Interaction Notes

Note 566

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Axial Backscattering from a Wide Angular Sector

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Abstract

A perfectly conducting angular sector is a canonical shape for scattering calculations. A previous paper has calculated the axial backscatter from a thin angular sector, obtaining an asymptotic result in the limit of $\psi \to 0$ where ψ is the half angle of the angular sector. Using a completely different technique the present paper calculates the axial backscatter for a wide angular sector, asymptotic in the limit as $\psi \to \pi / 2$.

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1. Introduction

In testing the physical-optics (PO) approximation for axial backscattering from cones it has been observed that for a circular perfectly conducting cone the PO result is quite good, at least asymptotically for small (near zero) and wide (near $\pi/2$) cone angles [3, 10]. However, for noncircular cones the disagreement can be large as shown for the thin angular sector (perfectly conducting), and more generally for the thin elliptic cone (perfectly conducting) [3, 4]. The present paper considers the wide angular sector as in Fig. 1.1 for ψ near $\pi/2$ (ψ' near 0). The angular sector lies on the xz plane ($y = 0$) with the z axis as the sector bisector (symmetry axis).

The incident fields are

$$
\begin{aligned}
\vec{E}^{(inc)}(\vec{r},t) &= E_0 \vec{1}_x f\left(t - \frac{\vec{1}_z \cdot \vec{r}}{c}\right) \\
\vec{H}^{(inc)}(\vec{r},t) &= \frac{E_0}{Z_0} \vec{1}_y f\left(t - \frac{\vec{1}_z \cdot \vec{r}}{c}\right) \\
c &= [\mu_0 \varepsilon_0]^{-\frac{1}{2}} \equiv \text{speed of light} \\
Z_0 &= \left[\frac{\mu_0}{\varepsilon_0}\right]^{\frac{1}{2}} \equiv \text{wave impedance of free space}
\end{aligned}
\tag{1.1}
$$

This gives a plane wave with

$$
\begin{aligned}
\vec{1}_i &= \vec{1}_z \equiv \text{direction of incidence} \\
\vec{1}_e &= \vec{1}_x \equiv \text{polarization}
\end{aligned}
\tag{1.2}
$$

While the waveform $f(t)$ is quite general we can specialize it (e.g., as a step function) at our convenience.

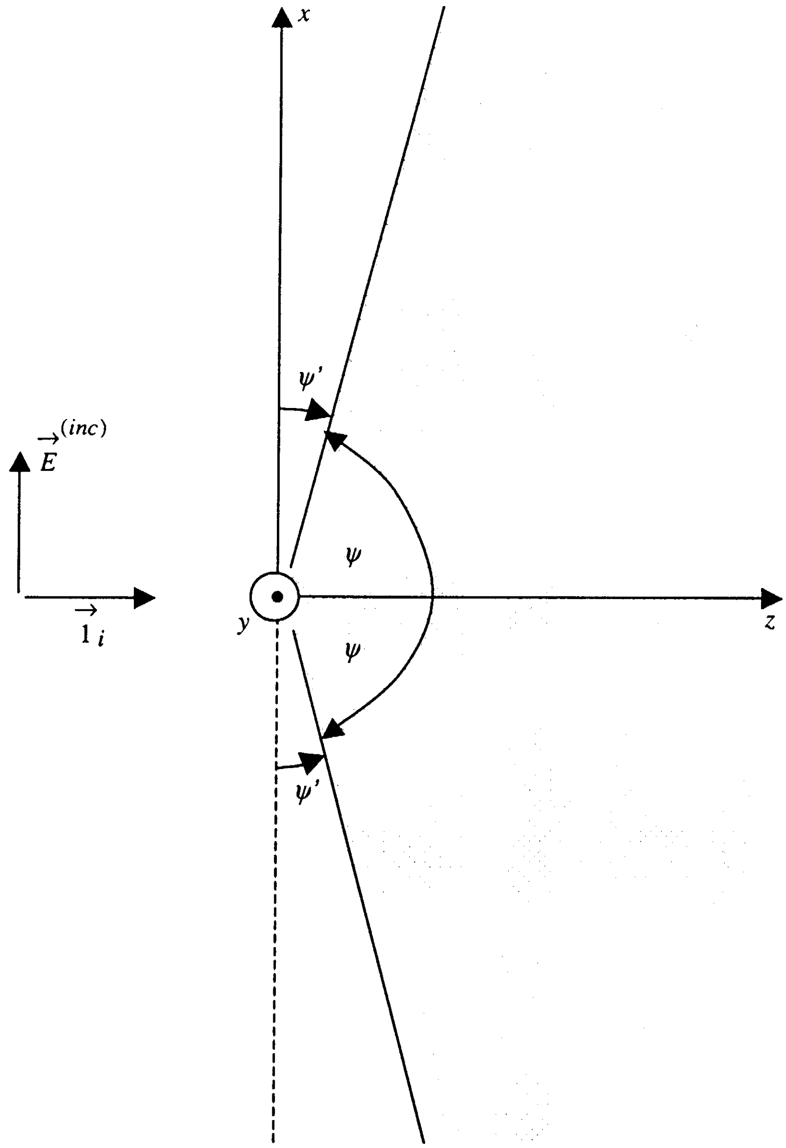


Fig. 1.1 Wide Perfectly Conducting Angular Sector

2. Backscattering Formulation

To calculate the backscattering let us fist express the far scattered field in terms of the surface current density (sheet current) as [3]

$$\vec{E}_f^{(sc)}(\vec{r},t) = -\frac{\mu_0}{4\pi r}\frac{\partial}{\partial t}\left[\overleftrightarrow{1}_z \bullet \int_S \vec{J}_s\left(\vec{r}_s, t + \frac{\vec{1}_z \bullet [\vec{r} - \vec{r}_s]}{c}\right) dS\right]$$

$\vec{r}_s \equiv$ coordinate on S, the angular sector

$$\vec{J}_s(\vec{r}_s,t) = \vec{1}_x \bullet \left[\vec{H}(\vec{r}_s,t)\Big|_{x=0_+} - \vec{H}(\vec{r}_s,t)\Big|_{x=0_-} \right]$$

= total surface current density on both sides of angular sector

$$\overleftrightarrow{1}_z = \overleftrightarrow{1} - \vec{1}_z \vec{1}_z \equiv \text{transverse (to } z\text{) dyadic} \tag{2.1}$$

$$\overleftrightarrow{1} = \vec{1}_x \vec{1}_x + \vec{1}_y \vec{1}_y + \vec{1}_z \vec{1}_z \equiv \text{identity}$$

Since $\vec{J}_s$ can have no y component the above reduces to

$$\vec{E}_f^{(sc)}(\vec{r},t) = -\frac{\mu_o}{4\pi r}\vec{1}_x \frac{\partial}{\partial t}\int_S J_{s_x}\left(\vec{r}_s, t + \frac{z - z_s}{c}\right) dS \tag{2.2}$$

leaving only an x component to contribute to the integral.

A general form for the backscattered far field is

$$\vec{E}_f^{(sc)}(\vec{r},t) = \frac{1}{4\pi r}\overleftrightarrow{\Lambda}_b(\vec{1}_z,t) \circ \vec{E}^{(inc)}\left(\vec{0}, t - \frac{r}{c}\right)$$

$\overleftrightarrow{\Lambda}_b(\vec{1}_z,t) \circ \equiv$ backscattering dyadic operator

$\circ \equiv$ convolution with respect to time (2.3)

$\overleftrightarrow{\Lambda}_b(\vec{1}_z,t) = \overleftrightarrow{\Lambda}_b^T(\vec{1}_z,t) \equiv$ scattering dyadic impulse response

Using our previous definition [3]

$$\overleftrightarrow{\Lambda}_b^{(1)}(\vec{1}_z,t) = \int_{0_-}^{t} \overleftrightarrow{\Lambda}_b(\vec{1}_z,t')\,dt' \equiv \text{scattering dyadic step response} \tag{2.4}$$

we can specialize the incident waveform as

$$f(t) = u(t) \tag{2.5}$$

to correspond to the results in Appendix A which uses a step-function incident wave. Then we have

$$\overleftrightarrow{\Lambda}_b^{(1)}(\vec{1}_z,t) = -\vec{1}_x\vec{1}_x\frac{\mu_0}{E_0}\frac{\partial}{\partial t}\int_S J_{s_x}\left(\vec{r}_s,t-\frac{z_s}{c}\right)dS \tag{2.6}$$

where J_{s_x} is now in response to a step incident wave of strength E_0. This also implies

$$\overleftrightarrow{\Lambda}_b(\vec{1}_z,t) = -\vec{1}_x\vec{1}_x\frac{\mu_0}{E_0}\frac{\partial^2}{\partial t^2}\int_S J_{s_x}\left(\vec{r}_s,t-\frac{z_s}{c}\right)dS \tag{2.7}$$

As a general result for cones (dilation symmetry [11]) we have

$$\begin{aligned}\overleftrightarrow{\Lambda}_b^{(1)}(\vec{1}_z,t) &= ct\,u(t)\overleftrightarrow{K}_b(\vec{1}_z)\\ \overleftrightarrow{\Lambda}_b(\vec{1}_z,t) &= c\,u(t)\overleftrightarrow{K}_b(\vec{1}_z)\end{aligned} \tag{2.8}$$

It is this constant dyadic $\overleftrightarrow{K}_b$ that we wish to calculate for small ψ'. From (2.7) we have

$$\overleftrightarrow{K}_b(\vec{1}_z)\,u(t) = -\vec{1}_x\vec{1}_x\frac{\mu_0}{c\,E_0}\frac{\partial^2}{\partial t^2}\int_S J_{s_x}\left(\vec{r}_s,t-\frac{z_s}{c}\right)dS \quad \text{for } t>0 \tag{2.9}$$

3. Backscattering Dyadic

The half-plane solution is developed in Appendix A. Let us use this as an approximation for small positive ψ'. For $\psi' \to 0$ the surface current density goes to the exact half-plane solution. Note now that the wave first reaches the edge at

$$t_0 = \frac{z_0}{c} = \frac{|x_s|}{c}\tan(\psi') > 0 \tag{3.1}$$

The surface current density is then delayed as a function of $|x|$. The wavefront is still at $z_s = ct$. Accordingly the parameter τ in Appendix A needs to be a function of $|x_s|$ as

$$\tau = \frac{c[t-t_0]}{z_s - z_0} = \frac{c[t-t_0]}{z_s - ct_0} \tag{3.2}$$

to be used in

$$J_{s_x}(\tau) = \frac{8}{\pi}\frac{E_0}{Z_0}\left[\frac{\tau-1}{2}\right]^{\frac{1}{2}} u(\tau-1) \tag{3.3}$$

This approximation is asymptotic as $\psi' \to 0$.

Noting that the maximum $|x_s|$ with surface current density at time $t > 0$ is where $t = t_0$, giving the integration limit

$$|x_s| = ct\cot(\psi') \tag{3.4}$$

then (2.9) becomes

$$K_{b_{x,x}}(\vec{1}_z)u(t) = \frac{16}{\pi\sqrt{2}c^2}\frac{\partial^2}{\partial t^2}\int_0^{ct\cot(\psi')}\int_0^{ct}\left[\frac{c[t-t_0]-z_s}{z_s - z_o} - 1\right]^{\frac{1}{2}} u\left(\frac{c[t-t_0]-z_s}{z_s - z_o} - 1\right) dz_s dx_s \tag{3.5}$$

with a factor of 2 accounting for the symmetry with respect to x_s.

Since t_0 and z_0 are not functions of z_s, let us consider that integral first. Substituting

$$\chi \equiv 2\frac{z_s - z_o}{c[t-t_o]-z_o} = 2\frac{z_s - z_o}{c[t-2t_o]} = 2\frac{z_s - ct_o}{c[t-2t_o]} \quad , \quad dz_s = \frac{c}{2}[t-2t_o]\,d\chi \tag{3.6}$$

gives

$$\begin{aligned}
&\int_{z_o}^{ct}\left[\frac{c[t-t_o]-z_s}{z_s-z_o} - 1\right]^{\frac{1}{2}} u\left(\frac{c[t-t_o]-z_s}{z_s-z_o} - 1\right) dz_s \\
&= u\left(\frac{ct}{z} - z_o\right)\int_{z_o}^{ct}\left[\frac{c[t-t_o]-z_s}{z_s-z_o} - 1\right]^{\frac{1}{2}} dz_s \\
&= \frac{1}{\sqrt{2}}c[t-2t_o]\,u(t-2t_o)\int_0^1\left[\chi^{-1}-1\right]^{\frac{1}{2}} d\chi
\end{aligned} \tag{3.7}$$

Using integral tables [9] we have

$$\int_0^1\left[\frac{1-\chi}{\chi}\right]^{\frac{1}{2}}\int_0^1 d\chi = \chi^{\frac{1}{2}}[1-\chi]^{\frac{1}{2}}\Bigg|_0^1 + \frac{1}{2}\int_0^1 \chi^{-\frac{1}{2}}[1-\chi]^{-\frac{1}{2}} d\chi = \arctan\left(\left[\frac{\chi}{1-\chi}\right]^{\frac{1}{2}}\right)\Bigg|_0^1 = \frac{\pi}{2} \tag{3.8}$$

Collecting terms we now have

$$\begin{aligned}
K_{b_{x,x}}(\vec{1}_z)\,u(t) &= -\frac{4}{c}\frac{\partial^2}{\partial t^2}\int_0^{ct\cot(\psi')}[t-2t_o]\,u(t-2t_o)\,dx_s \\
&= -\frac{4}{c}\frac{\partial^2}{\partial t^2}\int_0^{ct\cot(\psi')}\left[t-2\frac{x_s}{c}\tan(\psi')\right]u\left(t-2\frac{x_s}{c}\tan(\psi')\right)dx_s \\
&= -\frac{4}{c}\frac{\partial^2}{\partial t^2}\left[\int_0^{\frac{ct}{2}\cot(\psi')}\left[t-2\frac{x_s}{c}\tan(\psi')\right]dx_s\,u(t)\right]
\end{aligned} \tag{3.9}$$

Substituting

$$t' = t - 2\frac{x_s}{c}\tan(\psi') \quad , \quad dx_s = -\frac{c}{2}\cot(\psi')dt' \tag{3.10}$$

gives

$$K_{b_{x,x}}(\vec{1}_z)u(t) = -2\cot(\psi')\frac{\partial^2}{\partial t^2}\left[\int_0^t t'dt'u(t)\right] = -2\cot(\psi')\frac{\partial}{\partial t}\left[tu(t)\right] \tag{3.11}$$

Thus we have

$$\begin{aligned} K_{b_{x,x}}(\vec{1}_z) &= -2\cot(\psi') \quad \text{for } \psi' \to 0 \\ \overleftrightarrow{K}_b(\vec{1}_z) &= -2\cot(\psi')\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= -2\cot(\psi)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for } \psi' \to \frac{\pi}{2} \end{aligned} \tag{3.12}$$

in the usual two-dimensional form (x,y or h,v coordinates) for backscattering.

4. Concluding Remarks

The solution here has some interesting properties. Note the proportionality to $\cot(\psi')$ in (3.12). This can be contrasted to the $\cot^2(\psi')$ dependence for the wide circular cone [3, 10]. The difference between these two can be ascribed to the fact that the angular sector has an integral over the surface current density using only one transverse coordinate, while the circular cone has an integral over two transverse coordinates. As an alternate view consider that for $\psi' \to 0$ the angular sector tends to a half plane which scatters field proportional to $r^{-1/2}$, while the circular cone tends to a plane which scatters field proportional to r° [11] (which requires a more singular behavior of the r^{-1} term as $\psi' \to 0$).

So now we have solutions for the axial backscattering from both thin [4] and wide perfectly conducting angular sectors. This leaves the intermediate angles ψ to be solved. There exists a solution in terms of an infinite series of eigenfunctions [6-8]. However, this does not give simple analytic insights. Further development of "exact" analytical and numerical results would be helpful.

Appendix A. Surface Current Density for a Step-Function Plane Wave Incident on a Half Plane Parallel to the Half Plane and Perpendicular to the Edge

As an intermediate step in the backscattering calculation let us consider the canonical problem in which a step-function wave is normally incident on the edge of a perfectly conducting half plane as indicated in Fig. A.1. The direction of incidence is in the plane of the half plane. The coordinates in this appendix are chosen to correspond those in the article [5] on which the solution is based.

Figure A.2 shows the situation before and after the wave reaches the half plane. As indicated in Fig. A.2B there is a circular cylinder of radius ct containing both incident and scattered fields. The incident field takes the form

$$\vec{E}^{(inc)}(\vec{r},t) = E_o \vec{1}_{z'} u\left(t - \frac{\vec{1}_{x'} \bullet \vec{r}}{c}\right) \tag{A.1}$$

The incident plus scattered field (total field) takes the form using cylindrical (Ψ, ϕ, z) coordinates

$$\begin{aligned} \vec{E}(\vec{r},t) &= E_o \vec{1}_{z'} f\left(\frac{\Psi}{ct}, \phi\right) \quad \text{for } 0 \leq \Psi \leq ct \\ x' &= \Psi \cos(\phi) \\ y' &= \Psi \sin(\phi) \end{aligned} \tag{A.2}$$

The combination Ψ / ct expresses the dilation symmetry of this geometry [11].

Since there is only a z' component of the electric field, the problem basically scalarizes and involves only two dimensions. As indicated in Fig. A.2B the function $f\left(\frac{\Psi}{ct}, \phi\right)$ has boundary values 1 on $\Psi = ct$ and 0 (zero tangential electric field) on the x' axis for $0 \leq x' < ct$. The single electric-field component satisfies a scalar wave equation

$$\begin{aligned} \nabla^2 f - \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} &= 0 \\ \nabla^2 f &= \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} = \frac{1}{\Psi} \frac{\partial}{\partial \Psi}\left[\frac{\Psi \partial f}{\partial \Psi}\right] + \frac{1}{\Psi^2} \frac{\partial^2 f}{\partial \phi^2} \end{aligned} \tag{A.3}$$

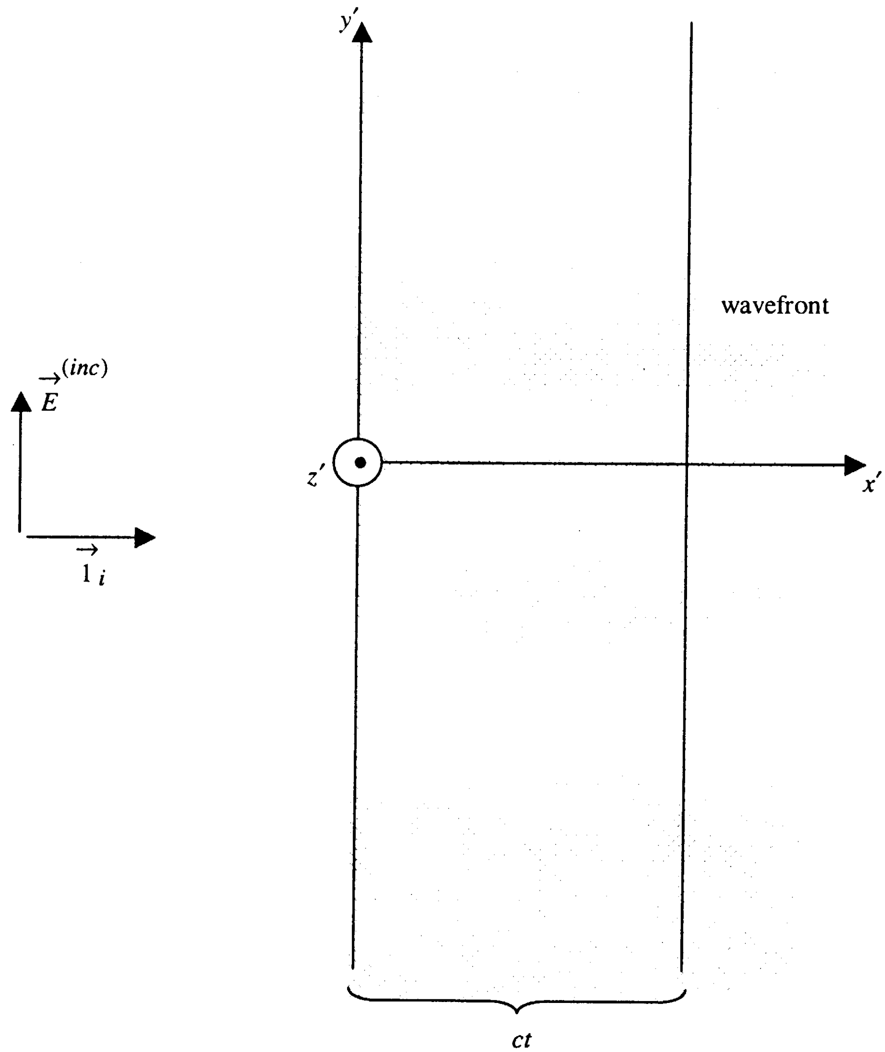
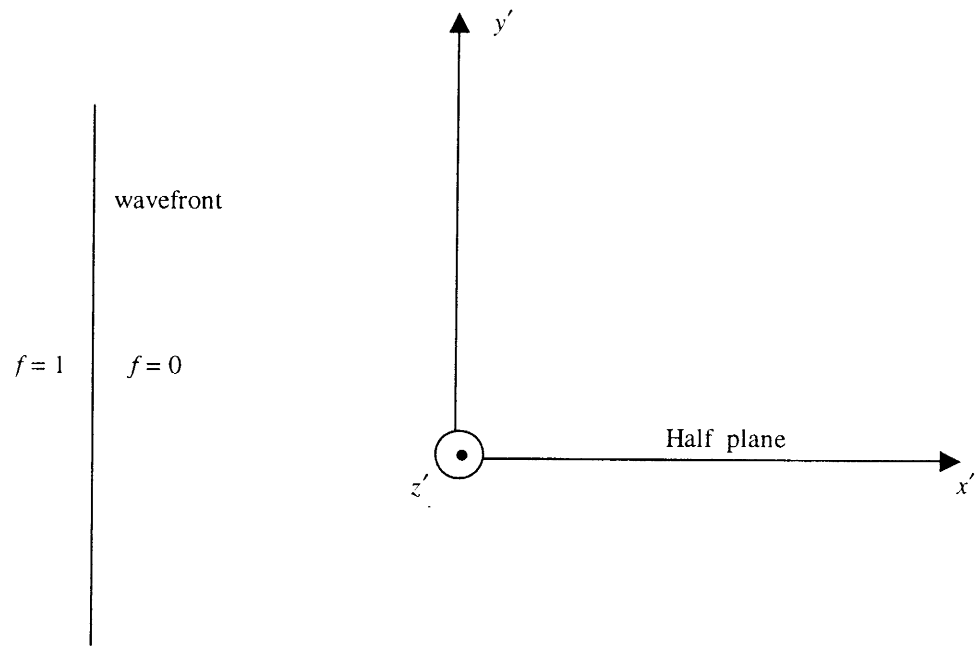
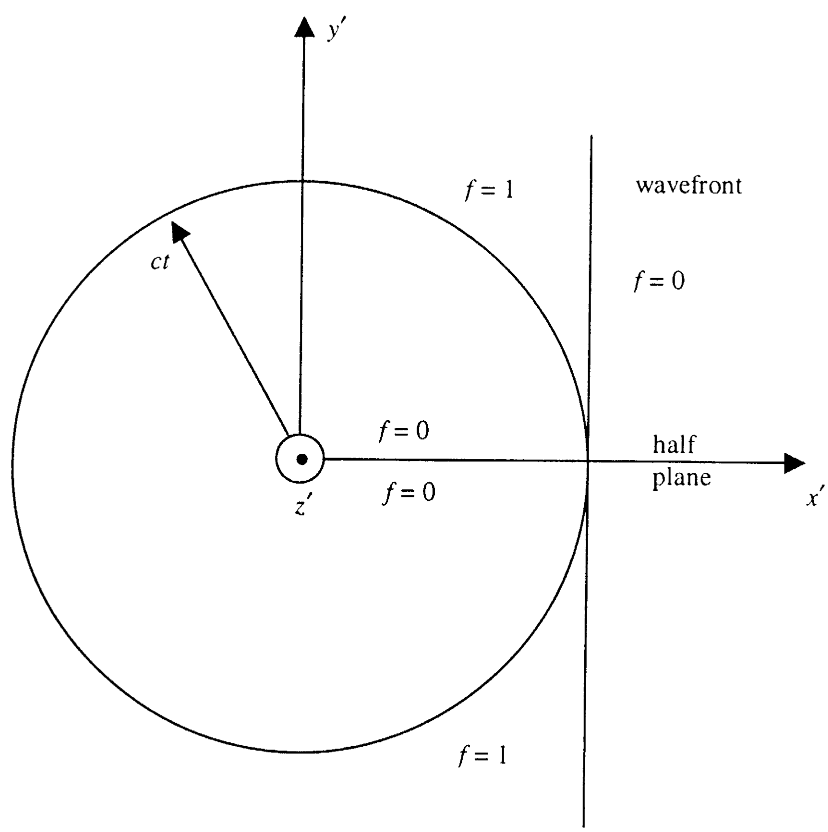


Fig. A.1 Interaction of Step-Function Plane Wave with Perfectly Conducting Half Plane



A. Wave for $t < 0$



B. Wave for $t > 0$

Fig. A.2 Two-Dimensional Scattering Problem for Incident Step-Function Wave

Following [1, 2, 5] define two new coordinates as

$$p = ct\left[1-\left[\frac{\Psi}{ct}\right]^2\right]^{\frac{1}{2}} \quad , \quad q = \left[1-\left[\frac{\Psi}{ct}\right]^2\right]^{\frac{1}{2}} \tag{A.4}$$

giving

$$\frac{\partial}{\partial p}\left[p^2\frac{\partial f}{\partial p}\right] - \frac{\partial}{\partial q}\left[\left[q^2-1\right]\frac{\partial f}{\partial q}\right] - \frac{1}{q^2-1}\frac{\partial^2 f}{\partial \phi^2} = 0 \tag{A.5}$$

However, f must be a function of only Ψ / ct and ϕ, giving

$$\frac{\partial}{\partial q}\left[\left[q^2-1\right]\frac{\partial f}{\partial q}\right] + \frac{1}{q^2-1}\frac{\partial^2 f}{\partial \phi^2} = 0 \tag{A.6}$$

Substitute

$$\xi = \left[\frac{q-1}{q+1}\right]^{\frac{1}{2}} = \left[\frac{ct}{\Psi} + \left[\left[\frac{ct}{\Psi}\right]^2 - 1\right]^{\frac{1}{2}}\right]^{-1} = \frac{ct}{\Psi} - \left[\left[\frac{ct}{\Psi}\right]^2 - 1\right]^{\frac{1}{2}} \tag{A.7}$$

giving

$$\frac{1}{\xi}\frac{\partial}{\partial \xi}\left[\xi\frac{\partial f}{\partial \xi}\right] + \frac{1}{\xi^2}\frac{\partial^2 f}{\partial \phi^2} = 0 \tag{A.8}$$

This is the Laplace equation in equivalent cylindrical (ξ, ϕ) coordinates. So we substitute

$$\zeta = \xi e^{j\phi} = \zeta_r + j\zeta_i \tag{A.9}$$

for solution by conformal transformation.

The general solution for the perfectly conducting wedge is given in [5]. Specializing this to the present case we have

$$f = \frac{1}{\pi}\arctan\left(\frac{-[1-\xi]}{2\xi^{\frac{1}{2}}\sin\left(\frac{\phi}{2}\right)}\right) - \frac{1}{\pi}\arctan\left(\frac{1-\xi}{2\xi^{\frac{1}{2}}\sin\left(\frac{\phi}{2}\right)}\right) \tag{A.10}$$

$$0 \le \xi \le 1 \quad , \quad 0 \le \phi \le 2\pi$$

where the arctan functions are each evaluated between 0 and π. Noting that the first arctan varies from $\pi/2$ to π we can replace this by π - arctan reversing the sign of the argument, giving

$$f = 1 - \frac{2}{\pi}\arctan\left(\frac{1-\xi}{2\xi^{\frac{1}{2}}\sin\left(\frac{\phi}{2}\right)}\right) \tag{A.11}$$

with the arctan ranging between 0 and $\pi/2$. There are other forms this solution can take [2], but for our special case the above is relatively simple.

We can find the magnetic field from the Maxwell equation

$$\frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu_o}\nabla\times\vec{E} = -\frac{1}{\mu_o}\left[\left[\frac{1}{\Psi}\frac{\partial}{\partial\phi}E_{z'}\right]\vec{1}_{\psi} - \left[\frac{\partial}{\partial\psi}E_{z'}\right]\vec{1}_{\phi}\right] \tag{A.12}$$

On the half plane

$$\begin{aligned} E_{z'}\big|_{\phi=0} &= 0 \\ \frac{\partial}{\partial t}\vec{J}_s &= 2\,\vec{1}_{\phi}\times\frac{\partial}{\partial t}\vec{H}\bigg|_{\phi=0_+} = 2\,\vec{1}_{z'}\frac{1}{\mu_o\Psi}\frac{\partial}{\partial\phi}E_{z'}\bigg|_{\phi=0_+} \\ \frac{\partial J_{s_{z'}}}{\partial t} &= \frac{2E_0}{\mu_o\Psi}\frac{\partial f}{\partial\phi}\bigg|_{\phi=0_+} \end{aligned} \tag{A.13}$$

Note the factor of two accounting for the magnetic field both above ($\phi = 0_+$) and below ($\phi = 0_-$) the half plane. From (A.11) we find

$$\frac{\partial f}{\partial \phi} = \frac{1}{\pi}\left[1 + \left[\frac{1-\xi}{2\xi^{\frac{1}{2}}\sin\left(\frac{\phi}{2}\right)}\right]^2\right]^{-1} \frac{1-\xi}{2\xi^{\frac{1}{2}}} \frac{\cos\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi}{2}\right)}$$

$$\left.\frac{\partial f}{\partial \phi}\right|_{\phi=0_+} = \frac{2}{\pi}\frac{\xi^{\frac{1}{2}}}{1-\xi} = \frac{2}{\pi}\left[\xi^{-\frac{1}{2}} - \xi^{\frac{1}{2}}\right]^{-1} \tag{A.14}$$

$$\frac{\partial J_{s_{z'}}}{\partial t} = \frac{4E_0}{\pi\mu_0\Psi}\left[\xi^{-\frac{1}{2}} - \xi^{\frac{1}{2}}\right]^{-1}$$

where this applies for $ct > \Psi$.

For convenience define

$$\tau = \frac{ct}{\Psi}\left(= \frac{ct}{x'} \text{ on half plane}\right)$$

$$\xi = \left[\tau + \left[\tau^2 - 1\right]^{\frac{1}{2}}\right]^{-1} = \tau - \left[\tau^2 - 1\right]^{\frac{1}{2}} \text{ for } \tau > 1 \tag{A.15}$$

$$\xi^{-\frac{1}{2}} - \xi^{\frac{1}{2}} = \left[\xi^{-1} - 2 + \xi\right]^{\frac{1}{2}} = \left[2[\tau - 1]\right]^{\frac{1}{2}} \text{ for } \tau > 1$$

Then we integrate

$$J_{s_{z'}}(\tau) = \frac{4E_0}{\pi\sqrt{2}\mu_0\Psi}\int_{\frac{\Psi}{c}}^{t}[\tau'-1]^{-\frac{1}{2}}dt' = \frac{4E_0}{\pi\sqrt{2}Z_0}\int_{1}^{\psi}[\tau'-1]^{-\frac{1}{2}}d\tau' = \frac{8E_0}{\pi Z_0}\left[\frac{\tau-1}{2}\right]^{\frac{1}{2}} u(\tau-1) \tag{A.16}$$

recognizing that for $\tau < 1$ the surface current density is zero. This has the expected singularity near the edge $(x' \to 0,\ \tau \to \infty)$. Near the wavefront $(x' = ct)$ this goes to zero as expected.

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