Interaction Notes

Note 571

01 July 2001

An Interpolation Technique for Analyzing Sections of Nonuniform Multiconductor Transmission Lines

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Abstract

The product integrals representing propagation on nonuniform multiconductor transmission lines (NMTLs) do not have closed form analytic expressions in the general case. This paper goes beyond the usual staircase approximation for dividing the NMTL into a set of uniform sections. Using an appropriate average value of the propagation supermatrix in a section (which gives an analytic product integral), a linear correction term is developed which can also be analytically evaluated if the propagation supermatrix is smoothly varying and not varying too much in a section.

This work was sponsored in part by the Air Force Office of Scientific Research, and in part by the Air Force Research Laboratory, Directed Energy Directorate.

1. Introduction

Much recent attention has been given to the solution of propagation on and coupling to a nonuniform multi-conductor transmission line (NMTL). The general telegrapher equations are

$$\begin{aligned}\frac{\partial}{\partial z}\left(\tilde{V}_n(z,s)\right) &= -\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{I}_n(z,s)\right) + \left(\tilde{V}_n^{(s)'}(z,s)\right)\\ \frac{\partial}{\partial z}\left(\tilde{I}_n(z,s)\right) &= -\left(\tilde{Y}'_{n,m}(z,s)\right)\cdot\left(\tilde{V}_n(z,s)\right) + \left(\tilde{I}_n^{(s)'}(z,s)\right)\end{aligned} \tag{1.1}$$

for N conductors plus reference (zero voltage). The vectors have N components and the matrices are N x N. The various terms are

$$\begin{aligned}\left(\tilde{V}_n(z,s)\right) &\equiv \text{voltage vector}\\ \left(\tilde{I}_n(z,s)\right) &\equiv \text{current vector}\\ \left(\tilde{V}_n^{(s)'}(z,s)\right) &\equiv \text{per-unit-length voltage-source vector}\\ \left(\tilde{I}_n^{(s)'}(z,s)\right) &\equiv \text{per-unit-length current-source vector}\\ \left(\tilde{Z}'_{n,m}(z,s)\right) &\equiv \text{per-unit-length impedance matrix (passive and symmetric (reciprocity))}\\ \left(\tilde{Y}'_{n,m}(z,s)\right) &\equiv \text{per-unit-length admittance matrix (passive and symmetric (reciprocity))}\\ \sim &\equiv \text{two-sided Laplace transform over time } t\\ s = \Omega + j\omega &\equiv \text{Laplace-tranform variable or complex frequency}\\ z &\equiv \text{spacial coordinate (meters) along transmission line}\end{aligned} \tag{1.2}$$

The two telegrapher equations are readily combined into a single equation with $2N$-component vectors and $2N$ x $2N$ matrices as

$$\begin{aligned}&\frac{\partial}{\partial z}\begin{pmatrix}\left(\tilde{V}_n(z,s)\right)\\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n(z,s)\right)\end{pmatrix}\\ &\quad= -\begin{pmatrix}\left(0_{n,m}\right) & \left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}_{n,m}(s)\right)\\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right) & \left(0_{n,m}\right)\end{pmatrix}\odot\begin{pmatrix}\left(\tilde{V}_n(z,s)\right)\\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n(z,s)\right)\end{pmatrix}\\ &\qquad+\begin{pmatrix}\left(\tilde{V}_n^{(s)'}(z,s)\right)\\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n^{(s)'}(z,s)\right)\end{pmatrix}\end{aligned}$$

$$\left(\tilde{Z}_{n,m}(s)\right) = \left(\tilde{Z}_{n,m}(s)\right)^T = \left(\tilde{Y}_{n,m}(s)\right)^{-1} \tag{1.3}$$

$\equiv$ normalizing impedance matrix chosen at our convenience (not a function of z)

The general solution to such an equation is found from the supermatrizant differential equation

$$\begin{aligned}
&\frac{\partial}{\partial z}\left(\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{v,v'}\right) = \left(\left(\tilde{\Gamma}_{n,m}(z,s)\right)_{v,v'}\right) \odot \left(\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{v,v'}\right) \\
&\left(\left(\tilde{\Gamma}_{n,m}(z,s)\right)_{v,v'}\right) = -\begin{pmatrix} \left(0_{n,m}\right) & \left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{Y}_{n,m}(s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(s)\right) & \left(0_{n,m}\right) \end{pmatrix} \\
&\left(\left(\tilde{U}_{n,m}(z_0,z_0)\right)_{v,v'}\right) = \left(\left(1_{n,m}\right)_{v,v'}\right) = \begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(1_{n,m}\right) \end{pmatrix}
\end{aligned} \tag{1.4}$$

(boundary condition)

This has a solution as the product integral

$$\left(\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{v,v'}\right) = \prod_{z_0}^{z} e^{\left(\left(\tilde{\Gamma}_{n,m}(z',s)\right)_{v,v'}\right)dz'} \tag{1.5}$$

The solution to (1.3) is then found as

$$\begin{aligned}
&\begin{pmatrix} \left(\tilde{V}_n(z,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(z,s)\right) \end{pmatrix} = \left(\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{v,v'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(z_0,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(z_0,s)\right) \end{pmatrix} \\
&\quad + \int_{z_0}^{z} \left(\left(\tilde{U}_{n,m}(z,z';s)\right)_{v,v'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n^{(s)'}(z',s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n^{(s)'}(z',s)\right) \end{pmatrix}
\end{aligned} \tag{1.6}$$

Our task is then to find the solution to (1.4), or equivalently solve the product integral (1.5).

Various techniques are available for exact solutions in special cases and approximate solutions in more general cases [4, 5]. While one can approximate an NMTL by a cascade of uniform MTL sections (staircase approximation, each section with a closed-form product integral as a matrix exponential), one would like a better approximation making a smooth transition from section to section (avoiding discrete reflections). For special constraints on $((\tilde{\Gamma}_{n,m}(z,s)))$ such as z-independent eigenvectors, or for equal modal speeds, special techniques have been developed for a smooth interpolation between section ends with various forms for appropriate eigenvalues (linear variation, exponential variation, etc.) possible [1, 3]. For an almost uniform MTL a perterbation solution with error estimate has also been developed [2].

In this paper we develop another interpolation scheme, applicable to general NMTL sections if the variation from end to end is not too large.

2. Splitting the Propagation Supermatrix

For the ℓth section of the NMTL set

$$\left(\left(\tilde{\Gamma}_{n,m}^{(\ell)}(z,s)\right)_{\nu,\nu'}\right) \equiv \left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\nu,\nu'}\right) + \left(\left(\tilde{\Gamma}_{n,m}^{(\ell,1)}(z,s)\right)_{\nu,\nu'}\right)$$
$$z_\ell \le z \le z_{\ell+1} \qquad \text{(line section)} \tag{2.1}$$

The first term (the reference value) is taken as independent of z by choosing some particular value in the ℓth section, or by some kind of average over the section. The second term then represents the deviation from the reference and generally varies along the section.

With this decomposition apply the product integral to the reference term (constant) as

$$\left(\left(\tilde{G}_{n,m}^{(\ell)}(z,z_\ell;s)\right)_{\nu,\nu'}\right) = \prod_{z_\ell}^{z} e^{\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\nu,\nu'}\right)dz'}$$
$$= e^{\left(\left(\tilde{\Gamma}_{(s)}^{(\ell,0)}\right)_{\nu,\nu'}\right)[z-z_\ell]}$$
$$\left(\left(\tilde{G}_{n,m}^{(\ell)}(z,z_\ell;s)\right)_{\nu,\nu'}\right)^{-1} = e^{-\left(\left(\tilde{\Gamma}_{(s)}^{(\ell,0)}\right)_{\nu,\nu}\right)[z-z_\ell]} \tag{2.2}$$

giving a closed form result. Then apply the sum rule of the product integral to give

$$\prod_{z_\ell}^{z} e^{\left(\left(\tilde{\Gamma}_{n,m}^{(\ell)}(z',s)\right)_{\nu,\nu'}\right)dz'} = \left(\left(\tilde{G}_{n,m}^{(\ell,0)}(z,z_\ell;s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{G}^{(\ell,1)}(z,z_\ell;s)\right)_{\nu,\nu'}\right)$$
$$\left(\left(\tilde{G}^{(\ell,1)}(z,z_\ell;s)\right)_{\nu,\nu'}\right) = \prod_{z_\ell}^{z} e^{\left(\left(\tilde{H}_{n,m}^{(\ell)}(z',z_\ell;s)\right)_{\nu,\nu'}\right)dz'} \tag{2.3}$$
$$\left(\left(\tilde{H}_{n,m}^{(\ell)}(z,z_\ell;s)\right)_{\nu,\nu'}\right) \equiv \left(\left(\tilde{G}^{(\ell,0)}(z,z_\ell;s)\right)_{\nu,\nu'}\right)^{-1} \odot \left(\left(\tilde{\Gamma}_{n,m}^{(\ell,1)}(z,s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{G}_{n,m}^{(\ell,0)}(z,z_\ell;s)\right)_{\nu,\nu'}\right)$$

In this form the remaining product integral is suitable for a perturbation solution if the deviation term in (2.1) is sufficiently small (almost uniform MTL [2]). Our procedure here, however, is somewhat different in that we wish to evaluate this term for special forms of interpolation in the section $z_\ell \leq z \leq z_{\ell+1}$.

As a next step let us diagonalize the reference propagation supermatrix as

$$
\begin{aligned}
\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\nu,\nu'}\right) &= \sum_{\beta=1}^{2N} \tilde{\gamma}_\beta(s)\left(\left(\tilde{r}_n^{(\ell)}(s)\right)_\nu\right)_\beta \left(\left(\tilde{\ell}_n^{(\ell)}(s)\right)_\nu\right)_\beta \\
\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{r}_n^{(\ell)}(s)\right)_\nu\right)_\beta &= \tilde{\gamma}_\beta(s)\left(\left(\tilde{r}_n^{(\ell)}(s)\right)_\nu\right)_\beta \\
\left(\left(\tilde{\ell}_n^{(\ell)}(s)\right)_\nu\right)_\beta \odot \left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\nu,\nu'}\right) &= \tilde{\gamma}_\beta(s)\left(\left(\tilde{\ell}_n^{(\ell)}(s)\right)_\nu\right)_\beta \\
\left(\left(\tilde{\ell}_n^{(\ell)}(s)\right)_\nu\right)_{\beta_1} \odot \left(\left(\tilde{r}_n^{(\ell)}(s)\right)_\nu\right)_{\beta_2} &= 1_{\beta_1,\beta_2} = \begin{cases} 1 & \text{for } \beta_1 = \beta_2 \\ 0 & \text{for } \beta_1 \neq \beta_2 \end{cases} \qquad \text{(biorthonormal)} \\
\left(\tilde{\gamma}_\beta(s)\right) &= \left(\left(\tilde{\ell}_n^{(\ell)}(s)\right)_\nu\right)_\beta \odot \left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{r}_n^{(\ell)}(s)\right)_\nu\right)_\beta
\end{aligned}
\tag{2.4}
$$

This, of course, assumes a complete set of right and left eigenvectors. For distinct $\tilde{\gamma}_\beta(s)$, this is assured, but for special cases of two or more equal eigenvalues one needs to consider each case. It should be noted that even in the case of all modal speeds (eigenvalues) the same, complete diagonalization has been achieved [1, 3].

Noting that (2.4) is independent of z, then the matrix exponential is evaluated as

$$
\begin{aligned}
\left(\left(\tilde{G}_{n,m}^{(\ell,0)}(z,z_\ell;s)\right)_{\nu,\nu'}\right)^{\pm 1} &= e^{\pm\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\nu,\nu'}\right)[z-z_\ell]} \\
&= \sum_{\beta=1}^{2N} e^{\pm\tilde{\gamma}_\beta(s)[z-z_\ell]}\left(\left(\tilde{r}_n^{(\ell)}(s)\right)_\nu\right)_\beta \left(\left(\tilde{\ell}_n^{(\ell)}(s)\right)_\nu\right)_\beta
\end{aligned}
\tag{2.5}
$$

This will allow a more convenient evaluation of the remaining product integral.

3. Interpolating the Propagation Supermatrix

Now let us consider special forms for the two terms in (2.1). We could choose the reference term as the value of the propagation supermatrix at any z in the interval $z_\ell \le z \le z_{\ell+1}$. Another approach, which we adopt here, is to choose an average of the two values at the section ends as

$$\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,1)}(s)\right)_{\nu,\nu'}\right) \equiv \frac{1}{2}\left[\left(\left(\tilde{\Gamma}_{n,m}^{(\ell)}(z_{\ell+1},s)\right)_{\nu,\nu'}\right) + \left(\left(\tilde{\Gamma}_{n,m}^{(\ell)}(z_\ell,s)\right)_{\nu,\nu'}\right)\right] \tag{3.1}$$

Then let us choose the deviation term in the factored form

$$\begin{aligned}\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,1)}(s)\right)_{\nu,\nu'}\right) &\simeq f^{(\ell)}(z)\left(\left(\tilde{C}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right)\\ \left(\left(\tilde{C}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right) &\equiv \frac{1}{2}\left[\left(\left(\tilde{\Gamma}^{(\ell)}(z_{\ell+1},s)\right)_{\nu,\nu'}\right) - \left(\left(\tilde{\Gamma}_{n,m}^{(\ell)}(z_\ell,s)\right)_{\nu,\nu'}\right)\right]\\ f^{(\ell)}(z_\ell) &= -1 \quad , \quad f^{(\ell)}(z_{\ell+1}) = 1\\ f^{(\ell)}(z) &= \text{monotone nondecreasing function in } z_\ell \le z \le z_{\ell+1}\end{aligned} \tag{3.2}$$

The form in (3.2) is an approximation valid for smooth variation of the propagation supermatrix in the interval. It is exact at $z = z_\ell\, , z_{\ell+1}$. For the terms in (1.4) we have

$$\left(\left(\tilde{\Gamma}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right) = -\begin{pmatrix} (0_{n,m}) & \frac{1}{2}\left[\left(\tilde{Z}'_{n,m}(z_{\ell+1},s)\right) + \left(\tilde{Z}'_{n,m}(z_\ell,s)\right)\right]\cdot\left(\tilde{Y}_{n,m}^{(\ell)}(s)\right) \\ \frac{1}{2}\left(\tilde{Z}_{n,m}^{(\ell)}(s)\right)\cdot\left[\left(\tilde{Y}'_{n,m}(z_{\ell+1},s)\right) + \left(\tilde{Y}'_{n,m}(z_\ell,s)\right)\right] & (0_{n,m})\end{pmatrix}$$

$$\left(\left(\tilde{C}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right) = -\begin{pmatrix} (0_{n,m}) & \frac{1}{2}\left[\left(\tilde{Z}'_{n,m}(z_{\ell+1},s)\right) - \left(\tilde{Z}'_{n,m}(z_\ell,s)\right)\right]\cdot\left(\tilde{Y}_{n,m}^{(\ell)}(s)\right) \\ \frac{1}{2}\left(\tilde{Z}_{n,m}^{(\ell)}(s)\right)\cdot\left[\left(\tilde{Y}'_{n,m}(z_{\ell+1},s)\right) - \left(\tilde{Y}'_{n,m}(z_\ell,s)\right)\right] & (0_{n,m})\end{pmatrix}$$

$$\left(\tilde{Z}_{n,m}^{(\ell)}(s)\right) = \left(\tilde{Y}_{n,m}^{(\ell)}(s)\right)^{-1} \equiv \text{normalizing impedance matrix for } \ell \text{ th section} \tag{3.3}$$

The variation with respect to z is then equivalently taken as a variation in the per-unit-length impedance and admittance matrices. The normalizing impedance matrix is typically defined in terms of these as a characteristic impedance matrix. For this purpose we can use

$$
\begin{aligned}
\left(\tilde{g}_{n,m}^{(\ell)}(s)\right) &= \frac{1}{2}\left[\left[\left(\tilde{Z}'_{n,m}(z_{\ell+1},s)\right) + \left(\tilde{Z}'_{n,m}(z_{\ell},s)\right)\right] \cdot \left[\left(\tilde{Y}'_{n,m}(z_{\ell+1},s)\right) + \left(\tilde{Y}'_{n,m}(z_{\ell},s)\right)\right]\right]^{\frac{1}{2}} \\
&\qquad \text{(positive real (p.r.) square root)} \\
\left(Z_{c_{h,m}}^{(\ell)}(s)\right) &= 2\left(\tilde{g}_{n,m}^{(\ell)}(s)\right) \cdot \left[\left(\tilde{Y}'_{n,m}(z_{\ell+1},s)\right) + \left(\tilde{Y}'_{n,m}(z_{\ell},s)\right)\right]^{-1} \\
&= \frac{1}{2}\left(\tilde{g}_{n,m}^{(\ell)}(s)\right)^{-1} \cdot \left[\left(\tilde{Z}'_{n,m}(z_{\ell+1},s)\right) + \left(\tilde{Z}'_{n,m}(z_{\ell},s)\right)\right] \\
&= \left(\tilde{Z}_{c_{h,m}}^{(\ell)}(s)\right)^{\mathrm{T}} = \left(Y_{c_{h,m}}^{(\ell)}(s)\right)^{-1}
\end{aligned}
\tag{3.4}
$$

(3.4

This is an appropriate average characteristic impedance matrix for the ℓth section.

There are various forms one might choose for $f^{(\ell)}(z)$. The simplest smooth function is

$$
f^{(\ell)}(z) = \frac{2z - z_{\ell+1} - z_{\ell}}{z_{\ell+1} - z_{\ell}} \tag{3.5}
$$

This is like the first-power term in a Taylor series expansion. The constant term is just the reference term. Of course, one could actually make a Taylor series expansion of $((\tilde{\Gamma}^{(\ell)}(z,s))_{\nu,\nu})$ about the center of the interval and have the first-power term as a constant matrix times z. However, this would not have the exact match at $z = z_{\ell}, z_{\ell+1}$.

4. Solution of the Second Product Integral

Now we have

$$\left(\left(\tilde{H}_{n,m}^{(\ell)}(z,z_\ell;s)\right)_{\nu,\nu'}\right) = f^{(\ell)}(z)\left(\left(\tilde{G}^{(\ell,0)}(z,z_\ell;s)\right)_{\nu,\nu'}\right)^{-1} \odot \left(\left(\tilde{C}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{G}_{n,m}^{(\ell)}(z,z_\ell;s)\right)_{\nu,\nu'}\right)$$

$$= f^{(\ell)}(z)\sum_{\beta=1}^{2N}\sum_{\beta'=1}^{2N} e^{-\left[\tilde{\gamma}_\beta^{(\ell)}(s)-\tilde{\gamma}_{\beta'}^{(\ell)}(s)\right][z-z_\ell]}\left(\left(\tilde{r}_n^{(\ell)}(s)\right)_\nu\right)_\beta\left(\left(\tilde{\ell}_n^{(\ell)}(s)\right)_\nu\right)_{\beta'}\tilde{a}_{\beta,\beta'}^{(\ell)}(s)$$

$$\tilde{a}_{\beta,\beta'}^{(\ell)}(s) \equiv \left(\left(\tilde{r}_n^{(\ell)}(s)\right)_\nu\right)_\beta \cdot \left(\left(\tilde{C}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right) \cdot \left(\left(\tilde{\ell}_n^{(\ell)}(s)\right)_\nu\right)_{\beta'} \tag{4.1}$$

So what we have is a set of constant dyadics times scalar functions of z to be product integrated. Unfortunately, not all β,β' pairs of $((\tilde{r}_n(s)_\nu)_\beta((\tilde{\ell}_n(s))_\nu)_{\beta'}$ commute.

An observation concerning $((\tilde{H}_{n,m}^{(\ell)}(z,z_\ell;s))_{\nu,\nu'})$ is that it is a similarity transform and hence has the same eigenvalues as $f^{(\ell)}(z)((\tilde{C}_{n,m}^{(\ell)}(s))_{\nu,\nu'})$. So if $((\tilde{C}_{n,m}^{(\ell)}(s))_{\nu,\nu'})$ is small compared to $((\tilde{\Gamma}_{n,m}^{(\ell)}(s))_{\nu,\nu'})$ then the product integral of $((\tilde{H}_{n,m}^{(\ell)}(z,z_\ell;s))_{\nu,\nu'})$ can be taken numerically with comparatively large steps (staircase approximation) through the ℓth section as compared a similar numerical product integral of $((\tilde{\Gamma}_{n,m}(z,s))_{\nu,\nu'})$ through the section. Of course, we have evaluated the product integral of $((\tilde{\Gamma}_{n,m}^{(\ell)}(s))_{\nu,\nu'})$ *analytically.* So one option is to numerically evaluate

$$\left(\left(\tilde{G}^{(\ell,1)}(z,z_\ell;s)\right)_{\nu,\nu'}\right) =$$

$$\prod_{z_\ell}^{z} e^{f^{(\ell)}(z')\left(\left(\tilde{G}_{n,m}^{(\ell,0)}(z',z_\ell;s)\right)_{\nu,\nu'}\right)^{-1} \odot \left(\left(\tilde{C}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{G}_{n,m}^{(\ell,0)}(z',z_\ell;s)\right)_{\nu,\nu'}\right)dz'} \tag{4.2}$$

in the form that the integrand is expressed in (4.1)

Another approach, provided $((\tilde{C}_{n,m}^{(\ell)}(s))_{\nu,\nu'})$ is sufficiently small, is to expand the product integral using the first few terms of the matrizant series [2] as

$$\left(\left(\tilde{G}^{(\ell,1)}(z,z_\ell;s)\right)_{\nu,\nu'}\right) =$$

$$\left(\left(1_{n,m}\right)_{\nu,\nu'}\right) + \int_{z_\ell}^{z} f^{(\ell)}(z')\left(\left(\tilde{G}^{(\ell,0)}(z',z_\ell;s)\right)_{\nu,\nu'}\right)^{-1} \odot \left(\left(\tilde{C}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{G}_{n,m}^{(\ell,0)}(z',z_\ell;s)\right)_{\nu,\nu'}\right) dz'$$

$$+ O\left(\left(\chi_{\max}\left|z-z_\ell\right|\right)^2\right) + \left|z-z_0\right| \to 0$$

$$\chi_{\max} = \text{maximum magnitude eigenvalue of} \left(\left(\tilde{C}_{n,m}^{(\ell)}(s)\right)_{\nu,\nu'}\right) \tag{4.3}$$

As one chooses $z_{\ell+1} - z_\ell$ smaller, then $\chi_{\max} \to 0$ for smooth variation of the propagation supermatirx. This points out that if there are any discontinuities in the propagation supermatrix, these should be placed at section end points.

For evaluating the correction term in (4.3) it is convenient to define

$$\begin{gathered} z_\ell^{(c)} \equiv \frac{z_{\ell+1}+z_\ell}{2} \equiv \text{section center}, \quad \Delta_\ell \equiv z_{\ell+1} - z_\ell \equiv \text{section length} \\ f^{(\ell)}(z) = \frac{2}{\Delta_\ell}\left[z - z_\ell^{(c)}\right], \quad z_\ell^{(c)} - z_\ell = \frac{\Delta_\ell}{2} \end{gathered} \tag{4.4}$$

Then we have

$$\tilde{F}_{\beta,\beta'}^{(\ell)}(z,z_\ell;s) = \int_{z_\ell}^{z} f(z') e^{-\left[\tilde{\gamma}_\beta(s)-\tilde{\gamma}_{\beta'}(s)\right]\left[z'-z_\ell\right]} dz'$$

$$= \frac{2}{\Delta_\ell} e^{-\left[\tilde{\gamma}_\beta(s)-\tilde{\gamma}_{\beta'}(s)\right]\left[z_\ell^{(c)}-z_\ell\right]} \int_{z_\ell}^{z} \left[z'-z_\ell^{(c)}\right] e^{-\left[\tilde{\gamma}_\beta(s)-\tilde{\gamma}_{\beta'}(s)\right]\left[z'-z_\ell^{(c)}\right]} dz'$$

$$= \frac{2}{\Delta_\ell} e^{-\left[\tilde{\gamma}_\beta(s)-\tilde{\gamma}_{\beta'}(s)\right]\frac{\Delta_\ell}{2}} \int_{z_\ell - z_\ell^{(c)}}^{z-z_\ell^{(c)}} z'' e^{-\left[\tilde{\gamma}_\beta(s)-\tilde{\gamma}_{\beta'}(s)\right]} dz''$$

$$= \frac{2}{\Delta_\ell} e^{-\left[\tilde{\gamma}_\beta(s)-\tilde{\gamma}_{\beta'}(s)\right]\frac{\Delta_\ell}{2}} e^{-\left[\tilde{\gamma}_\beta(s)-\tilde{\gamma}_{\beta'}(s)\right]z''} \left[-\frac{z''}{\tilde{\gamma}_\beta(s)-\tilde{\gamma}_{\beta'}(s)} - \frac{1}{\left[\tilde{\gamma}_\beta(s)-\tilde{\gamma}_{\beta'}(s)\right]^2}\right]_{z_\ell - z_\ell^{(c)}}^{z-z_\ell^{(c)}}$$

$$= \frac{2}{\Delta_\ell} e^{-\left[\tilde{\gamma}_\beta(s) - \tilde{\gamma}_{\beta'}(s)\right]\frac{\Delta_\ell}{2}} \left[e^{-\left[\tilde{\gamma}_\beta(s) - \tilde{\gamma}_{\beta'}(s)\right]\left[z - z_\ell^{(c)}\right]} \left[-\frac{z - z_\ell^{(c)}}{\tilde{\gamma}_\beta(s) - \tilde{\gamma}_{\beta'}(s)} - \frac{1}{\left[\tilde{\gamma}_\beta(s) - \tilde{\gamma}_{\beta'}(s)\right]^2} \right] \right.$$

$$\left. + e^{-\left[\tilde{\gamma}_\beta(s) - \tilde{\gamma}_{\beta'}(s)\right]\left[-\frac{\Delta_\ell}{2}\right]} \left[-\frac{1}{2} \frac{\Delta_\ell}{\tilde{\gamma}_\beta(s) - \tilde{\gamma}_{\beta'}(s)} + \frac{1}{\left[\tilde{\gamma}_\beta(s) - \tilde{\gamma}_{\beta'}(s)\right]^2} \right] \right]$$

$$= \frac{2}{\Delta_\ell} e^{-\left[\tilde{\gamma}_\beta(s) - \tilde{\gamma}_{\beta'}(s)\right]\left[z - z_\ell\right]} \left[-\frac{z - z_\ell^{(c)}}{\tilde{\gamma}_\beta(s) - \tilde{\gamma}_{\beta'}(s)} - \frac{1}{\left[\tilde{\gamma}_\beta(s) - \tilde{\gamma}_{\beta'}(s)\right]^2} \right]$$

$$- \frac{1}{\tilde{\gamma}_\beta(s) - \tilde{\gamma}_{\beta'}(s)} + \frac{2}{\Delta_\ell} \frac{1}{\left[\tilde{\gamma}_\beta(s) - \tilde{\gamma}_{\beta'}(s)\right]^2}$$

$$\text{for } \tilde{\gamma}_\beta(s) \neq \tilde{\gamma}_{\beta'}(s) \tag{4.5}$$

$$\tilde{F}_{\beta,\beta'}^{(\ell)}(z, z_\ell; s) = \frac{2}{\Delta_\ell} \int_{z_\ell}^{z} \left[z' - z_\ell^{(c)} \right] dz' = \frac{1}{\Delta_\ell} \left[z' - z_\ell^{(c)} \right]^2 \Bigg|_{z_\ell}^{z}$$

$$= \frac{1}{\Delta_\ell} \left[z - z_\ell^{(c)} \right]^2 - \frac{\Delta_\ell}{4}$$

$$\text{for } \tilde{\gamma}_\beta(s) = \tilde{\gamma}_{\beta'}(s)$$

With these results we have

$$\left(\left(\tilde{G}^{(\ell,1)}(z, z_\ell; s) \right)_{\nu,\nu'} \right) = \left(\left(1_{n,m} \right)_{\nu,\nu'} \right)$$

$$+ \sum_{\beta=1}^{2N} \sum_{\beta'=1}^{2N} \tilde{F}_{\beta,\beta'}^{(\ell)}(z, z_\ell; s)\, \tilde{a}_{\beta,\beta'}^{(\ell)} \left(\left(\tilde{r}_n^{(\ell)}(s) \right)_{\nu,\nu'} \right) \left(\left(\tilde{\ell}_n^{(\ell)}(s) \right)_{\nu,\nu'} \right) \tag{4.6}$$

with terms defined in (2.4), (4.1), and (4.5). Recalling from (2.3) and (2.5)

$$\left(\left(\tilde{U}_{n,m}(z, z_\ell; s) \right)_{\nu,\nu'} \right) = \prod_{z_\ell}^{z} e^{\left(\left(\tilde{\Gamma}_{n,m}^{(\ell)}(z', s) \right)_{\nu,\nu'} \right) dz'} = \left(\left(\tilde{G}_{n,m}^{(\ell,0)}(z, z_\ell; s) \right)_{\nu,\nu'} \right) \odot \left(\left(\tilde{G}_{n,m}^{(\ell,1)}(z, z_\ell; s) \right)_{\nu,\nu'} \right)$$

$$\left(\left(\tilde{G}_{n,m}^{(\ell,0)}(z, z_\ell; s) \right)_{\nu,\nu'} \right) = \sum_{\beta=1}^{N} e^{\gamma_\beta(s)[z - z_\ell]} \left(\left(\tilde{r}_n^{(\ell)}(s) \right)_\nu \right)_\beta \left(\left(\tilde{\ell}_n^{(\ell)}(s) \right)_\nu \right)_\beta \tag{4.7}$$

we now have the approximate solution for the product integral describing the NMTL in the ℓth line section. Returning to (1.6) this gives the solution for the voltages and currents on the NMTL with z_0 and z' replacing z_ℓ and $(\tilde{Z}_{n,m}(s))$ taken for the ℓth section.

In going from one section to the next (ℓ to $\ell+1$) one can cascade the results. If there are no source terms in (1.6) one can use

$$\begin{pmatrix} (\tilde{V}_n(z,s)) \\ \left(\tilde{Z}_{n,m}^{(\ell+1)}(s)\right) \cdot \left(\tilde{I}_n(z,s)\right) \end{pmatrix}$$
$$= \left(\left(\tilde{U}_{n,m}^{(\ell+1)}(z, z_{\ell+1}; s)\right)_{\nu,\nu'}\right) \odot \begin{pmatrix} (1_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & \left(\tilde{Z}_{n,m}^{(\ell+1)}(s)\right) \cdot \left(\tilde{Z}_{n,m}^{(\ell)}(s)\right)^{-1} \end{pmatrix} \tag{4.8}$$
$$\odot \left(\left(\tilde{U}_{n,m}^{(\ell)}(z_{\ell+1}, z_\ell; s)\right)_{\nu,\nu'}\right) \odot \begin{pmatrix} (\tilde{V}_n(z_\ell,s)) \\ \left(\tilde{Z}_{n,m}^{(\ell)}(s)\right) \cdot \left(\tilde{I}_n(z_\ell,s)\right) \end{pmatrix}$$
$$z_\ell \le z \le z_{\ell+1}$$

as discussed in [1]. This can be extended from any ℓ (e.g., 1 or 0) to anywhere along the NMTL to form a matrix description of any length of NMTL. Note the presence of the matrix to renormalize the normalizing impedance matrix in going from one section to the next.

5. Concluding Remarks

So now we have a scheme for interpolating the propagation supermatrix in sections of NMTLs which uses an average value to obtain a closed form product integral, followed by a linear correction which gives a good approximation provided the propagation supermatrix varies smoothly and only a little through each line section. This removes the problem of jump discontinuities (with associated reflections) when using a staircase approximation. If one has discontinuities in the NMTL one is analyzing, these can be placed at section boundaries so that one is not interpolating through such discontinuities.

In the present paper no restriction is made that the modal speeds be the same as in previous papers [1, 3]. The present paper then applies to the more general NMTL.

References

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