Interaction Notes

Note 596

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Scattering Matrices for Nonuniform Multiconductor Transmission Lines

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Abstract

A section of nonuniform multiconductor transmission line of finite length ℓ can be characterized by various forms of scattering matrix. If the transmission line is lossless special properties apply.

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1. Introduction

A recent paper [5] addresses some of the properties of the supermatrizant and its blocks, based on reciprocity, energy and norms, for nonuniform multiconductor transmission lines (NMTLs). Associated with such NMTLs are also scattering matrices. Such is the subject of the present paper.

As indicated in Fig. 1.1, we have a section of NMTL of length ℓ. At $z = 0$ this connects to a uniform MTL with characteristic impedance matrix $\left(Z_{n,m}^{(1)}\right)$ of size N x N and assumed lossless and symmetric (reciprocity). Similarly at $z = \ell$ another uniform MTL with characteristic impedance matrix $\left(Z_{n,m}^{(2)}\right)$ of size N x N is also connected.

As discussed in [1] one can define waves by combining voltage and current variables. For the left MTL we have

$$
\begin{aligned}
&\left(\tilde{V}_n^{(1)}(z,s)\right)_{\pm} = \left(\tilde{V}_n^{(1)}(z,s)\right) \pm \left(Z_{n,m}^{(1)}\right) \cdot \left(\tilde{I}_n^{(1)}(z,s)\right) \\
&\sim \; = \; \text{two-sided Laplace transform over time } t \\
&s \; = \; \Omega + j\omega \equiv \text{Laplace-transform variable or complex frequency} \\
&\left(\tilde{V}_n^{(1)}(0,s)\right)_{+} \equiv \text{wave incident on NMTL from left} \\
&\left(\tilde{V}_n^{(1)}(0,s)\right)_{-} \equiv \text{wave leaving from NMTL to left}
\end{aligned}
\tag{1.1}
$$

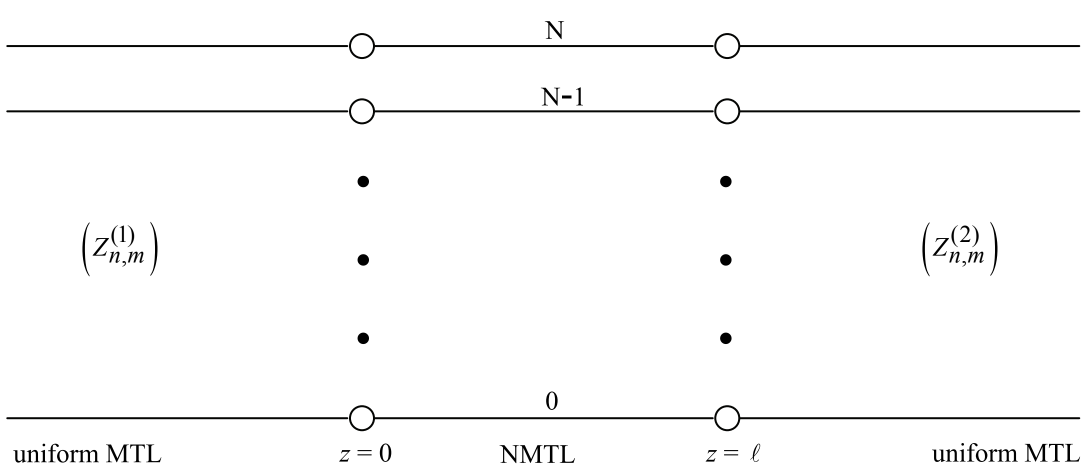

Fig. 1.1 Section of NMTL Connected to Two Lossless MTLs.

Similarly, we have for the right MTL

$$
\begin{aligned}
\left(\tilde{\mathrm{V}}_n^{(2)}(z,s)\right)_{\pm} &= \left(\tilde{\mathrm{V}}_n^{(2)}(z,s)\right) \pm \left(Z_{n,m}^{(2)}\right) \cdot \left(\tilde{I}_n^{(1)}(z,s)\right) \\
\left(\tilde{\mathrm{V}}_n^{(2)}(\ell,s)\right)_{+} &\equiv \text{wave leaving from NMTL to right} \\
\left(\tilde{\mathrm{V}}_n^{(2)}(\ell,s)\right)_{-} &\equiv \text{wave incident on NMTL from right}
\end{aligned}
\tag{1.2}
$$

The NMTL can then be described by a scattering matrix with

$$
\begin{pmatrix} \left(\tilde{\mathrm{V}}_n^{(1)}(0,s)\right)_{-} \\ \left(\tilde{\mathrm{V}}_n^{(2)}(\ell,s)\right)_{+} \end{pmatrix} = \left(\left(\tilde{S}_{n,m}(s)\right)_{\nu,\nu'}\right) \cdot \begin{pmatrix} \left(\tilde{\mathrm{V}}_n^{(1)}(0,s)\right)_{+} \\ \left(\tilde{\mathrm{V}}_n^{(2)}(\ell,s)\right)_{-} \end{pmatrix}
\tag{1.3}
$$

relating incoming incident) waves to outgoing waves.

For later use we have

$$
\begin{aligned}
\left(Z_{n,m}^{(1)}\right) &= \left(Z_{n,m}^{(1)}\right)^{1/2} = \left(Y_{n,m}^{(1)}\right)^{-1/2} = \left(y_{n,m}^{(1)}\right)^{-1} \\
\left(Z_{n,m}^{(2)}\right) &= \left(Z_{n,m}^{(2)}\right)^{1/2} = \left(Y_{n,m}^{(2)}\right)^{-1/2} = \left(y_{n,m}^{(2)}\right)^{-1}
\end{aligned}
\tag{1.4}
$$

This allows us to define a renormalized form of the waves [6] as

$$
\begin{aligned}
\left(\tilde{\mathrm{v}}_n^{\binom{1}{2}}(z,s)\right)_{\pm} &\equiv \left(y_{n,m}^{\binom{1}{2}}\right) \cdot \left(\tilde{\mathrm{V}}_n^{\binom{1}{2}}(z,s)\right)_{\pm} \\
&= \left(y_{n,m}^{\binom{1}{2}}\right) \cdot \left(\tilde{\mathrm{V}}_n^{\binom{1}{2}}(z,s)\right) \pm \left(z_{n,m}^{\binom{1}{2}}\right) \cdot \left(\tilde{I}_n^{\binom{1}{2}}(z,s)\right)
\end{aligned}
\tag{1.5}
$$

Corresponding to this we have a renormalized scattering matrix as

$$
\begin{pmatrix} \left(\tilde{\mathrm{v}}_n^{(1)}(0,s)\right)_- \\ \left(\tilde{\mathrm{v}}_n^{(2)}(\ell,s)\right)_+ \end{pmatrix} = \left(\left(\tilde{S}_{n,m}^{(R)}(s)\right)_{\nu,\nu'}\right) \cdot \begin{pmatrix} \left(\tilde{\mathrm{v}}_n^{(1)}(0,s)\right)_+ \\ \left(\tilde{\mathrm{v}}_n^{(2)}(\ell,s)\right)_- \end{pmatrix}
$$

$$
\left(\left(\tilde{S}_{n,m}(s)\right)_{\nu,\nu'}\right) = \begin{pmatrix} \left(z_{n,m}^{(1)}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(z_{n,m}^{(2)}\right) \end{pmatrix} \cdot \left(\left(\tilde{S}_{n,m}^{(R)}(s)\right)_{\nu,\nu'}\right) \cdot \begin{pmatrix} \left(y_{n,m}^{(1)}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(y_{n,m}^{(2)}\right) \end{pmatrix} \tag{1.6}
$$

See also [2] for some properties of these scattering matrices.

For the NMTL we have the combined telegrapher equations

$$
\frac{d}{dz}\begin{pmatrix} \left(\tilde{\mathrm{V}}_n(z,s)\right) \\ \left(Z_{n,m}\right) \cdot \left(\tilde{I}_n(z,s)\right) \end{pmatrix} =
$$

$$
-\begin{pmatrix} \left(0_{n,m}\right) & \left(\tilde{Z}'_{n,m}(z,s)\right) = \left(Y_{n,m}\right) \\ \left(Z_{n,m}\right) \cdot \left(\tilde{Y}'_{n,m}(z,s)\right) & \left(0_{n,m}\right) \end{pmatrix} \cdot \begin{pmatrix} \left(\tilde{\mathrm{V}}_n(z,s)\right) \\ \left(Z_{n,m}\right) \cdot \left(\tilde{I}_n(z,s)\right) \end{pmatrix}
$$

$$
+\begin{pmatrix} \left(\tilde{\mathrm{V}}_n^{(s)'}(z,s)\right) \\ \left(Z_{n,m}\right) \cdot \left(\tilde{I}_n^{(s)'}(z,s)\right) \end{pmatrix} \tag{1.7}
$$

$$
\left(Z_{n,m}\right) = \left(Z_{n,m}\right)^T = \left(Y_{n,m}\right)^{-1} \equiv \text{normalizing constant impedance matrix (chosen at our convenience)}
$$

For present purposes we set the distributed sources to zero. The associated matrizant differential equation is

$$
\frac{d}{dz}\left(\left(\tilde{G}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) = \left(\left(\tilde{\Gamma}_{n,m}(z,s)\right)_{\nu,\nu'}\right) \cdot \left(\left(\tilde{G}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right)
$$

$$
\left(\left(\tilde{\Gamma}_{n,m}(z,s)\right)_{\nu,\nu'}\right) = -\begin{pmatrix} \left(0_{n,m}\right) & \left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(Y_{n,m}\right) \\ \left(Z_{n,m}\right) \cdot \left(\tilde{Y}'_{n,m}(z,s)\right) & \left(0_{n,m}\right) \end{pmatrix} \tag{1.8}
$$

$$
\left(\left(\tilde{G}_{n,m}(z_0,z_0;s)\right)_{\nu,\nu'}\right) = \left(\left(1_{n,m}\right)_{\nu,\nu'}\right) \text{ (boundary condition)}
$$

The solution is the product integral

$$
\begin{aligned}
&\left(\left(\tilde{G}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) = \prod_{z_0}^{z} e^{\left(\left(\tilde{\Gamma}_{n,m}(z's)\right)_{\nu,\nu'}\right)dz'} \\
&tr\left(\left(\left(\tilde{\Gamma}_{n,m}(z,s)\right)_{\nu,\nu'}\right)\right) = 0 \ , \quad \det\left(\left(\left(\tilde{G}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right)\right) = 1
\end{aligned}
\tag{1.9}
$$

We will take z_0 as 0 and ℓ, the ends of the NMTL, for our present purposes.

2. Scattering Matrices

Consider an M-port network (lumped and/or distributed elements). Let it be characterized by an M x M impedance matrix

$$\left(\tilde{Z}_{n,m}^{(\mathrm{M})}(s)\right) = \left(\tilde{Z}_{n,m}^{(\mathrm{M})}(s)\right)^T = \left(\tilde{Y}_{n,m}^{(\mathrm{M})}(s)\right)^{-1} \tag{2.1}$$

This is driven by an external set of sources which we can think of as incident from an M-conductor (plus reference) uniform MTL of characteristic impedance

$$\left(Z_{n,m}^{(\mathrm{ext})}\right) = \left(Z_{n,m}^{(\mathrm{ext})}\right)^T = \left(Y_{n,m}^{(\mathrm{ext})}(s)\right)^{-1} \tag{2.2}$$

Which we take as real and frequency-independent. The port voltages and currents are related as

$$\left(\tilde{\mathrm{V}}_n(s)\right) = \left(\tilde{Z}_{n,m}^{(\mathrm{M})}(s)\right) = \left(\tilde{I}_n(s)\right) \tag{2.3}$$

with $\left(\tilde{I}_n(s)\right)$ taken as positive into the network. In turn wave variables are defined [1] as

$$\left(\tilde{\mathrm{V}}_n(s)\right)_{\pm} = \left(\tilde{\mathrm{V}}_n(s)\right) \pm \left(\tilde{Z}_{n,m}^{(\mathrm{ext})}(s)\right) \cdot \left(\tilde{I}_n(s)\right)$$

$+ \Rightarrow$ incident wave (2.4)

$- \Rightarrow$ reflected (scattered) wave

A scattering matrix can now be defined via

$$\left(\tilde{V}_n(s)\right)_{-} = \left(\tilde{S}_{n,m}^{(\mathrm{M})}(s)\right) \cdot \left(\tilde{V}_n(s)\right) \tag{2.5}$$

This scattering matrix can take several forms [1]. For convenience define

$$\left(\tilde{z}_{n,m}^{(\mathrm{M})}(s)\right) \equiv \left(\tilde{Z}_{n,m}^{(\mathrm{M})}(s)\right) \cdot \left(Y_{n,m}^{(\mathrm{ext})}\right) = \left(\tilde{y}_{n,m}^{(\mathrm{M})}(s)\right)^{-1} \tag{2.6}$$

Giving various equivalent forms as

$$
\begin{aligned}
\left(\tilde{S}_{n,m}^{(\mathrm{M})}(s)\right) &= \left[\left(\tilde{z}_{n,m}^{(\mathrm{M})}(s)\right) + \left(1_{n,m}\right)\right]^{-1} \cdot \left[\left(\tilde{z}_{n,m}^{(\mathrm{M})}(s)\right) - \left(1_{n,m}\right)\right] \\
&= \left[\left(\tilde{z}_{n,m}^{(\mathrm{M})}(s)\right) - \left(1_{n,m}\right)\right] \cdot \left[\left(\tilde{z}_{n,m}^{(\mathrm{M})}(s)\right) + \left(1_{n,m}\right)\right]^{-1} \\
&= \left[\left(1_{n,m}\right) + \left(\tilde{y}_{n,m}^{(\mathrm{M})}(s)\right)\right]^{-1} \cdot \left[\left(1_{n,m}\right) - \left(\tilde{y}_{n,m}^{(\mathrm{M})}(s)\right)\right] \\
&= \left[\left(1_{n,m}\right) - \left(\tilde{y}_{n,m}^{(\mathrm{M})}(s)\right)\right] \cdot \left[\left(1_{n,m}\right) + \left(\tilde{y}_{n,m}^{(\mathrm{M})}(s)\right)\right]^{-1}
\end{aligned}
\tag{2.7}
$$

Note that all the matrices in (2.7) have the same eigenvectors and commute.

From [5] for a lossless network we have

$$
\left(\tilde{Z}_{n,m}(-s)\right) = -\left(\tilde{Z}_{n,m}(s)\right) \qquad \text{(odd in } s\text{)} \tag{2.8}
$$

In (2.7) this gives

$$
\begin{aligned}
\left(\tilde{S}_{n,m}^{(\mathrm{M})}(-s)\right)^{-1} &= \left[-\left(z_{n,m}^{(\mathrm{M})}(s)\right) - \left(1_{n,m}\right)\right]^{-1} \cdot \left[-\left(z_{n,m}^{(\mathrm{M})}(s)\right) + \left(1_{n,m}\right)\right] \\
&= \left(\tilde{S}_{n,m}^{(\mathrm{M})}(s)\right) \\
\left(1_{n,m}\right) &= \left(\tilde{S}_{n,m}^{(\mathrm{M})}(-s)\right) \cdot \left(\tilde{S}_{n,m}^{(\mathrm{M})}(s)\right)
\end{aligned}
\tag{2.9}
$$

This is suggestive of unitary with $s = j\omega$, but not quite because $\left(\tilde{S}_{n,m}^{(\mathrm{M})}(s)\right)$ is not symmetric, in general.

Now consider the power delivered into the M-port. The real power can be expressed in a general form as

$$
\tilde{P}(s) = \frac{1}{2}\left[\left(\tilde{V}_n(s)\right) \cdot \left(\tilde{I}_n(-s)\right) + \left(\tilde{V}_n(-s)\right) \cdot \left(\tilde{I}_n(-s)\right)\right] \tag{2.10}
$$

Which for $s = j\omega$ must have

$$
\tilde{P}(j\omega) \geq 0 \tag{2.11}
$$

As a function of s (2.8) gives the analytic continuation into the *s* plane. We also note that

$$\tilde{P}(-s) = \tilde{P}(s) = \text{even function of } s \tag{2.12}$$

This can be cast in terms of the scattering matrix via

$$\begin{aligned}
8\,\tilde{P}(s) &= \left[\left(\tilde{V}_n(s)\right)_+ + \left(\tilde{V}_n(s)\right)_-\right] \cdot \left(Y_{n,m}^{(\text{ext})}\right) \cdot \left[\left(\tilde{V}_n(-s)\right)_+ - \left(\tilde{V}_n(-s)\right)_-\right] \\
&= \left[\left(\tilde{V}_n(-s)\right)_+ + \left(\tilde{V}_n(-s)\right)_-\right] \cdot \left(Y_{n,m}^{(\text{ext})}\right) \cdot \left[\left(\tilde{V}_n(s)\right)_+ - \left(\tilde{V}_n(-s)\right)_-\right] \\
4\,\tilde{P}(s) &= \left(\tilde{V}_n(s)\right)_+ \cdot \left(Y_{n,m}^{(\text{ext})}\right) \cdot \left(\tilde{V}_n(-s)\right)_+ - \left(\tilde{V}_n(s)\right)_- \cdot \left(Y_{n,m}^{(\text{ext})}\right) \cdot \left(\tilde{V}_n(-s)\right)_- \\
&= \left(\tilde{V}_n(s)\right)_+ \cdot \left[\left(Y_{n,m}^{(\text{ext})}\right) - \left(\tilde{S}_{n,m}(s)\right)^T \cdot \left(Y_{n,m}^{(\text{ext})}\right) \cdot \left(\tilde{S}_{n,m}(-s)\right)\right] \cdot \left(\tilde{V}_n(-s)\right)_+
\end{aligned} \tag{2.13}$$

This requires that we have the Hermitian matrix

$$\begin{aligned}
\left(\tilde{H}_{n,m}(j\omega)\right) &\equiv \left(Y_{n,m}^{(\text{ext})}\right) - \left(\tilde{S}_{n,m}^{(M)}(j\omega)\right)^T \cdot \left(Y_{n,m}^{(\text{ext})}\right) \cdot \left(\tilde{S}_{n,m}^{(M)}(-j\omega)\right) \\
&= \left(\tilde{H}_{n,m}(j\omega)\right)^\dagger \\
&= \left(Y_{n,m}^{(\text{ext})}\right) - \left(\tilde{S}_{n,m}^{(M)}(j\omega)\right)^T \cdot \left(Y_{n,m}^{(\text{ext})}\right) \cdot \left(\tilde{S}_{n,m}^{(M)}(-j\omega)\right)
\end{aligned} \tag{2.14}$$

Which is also termed self adjoint.

For a lossless network we have

$$\left(\tilde{H}_{n,m}(j\omega)\right) = \left(0_{n,m}\right) \tag{2.15}$$

And by analytic continuation

$$\left(\tilde{H}_{n,m}(s)\right) = \left(0_{n,m}\right) \tag{2.16}$$

Throughout the complex s plane. For such a lossless network we can rearrange (2.12) as

$$\left(1_{n,m}\right) = \left(Z_{n,m}^{(\text{ext})}\right) \cdot \left(\tilde{S}_{n,m}^{(\text{M})}(-s)\right)^T \cdot \left(Y_{n,m}^{(\text{ext})}\right) \cdot \left(\tilde{S}_{n,m}^{(\text{M})}(s)\right) \tag{2.17}$$

If we choose

$$\left(Z_{n,m}^{(\text{ext})}\right) = \text{constant times } \left(1_{n,m}\right) \tag{2.18}$$

then

$$\left(S_{n,m}(j\omega)\right) = \text{unitary matrix} \tag{2.19}$$

It is this last form that is often used in circuit theory [8].

More generally, let us try

$$\begin{aligned} \left(S_{n,m}^{(\text{M}')}(s)\right) &= \left(y_{n,m}^{(\text{ext})}\right) \cdot \left(\tilde{S}_{n,m}^{(\text{M})}(s)\right) \cdot \left(z_{n,m}^{(\text{ext})}\right) \\ \left(z_{n,m}^{(\text{ext})}\right) &\equiv \left(Z_{n,m}^{(\text{ext})}\right)^{1/2} \equiv \left(y_{n,m}^{(\text{ext})}\right)^{-1} \end{aligned} \tag{2.20}$$

Which in (2.14), gives

$$\begin{aligned} \left(1_{n,m}\right) &= \left(\tilde{S}_{n,m}^{(\text{M}')}(-s)\right)^T \cdot \left(\tilde{S}_{n,m}^{(\text{M}')}(s)\right) = \left(\tilde{S}_{n,m}^{(\text{M})}(s)\right) \cdot \left(\tilde{S}_{n,m}^{(\text{M}')}(-s)\right)^T \\ \left(\tilde{S}_{n,m}^{(\text{M}')}(j\omega)\right) &= \text{unitary matrix} \end{aligned} \tag{2.21}$$

This is a more general result than (2.18), (2.19), allowing various choices for $\left(Z_{n,m}^{(\text{ext})}\right)$.

Revisiting (2.5) we now have

$$\begin{aligned} \left(\tilde{S}_{n,m}^{(\text{M})}(s)\right) &= \left(z_{n,m}^{(\text{ext})}\right) \cdot \left(\tilde{S}_{n,m}^{(\text{M}')}(s)\right) \cdot \left(y_{n,m}^{(\text{ext})}\right) \\ \left[\left(y_{n,m}^{(\text{ext})}\right) \cdot \left(\tilde{V}_n(s)\right)_-\right] &= \left(\tilde{S}_{n,m}^{(\text{M}')}(s)\right) \cdot \left[\left(y_{n,m}^{(\text{ext})}\right) \cdot \left(\tilde{V}_n(s)\right)_+\right] \end{aligned} \tag{2.22}$$

Where the bracketed terms are the renormalized combined-voltage waves. This has the same form as (1.5) and following, allowing us to identify $\left(\tilde{S}_{n,m}^{(\text{M}')}(s)\right)$ with $\left(\tilde{S}_{n,m}^{(\text{R})}(s)\right)$ in Section 1.

Going back to (2.7) we have (with some manipulation)

$$
\begin{aligned}
\left(\tilde{S}_{n,m}^{(\mathrm{M}')}(s)\right) &= \left(y_{n,m}^{(\mathrm{ext})}\right) \cdot \left(\tilde{S}_{n,m}^{(\mathrm{M})}(s)\right) \cdot \left(z_{n,m}^{(\mathrm{ext})}\right) \\
&= \left[\left(y_{n,m}^{(\mathrm{ext})}\right) \cdot \left(\tilde{Z}_{n,m}^{(\mathrm{M})}\right) \cdot \left(y_{n,m}^{(\mathrm{ext})}\right) + \left(1_{n,m}\right)\right]^{-1} \\
&\quad \cdot \left[\left(y_{n,m}^{(\mathrm{ext})}\right) \cdot \left(\tilde{Z}_{n,m}^{(\mathrm{M})}\right) \cdot \left(y_{n,m}^{(\mathrm{ext})}\right) - \left(1_{n,m}\right)\right] \\
&= \left[\left(y^{(\mathrm{ext})}\right) \cdot \left(\tilde{Z}_{n,m}^{(\mathrm{M})}(s)\right) \cdot \left(y_{n,m}^{(\mathrm{ext})}\right) - \left(1_{n,m}\right)\right] \\
&\quad \cdot \left[\left(y_{n,m}^{(\mathrm{ext})}\right) \cdot \left(\tilde{Z}_{n,m}^{(\mathrm{M})}(s)\right) \cdot \left(y_{n,m}^{(\mathrm{ext})}\right) + \left(1_{n,m}\right)\right]^{-1} \\
&= \left[\left(1_{n,m}\right) + \left(z_{n,m}^{(\mathrm{ext})}\right) \cdot \left(\tilde{Y}_{n,m}^{(\mathrm{M})}(s)\right) \cdot \left(z_{n,m}^{(\mathrm{ext})}\right)\right]^{-1} \\
&\quad \left[\left(1_{n,m}\right) - \left(z_{n,m}^{(\mathrm{ext})}\right) \cdot \left(\tilde{Y}_{n,m}^{(\mathrm{M})}(s)\right) \cdot \left(z_{n,m}^{(\mathrm{ext})}\right)\right] \\
&= \left[\left(1_{n,m}\right) - \left(z_{n,m}^{(\mathrm{ext})}\right) \cdot \left(\tilde{Y}_{n,m}^{(\mathrm{M})}(s)\right) \cdot \left(z_{n,m}^{(\mathrm{ext})}\right)\right] \\
&\quad \cdot \left[\left(1_{n,m}\right) + \left(z_{n,m}^{(\mathrm{ext})}\right) \cdot \left(\tilde{Y}_{n,m}^{(\mathrm{M})}(s)\right) \cdot \left(z_{n,m}^{(\mathrm{ext})}\right)\right]^{-1}
\end{aligned}
\tag{2.23}
$$

Noting the commuting of the matrix combinations we also have

$$
\left(\tilde{S}_{n,m}^{(\mathrm{M}')}(s)\right) = \left(\tilde{S}_{n,m}^{(\mathrm{M}')}(s)\right)^{T} \quad \text{(symmetric)} \tag{2.24}
$$

As discussed in Appendix A, a lossless unitary symmetric matrix also has a complete set of M real eigenvectors with eigenvalues all of magnitude one.

The passive nature of the scattering Babanian matrix also gives [8]

$$
\begin{aligned}
&\left(1_{n,m}\right) - \left(\tilde{S}_{n,m}^{(\mathrm{M}')}(j\omega)\right)^{\dagger} \cdot \left(\tilde{S}_{n,m}'^{(\mathrm{M}')}(j\omega)\right) = \; = \text{positive semidefinite} \\
&1 - \sum_{n} \left|\tilde{S}_{n,m}^{(\mathrm{M}')}(j\omega)\right|^{2} \geq 0 \qquad \text{for each } m \\
&\left|\tilde{S}_{n,m}^{(\mathrm{M}')}(j\omega)\right| \leq 1
\end{aligned}
\tag{2.25}
$$

For a lossless network we also have

$$\sum_{n} \tilde{S}^{(\mathrm{M}')}_{n,m}(-j\omega)\, \tilde{S}^{(\mathrm{M}')}_{n',m}(j\omega) \;=\; 1_{n,n'} \tag{2.26}$$

Which extends into the complex plane as

$$\sum_{m} \tilde{S}^{(\mathrm{M}')}_{n,m}(-s)\, \tilde{S}_{n',m}(s) \;=\; 1_{n,n'} \tag{2.27}$$

As discussed in Appendix A, a lossless unitary symmetric matrix also has a complete set of M real eigenvectors with eigenvalues all of magnitude one.

3. Scattering Supermatrix for Section of NMTL

As discussed in Section 2 we need the scattering supermatirx in the form

$$
\begin{aligned}
\left(\left(\tilde{S}'_{n,m}(s)\right)_{\nu,\nu'}\right) &= \left(\left(y^{(\mathrm{ext})}_{n,m}\right)_{\nu,\nu'}\right) \cdot \left(\left(\tilde{S}'_{n,m}(s)\right)_{\nu,\nu'}\right) \cdot \left(\left(z^{(\mathrm{ext})}_{n,m}\right)_{\nu,\nu'}\right) \\
\left(\left(y^{(\mathrm{ext})}_{n,m}\right)_{\nu,\nu'}\right) &= \begin{pmatrix} \left(y^{(1)}_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(y^{(2)}_{n,m}\right) \end{pmatrix} = \left(\left(Y^{(\mathrm{ext})}_{n,m}\right)_{\nu,\nu'}\right)^{1/2} = \left(\left(z^{(\mathrm{ext})}_{n,m}\right)_{\nu,\nu'}\right) \\
\left(\left(z^{(\mathrm{ext})}_{n,m}\right)_{\nu,\nu'}\right) &= \begin{pmatrix} \left(z^{(1)}_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(z^{(2)}_{n,m}\right) \end{pmatrix} = \left(\left(Z^{(\mathrm{ext})}_{n,m}\right)_{\nu,\nu'}\right)^{1/2} \\
\left(Z^{(\nu)}_{n,m}\right) &= \left(Z^{(\nu)}_{n,m}\right)^{1/2} = \left(y^{(\nu)}_{n,m}\right)^{-1} \\
\left(y^{(\nu)}_{n,m}\right) &= \left(Y^{(\nu)}_{n,m}\right)^{1/2}
\end{aligned}
\tag{3.1}
$$

Since

$$
\left(\left(\tilde{S}'_{n,m}(s)\right)_{\nu,\nu'}\right) = \left(\left(\tilde{S}'_{n,m}(s)\right)_{\nu,\nu'}\right)^{T}
\tag{3.2}
$$

we have for the submatrices (each N x N)

$$
\begin{aligned}
\left(\tilde{S}'_{n,m}(s)\right)_{1,1} &= \left(\tilde{S}'_{n,m}(s)\right)^{T}_{1,1} \\
\left(\tilde{S}'_{n,m}(s)\right)_{2,2} &= \left(\tilde{S}'_{n,m}(s)\right)^{T}_{2,2} \\
\left(\tilde{S}'_{n,m}(s)\right)_{2,1} &= \left(\tilde{S}'_{n,m}(s)\right)^{T}_{1,2}
\end{aligned}
\tag{3.3}
$$

These are related to the other form of the scattering matrix as

$$
\left(\left(\tilde{S}'_{n,m}(s)\right)_{\nu,\nu'}\right) = \begin{pmatrix} \left(y^{(1)}_{n,m}\right) \cdot \left(\tilde{S}_{n,m}(s)\right)_{1,1} \cdot \left(z^{(1)}_{n,m}\right) & \left(y^{(1)}_{n,m}\right) \cdot \left(\tilde{S}_{n,m}(s)\right)_{1,2} \cdot \left(z^{(2)}_{n,m}\right) \\ \left(y^{(2)}_{n,m}\right) \cdot \left(\tilde{S}_{n,m}(s)\right)_{2,1} \cdot \left(z^{(1)}_{n,m}\right) & \left(y^{(2)}_{n,m}\right) \cdot \left(\tilde{S}_{n,m}(s)\right)_{1,1} \cdot \left(z^{(2)}_{n,m}\right) \end{pmatrix}
\tag{3.4}
$$

So the blocks are still relatively simple in form and have the symmetries given in (3.3). This relates renormalized wave variables as

$$\begin{pmatrix} \left(\mathrm{v}_n^{(1)}(0,s)\right)_- \\ \left(\tilde{\mathrm{v}}_n^{(2)}(\ell,s)\right)_+ \end{pmatrix} = \left(\left(\tilde{S}'_{n,m}(s)\right)_{\nu,\nu'} \right) \square \begin{pmatrix} \left(\tilde{\mathrm{v}}_n^{(1)}(0,s)\right)_+ \\ \left(\tilde{\mathrm{v}}_n^{(2)}(\ell,s)\right)_- \end{pmatrix}$$

$$\begin{pmatrix} \left(\mathrm{v}_n^{(1)}(0,s)\right)_- \\ \left(\tilde{\mathrm{v}}_n^{(2)}(\ell,s)\right)_+ \end{pmatrix} = \begin{pmatrix} \left(y_{n,m}^{(1)}\right) \square \left(\tilde{V}_n^{(1)}(0,s)\right)_- \\ \left(y_{n,m}^{(2)}\right) \square \left(\tilde{V}_n^{(2)}(\ell,s)\right)_+ \end{pmatrix} \tag{3.5}$$

$$\begin{pmatrix} \left(\tilde{\mathrm{v}}_n^{(1)}(0,s)\right)_+ \\ \left(\tilde{\mathrm{v}}_n^{(2)}(\ell,s)\right)_- \end{pmatrix} = \begin{pmatrix} \left(y_{n,m}^{(1)}\right) \square \left(\tilde{V}_n^{(1)}(0,s)\right)_+ \\ \left(y_{n,m}^{(2)}\right) \square \left(\tilde{V}_n^{(2)}(\ell,s)\right)_- \end{pmatrix}$$

The impedance supermatrix for the NMTL takes the form

$$\begin{pmatrix} \left(\tilde{V}_n(0,s)\right) \\ \left(\tilde{V}_n(\ell,s)\right) \end{pmatrix} = \left(\left(\tilde{Z}_{n,m}^{(N)}(s)\right)_{\nu,\nu'} \right) = \begin{pmatrix} \left(\tilde{I}_n(0,s)\right) \\ -\left(\tilde{I}_n(\ell,s)\right) \end{pmatrix}$$

$$\left(\left(\tilde{Z}_{n,m}^{(N)}(s)\right)_{\nu,\nu'} \right) = \left(\left(\tilde{Z}_{n,m}^{(N)}(s)\right)_{\nu,\nu'} \right)^T = \left(\left(\tilde{Y}_{n,m}^{(N)}(s)\right)_{\nu,\nu'} \right)^{-1} \tag{3.6}$$

= impedance supermatrix for the NMTL section

Note the use of $-(\tilde{I}_n(\ell,s))$ since we need currents *into* the section for impedance purposes.

There are various ways to calculate the $(\tilde{S}_{n,m}(s))_{\nu,\nu'}$ submatrices. For present illustration let us calculate the impedance or admittance supermatrix for (1.9), which gives

$$
\begin{pmatrix} \left(\tilde{V}_n\left(\ell,s\right)\right) \\ \left(Z_{n,m}\right) \cdot \left(\tilde{I}_n\left(\ell,s\right)\right) \end{pmatrix} = \left(\left(\tilde{G}_{n,m}\left(\ell,0;s\right)\right)_{\nu,\nu'}\right) \cdot \begin{pmatrix} \left(\tilde{V}_n\left(0,s\right)\right) \\ \left(Z_{n,m}\right) \cdot \left(\tilde{I}_n\left(0,s\right)\right) \end{pmatrix}
$$

$$
\begin{pmatrix} \left(\tilde{V}_n\left(0,s\right)\right) \\ \left(Z_{n,m}\right) \cdot \left(\tilde{I}_n\left(0,s\right)\right) \end{pmatrix} = \left(\left(\tilde{G}_{n,m}\left(0,\ell;s\right)\right)_{\nu,\nu'}\right) \cdot \begin{pmatrix} \left(\tilde{V}_n\left(\ell,s\right)\right) \\ \left(Z_{n,m}\right) \cdot \left(\tilde{I}_n\left(\ell,s\right)\right) \end{pmatrix} \tag{3.7}
$$

$$
\left(\left(\tilde{G}_{n,m}\left(0,\ell;s\right)\right)_{\nu,\nu'}\right) = \left(\left(\tilde{G}_{n,m}\left(\ell,0\right)\right)_{\nu,\nu'}\right)^{-1}
$$

Writing out the blocks we have

$$
\begin{aligned}
\left(\tilde{V}_n\left(\ell,s\right)\right) &= \left(\tilde{G}_{n,m}\left(\ell,0;s\right)\right)_{1,1} \cdot \left(\tilde{V}_n\left(0,s\right)\right) + \left(\tilde{G}_{n,m}\left(\ell,0;s\right)\right)_{1,2} \cdot \left(Z_{n,m}\right) \cdot \left(\tilde{I}_n\left(0,s\right)\right) \\
\left(\tilde{Z}_{n,m}\right) \cdot \left(\tilde{I}_n\left(\ell,s\right)\right) &= \left(\tilde{G}_{n,m}\left(\ell,0;s\right)\right)_{2,1} \cdot \left(\tilde{V}_n\left(0,s\right)\right) + \left(\tilde{G}_{n,m}\left(\ell,0;s\right)\right)_{2,2} \cdot \left(Z_{n,m}\right) \cdot \left(\tilde{I}_n\left(0,s\right)\right) \\
\left(\tilde{V}_n\left(0,s\right)\right) &= \left(\tilde{G}_{n,m}\left(0,\ell;s\right)\right)_{1,1} \cdot \left(\tilde{V}_n\left(\ell,s\right)\right) + \left(\tilde{G}_{n,m}\left(0,\ell;s\right)\right)_{1,2} \cdot \left(Z_{n,m}\right) \cdot \left(\tilde{I}_n\left(\ell,s\right)\right) \\
\left(Z_{n,m}\right) \cdot \left(\tilde{I}_n\left(0,s\right)\right) &= \left(\tilde{G}_{n,m}\left(0,\ell;s\right)\right)_{2,1} \cdot \left(\tilde{V}_n\left(\ell,s\right)\right) + \left(\tilde{G}_{n,m}\left(0,\ell;s\right)\right)_{2,2} \cdot \left(Z_{n,m}\right) \cdot \left(\tilde{I}_n\left(\ell,s\right)\right)
\end{aligned} \tag{3.8}
$$

From the first and third of these we have

$$
\begin{aligned}
\left(\tilde{V}_n\left(0,s\right)\right) = {} & \left(\tilde{G}_{n,m}\left(0,\ell;s\right)\right)_{1,1} \cdot \left(\tilde{G}_{n,m}\left(\ell,0;s\right)\right)_{1,1} \cdot \left(\tilde{V}_n\left(0,s\right)\right) \\
& + \left(\tilde{G}_{n,m}\left(0,\ell;s\right)\right)_{1,1} \cdot \left(\tilde{G}_{n,m}\left(\ell,0;s\right)\right)_{1,2} \cdot \left(\tilde{Z}_{n,m}\right) \cdot \left(\tilde{I}_n\left(0,s\right)\right) \\
& + \left(\tilde{G}_{n,m}\left(0,\ell;s\right)\right)_{1,2} \cdot \left(\tilde{Z}_{n,m}\right) \cdot \left(\tilde{I}_n\left(\ell,s\right)\right) \\
\left(\tilde{V}_n\left(\ell,s\right)\right) = {} & \left(\tilde{G}_{n,m}\left(\ell,0;s\right)\right)_{1,1} \cdot \left(\tilde{G}_{n,m}\left(0,\ell;s\right)\right)_{1,1} \cdot \left(\tilde{V}_n\left(\ell,s\right)\right) \\
& + \left(\tilde{G}_{n,m}\left(\ell,0;s\right)\right)_{1,2} \cdot \left(\tilde{Z}_{n,m}\right) \cdot \left(\tilde{I}_n\left(0,s\right)\right) \\
& + \left(\tilde{G}_{n,m}\left(\ell,0;s\right)\right)_{1,1} \cdot \left(\tilde{G}_{n,m}\left(0,\ell;s\right)\right)_{1,2} \cdot \left(\tilde{Z}_{n,m}\right) \cdot \left(\tilde{I}_n\left(\ell,s\right)\right)
\end{aligned} \tag{3.9}
$$

Defining

$$
\begin{aligned}
\left(\tilde{D}_{n,m}^{(0)}\left(\ell,s\right)\right) &\equiv \left(1_{n,m}\right) - \left(\tilde{G}_{n,m}\left(0,\ell;s\right)\right)_{1,1} \cdot \left(\tilde{G}_{n,m}\left(\ell,0;s\right)\right)_{1,1} \\
\left(\tilde{D}_{n,m}^{(\ell)}\left(\ell,s\right)\right) &\equiv \left(1_{n,m}\right) - \left(\tilde{G}_{n,m}\left(\ell,0;s\right)\right)_{1,1} \cdot \left(\tilde{G}_{n,m}\left(0,\ell;s\right)\right)_{1,1}
\end{aligned} \tag{3.10}
$$

we have the blocks of the impedance matrix as

$$
\begin{aligned}
\left(\tilde{Z}_{n,m}^{(\mathrm{N})}(s)\right)_{1,1} &= \left(\tilde{D}_{n,m}^{(0)}(\ell,s)\right)^{-1} \odot \left(\tilde{G}_{n,m}(0,\ell;s)\right)_{1,1} \odot \left(\tilde{G}_{n,m}(\ell,0;s)\right)_{1,2} \odot \left(Z_{n,m}\right) \\
\left(\tilde{Z}_{n,m}^{(\mathrm{N})}(s)\right)_{1,2} &= -\left(\tilde{D}_{n,m}^{(0)}(\ell,s)\right)^{-1} \odot \left(\tilde{G}_{n,m}(0,\ell;s)\right)_{1,2} \odot \left(Z_{n,m}\right) \\
\left(\tilde{Z}_{n,m}^{(\mathrm{N})}(s)\right)_{2,1} &= \left(\tilde{D}_{n,m}^{(\ell)}(\ell,s)\right)^{-1} \odot \left(\tilde{G}_{n,m}(\ell,0;s)\right)_{1,2} \odot \left(Z_{n,m}\right) \\
\left(\tilde{Z}_{n,m}^{(\mathrm{N})}(s)\right)_{2,2} &= -\left(\tilde{D}_{n,m}^{(\ell)}(\ell,s)\right)^{-1} \odot \left(\tilde{G}_{n,m}(\ell,0;s)\right)_{1,1} \odot \left(\tilde{G}_{n,m}(0,\ell;s)\right)_{1,2} \odot \left(Z_{n,m}\right)
\end{aligned}
\tag{3.11}
$$

Note that the submatrices with arguments $(0,\ell)$ can be found in terms of those with arguments $(\ell,0)$ (and conversely) from the formulae in Appendix B.

From the formulae in (2.7) this is converted to $((\tilde{S}_{n,m}(s))_{\nu,\nu'})$. From (3.4) this is converted to $((\tilde{S}'_{n,m}(s))_{\nu,\nu'})$. It is this last form which is unitary for $s = j\omega$ in the case of a lossless NMTL.

4. Junction of Two MTLs

If the length of the NMTL shrinks to zero, the formulae in (3.11) are not well behaved. For convenience, first let the NMTL be a uniform MTL so that

$$\left(\left(\tilde{G}_{n,m}(z,0;s)\right)_{\nu,\nu'}\right) = e^{\left(\left(\tilde{\Gamma}_{n,m}(0,s)\right)_{\nu,\nu'}\right)z} \tag{4.1}$$

And all derivatives of $((\tilde{\Gamma}_{n,m}(z,s))_{\nu,\nu'})$ are zero. The matrizant differential equation is

$$\frac{d}{dz}\left(\left(\tilde{G}_{n,m}(z,0;s)\right)_{\nu,\nu'}\right) = \left(\left(\tilde{\Gamma}_{n,m}(0,s)\right)_{\nu,\nu'}\right) \cdot \left(\left(\tilde{G}_{n,m}(z,0;s)\right)_{\nu,\nu'}\right) \tag{4.2}$$

From which we can evaluate derivatives recursively as

$$\frac{d^{p}}{dz^{p}}\left(\left(\tilde{G}_{n,m}(z,0;s)\right)_{\nu,\nu'}\right) = \left(\left(\tilde{\Gamma}_{n,m}(0,s)\right)_{\nu,\nu'}\right) \cdot \frac{d^{p-1}}{dz^{p-1}}\left(\left(\tilde{G}_{n,m}(z,0;s)\right)_{\nu,\nu'}\right) \tag{4.3}$$

Expanding the exponential matrix we have

$$\left(\left(\tilde{G}_{n,m}(z,0;s)\right)_{\nu,\nu'}\right) = \left(\left(1_{n,m}\right)_{\nu,\nu'}\right) = \sum_{p=1}^{\infty}\frac{1}{p!}\left(\left(\tilde{\Gamma}_{n,m}(0,s)\right)_{\nu,\nu'}\right)^{p} z^{p} \tag{4.4}$$

Noting that $((\tilde{\Gamma}_{n,m}(0,s))_{\nu,\nu'})$ has zero blocks on the diagonal, then this property holds for all odd p. Hence we can write

$$\begin{aligned}
&\left(\left(\tilde{G}_{n,m}(z,0;s)\right)_{1,1}\right) = \left(1_{n,m}\right) + \frac{1}{2}\left(\tilde{\gamma}_{n,m}(s)\right)^{2} z^{2} = O\left(z^{4}\right) \\
&\left(\tilde{\gamma}_{n,m}(s)\right)^{2} = \left(\tilde{Z}'_{n,m}(0,s)\right) \cdot \left(\tilde{Y}'_{n,m}(0,s)\right) = s^{2}\left(\tilde{L}'_{n,m}(0)\right) \cdot \left(\tilde{Y}'_{n,m}(0)\right) \text{ for usual lossless NMTL}
\end{aligned} \tag{4.5}$$

We also have for the off-diagonal block

$$\left(\tilde{G}_{n,m}(z,0;s)\right)_{1,2} = -\left(\tilde{Z}'_{n,m}(0,s)\right) \cdot \left(Y_{n,m}\right) z + O\left(z^{3}\right) \tag{4.6}$$

Substituting, we find from (3.10) and (3.11)

$$
\begin{aligned}
\left(\tilde{D}^{(0)}_{n,m}(\ell,s) \right) &= \left(1_{n,m} \right) - \left(\tilde{G}_{n,m}(0,\ell;s) \right)_{1,1} \cdot \left(\tilde{G}_{n,m}(\ell,0;s) \right)_{1,1} \\
&= -\left(\tilde{\gamma}_{n,m}(s) \right)^2 \ell^2 + \mathrm{O}\left(\ell^4 \right) = \tilde{D}^{(\ell)}_{n,m}(\ell,s) \\
\left(\left(Z^{(N)}_{n,m}(s) \right)_{\nu,\nu'} \right) &= -\left[\left(\tilde{\gamma}_{n,m}(s) \right)^2 \ell^2 + \mathrm{O}\left(\ell^4 \right) \right]^{-1} \cdot \left[\left(\tilde{Z}'_{n,m}(0,s) \right) \ell + \mathrm{O}\left(\ell^3 \right) \right] \otimes \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \\
&= \ell^{-1} \left(\tilde{Y}'_{n,m}(0,s) \right)^{-1} \otimes \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \mathrm{O}(\ell)
\end{aligned}
\tag{4.7}
$$

As we can see, the leading term is a singular supermatrix [4 (Appendix A)], introducing some difficulties in taking the limit as $\ell \to 0$.

Fortunately, we can directly solve for the scattering matrix for a zero-length section by imposing continuity of voltage and current through this junction, as

$$
\left(\tilde{V}^{(1)}_{n}(0,s) \right) = \left(\tilde{V}^{(2)}_{n}(0,s) \right), \quad \left(\tilde{I}^{(1)}_{n}(0,s) \right) = \left(\tilde{I}^{(2)}_{n}(0,s) \right)
\tag{4.8}
$$

The scattering matrix is defined by

$$
\begin{pmatrix}
\left(\tilde{V}^{(1)}_{n}(0,s) \right) - \left(Z^{(1)}_{n,m} \right) \cdot \left(\tilde{I}^{(1)}_{n}(0,s) \right) \\
\left(\tilde{V}^{(2)}_{n}(0,s) \right) + \left(\tilde{Z}^{(2)}_{n,m} \right) \cdot \left(\tilde{I}^{(2)}_{n}(0,s) \right)
\end{pmatrix}
= \left(\left(S_{n,m} \right)_{\nu,\nu'} \right) \cdot
\begin{pmatrix}
\left(\tilde{V}^{(1)}_{n}(0,s) \right) + \left(Z^{(1)}_{n,m} \right) \cdot \left(\tilde{I}^{(1)}_{n}(0,s) \right) \\
\left(\tilde{V}^{(2)}_{n}(0,s) \right) - \left(\tilde{Z}^{(2)}_{n,m} \right) \cdot \left(\tilde{I}^{(2)}_{n}(0,s) \right)
\end{pmatrix}
\tag{4.9}
$$

where it is now frequency independent. By setting the wave incident from the right to zero we find

$$
\begin{aligned}
\left(S_{n,m} \right)_{1,1} &= \left[\left(Z^{(2)}_{n,m} \right) - \left(Z^{(1)}_{n,m} \right) \right] \cdot \left[\left(Z^{(2)}_{n,m} \right) + \left(Z^{(1)}_{n,m} \right) \right]^{-1} \\
\left(S_{n,m} \right)_{2,1} &= 2 \left(Z^{(2)}_{n,m} \right) \cdot \left[\left(Z^{(2)}_{n,m} \right) + \left(Z^{(1)}_{n,m} \right) \right]^{-1}
\end{aligned}
\tag{4.10}
$$

Similarly, by setting the wave incident from the left to zero we find

$$
\begin{aligned}
\left(S_{n,m} \right)_{1,2} &= 2 \left(Z^{(1)}_{n,m} \right) \cdot \left[\left(Z^{(2)}_{n,m} \right) + \left(Z^{(1)}_{n,m} \right) \right]^{-1} \\
\left(S_{n,m} \right)_{2,2} &= \left[\left(Z^{(1)}_{n,m} \right) - \left(Z^{(2)}_{n,m} \right) \right] \cdot \left[\left(Z^{(2)}_{n,m} \right) + \left(Z^{(1)}_{n,m} \right) \right]^{-1}
\end{aligned}
\tag{4.11}
$$

From (3.4) we have

$$
\begin{aligned}
\left(S'_{n,m}\right)_{1,1} &= \left[\left(y^{(1)}_{n,m}\right)\cdot\left(Z^{(2)}_{n,m}\right)\cdot\left(y^{(1)}_{n,m}\right)-\left(1_{n,m}\right)\right]\cdot\left[\left(y^{(1)}_{n,m}\right)\cdot\left(Z^{(2)}_{n,m}\right)\cdot\left(y^{(1)}_{n,m}\right)+\left(1_{n,m}\right)\right]^{-1} \\
&= \left[\left(y^{(1)}_{n,m}\right)\cdot\left(Z^{(2)}_{n,m}\right)\cdot\left(y^{(2)}_{n,m}\right)+\left(1_{n,m}\right)\right]^{-1}\cdot\left[\left(y^{(1)}_{n,m}\right)\cdot\left(Z^{(2)}_{n,m}\right)\cdot\left(y^{(1)}_{n,m}\right)-\left(1_{n,m}\right)\right] \\
\left(S'_{n,m}\right)_{1,2} &= 2\left[\left(z^{(2)}_{n,m}\right)\cdot\left(y^{(1)}_{n,m}\right)+\left(y^{(2)}_{n,m}\right)\cdot\left(z^{(1)}_{n,m}\right)\right]^{-1} \\
\left(S'_{n,m}\right)_{2,1} &= 2\left[\left(y^{(1)}_{n,m}\right)\cdot\left(z^{(2)}_{n,m}\right)+\left(z^{(1)}_{n,m}\right)\cdot\left(y^{(2)}_{n,m}\right)\right]^{-1} \\
\left(S'_{n,m}\right)_{2,2} &= \left[\left(y^{(2)}_{n,m}\right)\cdot\left(Z^{(1)}_{n,m}\right)\cdot\left(y^{(2)}_{n,m}\right)-\left(1_{n,m}\right)\right]\cdot\left[\left(y^{(2)}_{n,m}\right)\cdot\left(Z^{(1)}_{n,m}\right)\cdot\left(y^{(2)}_{n,m}\right)+\left(1_{n,m}\right)\right]^{-1} \\
&= \left[\left(y^{(2)}_{n,m}\right)\cdot\left(Z^{(1)}_{n,m}\right)\cdot\left(y^{(2)}_{n,m}\right)+\left(1_{n,m}\right)\right]^{-1}\cdot\left[\left(y^{(2)}_{n,m}\right)\cdot\left(Z^{(1)}_{n,m}\right)\cdot\left(y^{(2)}_{n,m}\right)-\left(1_{n,m}\right)\right]
\end{aligned}
\tag{4.12}
$$

Evidently $((S'_{n,m})_{\nu,\nu'})$ is symmetric and real. From previous discussion it is also unitary.

5. Concluding Remarks

Scattering matrices can be formed in various ways, depending on the definitions of the incoming and outgoing waves. The traditional way involving volts ± current is not convenient when dealing with a length of NMTL. In this case it is convenient to normalize the current by an appropriate characteristic impedance matrix. However, the resulting scattering matrix for a lossless section of NMTL is not unitary.

By a renormalization procedure involving the square root of characteristic admittance and impedance matrices the resulting scattering matrix $((\tilde{S}'_{n,m}(s))_{\nu,\nu'})$ a lossless NMTL section is again unitary. As a special case we find that an NMTL of zero length has a real, symmetric, unitary form.

So now we have some general results for lossless NMTLs which are independent of the variation of $(L'_{n,m}(z))$ and $(C'_{n,m}(z))$ in the section of interest.

Appendix A. Unitary Matrices

A.1. General Case

Summarizing from [9], unitary N x N matrices have the equivalent statements:

(a) $(U_{n,m})$ is unitary

(b) $(U_{n,m})$ is nonsingular and $(U_{n,m})^{\dagger} = (U_{n,m})^{-1}$

(c) $(U_{n,m}) \cdot (U_{n,m})^{\dagger} = \left(1_{n,m}\right) = (U_{n,m})^{\dagger} \cdot (U_{n,m})$

(d) $(U_{n,m})^{\dagger}$ is unitary (adjoint $\dagger = T*$)

(e) columns of $(U_{n,m})$ are biorthonormal

(f) rows of $(U_{n,m})$ are biorthonormal (A.1)

(g) $\left(y_n\right) = (U_{n,m}) \cdot \left(x_n\right) \Rightarrow \left|\left(y_n\right)\right| = \left|\left(x_n\right)\right|$ (length preserving)

Stated in another way we have

$$\left(r_n^{(m)}\right) \equiv m\text{th row as a vector}$$

$$\left(r_n^{(m_1)}\right)^* \cdot \left(r_n^{(m_2)}\right) = 1_{m_1,m_2} \quad \text{(biorthonormal)}$$

$$\left(c_m^{(n)}\right) \equiv n\text{th column as a vector} \tag{A.2}$$

$$\left(c_m^{(n_1)}\right)^* \cdot \left(c_m^{(n_2)}\right) = 1_{n_1,n_2} \quad \text{(biorthonormal)}$$

Write an eigenvector equation as

$$u_\beta \left(x_n\right)_\beta = \left(U_{n,m}\right) \cdot \left(x_n\right)_\beta$$

$$u_\beta \left(y_n\right)_\beta = \left(y_n\right)_\beta \cdot \left(U_{n,m}\right) = \left(U_{n,m}\right)^T \cdot \left(y_n\right)_\beta$$

$$\left(x_n\right)_\beta \cdot \left(y_n\right)_{\beta_2} = 1_{\beta_1\beta_2} \quad \text{(biorthonormal)} \tag{A.3}$$

$$u_\beta^* \left(y_n\right)_\beta^* = \left(U_{n,m}\right)^{\dagger} \cdot \left(y_n\right)_\beta^*$$

$$u_\beta^* \left(U_{n,m}\right) \cdot \left(y_n\right)_\beta^* = \left(y_n\right)_\beta^*$$

From this we identify

$$
\begin{aligned}
\left(y_n\right)_\beta^* &= \left(x_n\right)_\beta \\
u_\beta^* &= u_\beta^{-1} \ , \ \ u_\beta^* u_\beta = 1 \\
\left|u_\beta\right| &= 1
\end{aligned}
\tag{A.4}
$$

So that all eigenvalues have unit magnitude (a not-usually-stated property).

We can now write a dyadic expansion as

$$
\left(U_{n,m}\right) = \sum_{\beta=1}^{N} u_\beta \ \left(x_n\right)_\beta \left(y_n\right)_\beta \tag{A.5}
$$

Noting that a unitary matrix is a special case of a normal matrix (one which commutes with its adjoint) then it also has a complex set of N independent eigenvectors [7], making (A.5) always possible.

A.2. Symmetric Unitary Matrices

Now let

$$
\begin{aligned}
\left(U_{n,m}\right)^T &= \left(U_{n,m}\right) \\
\left(U_{n,m}\right)^\dagger &= \left(U_{n,m}\right)^*
\end{aligned}
\tag{A.6}
$$

From the eigenvector equation (A.3) we have

$$
\begin{aligned}
u_\beta \left(x_n\right)_\beta &= \left(U_{n,m}\right) \cdot \left(x_n\right)_\beta = \left(x_n\right)_\beta \cdot \left(U_{n,m}\right) \\
\left(x_n\right)_\beta &= \left(y_n\right)_\beta \\
\left(x_n\right)_{\beta_1} \cdot \left(y_n\right)_{\beta_2} &= 1_{\beta_1,\beta_2} \qquad \text{(orthonormal)}
\end{aligned}
\tag{A.7}
$$

the left and right eigenvectors being the same (with scaling constant taken as unity).

Now from (A.4) we have

$$\left(y_n\right)_\beta = \left(x_n\right)_\beta^* = \left(x_n\right)_\beta \tag{A.8}$$

We have that

$$\left(x_n\right)_\beta = \text{real vector} \tag{A.9}$$

Consistent with our choice of scaling constant (the orthonormal requirement).

The dyadic expansion now takes the form

$$\left(U_{n,m}\right) = \sum_{\beta=1}^{N} u_\beta \left(x_n\right)_\beta \left(x_n\right)_\beta \tag{A.10}$$

Note that the u_β are still, in general, complex.

Appendix B. Supermatrix Inverse

The inverse of a supermatrix is given in terms of its submatrices (blocks). This can be found in various places [3, 7, 9]. We have

$$\left(\left(B_{n,m} \right)_{\nu,\nu'} \right) = \left(\left(A_{n,m} \right)_{\nu,\nu'} \right)^{-1} \ , \ \nu,\nu' = 1,2 \tag{B.1}$$

where these matrices are in symmetric compatable order (same division of both rows and columns). The diagonal blocks (1,1 and 2,2) are square. The submatrices of the inverse supermatrix are given by

$$\begin{aligned}
\left(B_{n,m} \right)_{1,1} &= \left[\left(A_{n,m} \right)_{1,1} - \left(A_{n,m} \right)_{1,2} \cdot \left(A_{n,m} \right)_{2,2}^{-1} \cdot \left(A_{n,m} \right)_{2,1} \right]^{-1} \\
\left(B_{n,m} \right)_{2,2} &= \left[\left(A_{n,m} \right)_{2,2} - \left(A_{n,m} \right)_{2,1} \cdot \left(A_{n,m} \right)_{1,1}^{-1} \cdot \left(A_{n,m} \right)_{1,2} \right]^{-1} \\
\left(B_{n,m} \right)_{1,2} &= \left[\left(A_{n,m} \right)_{1,1} - \left(A_{n,m} \right)_{1,2} \cdot \left(A_{n,m} \right)_{2,2}^{-1} \cdot \left(A_{n,m} \right)_{2,1} \right]^{-1} \cdot \left(A_{n,m} \right)_{1,2} \cdot \left(A_{n,m} \right)_{2,2}^{-1} \\
&= - \left(A_{n,m} \right)_{1,1}^{-1} \cdot \left(A_{n,m} \right)_{1,2} \cdot \left[\left(A_{n,m} \right)_{2,2} \cdot \left(A_{n,m} \right)_{2,1} \cdot \left(A_{n,m} \right)_{1,1}^{-1} \cdot \left(A_{n,m} \right)_{1,2} \right]^{-1} \\
\left(B_{n,m} \right)_{2,1} &= - \left[\left(A_{n,m} \right)_{2,2} - \left(A_{n,m} \right)_{2,1} \cdot \left(A_{n,m} \right)_{1,1}^{-1} \cdot \left(A_{n,m} \right)_{1,2} \right]^{-1} \cdot \left(A_{n,m} \right)_{2,1} \cdot \left(A_{n,m} \right)_{1,1}^{-1} \\
&= - \left(A_{n,m} \right)_{2,2}^{-1} \cdot \left(A_{n,m} \right)_{2,1} \cdot \left[\left(A_{n,m} \right)_{1,1} \cdot \left(A_{n,m} \right)_{1,2} \cdot \left(A_{n,m} \right)_{2,2}^{-1} \cdot \left(A_{n,m} \right)_{2,1} \right]^{-1}
\end{aligned} \tag{B.2}$$

If, in addition, the supermatrix is symmetric we have

$$\begin{aligned}
\left(\left(A_{n,m} \right)_{\nu,\nu'} \right) &= \left(\left(A_{n,m} \right)_{\nu,\nu'} \right)^{T} \\
\left(\left(B_{n,m} \right)_{\nu,\nu'} \right) &= \left(\left(B_{n,m} \right)_{\nu,\nu'} \right)^{T}
\end{aligned} \tag{B.3}$$

Which in terms of the submatrices gives

$$\begin{aligned}
\left(A_{n,m} \right)_{1,1} = \left(A_{n,m} \right)_{1,1}^{T} \ , \quad \left(A_{n,m} \right)_{2,2} = \left(A_{n,m} \right)_{2,2}^{T} \ , \quad \left(A_{n,m} \right)_{2,2} = \left(A_{n,m} \right)_{1,2}^{T} \\
\left(B_{n,m} \right)_{1,1} = \left(B_{n,m} \right)_{1,1}^{T} \ , \quad \left(B_{n,m} \right)_{2,2} = \left(B_{n,m} \right)_{2,2}^{T} \ , \quad \left(B_{n,m} \right)_{2,1} = \left(B_{n,m} \right)_{1,2}^{T}
\end{aligned} \tag{B.4}$$

The reader can take the transposes in (B.1) to verify these relations.

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