Interaction Notes

Note 555

June 1999

A Geometric Scattering Expansion of the Current on Conducting Surfaces and Transmission Line Structures

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Abstract

We derive a perturbative construction scheme for the current on one- and two-dimensional conducting geometries. These conductors can either be in the presence of electromagnetic sources or also be part of the source itself. The current is expressed in terms of a scattering expansion which turns out to be an expansion in the curvature of the conducting geometries. This geometric formalism allows one to perturbatively determine characteristic parameters of arbitrary transmission line structures.

1 Introduction

The coupling of electromagnetic fields into complex electronic systems plays an important role in EMC. Especially multiconductor transmission lines (MTLs) transport voltage- and current-perturbations to the entrance of sensitive devices. Such lines are usually led very closely to conducting walls, but they are seldom parallel to each other. Also the perturbation spectra contain more and more higher frequency contributions. Therefore it is desirable to have a concept at hand to treat beyond the ansatz of the telegrapher equations MTLs which are not uniform and may significantly radiate.

In the following we will propose such a concept. It is based on a perturbation series for the electromagnetic current on two- or one-dimensional conducting geometries. This perturbative approach is derived from an integral equation for the electric current. The key observation is that *it is possible to formulate the perturbation series directly as an expansion in the curvature of the conducting geometries.* Therefore the perturbation series is applicable to general conducting geometries, directly yielding corrections to geometries which *deviate* from a certain symmetry or uniformity. Such a direct connection of electromagnetic quantities to differential geometric quantities is, to a certain extent, familiar from the geometric theory of diffraction, see, e.g., [1], §8 or [2], §13. But to our knowledge this link has not yet been established in the framework of transmission line theory on a general basis. For the case of transient electromagnetic lens design, involving the propagation of TEM waves, a synthesis of electromagnetic theory and differential geometry is provided by [3].

2 Perfect conductors and electromagnetic waves - a perturbative scattering expansion

In this section we will describe electromagnetic fields and current distributions in the presence of conducting surfaces. We have in mind to later modify these conducting surfaces to transmission line structures. Our description is the modification of an expansion method which already turned out to be useful in the determination of the electromagnetic eigenmodes of conducting cavities [4].

The physical setup is the following: We start from a collection of conductors which are characterized by a certain geometry. We further assume *time harmonic* electromagnetic sources ρ_s and $\boldsymbol{J}_s$ that oscillate with frequency ω according to $\exp(-j\omega t)$. Due to the continuity equation $j\omega\rho_s(\boldsymbol{r}) = \boldsymbol{\nabla}\cdot\boldsymbol{J}_s$ the charge density ρ_s is directly related to the current density $\boldsymbol{J}_s$.

The electromagnetic fields emanating from the sources ρ_s and $\boldsymbol{J}_s$ will induce a current $\boldsymbol{J}_c$ and a charge distribution ρ_c on the conductors which, in turn, are sources of additional electromagnetic fields. We will first construct the induced current $\boldsymbol{J}_c$. Then the induced charge distribution ρ_c is determined via the continuity equation. The total current $\boldsymbol{J}$

and charge distribution ρ on the conductors are obtained as $\boldsymbol{J} = \boldsymbol{J}_s + \boldsymbol{J}_c$ and $\rho = \rho_s + \rho_c$, respectively.

We note that the distinction between source currents and induced currents is unambiguous as long as the sources and the conductors are spatially separated. This is not always the case since sources are often directly connected to conductors, e.g. in the case of a transmission line or antenna. Then the conductors themselves become primary electromagnetic sources. In such a situation we want to understand the source current as the lowest order contribution to the actual current which takes into account only the primary effects of the source but neglects secondary effects like the backreaction of a conductor on itself or the mutual influence of different conductors.

2.1 Solution of Maxwell's equations

We start from the wave equation for the electromagnetic vector potential $\boldsymbol{A}$ in the Lorentz gauge. Since time harmonic fields are assumed the wave equation assumes the form

$$\triangle \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} \tag{1}$$

where $k = \omega/c$. The current $\boldsymbol{J}$ is the sum of both the source current and the induced current, $\boldsymbol{J} = \boldsymbol{J}_s + \boldsymbol{J}_c$. The solution of (1) is given by

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{r}) &= \frac{\mu}{4\pi} \int \frac{e^{jk|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \, d^3r' =: \mu \int G(\boldsymbol{r},\boldsymbol{r}') \boldsymbol{J}(\boldsymbol{r}') \, d^3r' \\ &= \mu \int G(\boldsymbol{r},\boldsymbol{r}') \Big(\boldsymbol{J}_s(\boldsymbol{r}') + \boldsymbol{J}_c(\boldsymbol{r}') \Big) \, d^3r' \,, \end{aligned} \tag{2}$$

with $G(\boldsymbol{r},\boldsymbol{r}') = \exp(jk|\boldsymbol{r}-\boldsymbol{r}'|)/4\pi|\boldsymbol{r}-\boldsymbol{r}'|$ the Green function of free space. We obtain the magnetic field $\boldsymbol{H}$ and the magnetic field strength $\boldsymbol{B}$ as

$$\begin{aligned} \boldsymbol{H}(\boldsymbol{r}) &= \frac{1}{\mu} \boldsymbol{B}(\boldsymbol{r}) = \frac{1}{\mu} \nabla \times \boldsymbol{A}(\boldsymbol{r}) \\ &= \int \nabla_r \times \Big[G(\boldsymbol{r},\boldsymbol{r}') \Big(\boldsymbol{J}_s(\boldsymbol{r}') + \boldsymbol{J}_c(\boldsymbol{r}') \Big) \Big] \, d^3r' \\ &=: \boldsymbol{H}_s(\boldsymbol{r}) + \boldsymbol{H}_c(\boldsymbol{r}) \,. \end{aligned} \tag{3}$$

The field $\boldsymbol{H}_s(\boldsymbol{r})$ denotes the magnetic excitation that is *directly* generated by the source $\boldsymbol{J}_s$. The magnetic excitation due to the induced current $\boldsymbol{J}_c$ is reflected by $\boldsymbol{H}_c(\boldsymbol{r})$.

We observe that with the knowledge of $\boldsymbol{H}$ it is immediate to also obtain the electric field,

$$\varepsilon \boldsymbol{E} = \boldsymbol{D} = \frac{j}{\omega} \Big(\nabla \times \boldsymbol{H} - \boldsymbol{J} \Big) \,. \tag{4}$$

2.2 Construction of the induced current by means of a scattering expansion

The next step is to relate on the conductors the magnetic field $\boldsymbol{H}$ of Eq. (3) to the current $\boldsymbol{J}$. This requires a boundary condition which is obtained from the Maxwell equation $\nabla \times \boldsymbol{H} - \partial \boldsymbol{D}/\partial t = \boldsymbol{J}$ in a standard way. To this end one considers two three-dimensional regions, say, region 1 (e.g. a conductor) and region 2 (e.g. vacuum). Both regions are supposed to be separated by a two-dimensional surface. As a further assumption one requires $\boldsymbol{J}$ to be of the form of a *current line density* $\boldsymbol{k}$ that flows on this separating surface. It then follows that the magnetic field is discontinuous across the boundary according to the formula

$$\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{k} \tag{5}$$

where $\boldsymbol{n}$ denotes a normal vector pointing from region 1 to region 2 and $\boldsymbol{H}_1$, $\boldsymbol{H}_2$ denote the limiting value of $\boldsymbol{H}$ on the surface and within the respective region.

physical setup	boundary condition
1. general case	$\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{k}$
2. region 1 being a perfect conductor	$\boldsymbol{n} \times \boldsymbol{H}_2 = \boldsymbol{k}$
3. neither region 1 nor region 2 are perfect conductors	$\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = 0$

Tab. 1: Some different cases of the boundary condition which follow from the Maxwell equation $\nabla \times \boldsymbol{H} - \partial \boldsymbol{D}/\partial t = \boldsymbol{k}$. The third case results from the fact that there is no idealized surface current if neither region 1 nor region 2 are perfect conductors.

The characteristic of a perfect conductor is that it cannot sustain an oscillating magnetic field in its inside. Therefore, if region 1 is taken as a perfect conductor, $\boldsymbol{H}_1 = 0$ and (5) simplifies to $\boldsymbol{n} \times \boldsymbol{H}_2 = \boldsymbol{k}$, see also Tab. 1. For a good conductor this is also a satisfying approximation since the magnetic field $\boldsymbol{H}$ and the electric current $\boldsymbol{J}$ quickly decay within the skin depth, compare [5], Chap. 8.1. for a discussion of this point. Since we concentrate on conductors we write

$$\boldsymbol{H} = \boldsymbol{H}_2\,, \qquad \boldsymbol{H}_1 = 0\,, \qquad \text{(on the surface)}\,, \tag{6}$$

with $\boldsymbol{H} = \boldsymbol{H}_s + \boldsymbol{H}_c$ the total magnetic field on the surface. It follows from (6) and (5) that on the surface

$$\boldsymbol{n} \times \boldsymbol{H} = \boldsymbol{k} = \boldsymbol{k}_s + \boldsymbol{k}_c\,. \tag{7}$$

Now it is immediate to obtain with (3) an integral equation for the induced current $\boldsymbol{k}_c$:

$$\boldsymbol{k}_c(\boldsymbol{r}) \;=\; \boldsymbol{n}_r \times \Big(\boldsymbol{H}_s(\boldsymbol{r}) + \boldsymbol{H}_c(\boldsymbol{r})\Big) - \boldsymbol{k}_s(\boldsymbol{r})$$

$$
\begin{aligned}
&= \boldsymbol{n}_r \times \boldsymbol{H}_s(\boldsymbol{r}) - \boldsymbol{k}_s(\boldsymbol{r}) \\
&\qquad + \int \boldsymbol{n}_r \times \left[\boldsymbol{\nabla}_r \times \left(G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{k}_c(\boldsymbol{r}')\right)\right] d^2\sigma' \\
&=: \boldsymbol{k}_{1c} + \int \boldsymbol{n}_r \times \left[\boldsymbol{\nabla}_r \times \left(G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{k}_c(\boldsymbol{r}')\right)\right] d^2\sigma' . \qquad (8)
\end{aligned}
$$

Here the subscript r of $\boldsymbol{n}_r$ indicates that the normal vector $\boldsymbol{n}$ is taken at $\boldsymbol{r}$. The same notation applies to the gradient $\boldsymbol{\nabla}$. The two-dimensional volume element $d^2\sigma'$ indicates that the integral extends over a two-dimensional surface with integration variable $\boldsymbol{r}'$. The current

$$
\boldsymbol{k}_{1c} := \boldsymbol{n} \times \boldsymbol{H}_s - \boldsymbol{k}_s \qquad (9)
$$

constitutes the first order contribution to the induced current. This is graphically displayed in Fig. 1.

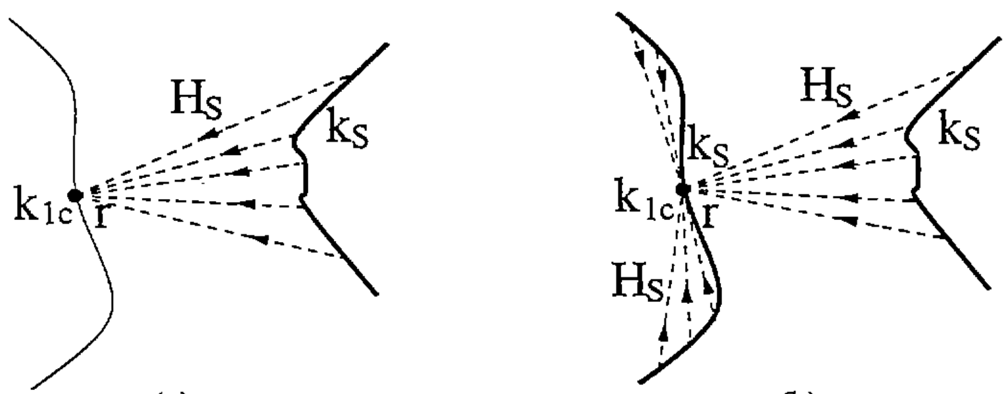

Fig. 1: The contribution $\boldsymbol{k}_{1c}(\boldsymbol{r})$ on the surface of a conductor is determined from primary magnetic fields (dashed lines) that lead without scattering from a source (thick line) to $\boldsymbol{r}$. In (a) the conductor (thin line) and the source are separated. If the conductor itself belongs to the source also its primary magnetic fields and the source current $\boldsymbol{k}_s(\boldsymbol{r})$ must be considered according to (9).

Higher order contributions are obtained by iteration: If we symbolically define the functional

$$
F ... := \int \boldsymbol{n}_r \times \left[\boldsymbol{\nabla}_r \times \left(G(\boldsymbol{r},\boldsymbol{r}') ... (\boldsymbol{r}')\right)\right] d^2\sigma' \qquad (10)
$$

the expansion of $\boldsymbol{k}_c$ is given by

$$
\begin{aligned}
\boldsymbol{k}_c &= \boldsymbol{k}_{1c} + \boldsymbol{k}_{2c} + \boldsymbol{k}_{3c} + \cdots \\
&= \boldsymbol{k}_{1c} + F\boldsymbol{k}_{1c} + F^2\boldsymbol{k}_{1c} + \cdots . \qquad (11)
\end{aligned}
$$

The functional F can be written in a slightly different way: We have

$$
\boldsymbol{\nabla}_r \times \left(G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{k}_{1c}(\boldsymbol{r}')\right) = \left(\boldsymbol{\nabla}_r G(\boldsymbol{r},\boldsymbol{r}')\right) \times \boldsymbol{k}_{1c}(\boldsymbol{r}') \qquad (12)
$$

and, since $G(\boldsymbol{r},\boldsymbol{r}') = G(|\boldsymbol{r}-\boldsymbol{r}'|)$, the gradient on the right hand side of the last equation simplifies to

$$\nabla_r G(\boldsymbol{r},\boldsymbol{r}') = \frac{\partial G(|\boldsymbol{r}-\boldsymbol{r}'|)}{\partial|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{e}_{r-r'} = G'(|\boldsymbol{r}-\boldsymbol{r}'|)\boldsymbol{e}_{r-r'}\,, \tag{13}$$

where the prime at G indicates differentiation with respect to the argument and $\boldsymbol{e}_{r-r'}$ denotes a unit vector pointing from $\boldsymbol{r}'$ to $\boldsymbol{r}$. Therefore equation (10) reduces to the expression

$$F... = \int G'(\boldsymbol{r},\boldsymbol{r}')\,\boldsymbol{n}_r \times \Big(\boldsymbol{e}_{r-r'} \times ...(\boldsymbol{r}')\Big)\,d^2\sigma'\,. \tag{14}$$

The second order contribution to the induced current $\boldsymbol{k}_c$ can be written as

$$\boldsymbol{k}_{2c} = F\boldsymbol{k}_{1c} = \int G'(\boldsymbol{r},\boldsymbol{r}')\,\boldsymbol{n}_r \times \Big(\boldsymbol{e}_{r-r'} \times \boldsymbol{k}_{1c}(\boldsymbol{r}')\Big)\,d^2\sigma'\,. \tag{15}$$

As is evident from (15) and Fig. 2, it takes into account magnetic fields which got scattered *once* at the conductors, with the electromagnetic field propagating freely in-between.

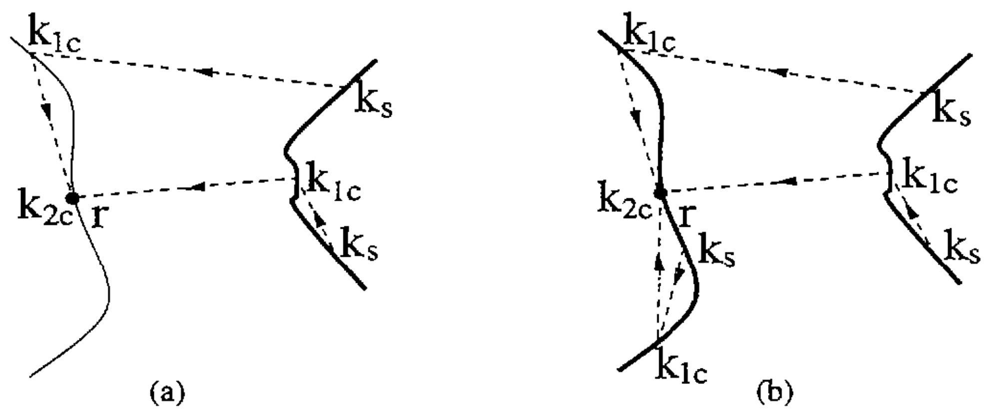

Fig. 2: The second order contribution $\boldsymbol{k}_{2c}(\boldsymbol{r})$ of the induced currents is obtained from secondary magnetic fields generated by the first order current $\boldsymbol{k}_{1c}$. The first order current itself is generated by primary magnetic fields which emanate from sources $\boldsymbol{k}_s$.

In a similar fashion we obtain the third order contribution

$$\begin{aligned}\boldsymbol{k}_{3c}(\boldsymbol{r}) &= \int G'(\boldsymbol{r},\boldsymbol{r}')\,\boldsymbol{n}_r \times \Big(\boldsymbol{e}_{r-r'} \times \boldsymbol{k}_{2c}(\boldsymbol{r}')\Big)\,d^2\sigma' \\ &= \int\!\!\int G'(\boldsymbol{r},\boldsymbol{r}')G'(\boldsymbol{r}',\boldsymbol{r}'')\,\boldsymbol{n}_r \times \Big(\boldsymbol{e}_{r-r'} \times \\ &\qquad \Big[\boldsymbol{n}_{r'} \times \Big(\boldsymbol{e}_{r'-r''} \times \boldsymbol{k}_{1c}(\boldsymbol{r}'')\Big)\Big]\Big)\,d^2\sigma''\,d^2\sigma'.\end{aligned} \tag{16}$$

It describes the effect of magnetic fields that got scattered *twice* at the conductors, compare Fig. 3.

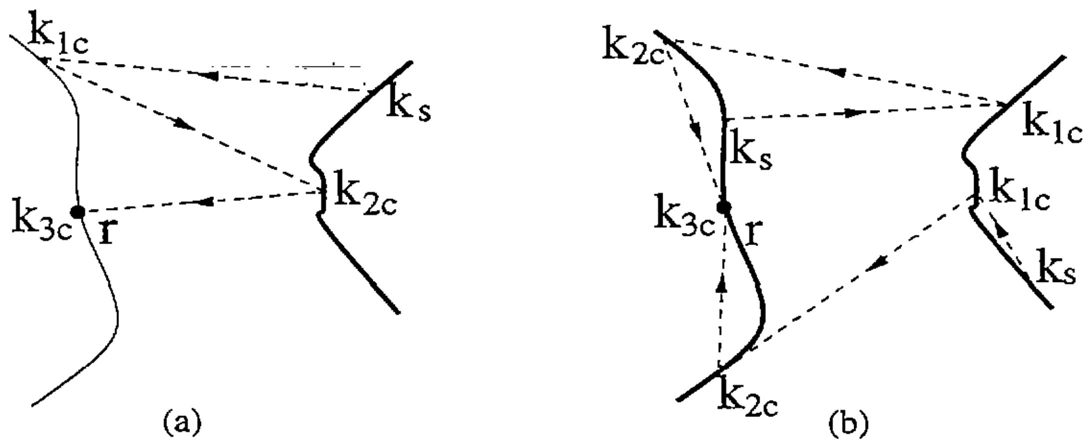

Fig. 3: The third order contribution $k_{3c}(r)$ of the induced currents is obtained from tertiary magnetic fields that experienced two scatterings at the conductors.

Therefore we arrive at the conclusion that the expansion (11) constitutes a *scattering expansion*. The nth order of the expansion describes the contributions from electromagnetic fields which got scattered $(n-1)$ times, with free propagation of the electromagnetic field inbetween. As is clear from the form of the functional F it is necessary to integrate over *all* possible paths with $(n-1)$ scatterings in order to obtain the nth order of the induced current.

3 The scattering expansion as an expansion in terms of curvature

In this section we want to show that the expansion (11) is not only an expansion in the number of scatterings but that it can also be viewed as an expansion in the curvature of the scattering surface. Before we show this property we have to recall some definitions from the differential geometry of curves and surfaces. A more rigorous introduction into this subject can be found in [6].

3.1 The curvature of curves and surfaces

To begin with we recall the notion of the curvature of a curve c. For concreteness we take c as a differentiable map from some interval of the real numbers into three-dimensional Euclidean space, i.e., $c : [a,b] \longrightarrow R^3$. This mapping takes a parameter $s \in [a,b]$ onto a point determined by $c(s) \in R^3$. It is always possible to choose the parameter s in a way such that the tangent $c'(s)$ is normalized to unity, $|c'(s)| = 1$ for all $s \in [a,b]$.

If $c'(s)$ is normalized to unity the value $|c''(s)|$ measures the angle between neighboring tangents $c'(s)$ and $c'(s+ds)$. This motivates to take $|c''(s)|$ as a measure of the directional change of $c'(s)$ and thus as a measure for a directional change of c. Therefore, if $c'(s)$ is normalized to unity, the quantity

$$\kappa(s) := |c''(s)| \tag{17}$$

is defined to be the *curvature* of c at s. We note that, due to $c'(s) \cdot c'(s) = 1$, the acceleration $c''(s)$ is perpendicular to the curve c, $c''(s) \cdot c'(s) = 0$. Thus it can be written as $c''(s) = \kappa(s)n(s)$ with n a normal vector of c.

The geometry of a curve in three-dimensional space is not only characterized by its curvature but also by its *torsion* τ. Torsion can be introduced in the following way: Consider the so-called *binormal* vector $b = c' \times n$. The quantity $|b'(s)|$ measures how fast the curve c bends away from the plane spanned by c' and n at s. We have

$$b' = c'' \times n + c' \times n' = c' \times n', \tag{18}$$

that is, b' is orthogonal to c' and b. Therefore we can write

$$b'(s) =: \tau(s)n(s). \tag{19}$$

This equation defines the torsion τ. Both curvature and torsion completely determine the local geometry of the curve c. We remark that the orthonormal frame $\{c', n, b\}$ is often referred to as the *Frenet frame* of the curve c.

Next we would like to generalize the notion of the curvature of a curve to the notion of the curvature of a surface S. We will restrict ourselves to two-dimensional surfaces in three-dimensional Euclidean space and assume that they are smooth and differentiable. It is clear that we can define a curve as a submanifold of some surface. This is simply the definition of a curve on a surface. Let us now concentrate on a specific point p of some surface S and consider all differentiable curves on S through p that are given by the intersection of a normal plane at p with S. For any such curve the curvature $\kappa(p)$ is defined as in Eq. (17). It can be proven [6] that the set of all possible values $\kappa(p)$ has exactly one maximum $\kappa_1(p)$ and one minimum $\kappa_2(p)$. (We remark that this includes the degenerate case $\kappa_1(p) = \kappa_2(p)$.) The functions κ_1 and κ_2 are called the *principal curvatures* of S at p. They completely describe the local geometry of S. The unit tangent vectors at p which are tangent to two curves c_1, c_2 with curvature $\kappa_1(p)$, $\kappa_2(p)$, respectively, are called the *principal axes.* With regard to the principal curvatures κ_1 and κ_2 one can define principal radii $r_1 := 1/\kappa_1$ and $r_2 := 1/\kappa_2$. In an intuitive sense, circles with radii r_1 and r_2 represent the maximum and minimum amount of curvature at a specific point of a surface.

3.2 Curvature expansion of the induced current on surfaces

We have seen that the functional F acts according to

$$\begin{aligned} \boldsymbol{k}_{(n+1)c}(\boldsymbol{r}) &= (F\boldsymbol{k}_{nc})(\boldsymbol{r}) \\ &= \int G'(\boldsymbol{r},\boldsymbol{r}')\,\boldsymbol{n}_r \times \left(\boldsymbol{e}_{r-r'} \times \boldsymbol{k}_{nc}(\boldsymbol{r}')\right) d^2\sigma' \end{aligned} \tag{20}$$

for $n \geq 1$. The integration domain is the support of $\boldsymbol{k}_{nc}$ which is given by the conducting surfaces. Since the current $\boldsymbol{k}_{nc}$ is flowing on these surfaces it is tangent to them. If we specialize on a (flat) plane as support of $\boldsymbol{k}_{nc}$ the unit vector $\boldsymbol{e}_{r-r'}$ is also a tangent vector of the surface and the cross product $\boldsymbol{e}_{r-r'} \times \boldsymbol{k}_{nc}$ does either vanish or is parallel to $\boldsymbol{n}_r$. In both cases the integrand of (20) vanishes and it follows that no induced current $\boldsymbol{k}_{(n+1)c}$ exists for $n \geq 1$. However, if the support is a curved surface the vectors $\boldsymbol{k}_{nc}$ and $\boldsymbol{e}_{r-r'}$ will, in general, not lie in the same tangent plane and we expect contributions to the induced currents of higher order.

We will now calculate the relation between the functional F and the curvature of the supporting surface in an approximate way. To this end we evaluate the integral $(F\boldsymbol{k}_{nc})(\boldsymbol{r})$ in the vicinity of $\boldsymbol{r}$. This is motivated by the expectation that the derivative $G'(\boldsymbol{r},\boldsymbol{r}')$ of the free Green function will significantly contribute to the integral only as long as $\boldsymbol{r}$ and $\boldsymbol{r}'$ are fairly close together. Furthermore we will assume that the difference $|\boldsymbol{r}-\boldsymbol{r}'|$ is rather small in comparison to both principal curvature radii of the surface.

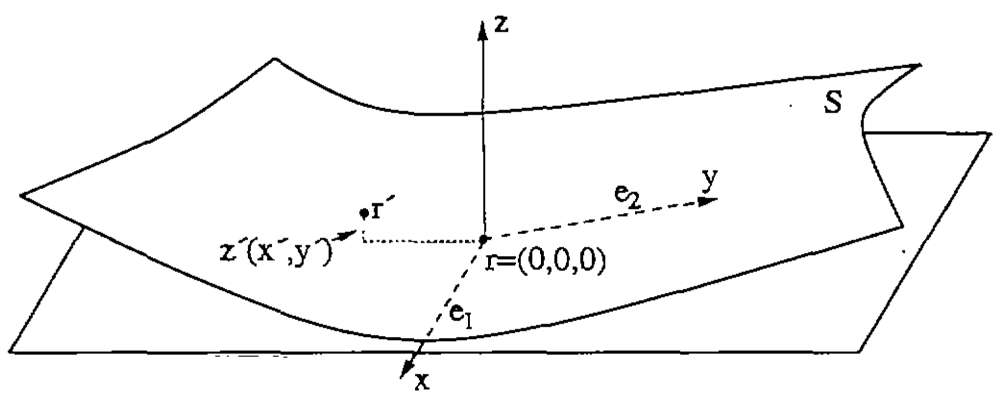

Fig. 4: The approximation of S at $\boldsymbol{r}$ by means of a tangent plane. The principal axis $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ determine a coordinate system with its origin at $\boldsymbol{r}$. If the point at $\boldsymbol{r}'$ is close to $\boldsymbol{r}$ its z-component z' can be written as a function of x' and y'.

Now we consider a point on a surface S which is described by the vector $\boldsymbol{r}$ of three-dimensional space. Since S is supposed to be differentiable there is a well defined tangent plane at $\boldsymbol{r}$, compare Fig. 4. Now we define a coordinate system where $\boldsymbol{r}$ has coordinates $(0,0,0)$ and the principal axes $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ determine an $x-$ and $y-$axis, respectively. Both unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ lie in the tangent plane at $\boldsymbol{r}$ but they are not necessarily orthogonal to each other. The coordinate system is completed by a vector which is normal to the tangent plane. This defines a z-axis. We can fix its direction if we assume that it points from a region 1 to a region 2, i.e., it is directed in the same way as the normal vector $\boldsymbol{n}$ of the previous section, cf. (5).

In the neighborhood of $\boldsymbol{r}$ the $z-$coordinate of a point on S can be written as a differentiable function of x and y, i.e., $z = h(x, y)$. This is, the function h determines the height of the projection between a point on S and the tangent plane at $\boldsymbol{r}$. A Taylor expansion of h around $x = y = 0$ yields

$$\begin{aligned} h(x,y) &= h(0,0) + \frac{\partial h}{\partial x}(0,0)\,x + \frac{\partial h}{\partial y}(0,0)\,y \\ &+\frac{1}{2}\Big(\frac{\partial^2 h}{\partial x^2}(0,0)\,x^2 + 2\frac{\partial^2 h}{\partial x \partial y}(0,0)\,xy + \frac{\partial^2 h}{\partial y^2}(0,0)\,y^2\Big) + \cdots \end{aligned} \tag{21}$$

for $x, y \ll 1$. This expression can be simplified: We have $h(0,0) = 0$ and also the first derivatives vanish since the $x-$ and $y-$axis lie in the tangent plane at the origin. Finally it can be shown [6] that

$$\frac{\partial^2 h}{\partial x^2}(0,0) = \kappa_1(0,0)\,, \quad \frac{\partial^2 h}{\partial y^2}(0,0) = \kappa_2(0,0)\,, \quad \frac{\partial^2 h}{\partial x \partial y}(0,0) = 0\,. \tag{22}$$

This is due to the fact that, by definition, the $x-$ and $y-$axis point along the direction of the principal axis. Accordingly, the expansion (21) can be expressed in terms of the principal curvatures κ_1 and κ_2 as

$$h(x,y) = \frac{1}{2}\Big(\kappa_1(0,0)x^2 + \kappa_2(0,0)y^2\Big) + \cdots \tag{23}$$

We are now in a position to approximate the expression

$$(F\boldsymbol{k}_{nc})(\boldsymbol{r}) = \int G'(\boldsymbol{r},\boldsymbol{r}')\,\boldsymbol{n}_r \times \Big(\boldsymbol{e}_{r-r'} \times \boldsymbol{k}_{nc}(\boldsymbol{r}')\Big)d^2\sigma' \tag{24}$$

to the desired extent. We concentrate on the vectorial expression in the integrand and use the identity

$$\begin{aligned} \boldsymbol{n}_r \times \Big(\boldsymbol{e}_{r-r'} \times \boldsymbol{k}_{nc}(\boldsymbol{r}')\Big) &= \boldsymbol{e}_{r-r'}\Big(\boldsymbol{n}_r \cdot \boldsymbol{k}_{nc}(\boldsymbol{r}')\Big) \\ &\quad -\boldsymbol{k}_{nc}(\boldsymbol{r}')\Big(\boldsymbol{n}_r \cdot \boldsymbol{e}_{r-r'}\Big)\,. \end{aligned} \tag{25}$$

The second term on the right hand side can be easily evaluated: In the xyz-coordinate system we have $\boldsymbol{n}_r = (0,0,1)$ and the unit vector $\boldsymbol{e}_{r-r'}$ is given by $\boldsymbol{e}_{r-r'} = (-x', -y', -z')/r'$ with $r' = |\boldsymbol{r}'|$. It follows with (23)

$$\begin{aligned} \boldsymbol{k}_{nc}(\boldsymbol{r}')\Big(\boldsymbol{n}_r \cdot \boldsymbol{e}_{r-r'}\Big) &= -\frac{z'}{r'}\,\boldsymbol{k}_{nc}(\boldsymbol{r}') \\ &= -\frac{1}{2r'}\Big(\kappa_1\,x'^2 + \kappa_2\,y'^2\Big)\,\boldsymbol{k}_{nc}(\boldsymbol{r}')\,. \end{aligned} \tag{26}$$

For notational convenience we omit here and in the following the argument $(0,0)$ of the principal curvatures.

The evaluation of the term $\boldsymbol{e}_{r-r'}(\boldsymbol{n}_r \cdot \boldsymbol{k}_{nc}(\boldsymbol{r}'))$ is a bit more involved since we do not explicitly know the direction of $\boldsymbol{k}_{nc}(\boldsymbol{r}')$ which, in view of the first order contribution (9), will be determined by the so far unspecified source $\boldsymbol{k}_s$. We only know that $\boldsymbol{k}_{nc}(\boldsymbol{r}')$ lies in the tangent plane at $\boldsymbol{r}'$. To make this statement more explicit we first obtain from $\boldsymbol{r}' = (x', y', 1/2(\kappa_1 x'^2 + \kappa_2 y'^2))$ a normalized tangent vector $\boldsymbol{e}_{r'}$ at $\boldsymbol{r}'$ as

$$\boldsymbol{e}_{r'} = \frac{1}{\sqrt{2}}(1, 1, \kappa_1 x' + \kappa_2 y'), \tag{27}$$

where we remind us that $\kappa_1 x'$ and $\kappa_2 y'$ are small quantities (since we assumed $x' \lll r_1$ and $y' \lll r_2$) such that quadratic contributions can be neglected,

$$|\boldsymbol{e}_{r'}| = \frac{1}{\sqrt{2}}\sqrt{2 + (\kappa_1 x' + \kappa_2 y')^2} \simeq \left(1 + \frac{1}{4}(\kappa_1 x' + \kappa_2 y')^2\right) \simeq 1. \tag{28}$$

In a similar way we find a normal vector $\boldsymbol{n}$ at $\boldsymbol{r}'$,

$$\boldsymbol{n}_{r'} = (-\kappa_1 x', -\kappa_2 y', 1), \qquad \text{with} \quad \boldsymbol{e}_{r'} \cdot \boldsymbol{n}_{r'} = 0. \tag{29}$$

Now we can write $\boldsymbol{k}_{nc}(\boldsymbol{r}')$ as

$$\boldsymbol{k}_{nc}(\boldsymbol{r}') = k_{nc}(\boldsymbol{r}')\, \boldsymbol{e}^{\theta}_{r'} \tag{30}$$

where $\boldsymbol{e}^{\theta}_{r'}$ denotes a unit tangent vector at $\boldsymbol{r}'$ which is obtained from $\boldsymbol{e}_{r'}$ by a rotation about an angle θ with the rotation axis given by $\boldsymbol{n}_{r'}$. To construct $\boldsymbol{e}^{\theta}_{r'}$ we observe that $\boldsymbol{n}_r$ and $\boldsymbol{n}_{r'}$ are related by the infinitesimal rotation

$$\boldsymbol{n}_{r'} = \boldsymbol{R}_{n \to n'}\, \boldsymbol{n}_r\,, \tag{31}$$

with

$$\boldsymbol{R}_{n\to n'} = \begin{pmatrix} 1 & 0 & -\kappa_1 x' \\ 0 & 1 & -\kappa_2 y' \\ \kappa_1 x' & \kappa_2 y' & 1 \end{pmatrix}, \tag{32}$$

$$\boldsymbol{R}^{-1}_{n\to n'} = \boldsymbol{R}_{n'\to n} = \begin{pmatrix} 1 & 0 & \kappa_1 x' \\ 0 & 1 & \kappa_2 y' \\ -\kappa_1 x' & -\kappa_2 y' & 1 \end{pmatrix}. \tag{33}$$

Furthermore, a finite rotation $\boldsymbol{R}^{\theta}_{n}$ about an angle θ with rotation axis $\boldsymbol{n}_r$ can be represented by

$$\boldsymbol{R}^{\theta}_{n} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{34}$$

We calculate the explicit expression for a rotation $\boldsymbol{R}^{\theta}_{n'}$ about an angle θ with rotation axis $\boldsymbol{n}_{r'}$, that is we calculate $\boldsymbol{R}^{\theta}_{n'} = \boldsymbol{R}_{n' \to n} \boldsymbol{R}^{\theta}_{n} \boldsymbol{R}_{n \to n'}$, and use the result to obtain to first order in the curvature

$$\begin{aligned} \boldsymbol{e}^{\theta}_{r'} &= \boldsymbol{R}^{\theta}_{n'} \boldsymbol{e}_{r'} \\ &= \Big(\cos\theta - \sin\theta,\, \sin\theta + \cos\theta, \\ &\qquad \kappa_1 x' (\cos\theta - \sin\theta) + \kappa_2 y' (\sin\theta + \cos\theta) \Big) . \end{aligned} \tag{35}$$

This expression has to be plugged into (30) to yield the desired form of $\boldsymbol{k}_{nc}(\boldsymbol{r}')$. Now it is easy to finally arrive at the result

$$\begin{aligned} &\boldsymbol{e}_{r-r'} \Big(\boldsymbol{n}_r \cdot \boldsymbol{k}_{nc}(\boldsymbol{r}') \Big) = \\ &\boldsymbol{k}_{nc}(\boldsymbol{r}') \Big(\kappa_1 x' (\cos\theta - \sin\theta) + \kappa_2 y' (\sin\theta + \cos\theta) \Big) \boldsymbol{e}_{r-r'} . \end{aligned} \tag{36}$$

Since $\boldsymbol{e}_{r-r'} = (-x', -y', -1/2(\kappa_1 x'^2 + \kappa_2 y'^2))/r'$ the z-component of (36) is of second order in the curvature and thus negligible in our approximation.

We return to the vectorial expression (25) and express it by means of the results (26) and (36). In (26) we may also substitute $k_{nc}(\boldsymbol{r}')\boldsymbol{e}^{\theta}_{r'}$ for $\boldsymbol{k}_{nc}(\boldsymbol{r}')$. This yields

$$\boldsymbol{n}_r \times \Big(\boldsymbol{e}_{r-r'} \times \boldsymbol{k}_{nc}(\boldsymbol{r}') \Big) = \tag{37}$$

$$\begin{aligned} k_{nc}(\boldsymbol{r}') \Big(\Big[&\kappa_1 x' (\cos\theta - \sin\theta) + \kappa_2 y' (\sin\theta + \cos\theta) \Big] \boldsymbol{e}_{r-r'} \\ &+ \frac{1}{2r'} (\kappa_1 x'^2 + \kappa_2 y'^2)\, \boldsymbol{e}^{\theta}_{r'} \Big) \end{aligned} \tag{38}$$

We replace the unit vectors by their explicit coordinate expressions and obtain after some rearranging

$$\begin{aligned} \boldsymbol{n}_r \times \Big(\boldsymbol{e}_{r-r'} \times \boldsymbol{k}_{nc}(\boldsymbol{r}') \Big) &= \frac{k_{nc}(\boldsymbol{r}')}{r'} \Big[\\ &\kappa_1 (x'^2 (\sin\theta - \cos\theta), x'^2 (\sin\theta + \cos\theta) - 2x'y' (\sin\theta - \cos\theta), 0) \\ &- \kappa_2 (y'^2 (\sin\theta - \cos\theta) - 2x'y' (\sin\theta + \cos\theta), y'^2 (\sin\theta + \cos\theta), 0) \Big] \\ &=: \frac{k_{nc}(\boldsymbol{r}')}{r'} \Big(\kappa_1 \boldsymbol{v}^{\theta}_{nc}(\boldsymbol{r}') + \kappa_2 \boldsymbol{w}^{\theta}_{nc}(\boldsymbol{r}') \Big) . \end{aligned} \tag{39}$$

The vectors $\boldsymbol{v}^{\theta}_{nc}(\boldsymbol{r}')$ and $\boldsymbol{w}^{\theta}_{nc}(\boldsymbol{r}')$ are introduced as abbreviations.

We are now at the end of our calculation. The relation between the higher order induced currents, as given by (20), is determined by means of (39) as

$$\boxed{\boldsymbol{k}_{(n+1)c}(\boldsymbol{r}) = \int \frac{G'(\boldsymbol{r}, \boldsymbol{r}')}{r'} k_{nc}(\boldsymbol{r}') \Big(\kappa_1 \boldsymbol{v}^{\theta}_{nc}(\boldsymbol{r}') + \kappa_2 \boldsymbol{w}^{\theta}_{nc}(\boldsymbol{r}') + \cdots \Big)\, d^2\sigma'.} \tag{40}$$

Here the dots indicate that the integration domain involves distances only up to second order. Third and higher order terms, which involve derivatives of the principal curvatures, can also be obtained in a straightforward way. It is thus evident that the scattering expansion of this section is an expansion in the principal curvatures of the scattering surface.

3.3 Curvature expansion of induced currents on wire structures

In this subsection we want to specialize the scattering expansion to the case of a scattering surface which can be approximated by a one-dimensional structure. One could think of such an approximation as being obtained from the deformation of a surface to a tube, the radius of which is getting smaller and smaller such that the tube approaches the form of a thin wire. This case becomes important in view of transmission line theory since transmission line structures are often represented by wires of small radius. For such a situation we expect a simplification of the relevant formulas. The main reason for this is the observation that on a one-dimensional structure the direction of $\boldsymbol{k}_{nc}(\boldsymbol{r}')$ is completely determined by the one-dimensional tangent space at $\boldsymbol{r}'$. Therefore there is no need to introduce, as in the last subsection, a free parameter θ which has to be calculated from the specific physical situation.

On the other hand side, a "thin-wire" approximation requires a careful limiting process, if properly done. Such a process typically involves an integration over a suitable cross section of the conductor to reduce its relevant dimension from two to one. We will assume in the following that such an integration can be performed and results in a constant factor a of physical dimension length. For the case of circular wire structures a is sometimes taken as $2\pi r_w$ with r_w the radius of the wire. However, we are aware of the fact that a determination of a will in general require a nontrivial physical justification.

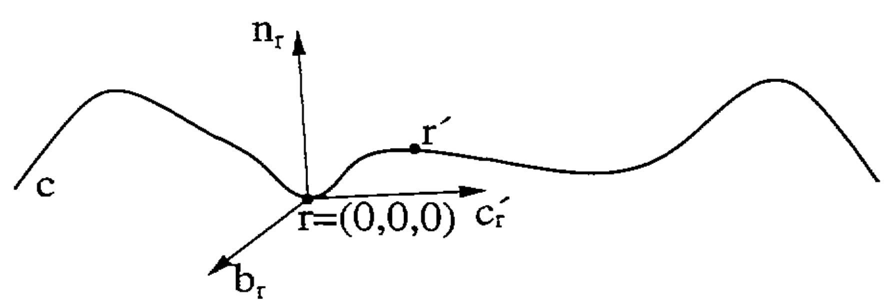

Fig. 5: The Frenet frame $\{\boldsymbol{c}'_r, \boldsymbol{n}_r, \boldsymbol{b}_r\}$ determines a coordinate system with its origin at $\boldsymbol{r}$.

As for the case of a scattering surface we first wish to evaluate the expression $\boldsymbol{n}_r \times (\boldsymbol{e}_{r-r'} \times \boldsymbol{k}_{nc}(\boldsymbol{r}'))$ in an approximate manner in order to use the result to evaluate the functional F. To this end we introduce a curve $\boldsymbol{c}$ and focus on a point of $\boldsymbol{c}$ that is determined by the vector $\boldsymbol{r}$ of three-dimensional space. This point we take as origin of a coordinate system which is spanned by the Frenet frame $\{\boldsymbol{c}'_r, \boldsymbol{n}_r, \boldsymbol{b}_r\}$ at $\boldsymbol{r}$, cf. Fig. 5.

As long as $\boldsymbol{r}'$ is fairly close to $\boldsymbol{r}$ we can Taylor expand $\boldsymbol{c}_{r'}$ around $\boldsymbol{r}$. We assume that $\boldsymbol{c}$ is parametrized by the parameter s with $\boldsymbol{c}(0) = \boldsymbol{c}_r$ and $\boldsymbol{c}(s) = \boldsymbol{c}_{r'}$ for $s \lll 1$. A Taylor expansion yields up to third order

$$\boldsymbol{c}(s) = \boldsymbol{c}(0) + \boldsymbol{c}'(0)s + \frac{\boldsymbol{c}''(0)}{2}s^2 + \frac{\boldsymbol{c}'''(0)}{6}s^3 + \dots . \tag{41}$$

We have

$$\begin{aligned} \boldsymbol{c}''(0) &= \kappa(0)\boldsymbol{n}_r , \qquad (42) \\ \boldsymbol{c}'''(0) &= \kappa'(0)\boldsymbol{n}_r + \kappa(0)\boldsymbol{n}'_r \\ &= \kappa'(0)\boldsymbol{n}_r - \kappa^2(0)\boldsymbol{c}'_r - \kappa(0)\tau(0)\boldsymbol{b}_r , \qquad (43) \end{aligned}$$

where the last equality follows from $\boldsymbol{n}' = -\kappa\boldsymbol{c}' - \tau\boldsymbol{b}$. We omit in the following the argument (0) of κ, τ and write (41) as

$$\boldsymbol{c}(s)-\boldsymbol{c}(0)=\Big(s-\frac{\kappa^2 s^3}{6}\Big)\boldsymbol{c}'_r + \Big(\frac{\kappa s^2}{2}+\frac{\kappa' s^3}{6}\Big)\boldsymbol{n}_r - \frac{\kappa\tau s^3}{6}\boldsymbol{b}_r + \cdots . \tag{44}$$

Since the Frenet frame at $\boldsymbol{r}$ is taken as coordinate system we have $\boldsymbol{c}'_r = (1,0,0)$, $\boldsymbol{n}_r = (0,1,0)$, and $\boldsymbol{b}_r = (0,0,1)$. This yields to third order

$$\boldsymbol{c}(s) - \boldsymbol{c}(0) = \Big(s - \frac{\kappa^2 s^3}{6}, \frac{\kappa s^2}{2} + \frac{\kappa' s^3}{6}, -\frac{\kappa\tau s^3}{6}\Big) \tag{45}$$

and

$$\boldsymbol{c}'(s) = \Big(1 - \frac{\kappa^2 s^2}{2}, \kappa s + \frac{\kappa' s^2}{2}, -\frac{\kappa\tau s^2}{2}\Big) . \tag{46}$$

The vector $\boldsymbol{c}'$ is actually not normalized to unity but rather of length $1 + \kappa^2 s^2 + \kappa\kappa' s^3/2$. However, in the following a normalization would only lead to corrections of at least fourth order. Thus we can omit the explicit normalization.

Now we use $\boldsymbol{e}_{r-r'} = (\boldsymbol{c}(s) - \boldsymbol{c}(0))/r'$ and calculate, up to third order, the terms

$$\begin{aligned} \boldsymbol{e}_{r-r'}\Big(\boldsymbol{n}_r \cdot \boldsymbol{k}_{nc}(\boldsymbol{r}')\Big) &= k_{nc}(\boldsymbol{r}')\Big(\kappa s + \frac{\kappa' s^2}{2}\Big)\boldsymbol{e}_{r-r'} \\ &= \frac{k_{nc}(\boldsymbol{r}')}{r'}\Big(\kappa s^2 + \frac{\kappa' s^3}{2}, \frac{\kappa^2 s^3}{2}, 0\Big) \qquad (47) \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{k}_{nc}(\boldsymbol{r}')\Big(\boldsymbol{n}_r \cdot \boldsymbol{e}_{r-r'}\Big) &= \frac{1}{r'}\Big(\frac{\kappa s^2}{2} + \frac{\kappa' s^3}{6}\Big)\boldsymbol{k}_{nc}(\boldsymbol{r}') \\ &= \frac{k_{nc}(\boldsymbol{r}')}{r'}\Big(\frac{\kappa s^2}{2} + \frac{\kappa' s^3}{6}, \frac{\kappa^2 s^3}{2}, 0\Big) . \qquad (48) \end{aligned}$$

This yields

$$
\begin{aligned}
\boldsymbol{e}_{r-r'}\Big(\boldsymbol{n}_r\cdot\boldsymbol{k}_{nc}(\boldsymbol{r}')\Big) - \boldsymbol{k}_{nc}(\boldsymbol{r}')\Big(\boldsymbol{n}_r\cdot\boldsymbol{e}_{r-r'}\Big)& \\
&= \frac{k_{nc}(\boldsymbol{r}')}{r'}\Big(\frac{\kappa s^2}{2}+\frac{\kappa' s^3}{3},0,0\Big) \\
&= \frac{k_{nc}(\boldsymbol{r}')}{r'}\Big(\frac{\kappa s^2}{2}+\frac{\kappa' s^3}{3}\Big)\boldsymbol{c}'_r\,. \qquad (49)
\end{aligned}
$$

Finally, we obtain the desired relation between the higher order induced current which is the one-dimensional analogue of (40):

$$
\boldsymbol{k}_{(n+1)c}(\boldsymbol{r}) = a\int \frac{G'(\boldsymbol{r},\boldsymbol{r}')}{r'}k_{nc}(\boldsymbol{r}')\Big(\frac{\kappa s^2}{2}+\frac{\kappa' s^3}{3}+\cdots\Big)\,\boldsymbol{c}'_r\,ds\,. \qquad (50)
$$

The dots remind us of the fourth and higher order contributions of s which are neglected in the approximation considered. We note that in the coordinate system applied we have $\boldsymbol{r}=(0,0,0)$ and $\boldsymbol{r}'=\boldsymbol{r}'(s), r'=r'(s)$. Since $\boldsymbol{k}_{(n+1)c}(\boldsymbol{r}) = k_{(n+1)c}(\boldsymbol{r})\,\boldsymbol{c}'_r$ we can write (50) in the simplified form

$$
\boxed{k_{(n+1)c}(\boldsymbol{r}) = a\int \frac{G'(\boldsymbol{r},\boldsymbol{r}')}{r'}k_{nc}(\boldsymbol{r}')\Big(\frac{\kappa s^2}{2}+\frac{\kappa' s^3}{3}+\cdots\Big)\,ds\,.} \qquad (51)
$$

To second order in s this is readily seen to be an expansion in the curvature κ. The third order contribution involves the derivative κ' of the curvature. It is straightforward to also obtain fourth and higher order contributions which then also include the torsion τ.

4 Concluding remarks

With (40) and (51) we have gained recursion formulas which directly yield a calculation scheme for those induced currents that are due to nontrivial geometries of the underlying conductors. In our approach these geometries are effectively characterized by means of coordinate independent scalar quantitites, such as κ_1, κ_2, and κ. In particular, recursion formula (51) is of a rather simple form since all vectorial expressions are reduced to scalars.

The recursive construction of the induced current immediately yields a recursive construction of the induced electromagnetic fields since we have

$$
\frac{1}{\mu}\boldsymbol{B}_{nc}(\boldsymbol{r}) = \boldsymbol{H}_{nc}(\boldsymbol{r}) \;=\; \boldsymbol{\nabla}\times\int G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{k}_{nc}(\boldsymbol{r}')\,d^2\sigma'\,, \qquad (52)
$$

and, outside the conductor where $\boldsymbol{k}=\boldsymbol{J}=0$,

$$
\boldsymbol{D}_{nc} = \varepsilon\boldsymbol{E}_{nc}(\boldsymbol{r}) \;=\; \frac{j}{\omega}\boldsymbol{\nabla}\times\boldsymbol{H}_{nc}(\boldsymbol{r})\,. \qquad (53)
$$

These fields enter, in turn, the definition of characteristic per-unit-length parameters of nonuniform MTLs [7], such that, finally, a recursive construction of these characteristics is possible. Corresponding calculations are currently under investigation and will be reported on in a forthcoming paper.

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