partial}{\partial z_i'} \left(G(\boldsymbol{r},\boldsymbol{r}_i')\cdot\rho_{i,tot}(\boldsymbol{r}_i') \right) \sqrt{\left(\frac{dx_i'}{dz_i'}\right)^2 + \left(\frac{dy_i'}{dz_i'}\right)^2 + 1}\; dz_i' \\
&= - \boldsymbol{u}_z \int_{z=0}^{z_L} \frac{\partial}{\partial z_i'} \left(G(\boldsymbol{r},\boldsymbol{r}_i')\cdot\rho_{i,tot}(\boldsymbol{r}_i') \sqrt{\left(\frac{dx_i'}{dz_i'}\right)^2 + \left(\frac{dy_i'}{dz'}\right)^2 + 1} \right) dz_i' \\
&= \boldsymbol{u}_z \left[- \left(G(\boldsymbol{r},\boldsymbol{r}_i')\cdot\rho_{i,tot}(\boldsymbol{r}_i') \sqrt{\left(\frac{dx_i'}{dz_i'}\right)^2 + \left(\frac{dy_i'}{dz_i'}\right)^2 + 1} \right)_{\boldsymbol{r}_i(z_L)} \right. \\
&\qquad \left. + \left(G(\boldsymbol{r},\boldsymbol{r}_i')\cdot\rho_{i,tot}(\boldsymbol{r}_i') \sqrt{\left(\frac{dx_i'}{dz_i'}\right)^2 + \left(\frac{dy_i'}{dz_i'}\right)^2 + 1} \right)_{\boldsymbol{r}_i(z_0)} \right]
\end{aligned}
\qquad \text{(A1.5)}
$$

(note: in case of the x- or y- component we would have chosen $t=x$ or $t=y$, respectively.)

Since the argument of the square root in line 1 is a constant with respect to the partial differentiation it can by put into the brackets of line 2. The evaluation of the integral is now straight forward. It is a standard integral known from potential theory. The expressions put in square brackets, line 3 and 4, are identical with the z-component of the vector function $\boldsymbol{F}_i(\boldsymbol{r},\boldsymbol{r}_i(z_{\text{fin}}))$ as postulated in (A1.1). z_{fin} stands for z_0 and z_L, respectively.

Next the x- and y-component are evaluated by analogy. In principle we can proceed as we have already done in (A1.5). We should, however, pay attention to the fact that $y_i(x)$ and $z_i(x)$ or $x_i(y)$ and $z_i(y)$, respectively, need not necessarily be unique functions, whereas for $x(z)$ and $y(z)$ uniqueness along the integration path has been required, explicitly. We circumvent this problem by cutting the integration path in intervals where uniqueness is fulfilled. Then the rear-end-term of any interval will always be compensated by the front-end-term of the subsequent interval. Finally the remaining terms are those which correspond to the front end and the rear end of the total conductor.

$$
\boldsymbol{F}_i(\boldsymbol{r},\boldsymbol{r}_i(z_{fin})) = \boldsymbol{u}_x\cdot F_{i,x}(\boldsymbol{r},\boldsymbol{r}_i(z_{fin})) + \boldsymbol{u}_y\cdot F_{i,y}(\boldsymbol{r},\boldsymbol{r}_i(z_{fin})) + \boldsymbol{u}_z\cdot F_{i,z}(\boldsymbol{r},\boldsymbol{r}_i(z_{fin})) \qquad with
$$

$$
\begin{aligned}
F_{i,x}(\boldsymbol{r},\boldsymbol{r}_i(z_{fin})) &= \left(G(\boldsymbol{r},\boldsymbol{r}_i')\cdot\rho_{i,tot}(\boldsymbol{r}_i') \sqrt{1 + \left(\frac{dy_i'}{dx_i'}\right)^2 + \left(\frac{dz_i'}{dx_i'}\right)^2} \right)_{\boldsymbol{r}_i = \boldsymbol{r}_i(z_{fin})} \\
F_{i,y}(\boldsymbol{r},\boldsymbol{r}_i(z_{fin})) &= \left(G(\boldsymbol{r},\boldsymbol{r}_i')\cdot\rho_{i,tot}(\boldsymbol{r}_i') \sqrt{\left(\frac{dx_i'}{dy_i'}\right)^2 + 1 + \left(\frac{dz_i'}{dy_i'}\right)^2} \right)_{\boldsymbol{r}_i = \boldsymbol{r}_i(z_{fin})} \\
F_{i,z}(\boldsymbol{r},\boldsymbol{r}_i(z_{fin})) &= \left(G(\boldsymbol{r},\boldsymbol{r}_i')\cdot\rho_{i,tot}(\boldsymbol{r}_i') \sqrt{\left(\frac{dx_i'}{dz_i'}\right)^2 + \left(\frac{dy_i'}{dz_i'}\right)^2 + 1} \right)_{\boldsymbol{r}_i = \boldsymbol{r}_i(z_{fin})}
\end{aligned}
\qquad \text{(A1.6)}
$$

The vector function $\boldsymbol{F}_i(\boldsymbol{r}, \boldsymbol{r}_i(z_{\text{fin}}))$ is now completely defined.

Since we will first study the propagation of signals far apart from the ends of the transmission line we drop the end terms, temporarily. They will be taken up again when the influence of the sources at the ends shall be discussed.

References

[1] A.T. Starr, The Nonuniform Transmission Line, Proc. I.R.E., vol. 20, pp. 1052-1063, June, 1932.

[2] C. R. Burrows, The Exponential Transmission Line, Bell. Syst. Techn. J. vol.17, pp.555-573, Oct. 1938.

[3] H.A. Wheeler, Transmission lines with exponential taper, Proc. I.R.E., vol. 27, pp.65-71, Jan. 1939.

[4] H.J.Scott, The Hyperbolic Transmission Line as a Matching Section, Proc. I.R.E., vol. 41, pp. 1654-1657, Nov. 1953.

[5] N. H. Younan, B.L. Cox, C.D. Taylor, W.D. Prather, An Exponentially Tapered Transmission Line Antenna, IEEE Trans. EMC, Vol 36, pp.141-144, May 1994.

[6] C.E. Baum, T.K. Liu, and F.M. Tesche, On the Analysis of General Multicoductor Transmission-Line Networks, Interaction Note 350, November 1978.

[7] J. Nitsch, C.E. Baum, and R.Sturm, Analytical Treatment of Circulant Nonuniform Multiconductor Transmission Lines, IEEE Trans. EMC, vol.354, pp 28-38, Feb. 1992.

[8] C.E. Baum, Approximation of Nonuniform Multiconductor Transmission Lines by Analytically Solvable Sections, Interaction Note 490, October 1992.

[9] C.E. Baum, Generalization of the BLT Equation, Interaction Note 511, April 1995.

[10] C.E. Baum, J.B. Nitsch, R.J. Sturm, Analytical Solution for Uniform and Nonuniform Transmission Lines with Sources, pp. 433-464, in W. Ross Stone (ed.), Review of Radio Science 1993-1996, Oxford University Press, 1996.

[11] C.E. Baum, J.B. Nitsch, R.J. Sturm, Nonuniform Multiconductor Transmission Lines, Interaction Notes 516, Feb. 1996.

[12] J.B. Nitsch, Exact Analytical Solution for Nonuniform Multiconductor Transmission Lines with the Aid of the Solution of a Corresponding Matrix Riccati Equation, Interaction Note 534, October 1998.