Interaction Notes

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Propagation of Current Waves along Quasi-Periodical Thin-Wire Structures: Accounting of Radiation Losses

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Abstract

The homogeneous problem of current propagation along a thin wire of arbitrary geometric form near ground is reduced use of the Full-Wave Transmission Line Theory [8-10] to a Schrödinger-like differential equation, with a "potential" depending on both the geometry of the wire and frequency. The "potential" is a complex – valued quantity that corresponds to either radiation losses in the framework of electrodynamics or to the absorption of particles in the framework of quantum mechanics. If the wire structure is quasi periodical, i.e., it consists of a finite number of identical sections, the "potential" can be approximately represented as a set of periodically arranged identical potentials. We use the formalism of transfer matrices and find an analytical expression for the transmission coefficient of the finite number of periodically located non-uniformities which also contains the scattering data of one non-uniformity. The obtained result yields the possibility to investigate forbidden and allowed frequency zones which are a typical feature of periodic structure.

Content

I. Introduction

An analysis of propagation of current waves along periodical structures in different radio-technical and electro-technical applications becomes necessary. These periodical structures show non-trivial electrodynamical properties and sometimes can be used to build filters, antennas and HPM sources. Moreover, because of the simplicity of building periodical thin-wire structures, they can serve as elements for the design of meta-materials.

The periodical infinite structures are studied in a number of papers [1,2] from an electrodynamical point of view. The propagation of current along infinite periodical transmission lines also can be applied to the periodical excitation of mechanical systems (parametrical resonances) [3]. However, in real life one deals with quasi-periodical systems: systems, which consist of a finite number of identical sections.

In the previous papers [4,25] we investigated the propagation of current waves along a wire without any losses (radiation, ohmic, dielectric, etc.). We transformed the equations of non-uniform transmission line theory (with length dependent inductance and capacitance per-unit length) to a second order differential equation for some auxiliary function, which is simply connected with the current. The equation looks like a usual one-dimensional Schrödinger equation in quantum mechanics, which potential can be calculated using the per-unit length transmission line parameters. The "potential" decays to zero at plus and minus infinity and the "energy" of the "particle" is positive, thus we deal with a one-dimensional quantum mechanical scattering problem [5] and can apply powerful and well-developed mathematical methods to investigate such problems. For the quasi-periodical wiring system the potential is also quasi-periodical. Using the formalism of the transfer matrix [6] we find an analytical expression for the transmission coefficient of the finite number of periodically located non-uniformities which also contains the scattering data for one non-uniformity. Depending on the frequency the absolute value of the transmission coefficient oscillates. This corresponds to forbidden and allowed frequency zones which are typical for periodical structures [7].

However, the usually used wiring systems have different kinds of losses: dielectric losses, ohmic losses, and radiation losses. In the present paper, we generalized the formalism [4,25] for such systems. We consider quasi-periodical structures, which consist of a wire with finite conductivity coated by lossy dielectric insulation. For this system, for high frequencies, when radiation losses can become substantial, the simple approach to a non-uniform transmission line used in [4] is not applicable, and we have to apply a general Full Wave Transmission Line theory (FWTL) [8,9,10], which has to be modified for the case of dielectric and ohmic losses.

From the first step in the second Section of this report, we obtain boundary conditions for the first and second Mixed Potential Integral Equation (MPIE), as for the potential, as well as for the tangential electrical field on the boundary of the wire. Then we use the obtained MPIE and general techniques described in [11] to derive simple analytical expressions for the global parameters of the Full-Wave Transmission Line theory. Using the global parameters the FWLT equations are reduced to the Schrödinger-like equation for some auxiliary function connected with the current in a simple way.

However, now the potential in the Schrödinger-like equation is, in contrast to the lossless case, complex-valued. The explicit connection between the losses and the imaginary component of the potential is established at the beginning of the third Section. Hereafter, we generalize the formalism of the transfer matrix for lossy systems and derive an analytical expression for the transmission coefficient of the finite number of periodically located lossy non-uniformities, which also contains the scattering data for one non-uniformity. Next, the

connection between the transformation parameters of quasi-periodical systems and parameters of corresponding periodical systems (quasi-pulse) is established.

In the fourth Section, we consider a numerical example of the quasi-periodical line with ohmic, dielectric and radiation losses. It will be shown, that the zone structure, which exists in the lossless case holds also for the lossy case. Again, in dependence on the frequency the absolute value of the total propagation coefficient through the chain oscillates. That corresponds to forbidden and allowed frequency zones. For the considered example, both dielectric and ohmic losses are negligible; however, the radiation losses decrease essentially the propagation coefficient through the chain for the allowed frequency zones.

In conclusion, we formulate some directions for future investigations.

2. Equation for the current in a wiring system with losses.

2.1. Mixed-Potential Integral Equations (MPIE) for the wiring system with losses.

In this Section, we obtain the Mixed-Potential Integral Equations (MPIE) for the coated wire with ohmic losses, which later will be a basis for the evaluation of the FWLT equations with global parameters. The system of MPIE for a perfectly conducting uncoated thin wire consists of two integral equations [11] which couple the total tangential current induced in the wire, and the potential (we use the Lorenz gauge) on the boundary of the wire. The first equation is a zero boundary condition for the tangential component of the total electric field (exciting external field plus the scattered field generated by the induced current). The second one is just an integral expression for the scalar potential in the Lorenz gauge in terms of the total tangential current induced in the wire. It is obvious, that both of them have to be changed for the coated wire with finite conductivity. Because these effects turn out to be small, we will treat them separately: i.e., for the first equation we consider an uncoated wire with a finite conductivity, and for the second one we consider the coated perfectly conducting wire.

2.1.1 The second MPIE equation: Boundary condition for the coated wire.

In this Section, we would like to obtain an expression for the potential on the boundary of the metallic kernel of the coated wire if the potential on the boundary of the uncoated wire is known. Let us consider the scattering by a thin dielectrically coated wire. A uniform coating of thickness $(b-a)$ is placed over a perfectly conducting wire with radius a. We assume, that the wire is thin and smooth, namely that $b << \lambda, 1/\mathcal{K}$, where λ is the wavelength of the exciting electromagnetic field and $\mathcal{K}$ is the curvature of the line along the wire axis. Under such conditions to establish the connection between potentials on the boundary of coated and non-coated wires within the neighborhood of the wire ($\rho, |l-l'| << \lambda, 1/\mathcal{K}$, where ρ is the distance from the wire in the local coordinate system, l is the length along wire axis), one can consider an electrostatic problem for the straight wire (see Fig. 1).

This electrostatic problem can be formulated as follows. There is a distributed charge on the boundary of the wire (with density per unit square $q_s = q/2\pi a$ which corresponds to the density per-unit length q). In the first case, the wire is uncoated, in the second case the wire is coated by a dielectric layer with dielectric permittivity ε. The task is to find the difference between the potentials on the boundary of the wire φ^0 and φ^ε.
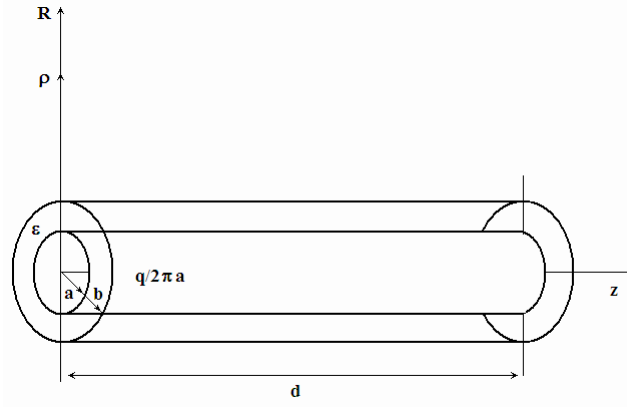
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Fig. 2.1: The electrostatic problem for a coated wire.

This problem can be solved using Gauss´ theorem in electrostatics. We consider a section of the cylinder with length d and some auxiliary cylindrical shell with radius ρ. According to Gauss´ theorem (where q_V is the volume density of charge)

$$\int\limits_V div\,\vec{D}\,dV = \int\limits_V q_V\,dV = \int\limits_S q_S\,dS = \int\limits_0^d qdz = qd \qquad (2.1.1.1)$$

taking into account that $\int\limits_V div\,\vec{D}\,dV = \int\limits_S \vec{D}\,d\vec{S} = 2\pi\rho dD$ $\qquad (2.1.1.2)$

(where an obvious cylindrical symmetry of the electrical field distribution is used) one can obtain an equation for the electrical displacement

$$D = \frac{q}{2\pi\rho} \qquad (2.1.1.3)$$

and for the electrical fields, both, in the first and second case:

$$E_\rho^0 = \frac{q}{2\pi\varepsilon_0\rho}, \qquad a < \rho \qquad (2.1.1.4)$$

$$E_\rho^\varepsilon = \begin{cases} \dfrac{q}{2\pi\varepsilon_0\varepsilon\rho}, & a < \rho < b \\[2ex] \dfrac{q}{2\pi\varepsilon_0\rho}, & \rho > b \end{cases} \qquad (2.1.1.5)$$

If now we choose as reference point for the potential some point R ($b << R << \lambda, 1/K$) we can write for the potentials on the boundary of the wire in both cases and for their difference:

$$\varphi^0 = -\int\limits_R^a E_\rho^0 d\rho \qquad (2.1.1.6)$$

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$$\varphi^\varepsilon = -\int\limits_R^a E_\rho^\varepsilon d\rho \qquad (2.1.1.7)$$

$$\varphi^\varepsilon - \varphi^0 = -\int\limits_R^a (E_\rho^\varepsilon - E_\rho^0) d\rho = \int\limits_b^a \left(\frac{q}{2\pi\varepsilon\varepsilon_0\rho} - \frac{q}{2\pi\varepsilon_0\rho} \right) d\rho = -\frac{q}{2\pi\varepsilon_0} \frac{\varepsilon-1}{\varepsilon} \ln\left(\frac{b}{a}\right) \qquad (2.1.1.8)$$

$$\text{or} \qquad \varphi^\varepsilon = \varphi^0 - \frac{q}{2\pi\varepsilon_0} \frac{\varepsilon-1}{\varepsilon} \ln\left(\frac{b}{a}\right) \qquad (2.1.1.9)$$

If we now return to the initial electrodynamic case, we can consider the potential φ^0 as the potential of the uncoated wire, and use the expression for the charge per-unit length which follows from the continuity equation

$$q = \frac{j}{\omega} \frac{\partial I(l)}{\partial l} \qquad (2.1.1.10)$$

we can obtain the desired connection between the potentials:

$$\varphi^\varepsilon(l) = \varphi^0(l) + \frac{1}{j\omega\widetilde{C}'} \frac{\partial I(l)}{\partial l} \qquad (2.1.1.11)$$

where some auxiliary capacitance is introduced

$$\widetilde{C}' = \frac{2\pi\varepsilon_0\varepsilon}{\varepsilon-1} \frac{1}{\ln(b/a)} \qquad (2.1.1.12)$$

Now we consider a thin wire of arbitrary geometric form, $\vec{r}(l)$ (where l is the coordinate of the wire's axis), near the perfectly conducting ground, which is excited by an external field $\vec{E}^i(\vec{r})$. Assuming, as usual in the thin-wire approximation, that the current $I(l)$ flows along the wire axis, and using the continuity equation (2.1.1.10) we can obtain the following equation for the scalar potential (in the Lorenz gauge) on the boundary of the uncoated wire:

$$\varphi^0(l) = -\frac{1}{j\omega 4\pi\varepsilon_0} \int\limits_0^L g_I^C(l,l') \frac{\partial I(l')}{\partial l'} dl' \qquad (2.1.1.13)$$

Here $L = 1/2 \oint dl'$ is half of the length of the complete closed loop. The function $g_C^I(l,l')$ is the Green's function along the curved line for the scalar potential, which takes into account the reflection of the ground plane:

$$g_I^C(l,l') = \frac{e^{-jk\sqrt{[\vec{r}(l)-\vec{r}(l')]^2+a^2}}}{\sqrt{[\vec{r}(l)-\vec{r}(l')]^2+a^2}} - \frac{e^{-jk\sqrt{[\vec{r}(l)-\vec{\tilde{r}}(l')]^2+a^2}}}{\sqrt{[\vec{r}(l)-\vec{\tilde{r}}(l')]^2+a^2}} \qquad (2.1.1.14)$$

Here $\vec{\tilde{r}}$ is the radius vector of the axis mirrored by the ground plane, and a is the radius of the wire.

Using (2.1.14) and (2.1.1.11) we can write the desired second MPIE equation for the potential of the coated wire

$$\int_0^L g_I^C(l,l') \frac{\partial I(l')}{\partial l'} dl' - \frac{4\pi\varepsilon_0}{\widetilde{C}'} \frac{\partial I(l)}{\partial l} + 4\pi\varepsilon_0 j\omega\varphi^\varepsilon(l) = 0 \qquad (2.1.1.15)$$

Here the auxiliary capacitance $\widetilde{C}'$ is defined by eq. (2.1.1.12) and contains the dielectric permittivity ε. If the lossy dielectric is considered, the second term in equation (2.1.1.15) is responsible for the dielectric losses.

2.1.2 The first MPIE equation: Boundary condition for the finite conductive wire.

Again we consider a thin wire of arbitrary geometric form near the perfectly conducting ground. It is excited by an external field $\vec{E}^i(\vec{r})$. In the previous consideration for the perfectly conducting wire [11], we assumed that the total (initial plus scattered) tangential electrical field on the boundary of the wire is zero, which lead to the first MPIE equation

$$E_l^{tot}(l) = \vec{e}_l(l)\left(\vec{E}^i(l) + \vec{E}^{sc}(l)\right) = E_l^i(l) - j\omega A_l(l) - \frac{\partial\varphi(l)}{\partial l} = 0 \qquad (2.1.2.1)$$

where E_l^i and A_l are tangential components of the exciting fields and vector potential respectively. Using the expression for the vector potential one can re-write (2.1.2.1) as

$$\frac{\partial\varphi(l)}{\partial l} + j\omega\frac{\mu_0}{4\pi}\int_0^L g_I^L(l,l')I(l')dl' = E_l^i(l) \qquad (2.1.2.2)$$

where the function $g_I^L(l,l')$ is the Green's function for the tangential component of the vector potential in the Lorenz gauge along the curved line, which takes into account the reflection of the ground plane:

$$g_I^L(l,l') = \vec{e}_l(l)\cdot\vec{e}_l(l')\frac{e^{-jk\sqrt{[\vec{r}(l)-\vec{r}(l')]^2+a^2}}}{\sqrt{[\vec{r}(l)-\vec{r}(l')]^2+a^2}} - \vec{e}_l(l)\cdot\vec{\tilde{e}}_l(l')\frac{e^{-jk\sqrt{[\vec{r}(l)-\vec{\tilde{r}}(l')]^2+a^2}}}{\sqrt{[\vec{r}(l)-\vec{\tilde{r}}(l')]^2+a^2}}$$

$$(2.1.2.3)$$

The unit tangential vector $\vec{e}_l(l) = \partial\vec{r}(l)/\partial l$ of the curve is taken along the wire axis, $\vec{\tilde{r}}(l)$ is the radius vector mirrored by the ground plane, and $\vec{\tilde{e}}_l(l) = \partial\vec{\tilde{r}}(l)/\partial l$ is the corresponding unit tangential vector.

If the transmission line is not perfectly conducting, the total tangential electric field on the conductor in not zero. In this case the boundary condition (2.1.2.1) (or the equation

(2.1.2.2)) can be modified to take into account the presence of a finite electrical conductivity of the wire. This is done by introducing a surface impedance approximation, which relates the total local electric field on the surface of the wire to the total tangential current flowing on the wire at the same point [1]. This relationship is expressed as

$$E_l^{tot}(l) = E_l^i(l) - j\omega A_l(l) - \frac{\partial \varphi(l)}{\partial l} = Z_w' I(l) \tag{2.1.2.4}$$

Here $Z_w'(j\omega)$ is the per-unit-length impedance of the wire[12,13]:

$$Z_w'(j\omega) = -\frac{Z_{cw}}{2\pi a} \frac{I_0(\gamma_w a)}{I_1(\gamma_w a)} \tag{2.1.2.5}$$

In eq. (2.1.2.5) the functions $I_0(x)$ and $I_1(x)$ are modified Bessel functions, Z_{cw} is the wave impedance in the conducting composition of the wire given by

$$Z_{cw} = \sqrt{\frac{j\omega\mu_0}{\sigma_w}} \tag{2.1.2.6}$$

and the term γ_w is the propagation constant in the wire material,

$$\gamma_w = \sqrt{j\omega\mu_0\sigma_w} \tag{2.1.2.7}$$

The electrical conductivity of the wire is denoted by σ_w. In this discussion, the wire is assumed nonmagnetic, with unit permittivity.

Various approximations to this wire impedance are possible [14]. At low frequencies, where $|\gamma_w a| << 1$, this impedance is given by

$$Z_w'(j\omega) \approx \frac{1}{\pi a^2 \sigma_w} + j\omega \frac{\mu_0}{8\pi} \approx R_{DC} + j\omega L_{in} \tag{2.1.2.8}$$

and at high frequencies, where $|\gamma_w a| >> 1$, this impedance becomes

$$Z_w'(j\omega) \approx \frac{1+j}{2\pi a} \sqrt{\frac{\mu_0 \omega}{2\sigma_w}} \tag{2.1.2.9}$$

With the introduction of the wire impedance in the expression for the tangential electrical field (2.1.2.4) and applying the same steps as done previously, the following first MPIE for the lossy wire results:

$$\frac{\partial \varphi(l)}{\partial l} + j\omega \frac{\mu_0}{4\pi} \int_0^L g_I^L(l,l')I(l')dl' + Z_w' I(l) = E_l^i(l) \tag{2.1.2.10}$$

These two MPIE (2.1.2.10) and (2.1.1.15) are the main results of the two first sub-Sections.

It is possible to show, that for the partial case – a straight finite wire in the free space, the second order Pocklington-like integro-differential equation for the induced current, which can be derived from (2.1.2.10) and (2.1.1.15) for the case Z'_w, coincides with results of papers [15], [16] for the coated thin straight wire.

2.2 Full-Wave Transmission Line (FWTL) equations for the wiring system with losses. Iteration approach for the global parameters.

The two MPIE obtained in the previous sub-Sections are a starting point for the derivation of the global parameters in the Full-Wave Transmission Line Theory (FWLT). In this Section we shortly outline the derivation, referring the reader to [11],[17],[18].
Let us consider again the system of MPIE for the thin coated, lossy wire of arbitrary geometric form (2.2.1 a,b)

$$
\begin{cases}
\dfrac{\partial \varphi(l)}{\partial l} + j\omega \dfrac{\mu_0}{4\pi} \int_0^L g_I^L(l,l') I(l')dl' + Z'_w I(l) = E_l^i(l) \\[4mm]
\int_0^L g_I^C(l,l') \dfrac{\partial I(l')}{\partial l'} dl' - \dfrac{4\pi\varepsilon_0}{\widetilde{C}'} \dfrac{\partial I(l)}{\partial l} + 4\pi\varepsilon_0 j\omega\varphi(l) = 0
\end{cases}
\qquad (2.2.1\ a,b)
$$

In order to define the global generalized transmission line parameters, we consider an excitation of the transmission line by point sources with arbitrary dimensionless amplitudes U_1 and U_2 located at the beginning and at the end of the line, which corresponds to the tangential component of the exciting electrical field:

$$
E_l^i(l) = (U_1 \delta(l-\Delta) + U_2 \delta(l-L+\Delta)) \cdot 1V \qquad \text{with} \qquad \Delta \to 0 \qquad (2.2.2)
$$

It is possible to show, that the excitation field for the loaded line formally can be reduced to the same equation [11,17,18]. For example, when the system is loaded by the impedance Z_2 at the right terminal, then the corresponding constant is

$$
U_2 = -I(L)Z_2 / 1V \qquad (2.2.3)
$$

Let now the functions $I_1(l)$, $I_2(l)$ and $\varphi_1(l)$, $\varphi_2(l)$ be solutions of the system (2.2.1 a, b) for the current and the potential with sources with of amplitude 1 V: $\delta(l-\Delta)$, $\delta(l-L+\Delta)$ located in the points Δ and $L-\Delta$, correspondingly. Due to the linearity of the considered problem we can write the solution for the total induced current as

$$
I(l) = U_1 I_1(l) + U_2 I_2(l) \qquad (2.2.4)
$$

For the potential $\varphi(l)$ along the wire we find a similar equation:

$$
\varphi(l) = U_1 \varphi_1(l) + U_2 \varphi_2(l) \qquad (2.2.5)
$$

Now we are ready to look for the system of differential equations (Full Wave Transmission Line equations, FWTL) for the potential and current outside the source region in the Transmission - Line - Theory - like form.

$$\begin{cases} \dfrac{d\varphi(l)}{dl} + j\omega P_{11}(l)I(l) + j\omega P_{12}(l)\varphi(l) = 0 \\[3mm] \dfrac{dI(l)}{dl} + j\omega P_{21}(l)\varphi(l) + j\omega P_{22}(l)I(l) = 0 \end{cases} \qquad (2.2.6)$$

To do that, we begin to use a matrix notation (this way of solution can be generalized for the multiconductor case). We introduce a column-vector $\underset{\downarrow}{x}$, which components are potential and current, as for the partial, as well as for the general solution :

$$\underset{\downarrow}{x_1} := \begin{bmatrix} \varphi_1(l) \\ I_1(l) \end{bmatrix}; \qquad \underset{\downarrow}{x_2} := \begin{bmatrix} \varphi_2(l) \\ I_2(l) \end{bmatrix}; \qquad \underset{\downarrow}{x} := \begin{bmatrix} \varphi(l) \\ I(l) \end{bmatrix} \qquad (2.2.7)$$

Now the equations (2.2.4) – (2.2.5) can be written in the following form:

$$\underset{\downarrow}{x} = U_1 \underset{\downarrow}{x_1} + U_2 \underset{\downarrow}{x_2} \qquad (2.2.8)$$

and, instead of (2.2.6) we have:

$$\frac{d}{dl}\underset{\downarrow}{x} + j\omega[P]\underset{\downarrow}{x} = 0, \text{ where } [P(j\omega,l)] = \begin{bmatrix} P_{11}(j\omega,l), & P_{12}(j\omega,l) \\ P_{21}(j\omega,l), & P_{22}(j\omega,l) \end{bmatrix}, \qquad (2.2.9 \text{ a,b})$$

Eqs. (2.2.9 a) and (2.2.8) mean that

$$U_1(\frac{d}{dl}\underset{\downarrow}{x_1} + j\omega[P]\underset{\downarrow}{x_1}) + U_2(\frac{d}{dl}\underset{\downarrow}{x_2} + j\omega[P]\underset{\downarrow}{x_2}) = 0 \qquad (2.2.10)$$

since the U_1 and U_2 have arbitrary values in (2.2.10) we derive

$$\frac{d}{dl}\underset{\downarrow}{x_1} + j\omega[P]\underset{\downarrow}{x_1} = 0 \qquad (2.2.11 \text{ a})$$

$$\frac{d}{dl}\underset{\downarrow}{x_2} + j\omega[P]\underset{\downarrow}{x_2} = 0 \qquad (2.2.11 \text{ b})$$

After an introduction of a matrix notation for the partial solutions

$$[X] := \begin{bmatrix} \underset{\downarrow}{x_1}, \underset{\downarrow}{x_2} \end{bmatrix} = \begin{bmatrix} \varphi_1(l) & \varphi_2(l) \\ I_1(l) & I_2(l) \end{bmatrix} \qquad (2.2.12)$$

eq. (2.2.11 a,b) can be written as

$$\frac{d}{dl}[X] + j\omega[P] \cdot [X] = 0 \tag{2.2.13}$$

If the matrix of partial solutions $\widehat{x}$ is non-degenerated, i.e.

$$\det[X] = \varphi_1(l)I_2(l) - \varphi_2(l)I_1(l) := -\Delta_{I,\varphi} \neq 0 \tag{2.2.14}$$

than the solution for the matrix $[P]$ can be written in the following form

$$[P(l)] = -\frac{1}{j\omega}\frac{d[X]}{dl} \cdot [X]^{-1} \tag{2.2.15}$$

In the usually used scalar notations (2.2.15) can be written as

$$P_{11}(l) = \frac{1}{j\omega}\frac{\varphi_1'(l)I_2(l) - I_1(l)\varphi_2'(l)}{\Delta_{I,\varphi}} \tag{2.2.16 a}$$

$$P_{12}(l) = -\frac{1}{j\omega}\frac{\varphi_1'(l)\varphi_2(l) - \varphi_1(l)\varphi_2'(l)}{\Delta_{I,\varphi}} \tag{2.2.16 b}$$

$$P_{21}(l) = \frac{1}{j\omega}\frac{I_1'(l)I_2(l) - I_1(l)I_2'(l)}{\Delta_{I,\varphi}}$$
$$\tag{2.2.16 c}$$

$$P_{22}(l) = -\frac{1}{j\omega}\frac{I_1'(l)\varphi_2(l) - \varphi_1(l)I_2'(l)}{\Delta_{I,\varphi}} \tag{2.2.16}$$

It is easy to show, that if we do not start the calculation from the functions $I_1(l)$, $I_2(l)$, $\varphi_1(l)$, $\varphi_2(l)$, but from some non-degenerated linear combinations of them: $\widetilde{I}_1(l)$, $\widetilde{I}_2(l)$, $\widetilde{\varphi}_1(l)$, $\widetilde{\varphi}_2(l)$ (or $\widetilde{\widehat{X}}$ in the matrix form) with

$$[\widetilde{X}] = [X] \cdot [\alpha], \text{ where } \det[\alpha] \neq 0 \tag{2.2.17}$$

the result for the matrix $[P(l)]$ will be kept the same

$$[\widetilde{P}(l)] = -\frac{1}{j\omega}\frac{d([X] \cdot [\alpha])}{dl} \cdot ([X] \cdot [\alpha])^{-1} = -\frac{1}{j\omega}\frac{d[X]}{dl} \cdot [\alpha] \cdot [\alpha]^{-1} \cdot [X]^{-1} =$$
$$= -\frac{1}{j\omega}\frac{d[X]}{dl} \cdot [X]^{-1} = [P(l)] \tag{2.2.18}$$

Thus, we have shown that the system of MPIE (2.2.1), the solution of which is defined by two independent constants can be explicitly reduced to the differential equations (2.2.6), (2.2.9 a) with parameters (2.2.15), (2.2.16). These parameters (global parameters in the Full-Wave

Transmission Line Theory or the parameters of "Maxwellian circuits") are complex valued, and describe the radiation of the system [11]. They depend on the geometry of the system, and therefore on the local parameter l along the line. This fact was established earlier in [8-10] with the method of the product integral, and (up to notation) in [19] by processing the numerical solution for the current and potential with the Method of Moments.

The solution of the system (2.2.6) with parameter matrix $\hat{P}(l)$ (2.2.16) and usually used boundary conditions for the currents and voltages (differences of potentials for the small Δ) in the points $l = \Delta$ and $l = L - \Delta$ yields the current and voltage distributions along the line for arbitrary given values of the terminal sources and/or loads. The procedure is convenient, when the exact values of the functions $I_1(l)$, $I_2(l)$, $\varphi_1(l)$, $\varphi_2(l)$ are known, from analytical [11,18] or numerical [19] solutions.

Another way to obtain the matrix of global parameters apriorily is to organize some iteration procedure for this matrix. Generally, at the zero steps, the approximate solution of the system (2.2.6) is defined. Then this solution is used to find the corresponding parameters, etc. In [8-10] as zero iteration the static distributions for the current and potential are used, and the first iteration for the parameters was obtained after some numerical procedure.

Another procedure, which is based on the thickness of the wire, was proposed in [11], where, at the zero step the MPIE (2.2.1) (within the logarithmical accuracy) was reduced to the classical TL system with constant parameters. The solution of this system with sources (2.2.2) yields the current of the first iteration, $I_1(l)$ and $I_2(l)$, the linear combination of which (up to the constant factor) can be represented as forward and backward propagating waves

$$I_1^{(1)}(l) = \exp(-jkl) \qquad\qquad I_2^{(1)}(l) = \exp(jkl) \qquad\qquad (2.2.19)$$

However, for the scalar potential in the first iteration and for its derivative, the exact equations (2.2.1 a,b) are used. After some straightforward calculations, we obtain the global parameter matrix in the first order approximation:

$$P_{11}^{(1)}(l) = c \frac{L'_+(l) - L'_-(l)}{\dfrac{1}{C'_+(l)} + \dfrac{1}{C'_-(l)} - \dfrac{2}{\widetilde{C}'}} \qquad\qquad (2.2.20\ a)$$

$$P_{12}^{(1)}(l) = \frac{\left(L'_+(l) + \dfrac{Z'_w}{j\omega}\right)\left(\dfrac{1}{C'_-(l)} - \dfrac{1}{\widetilde{C}'}\right) + \left(L'_-(l) + \dfrac{Z'_w}{j\omega}\right)\left(\dfrac{1}{C'_+(l)} - \dfrac{1}{\widetilde{C}'}\right)}{\dfrac{1}{C'_+(l)} + \dfrac{1}{C'_-(l)} - \dfrac{2}{\widetilde{C}'}} \qquad\qquad (2.2.20\ b)$$

$$P_{21}^{(1)}(l) = \frac{2}{\dfrac{1}{C'_+(l)} + \dfrac{1}{C'_-(l)} - \dfrac{2}{\widetilde{C}'}} \qquad\qquad (2.2.20\ c)$$

$$P_{22}^{(1)}(l) = -\frac{1}{c} \frac{\dfrac{1}{C_+'(l)} - \dfrac{1}{C_-'(l)}}{\dfrac{1}{C_+'(l)} + \dfrac{1}{C_-'(l)} - \dfrac{2}{\widetilde{C}'}} \tag{2.2.20 d}$$

In eqs. (2.2.20) we have used the following expressions for the "forward" and "backward" inductance and capacitance of the first order, respectively:

$$L_\pm'(l) = \frac{\mu_0}{4\pi} \int_0^L g_0^L(l,l') \exp(\mp jk(l'-l)) dl' \tag{2.2.21}$$

$$C_\pm'(l) = \frac{4\pi\varepsilon_0}{\displaystyle\int_0^L g_0^C(l,l') \exp(\mp jk(l'-l)) dl'} \tag{2.2.22}$$

For the low-frequency case ($k \to 0$) we find

$$L_+'(l) = L_-'(l) = L_0'(l) := \frac{\mu_0}{4\pi} \int_0^L \left\{ \frac{\vec{e}_l(l) \cdot \vec{e}_l(l')}{\sqrt{[\vec{r}(l) - \vec{r}(l')]^2 + a^2}} - \frac{\vec{e}_l(l) \cdot \vec{\tilde{e}}_l(l')}{\sqrt{[\vec{r}(l) - \vec{\tilde{r}}(l')]^2 + a^2}} \right\} dl' \tag{2.2.23}$$

$$C_+'(l) = C_-'(l) = C_0'(l) := \frac{4\pi\varepsilon_0}{\displaystyle\int_0^L \left\{ \frac{1}{\sqrt{[\vec{r}(l) - \vec{r}(l')]^2 + a^2}} - \frac{1}{\sqrt{[\vec{r}(l) - \vec{\tilde{r}}(l')]^2 + a^2}} \right\} dl'} \tag{2.2.24}$$

The quantities $L_0'(l)$ and $C_0'(l)$ constitute the real, low-frequency length dependent inductance and capacitance per unit length for the lossless uncoated transmission line [9]. Then, using (2.2.20) for our case we obtain the parameter matrix in the classical anti-diagonal form for the coated wire with losses.

$$\left[P^{(1)}(l) \right]_{k \to 0} = \begin{bmatrix} 0 & L_0'(l) + Z_w'/j\omega \\ C_{0\varepsilon}' & 0 \end{bmatrix} \tag{2.2.25}$$

where we have introduced the per-unit length capacitance for the coated wire

$$C_{0\varepsilon}' = \frac{1}{\dfrac{1}{C_0'(l)} - \dfrac{1}{\widetilde{C}'}} \tag{2.2.26}$$

3. Propagation of Current Waves along Quasi-Periodical Wiring Structures: a Quantum Mechanical Analogy.

3.1 FWTL and Schrödinger-like equations

We consider a long lossless thin conductor of arbitrary geometric form above a perfectly conducting ground, where the non-uniformities are located in the central part of the conductor. We assume, that the sources are located at the left end of the wire (at minus infinity) and that the reflection is absent from the right end of the wire (plus infinity). As was shown in the Section 2, the current and potential along the line are described by the FWTL (2.2.6) with the matrix of global parameters $[P(l)]$. In our consideration the Lorenz gauge for the potential is used, but, of course, we can use any another gauge, for example, the Coulomb gauge. However, for the current we obtain a gauge independent differential equation of second order [19,11]:

$$I''(l) + U_M(l)I'(l) + T_M(l)I(l) = 0 \qquad (3.1.1)$$

Here l is the length-parameter of the curve taken along the wire axis, $U_M(l)$ is the complex damping function, $T_M(l)$ corresponds to the square of the propagation constant. These parameters are connected with the global parameters of the MPIE [11] and they also depend on frequency and on the geometry of the system.

$$U_M(l) = -\frac{d}{dl}\ln(P_{21}) + j\omega(P_{11} + P_{22}) \qquad (3.1.2)$$

$$T_M(l) = j\omega P_{21}\frac{d}{dl}\left(\frac{P_{22}}{P_{21}}\right) - \omega^2 \det[P] \qquad (3.1.3)$$

To reduce eq. (3.1.1) to the form convenient for further analysis, we eliminate the first derivative by introduction of a new unknown function $\psi(l)$:

$$I(l) = f(l)\psi(l) \qquad (3.1.4)$$

$$f(l) = \exp\left(-\frac{1}{2}\int_{-\infty}^{l} U_M(l')dl'\right) = \sqrt{\frac{P_{21}(l)}{P_{21}(-\infty)}}\exp\left(-\frac{j\omega}{2}(P_{11}(l) + P_{22}(l))\right) \qquad (3.1.5)$$

The function $\psi(l)$ satisfies the differential equation of second order (3.1.6)

$$\psi''(l) + k^2(l)\psi(l) = 0; \qquad (3.1.6)$$

$$k^2(l) := T_M(l) - \frac{1}{2}\frac{dU_M(l)}{dl} - \frac{U_M(l)}{4} \qquad (3.1.7)$$

To consider the wiring structure at the uniform ends we introduce a "potential" function $u(l)$ as follows

$$\lim_{l \to \pm\infty} \left(k^2(l) \right) = k^2 = \frac{\omega^2}{c^2} > 0 \qquad (3.1.8)$$

$$u(l) = k^2 - k^2(l) \qquad (3.1.9)$$

$$\lim_{l \to \pm\infty} \left(u(l) \right) = 0 \qquad (3.1.10)$$

$$\psi'' + \left(k^2 - u(l) \right) \psi(l) = 0 \qquad (3.1.11)$$

Eq. (3.1.11) looks like a Schrödinger equation in non-relativistic quantum mechanics with the "potential" $u(l)$ [5]. As well as the parameters $T_M(l)$ and $U_M(l)$ the "potential" $u(l)$ depends on the geometry of the wire and on the frequency. For low frequencies, $kh \leq 1$ (where h is the height of the wire at $l = \pm\infty$, using (2.2.23)-(2.2.25) and (3.1.2)-(3.1.7)) one can show, that the "potential" can expressed as

$$u(l) = \frac{\omega^2}{c^2} - \omega^2 \left(L_0'(l) + Z_w' / j\omega \right) C_{0\varepsilon}'(l) - \frac{1}{2} \frac{1}{C_{0\varepsilon}'(l)} \frac{d^2 C_{0\varepsilon}'(l)}{d l^2} + \frac{3}{4} \frac{1}{\left(C_{0\varepsilon}'(l) \right)^2} \left(\frac{d C_{0\varepsilon}'(l)}{d l} \right)^2 \qquad (3.1.12)$$

From (3.1.12) it is obvious that for the lossless line the "potential" is real and eq. (3.1.11) describes the one-dimensional scattering of a quantum mechanical particle, when the number of particles is kept constant [5].
In the general frequency case, when we have to use in (3.1.2), (3.1.3), (3.1.7) the results of eqs. (2.2.20), the "potential" becomes:

$$u(l) = k^2 - j\omega P_{21} \frac{d}{dl}\left(\frac{P_{22}}{P_{21}} \right) + \omega^2 \det[\widehat{P}] - \frac{1}{2} \frac{d^2 \ln(P_{21})}{dl^2} + $$

$$\qquad (3.1.13)$$

$$+ j\omega \frac{d}{dl}\left(P_{11} + P_{22} \right) + \frac{1}{4}\left(j\omega(P_{11} + P_{22}) - \frac{d}{dl}\ln(P_{21}) \right)^2$$

Now (as well as for the low frequency case with ohmic or dielectric losses) the "potential" is a complex-valued quantity that corresponds to losses in the initial electrodynamics problem and corresponds to absorption of the particles in the quantum mechanical analogy.

3.2 One-dimensional scattering problem. Reflection and transmission coefficients for the lossy systems.

In the previous sub-Section we have shown, that the homogeneous electrodynamical problem is equivalent to the one-dimensional quantum mechanical scattering problem, where the particle comes from minus infinity, scatters at the potential, becomes partially absorbed in the potential region, propagates partially through the potential and also is partially reflected by the potential. The complex quantum mechanical amplitudes of these processes are described by the complex reflection R and transmission D coefficients (see Fig. 3.1)
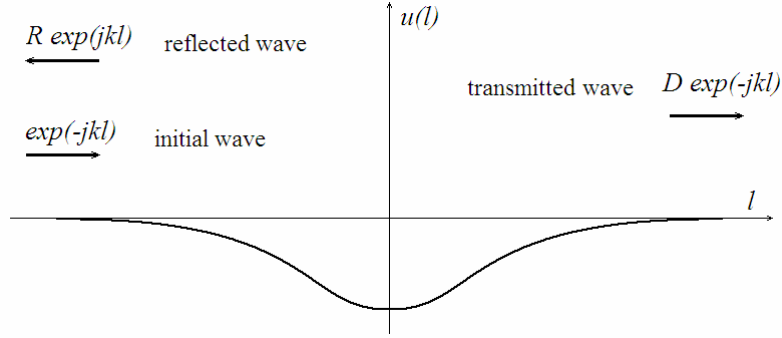
Fig. 3.1: One-dimensional quantum-mechanical scattering problem

$$\psi(l) = \begin{cases} \exp(-jkl) + R\exp(jkl) & \text{for} \quad l \to -\infty \\ D\exp(-jkl) & \text{for} \quad l \to \infty \end{cases} \qquad (3.2.1)$$

These coefficients become very important in this report and we now describe their properties in more detail.

For symmetrical scattering "potential", $u(-l) = u(l)$, these coefficients are the same for the left and for the right scattering problem.

It is possible to show that for the low-frequency lossless case (where the "potential" $u(l)$ is real) these coefficients satisfy the following equations [4]:

$$|R|^2 + |D|^2 = 1 \qquad (3.2.2)$$

$$\mathrm{Re}\{RD^*\} = 0 \qquad (3.2.3)$$

However, for the complex "potential" corresponding to radiation and (or) ohmic and dielectric losses the equations (3.2.2)-(3.2.3) are not valid. The imaginary part of the "potential" now is responsible for the losses. Let us establish this dependence.

If we have a uniform line with current waves propagating in both directions (and the field around wire is a TEM wave)

$$I(l) = I_1\exp(-jkl) + I_2\exp(jkl) \qquad (3.2.4)$$

the time averaged power propagating along the uniform line in the positive direction can be written as

$$W = \frac{Z_C}{2}\left(|I_1|^2 - |I_2|^2\right) \qquad (3.2.5)$$

Eq. (3.2.5) can be represented as

$$W(l) = \frac{Z_c}{4jk}\left\{I(l)\frac{\partial I^*(l)}{\partial l} - I^*(l)\frac{\partial I(l)}{\partial l}\right\} \qquad (3.2.6)$$

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Let us now define the value $W(l)$ by eq. (3.2.6) not only for the two asymptotic regions $(l \to \pm\infty)$ but also for the intermediate region (region of interaction). Now we can express the energy losses of the current during the scattering process (caused by radiation, ohmic or dielectric losses) using the law of energy conservation

$$W_{loss} = -\left(W(\infty) - W(-\infty)\right) \tag{3.2.7}$$

Using the representation of the current through the ψ-function (3.1.4) we can write for the quantity $W(l)$:

$$W(l) = \frac{Z_C}{4jk} \exp\left\{ \operatorname{Re} \int_{-\infty}^{l} U_M(l')dl' \right\} \left\{ \psi(l) \frac{\partial \psi^*(l)}{\partial l} - \psi^*(l) \frac{\partial \psi(l)}{\partial l} + \operatorname{Im}(U_M(l)) |\psi(l)|^2 \right\} \tag{3.2.8}$$

For a symmetrical wiring system, for example for the wire with vertical coordinate $x(l)$ and horizontal coordinate $z(l)$, which are given by the relations

$$\vec{r}(l) = (x(l), 0, z(l)); \qquad\qquad \vec{r}(-l) = (x(l), 0, -z(l)) \tag{3.2.9 a,b}$$

We find after some combersome calculation with the aid of the technique from Section 2, the following symmetry properties for the global FWTL parameters:

$$P_{11}(-l) = -P_{11}(l) \tag{3.2.10 a,b,c,d}$$

$$P_{12}(-l) = P_{12}(l)$$

$$P_{21}(-l) = P_{21}(l)$$

$$P_{22}(-l) = -P_{22}(l)$$

Now, using the definition of the parameter $U_M(l)$ (3.1.2) and eq. (3.2.10) we can find that

$$U_M(-\infty) = U_M(\infty) = 0, \qquad\qquad \int_{-\infty}^{\infty} U_M(l')dl' = 0 \tag{3.2.11 a,b}$$

For eq. (3.2.8) we then obtain

$$W(l) \underset{l \to \pm\infty}{=} \frac{Z_C}{4jk} \left\{ \psi(l) \frac{\partial \psi^*(l)}{\partial l} - \psi^*(l) \frac{\partial \psi(l)}{\partial l} \right\} \tag{3.2.12}$$

and, consequentely

$$W_{loss} = \frac{Z_C}{2} |I_0|^2 \cdot \left(1 - |R|^2 - |D|^2\right) \tag{3.2.13}$$

For the lossless case we obtain the obvious answer: $W_{loss} = 0$. We will make some standard manipulation with the Schrödinger equation (3.1.11) for the lossy case to obtain the term in the bracket (3.1.13).

Let us consider the Schrödinger equation (3.1.11) for the function $\psi(l)$ and for the complex conjugate function ψ^*. Multiply them, correspondingly, by ψ^* and by ψ, then subtract the second from the first equation:

$$d^2\psi(l)/dl^2 + \left(k^2 - u(l)\right)\psi(l) = 0 \qquad \cdot \psi^*(l)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \qquad\qquad\qquad (3.2.14\ a,b)$$
$$d^2\psi^*(l)/dl^2 + \left(k^2 - u^*(l)\right)\psi^*(l) = 0 \qquad \cdot \psi(l)$$

As result, we have

$$0 = \psi^*(l)\frac{d^2\psi(l)}{dl^2} - \psi(l)\frac{d^2\psi^*(l)}{dl^2} - u(l)\psi(l)\psi^*(l) + u^*(l)\psi^*(l)\psi(l) =$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.2.15)$$
$$\frac{d}{dl}\left\{\psi^*(l)\frac{d\psi(l)}{dl} - \psi(l)\frac{d\psi^*(l)}{dl}\right\} - 2j\,\text{Im}(u(l))\cdot|\psi(l)|^2 = 0$$

Integrating (3.2.15) from $-\infty$ to ∞ and using (3.2.1) yields:

$$\left\{\psi^*(l)\frac{d\psi(l)}{dl} - \psi(l)\frac{d\psi^*(l)}{dl}\right\}\Bigg|_{-\infty}^{\infty} = 2j\int_{-\infty}^{\infty}\text{Im}(u(l'))\cdot|\psi(l')|^2\,dl' = 2jk\left(1 - |R|^2 - |D|^2\right)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.2.16)$$

Taking into account (3.2.16) and (3.2.13), we finally have

$$W_{loss} = \frac{Z_C}{2}|I_0|^2\left(1 - |R|^2 - |D|^2\right) = \frac{Z_C}{2}|I_0|^2\frac{1}{k}\int_{-\infty}^{\infty}|\psi(l)|^2\,\text{Im}(u(l))dl \qquad (3.2.17)$$

The equation (3.2.17) gives the desired connection between the imaginary part of the "potential" and the losses in the wire.

3.3 Transfer matrices for the one-dimensional scattering problem.

In this sub-Section, we consider another, more general approach to describe one-dimensional scattering, which is given by the method of transfer matrix [6]. In this method, we consider waves propagating in positive and negative directions with different amplitudes, both from the left and right sides of the potential (3.3.1) (see Fig. 3.2)

$$\psi(l) = \begin{cases} C_1 \exp(-jkl) + C_2 \exp(jkl) & \text{for} \quad l \to -\infty \\ \widetilde{C}_1 \exp(-jkl) + \widetilde{C}_2 \exp(jkl) & \text{for} \quad l \to \infty \end{cases} \tag{3.3.1}$$
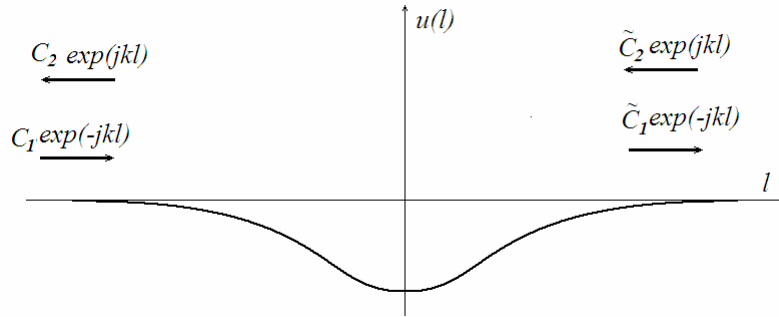

Fig. 3.2: On the definition of the transfer matrix $\widehat{S}$.

Because the considered problem is linear, there exists a linear connection between the asymptotic amplitudes of the potential. This connection is realized by 2×2 transfer matrix

$$\widehat{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \tag{3.3.2}$$

which links the column vectors of the wave amplitudes:

$$\underset{\downarrow}{\widetilde{C}} := \begin{pmatrix} \widetilde{C}_1 \\ \widetilde{C}_2 \end{pmatrix} = \widehat{S} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} =: \widehat{S} \underset{\downarrow}{C} \tag{3.3.3}$$

Let us now express the transfer matrix in terms of reflection and transmission coefficients R and D, respectively. For the left side scattering problem (Fig. 3.1), have the following components:

$$\underset{\downarrow}{C} := \begin{pmatrix} 1 \\ R \end{pmatrix} \qquad \text{and} \qquad \underset{\downarrow}{\widetilde{C}} := \begin{pmatrix} D \\ 0 \end{pmatrix} \tag{3.3.4 a,b}$$

Similarly, for the right hand side scattering problem, when the particle incidents from plus infinity, we have

$$\underset{\downarrow}{C} := \begin{pmatrix} 0 \\ D \end{pmatrix} \qquad \text{and} \qquad \underset{\downarrow}{\widetilde{C}} := \begin{pmatrix} R \\ 1 \end{pmatrix} \tag{3.3.5 a,b}$$

Substitute (3.3.5) into (3.3.3), and find two components of the matrix $\widehat{S}$:

$$S_{12} = \frac{R}{D} \qquad \text{and} \qquad S_{22} = \frac{1}{D} \tag{3.3.6 a,b}$$

Equations for the two other components of the transfer matrix are obtained after substitution of the (3.3.4) into (3.3.3)

$$\begin{pmatrix} D \\ 0 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} 1 \\ R \end{pmatrix} \qquad (3.3.7)$$

or,
$$D = S_{11} + S_{12}R = S_{11} + \frac{R^2}{D}$$
$$0 = S_{21} + S_{22}R = S_{21} + \frac{R}{D} \qquad (3.3.8 \text{ a,b})$$

(3.3.8) and (3.3.6) yield the desired equation for the matrix $\widehat{S}$

$$\widehat{S} = \begin{pmatrix} \dfrac{D^2 - R^2}{D} & \dfrac{R}{D} \\ -\dfrac{R}{D} & \dfrac{1}{D} \end{pmatrix} \qquad (3.3.9)$$

One can easily check that

$$\det \widehat{S} = 1 \qquad (3.3.10)$$

For a real potential (which corresponds to a lossless system) we use the properties of the reflection and transmission coefficient (3.2.2)-(3.2.3) to show, that [4,25]

$$\widehat{S} = \begin{pmatrix} \dfrac{1}{D^*} & \dfrac{R}{D} \\ \dfrac{R^*}{D^*} & \dfrac{1}{D} \end{pmatrix} = \begin{pmatrix} \dfrac{e^{j\varphi_D}}{|D|} & j(-1)^n \dfrac{\sqrt{1-|D|^2}}{|D|} \\ -j(-1)^n \dfrac{\sqrt{1-|D|^2}}{|D|} & \dfrac{e^{j\varphi_D}}{|D|} \end{pmatrix} \qquad (3.3.11)$$

where $D = |D|e^{j\varphi_D}$, and the numbers $n = 0,1,2,\ldots$ correspond to the number of bounded energy states in the potential (3.1.12).

Also, for the lossless system we have

$$\widehat{S}^* = \widehat{S}^{-1} \qquad (3.3.12)$$

The properties of the transfer matrix for the lossless case can be written in different ways:

$$S_{11} = S_{22}^*; \qquad S_{12} = S_{21}^*; \qquad \det \widehat{S} = |S_{11}|^2 - |S_{21}|^2 = 1 \qquad (3.3.13 \text{ a,b,c})$$

Therefore, the transfer matrix $\widehat{S}$ in this case belongs to the $SU(1,1)$ group of matrices [6,20]. During multiplication this matrix keeps the value

$$|C_1|^2 - |C_2|^2 = \left|\widetilde{C}_1\right|^2 - \left|\widetilde{C}_2\right|^2 \sim W = const \tag{3.3.14}$$

constant, which physically corresponds to the "particle flux" (power) conservation along the line.

For the free "particle" (current wave) propagation without any scattering, we also can introduce the transfer matrix. In this case the translation reduces to the change of the coordinate l. To see this we consider the free propagation waves for two origins of coordinates l_1 and l_2 (see Fig. 3.3):
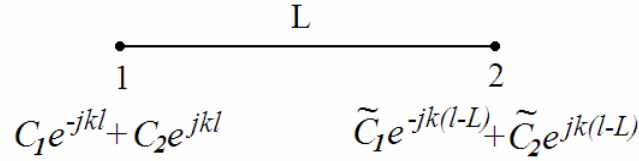

Fig. 3.3: Free propagation of the current wave

$$C_1 e^{-jk(l-l_1)} + C_2 e^{jk(l-l_1)} = \underbrace{C_1 e^{-jk(l_2-l_1)}}_{\widetilde{C}_1} e^{-jk(l-l_2)} + \underbrace{C_2 e^{jk(l_2-l_1)}}_{\widetilde{C}_2} e^{jk(l-l_2)} \tag{3.3.15}$$

From the (3.3.15) the transfer matrix for the free propagation is derived as:

$$\begin{pmatrix} \widetilde{C}_1 \\ \widetilde{C}_2 \end{pmatrix} = \begin{pmatrix} e^{-jk(l_2-l_1)} & 0 \\ 0 & e^{jk(l_2-l_1)} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} =: \widehat{T}(l_2, l_1) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \tag{3.3.16}$$

Of course, the transfer matrix for the free propagation also belongs to the SU(1,1) group of matrices.

3.4 Transfer – matrix formalism for the quasi-periodical system with losses.

Let us now assume that the quasi-periodical wiring structure is formed by a finite number of identical sub-elements (see Section 4). We assume that the corresponding potential in equation (3.1.11) can be represented as a set of periodically arranged identical potentials[1] :

We now consider the propagation through a chain, consisting of N potentials, separated by asymptotic regions, where the potential is approximately zero (see Fig. 3.4). Let us assume that the column vector on the right side of the quasi-periodical system (we assume now, that the beginning of the coordinates is in the center of the first potential) is $\underset{\downarrow}{C}$.

After the first scattered the column vector becomes $\widehat{S}_1 \underset{\downarrow}{C}$. Changing the coordinate origin by

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[1] It is possible to show, that this assumption better satisfies for the case of low frequencies, but also for high frequencies, when radiation does not dominate, this assumption is also approximately valid. Moreover, the more sub-elements are considered the better satisfies this assumption.

the free propagation transfer matrix $\widehat{T}_{21} := \widehat{T}(l_2, l_1)$ to the right side of the second potential, we have a column-vector $\widehat{T}_{21}\widehat{S}_1 \underset{\downarrow}{C}$. Repeating this process up to the last potential, we can write for the total transfer matrix, with the beginning of coordinates in the center of the last potential
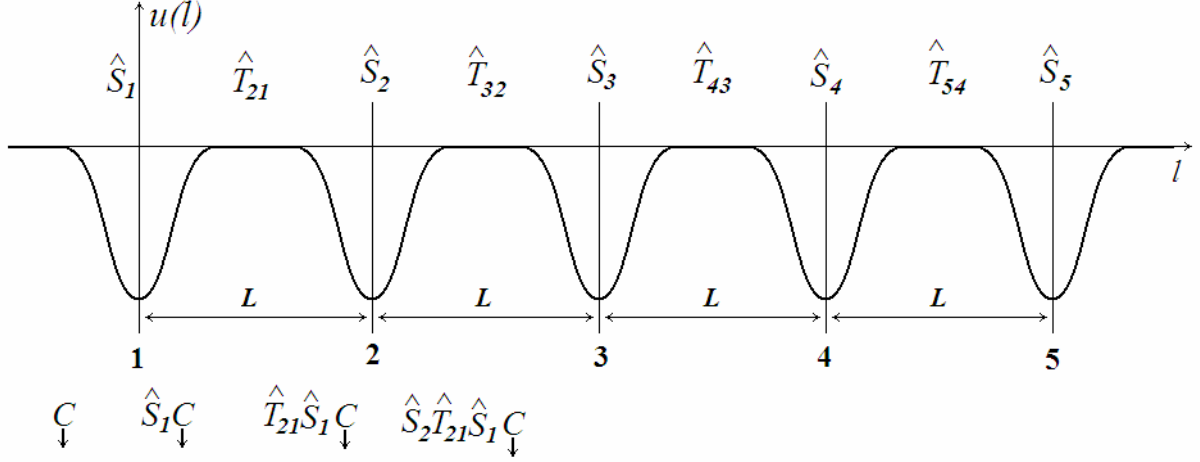

Fig. 3.4: Propagation through the quasi-periodical chain of potentials $N = 5$.

$$\widehat{T}^t(l_N, l_1) := \widehat{T}(l_N, l_{N+1}) \prod_{n=1}^{N} \widehat{T}(l_{n+1}, l_n)\widehat{S}_n \qquad (3.4.1)$$

After the return to the beginning of the coordinates to the center of the first potential we have for the corresponding transfer matrix[2]

$$\widehat{S}_{\Sigma(N)} = \widehat{T}(l_1, l_{N+1}) \prod_{n=1}^{N} \widehat{T}(l_{n+1}, l_n)\widehat{S}_n \qquad (3.4.2)$$

For the case of periodically arranged identical potentials eq. (3.4.2) can be re–written in the following form

$$\widehat{S}_{\Sigma(N)} = \widehat{T}(0, NL)\left(\widehat{T}(L,0)\widehat{S}\right)^N \qquad (3.4.3)$$

Now, it is more convenient to remove the coordinate origin in the centre of the quasi-periodical system. This can be done if the corresponding total potential is symmetrical, and

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[2] The matrix (3.4.1) is connected with an explicit form of the so-called product integral [9] (which yields the solution of the second order – differential equation (3.1.11)) for the case when the regions of interaction are divided by the regions of zero potential.
We mention here that the set of transfer matrices of the form (3.4.1) form a group:

       There is defined a product operation $\widehat{T}^t(l_3, l_2) \cdot \widehat{T}^t(l_2, l_1) = \widehat{T}^t(l_3, l_1)$;

       There exists a unit element $\widehat{T}^t(l_1, l_1) = \widehat{I}$;

       Each element has inverse element $\left(\widehat{T}^t(l_2, l_1)\right)^{-1} = \widehat{T}^t(l_1, l_2)$; $\widehat{T}^t(l_2, l_1)\widehat{T}^t(l_1, l_2) = \widehat{I}$

left and right reflection and transmission coefficients are the same. In this case, eq. (3.4.3) can be re-written as

$$\widehat{S}_{\Sigma(N)} = \widehat{T}(0,(N+1)L/2)\left(\widehat{T}(L,0)\widehat{S}\right)^N \widehat{T}(-(N-1)L/2,0) \tag{3.4.4}$$

or, using (3.3.2) and (3.3.9)

$$\widehat{S}_{\Sigma(N)} = \begin{pmatrix} \dfrac{D_{\Sigma(N)}^2 - R_{\Sigma(N)}^2}{D_{\Sigma(N)}} & \dfrac{R_{\Sigma(N)}}{D_{\Sigma(N)}} \\[3mm] -\dfrac{R_{\Sigma(N)}}{D_{\Sigma(N)}} & \dfrac{1}{D_{\Sigma(N)}} \end{pmatrix} \tag{3.4.5}$$

The knowledge of the total transfer matrix gives us the possibility to obtain an equation for the total transmission coefficient in explicit form, expressed by the scattering data for one potential. To calculate the total transfer matrix, we would like to obtain a simple analytical expression in the Nth power in eq. (3.4.4). For this purpose, it is enough to find an exponential representation of the matrix $\widehat{T}(L,0)\widehat{S}$ with some additive parameter [3].

Let us begin with the case of a lossless potential [4]. In this case, one can observe that the matrix $\widehat{T}(L,0)\widehat{S}$, as well as both of the factors belong to the $SU(1,1)$ group.
Now, one may remember that for the matrix of finite rotation for the spin of particles with spin $1/2$, which belongs to the $SU(2)$ group

$$\widehat{U} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}; \qquad \det\widehat{U} = 1; \qquad U_{11} = U_{22}^*; \qquad U_{12} = -U_{21}^* \tag{3.4.6}$$

the exponential representation becomes [5][4]:

$$\widehat{U} = \exp\left(j\varphi\vec{n}\,\widehat{\vec{\sigma}}/2\right) = \widehat{I}\cos(\varphi/2) + j\vec{n}\,\widehat{\vec{\sigma}}\sin(\varphi/2) \tag{3.4.7}$$

where φ is a real additive parameter – the angle of rotation around the unit vector $\vec{n}$ ($\vec{n}$ has real components and $\vec{n}^2 = 1$). $\widehat{\vec{\sigma}}$ is one vector of the Pauli matrices, which form together with the unit matrix $\widehat{I}$, a basis for 2×2 matrices

$$\widehat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \widehat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \widehat{\sigma}_y = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}; \qquad \widehat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad \tag{3.4.8}$$

---

[3] The parameter χ, which is the argument of the group element $g(\chi)$, is called an additive parameter, if for any two elements of this group $g(\chi_1)\cdot g(\chi_2) = g(\chi_1 + \chi_2)$.

[4] The second equality in (3.4.7) for the unit vector $\vec{n}$ can be obtained, using the anti-commutation properties of the Pauli matrices: $\widehat{\sigma}_i\widehat{\sigma}_j + \widehat{\sigma}_j\widehat{\sigma}_i = \delta_{i,j}$ and the definition of the matrix exponent from the series.

The properties of the group $SU(2)$ are similar to the properties of the group $SU(1,1)$, with the exception that a $SU(2)$ transformation keeps the following value constant:

$$|C_1|^2 + |C_2|^2 = |\widetilde{C}_1|^2 + |\widetilde{C}_2|^2 = const, \tag{3.4.9}$$

instead of (3.3.14)

Therefore, we look for an exponential representation of the matrix $\widehat{T}(L,0)\widehat{S}$ with different values of the parameters. In [4] it is shown that this approach is successful, however, the "angle of rotation φ", as well as some components of the "unit vector" $\vec{n}$ can be complex-valued.

Now we apply the same approach to the general case of a transfer matrix for the lossy system (not the $SU(1,1)$ case!). We try to look for the representation of the matrix $\widehat{T}(L,0)\widehat{S}$ in the form (3.4.6), namely

$$\widehat{T}(L,0)\widehat{S} = \exp\left(j\varphi\vec{n}\widehat{\vec{\sigma}}\right) = \widehat{I}\cos(\varphi) + j\vec{n}\widehat{\vec{\sigma}}\sin(\varphi) \tag{3.4.10}$$

(in the eq. (3.4.10) we use the parameter φ instead $\varphi/2$ for convenience).

Now we try to extract the parameter φ and n_x, n_y, n_z from the eq. (3.4.10).

The left side of (3.4.10) is

$$\widehat{T}(L,0)\widehat{S} := \begin{pmatrix} \widehat{T}_{11}^t(L,0) & \widehat{T}_{12}^t(L,0) \\ \widehat{T}_{21}^t(L,0) & \widehat{T}_{22}^t(L,0) \end{pmatrix} = \begin{pmatrix} e^{-jkL}\dfrac{(D^2 - R^2)}{D} & e^{-jkL}\dfrac{R}{D} \\ -e^{jkL}\dfrac{R}{D} & e^{jkL}\dfrac{1}{D} \end{pmatrix} \tag{3.4.11}$$

The right side of (3.4.10) can be written as

$$\exp\left(j\varphi\vec{n}\widehat{\vec{\sigma}}\right) = \widehat{I}\cos(\varphi) + j\vec{n}\widehat{\vec{\sigma}}\sin(\varphi) =$$

$$\widehat{I}\cos(\varphi) + jn_x\widehat{\sigma}_x\sin(\varphi) + jn_y\widehat{\sigma}_y\sin(\varphi) + jn_z\widehat{\sigma}_z\sin(\varphi) = \tag{3.4.12}$$

$$\begin{pmatrix} \cos\varphi + jn_z\sin\varphi & (jn_x + n_y)\sin\varphi \\ (jn_x - n_y)\sin\varphi & \cos\varphi - jn_z\sin\varphi \end{pmatrix}$$

We can find parameters of the exponential representation by comparison of (3.4.11) and (3.4.12). The additive parameter φ can be found by taking the sum of the diagonal elements:

$$\cos\varphi = \frac{1}{2}\left(T_{11}^t(L,0) + T_{22}^t(L,0)\right) = \frac{1}{2D}\left(e^{-jkL}(D^2 - R^2) + e^{jkL}\right) := \alpha \tag{3.4.13}$$

The coordinates of the "vector" n_x and n_y can be found by summation and subtraction of the non-diagonal elements, correspondingly:

$$j n_x \sin \varphi = \frac{1}{2}\left(T_{12}^t(L,0) + T_{21}^t(L,0)\right) = -j \frac{R}{D}\sin kL \qquad \text{or} \qquad n_x = -\frac{R}{D}\frac{\sin kL}{\sin \varphi} \quad (3.4.14)$$

$$n_y \sin \varphi = \frac{1}{2}\left(T_{12}^t(L,0) - T_{21}^t(L,0)\right) = \frac{R}{D}\cos kL \qquad \text{or} \qquad n_y = \frac{R}{D}\frac{\cos kL}{\sin \varphi} \qquad (3.4.15)$$

The z-component of the "vector" $\vec{n}$ can be found by subtraction of the diagonal elements:

$$j n_z \sin \varphi = \frac{1}{2}\left(T_{11}^t(L,0) - T_{22}^t(L,0)\right) = \frac{e^{-jkL}(D^2 - R^2) - e^{jkL}}{2D}$$

$$n_z = \frac{e^{-jkL}(D^2 - R^2) - e^{jkL}}{2jD\sin \varphi} \qquad (3.4.16)$$

We note that different from the lossless case, where the parameter φ is real or pure imaginary, in the lossy case this parameter is complex in any case (see also the Section 3.5). As in the case of a lossless potential [4], the vector $\vec{n}$ now is complex–valued[5] and (it is possible to show) it squared value becomes one:

$$\vec{n}^2 = 1. \qquad (3.4.17)$$

Now, having the matrix $\widehat{T}(L,0)\widehat{S}$ in the exponential form with additive parameter φ, we can easily write for the Nth power of it:

$$\left(\widehat{T}(L,0)\widehat{S}\right)^N = \left\{\exp(j\varphi \vec{n}\vec{\sigma})\right\}^N = \exp(jN\varphi \vec{n}\vec{\sigma}) = \begin{pmatrix} \cos N\varphi + jn_z \sin N\varphi & (jn_x + n_y)\sin N\varphi \\ (jn_x - n_y)\sin N\varphi & \cos N\varphi - jn_z \sin N\varphi \end{pmatrix} =$$

$$\begin{pmatrix} \cos N\varphi + \beta \dfrac{\sin N\varphi}{\sin \varphi} & e^{-jkL}\dfrac{R}{D}\dfrac{\sin N\varphi}{\sin \varphi} \\[2mm] -e^{jkL}\dfrac{R}{D}\dfrac{\sin N\varphi}{\sin \varphi} & \cos N\varphi - \beta \dfrac{\sin N\varphi}{\sin \varphi} \end{pmatrix} \qquad (3.4.18)$$

were we have introduced the notation

$$\beta := \frac{1}{2}\left(T_{11}^t(L,0) - T_{22}^t(L,0)\right) = \frac{1}{2D}\left(e^{-jkL}(D^2 - R^2) - e^{jkL}\right) \qquad (3.4.19)$$

And, after simple matrices multiplications, we have for the transfer matrix (3.4.4)

$$\widehat{S}_{\Sigma(N)} = \widehat{T}(0,(N+1)L/2)\left(\widehat{T}(L,0)\widehat{S}\right)^N \widehat{T}(-(N-1)L/2,0) =$$

---

[5] The complexity of the components of the unit vector $\vec{n}$, in contrast to the real unit vector for the operation of finite rotation, is caused by the fact that the transfer matrix $\widehat{T}^t(L,0)$ now is not hermitian.

$$\begin{pmatrix} e^{jkLN}\left(\cos N\varphi + jn_z \sin N\varphi\right) & e^{jkL}\left(jn_x + n_y\right)\sin N\varphi \\ e^{-jkL}\left(jn_x - n_y\right)\sin N\varphi & e^{-jkLN}\left(\cos N\varphi - jn_z \sin N\varphi\right) \end{pmatrix} = \tag{3.4.20}$$

$$\begin{pmatrix} e^{jkLN}\left(\cos N\varphi + \beta\dfrac{\sin N\varphi}{\sin \varphi}\right) & \dfrac{R}{D}\dfrac{\sin N\varphi}{\sin \varphi} \\ -\dfrac{R}{D}\dfrac{\sin N\varphi}{\sin \varphi} & e^{-jkLN}\left(\cos N\varphi - \beta\dfrac{\sin N\varphi}{\sin \varphi}\right) \end{pmatrix}$$

Comparing (3.4.18) with the equation for the total transfer matrix, expressed through the total reflection and transmission coefficients (3.4.5), we can find for these coefficients:

$$R_{\Sigma(N)} = \frac{R}{D}\frac{\sin N\varphi}{\sin \varphi}\frac{e^{jkLN}}{\cos N\varphi - \beta\dfrac{\sin N\varphi}{\sin \varphi}} \; ; \quad D_{\Sigma(N)} = \frac{e^{jkLN}}{\cos N\varphi - \beta\dfrac{\sin N\varphi}{\sin \varphi}} \tag{3.4.21 a,b}$$

Eq. (3.4.21) can be rewritten in another form using the definition of Chebyshev polynomials of the first, $T_N(x)$, and second, $U_N(x)$, kind [21].

$$\cos N\varphi = \cos\left(N\arccos(\alpha)\right) = T_N(\alpha) \tag{3.4 22 a,b}$$

$$\sin N\varphi = \sin\left(N\arccos(\alpha)\right) = (1-\alpha^2)U_{N-1}(a) = \sin \varphi\, U_{N-1}(\alpha)$$

$$R_{\Sigma(N)} = \frac{R}{D}\frac{e^{jkLN}U_{N-1}(\alpha)}{T_N(\alpha) - \beta U_{N-1}(\alpha)} \; ; \qquad D_{\Sigma(N)} = \frac{e^{jkLN}}{T_N(\alpha) - \beta U_{N-1}(\alpha)} \; ; \tag{3.4.23 a,b}$$

Using the definition of the auxiliary values α (3.4.13) and β (3.4.19) we can rewrite the equations for the total reflection and transmission coefficients of the periodical chain of one-dimension scatterers in the explicit form:

$$R_{\Sigma(N)} = \frac{R}{D}\frac{e^{jkLN}U_{N-1}\left(\dfrac{e^{-jkL}(D^2 - R^2) + e^{jkL}}{2D}\right)}{T_N\left(\dfrac{e^{-jkL}(D^2 - R^2) + e^{jkL}}{2D}\right) - \dfrac{e^{-jkL}(D^2 - R^2) - e^{jkL}}{2D}U_{N-1}\left(\dfrac{e^{-jkL}(D^2 - R^2) + e^{jkL}}{2D}\right)} \tag{3.4.24}$$

$$D_{\Sigma(N)} = \frac{e^{jkLN}}{T_N\left(\dfrac{e^{-jkL}(D^2 - R^2) + e^{jkL}}{2D}\right) - \dfrac{e^{-jkL}(D^2 - R^2) - e^{jkL}}{2D}U_{N-1}\left(\dfrac{e^{-jkL}(D^2 - R^2) + e^{jkL}}{2D}\right)} \tag{3.4.25}$$

The equations (3.4.24) –(3.2.25) are very important results of the present report.

We note, that the eq. for the Nth power of the matrix $\hat{T}^t(L,0)$ (3.4.18) and all formulae following for the reflection and transmission coefficient can be also obtained if one uses the standard methods to diagonalize matrices [22], for matrices with unit determinant. The present way, however, is clearer and can be a basis for future generalization in the case of multiconductor wires.

Let us now shortly investigate some special cases of eqs. (3.2.24)-(3.2.25).
For the case of a real potential (lossless line), using eqs. (3.2.2) and (3.2.3), we can write

$$\alpha = \cos(\varphi) = \mathrm{Re}\left(\frac{e^{jkL}}{D}\right); \qquad\qquad \beta = -\mathrm{Im}\left(\frac{e^{jkL}}{D}\right) \qquad\qquad (3.4.26\ a,b)$$

and, hereafter, (3.24)-(3.25) are reduced to the result of previous report [4]:

$$R_{\Sigma(N)} = \frac{R}{D}\,\frac{e^{jkLN} U_{N-1}\left(\mathrm{Re}\left(\dfrac{e^{jkL}}{D}\right)\right)}{T_N\left(\mathrm{Re}\left(\dfrac{e^{jkL}}{D}\right)\right) + j\,\mathrm{Im}\left(\dfrac{e^{jkL}}{D}\right) U_{N-1}\left(\mathrm{Re}\left(\dfrac{e^{jkL}}{D}\right)\right)} \qquad\qquad (3.4.27)$$

$$D_{\Sigma(N)} = \frac{e^{jkLN}}{T_N\left(\mathrm{Re}\left(\dfrac{e^{jkL}}{D}\right)\right) + j\,\mathrm{Im}\left(\dfrac{e^{jkL}}{D}\right) U_{N-1}\left(\mathrm{Re}\left(\dfrac{e^{jkL}}{D}\right)\right)} \qquad\qquad (3.4.28)$$

For the case of one scatterer $N=1$ (potential of general view), we use the values of Chebyshev polynomials ($T_1(x)=x$, $U_0(x)=1$) and find from eq. (3.4.22) an obvious result

$$D_{\Sigma(1)} = D; \qquad\qquad R_{\Sigma(1)} = R; \qquad\qquad (3.4.29\ a,b)$$

For the case of two scatterers $N=2$ using the values of the Chebyshev polynomials ($T_2(x)=2x^2-1, U_1(x)=2x$) it is possible to find

$$D_{\Sigma(2)} = \frac{D^2}{1-R^2 e^{-2jkL}}; \qquad\qquad R_{\Sigma(2)} = R\,e^{jkL} + \frac{RD^2 e^{-jkL}}{1-R^2 e^{-2jkL}}; \qquad\qquad (3.4.30\ a,b)$$

which can also be obtained by the application of Feynman diagrams for the one dimensional scattering[6].

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[6] During this calculation we use the facts that the coefficients R and D are quantum-mechanical complex amplitudes of the reflection and transmission events of one-center scattering. The complex amplitude of the free propagation between points l_2 and l_1 is given by the exponent function $\exp(-jk(l_2-l_1))$. After that one can consider different quantum mechanical processes, which lead to the propagation through or reflection from the chain of two potentials (penetration through the first potential, propagation between potentials, propagation

If the potential, which corresponds to one separate non-uniformity, is not easy to penetrate, $|R| \approx 1$, $|D| << 1$ one can observe a resonance scattering. To make a short quality investigation of this phenomena we consider the eq. (3.4.30a) and note that the one-potential reflection and transmission coefficients D and R have more slowly frequency dependence in comparison with the exponential function $\exp(-2jkL)$. Let us investigate now the frequency dependence of the transmission coefficient $D_{\Sigma(2)}$. For the main frequency region, when $|2kL - 2\varphi_R - 2\pi n| \sim 1$ (where φ_R is a phase of the transmission coefficient R, $n = 1,2,3..$) the denominator in (3.4.27) has an order of magnitude one, and, by this way, the penetration through the two-potential chain is small $|D_{\Sigma(2)}| \sim |D^2| << 1$. However, in the narrow frequency bands, when $|2kL - 2\varphi_R - 2\pi n| << 1$, one can observe a resonant scattering. By the introduction of the detuning $\Delta k_n = 2kL - 2\varphi_R - 2\pi n$ of the nth resonance one can write for the frequency dependence of the transmission coefficient in the neighborhood of this resonance the next equation:

$$D_{\Sigma(2)} \approx \frac{D^2}{1 - |R|^2 + 2j\Delta k_n |R|^2} \qquad (3.4.31)$$

The eq. (3.4.31) describes a typical resonance frequency curve. For the zero value of the frequency tuning a value of the propagation coefficient strongly increases

$$(D_{\Sigma(2)})_{res} \approx \frac{D^2}{1 - |R|^2} \qquad (3.4.32)$$

For the case of the lossless potential $(D_{\Sigma(2)})_{res} \approx D^2 / |D|^2$ and it absolute value is one.

One can show [23] that in the case of resonance scattering the particle is "jammed" between these potential pits (barriers), and has multiple re-reflections. It spends a long time inside the chain of potentials that leads to the increase of the wave function amplitude inside the chain. In the electrodynamics language, during the resonant scattering, the current amplitude between two scatterers increases. If we deal with lossy systems, the losses of any nature (ohmic, radiation, etc.) have to increase. This, for example, can be important for the intensity of radiation of the system and for the resistance of the system with respect to ohmic heating.

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through the second potential, reflection from the first and second potential) and use the quantum mechanical axioms, which state:
1. The amplitude of two independent events is the sum of their amplitudes;
2. The amplitude of two events in sequence is a product of their single amplitudes.

3.5 Allowed and forbidden frequency zones. Connection of the parameters of quasi-periodical and periodical systems.

In the present Section, we use the obtained results to investigate the transmission coefficients and to establish a connection between parameters of quasi-periodical and periodical systems.

First, we consider the lossless case. For such systems the equation of the "rotation angle" φ (main branch) can be written ((3.4.26 a)) for different magnitudes of the value α as

$$\varphi = \begin{cases} j\,\text{arccosh}\big(\text{Re}(e^{jkL}D^{-1})\big), & \text{Re}(e^{jkL}D^{-1}) \geq 1 \\ \arccos\big(\text{Re}(e^{jkL}D^{-1})\big), & -1 \leq \text{Re}(e^{jkL}D^{-1}) \leq 1 \\ \pi - j\,\text{arccosh}\big(\text{Re}(e^{jkL}D^{-1})\big), & \text{Re}(e^{jkL}D^{-1}) \leq -1 \end{cases} \qquad (3.5.1)$$

If the wave number (frequency) is such, that the parameter $-1 \leq \cos\varphi = \text{Re}(e^{jkL}D^{-1}) \leq 1$, we have in the denominator of (3.4.21 b) oscillating functions, and the total propagation coefficient is of the order of magnitude one. In the opposite limiting cases $\text{Re}(e^{jkL}D^{-1}) \geq 1$ or $\text{Re}(e^{jkL}D^{-1}) \leq 1$, we have the hyperbolical function in the denominator and the propagation coefficient is exponentially damped $\left|D_{\Sigma(N)}\right| \sim \exp(-N\varphi)$ (for $N \gg 1$), and the reflection coefficient is about one. Thus, we have shown that allowed and forbidden zones appear for the finite chain of potentials. These frequency zones are called allowed and forbidden, correspondingly, because in the allowed zone for $N \to \infty$ the particle can penetrate inside the semi-infinite chain and cannot be in the forbidden zone. The existence of the allowed and forbidden zones is well known in solid-state physics for an infinite periodical potential [7]. The good penetration through the finite chain of potentials for the allowed zone can be physically explained as a resonance scattering on the quasi-stationary energy levels of the chain of potentials, which appear because of the splitting of the quasi-stationary energy levels in the system of two potentials pits (barriers) (see the end of the previous Section). In this case, again one has $|R| \approx 1$, $|D| \ll 1$, the particle is "jammed" between these potential pits (barriers) and has multiple re-reflections. Again, the wave function (current amplitude) strongly increases, but for the chain the increase can be much stronger in comparison with the case of two potentials. This phenomena, numerically was investigated in [4] and can serve as basis to construct radiating devices, as well as to investigate the resistance of a periodical system with respect to ohmic heating.

It is possible to show that the inclusion of radiation losses reduces the penetration in the allowed zones, (see Section 4 of the present report), but the structure of the allowed and forbidden zones remains valid.

Now we establish the connection between the parameters of the propagation of the particle through the infinite chain and investigate parameters of one-center scattering.
If the particle propagates through the infinite chain of potentials with period L,

$$u(l + L) = u(L) \qquad (3.5.2)$$

It´s wave function can be represented as [7] (Floquet theorem):

$$\psi(l) = \exp(-jKl) \cdot \Psi(l), \tag{3.5.3}$$

$$\Psi(l+L) = \Psi(l) \tag{3.5.4}$$

where the function $\Psi(l)$ is periodical with period L, and the parameter K, called quasi-pulse (more exactly quasi-wave number), characterizes the translation properties of the wave function (it may be positive as well as negative).

$$\psi(l+L) = \exp(-jK(l+L)) \cdot \Psi(l+L) = \exp(-jKL)\psi(l) \tag{3.5.5}$$

Remember that in our case of periodical potential pits (barriers), separated by asymptotic regions, the wave function can be represented in the asymptotic region as

$$\psi(l) = C_1 \exp(-jkl) + C_2 \exp(jkl) \tag{3.5.6}$$

The transfer matrix $\widehat{T}^t(L,0)$ for the column vector $\underset{\downarrow}{C}$ for one period translation is given by eq. (3.4.1). On the other hand, eq. (3.5.5) yields the following expression for the one-period transfer matrix

$$\widehat{T}^t(L,0) = \exp(-jKL)\widehat{I} \tag{3.5.7}$$

Equalizing the transfer matrices (3.4.1) and (3.5.7) leads to a homogeneous linear system for the column vectors:

$$\begin{pmatrix} e^{-jkL} \dfrac{(D^2-R^2)}{D} & e^{-jkL} \dfrac{R}{D} \\ -e^{jkL} \dfrac{R}{D}, & e^{jkL} \dfrac{1}{D} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = e^{-jKL}\widehat{I} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \tag{3.5.8}$$

which is solvable, if

$$\det \begin{pmatrix} e^{-jkL} \dfrac{(D^2-R^2)}{D} - e^{-jKL} & e^{-jkL} \dfrac{R}{D} \\ -e^{jkL} \dfrac{R}{D}, & e^{jkL} \dfrac{1}{D} - e^{-jKL} \end{pmatrix} \neq 0 \tag{3.5.9}$$

This equation yields as a result for the quasi-pulse

$$\cos(KL) = \frac{e^{-jkL}(D^2-R^2) + e^{jkL}}{2D} = \cos(\varphi) \tag{3.5.10}$$

The equation (3.5.10) establishes the connection between the quasi-pulse (quasi-wave number) K and the usual wave number $k = \omega/c$. It is the so-called dispersion equation [7].

From the (3.5.10) and (3.5.1) one can observe, that for the allowed zones the quasi-pulse is real (for the lossless potential) and the particle can propagate along the infinite chain.

For the forbidden zones the quasi – pulse is imaginary and the propagation disappears. In case of a lossy potential the imaginary part of the propagation constant influences the partical propagation through the allowed zones by some decrement.

Now we want to add a few words about the propagation of energy in the infinite periodical system. Using the formulae from Section 3.2, one can obtain the following equation (in the asymptotic regions) for the averaged power propagating along the line

$$W = \frac{Z_C |I_0|^2}{2} \left(|C_1|^2 - |C_2|^2 \right) \tag{3.5.11}$$

Hereafter, it is necessary to use the connection of the coefficients C_1 and C_2 from eq. (3.5.8). Omitting quite awkward investigations, we formulate here the finite results: for some (even) allowed zones the direction of the propagation of the phase – the sign of the quasi-pulse K coincides with the direction of the propagation of the energy – have the sign of (3.5.11). For other (odd) allowed zones, these directions are opposite. It seems to be that this fact is connected with the experimentally established [24] connection of the phase and group velocity of periodically loaded transmission lines, when, for the some frequency bands, they have opposite directions.

4. Numerical example. Relative importance of different lossy mechanisms.

In this Section, we consider a specific example and apply the developed method. We consider a wire of radius $a = 1$ cm, which came from minus infinity at a height of $h = 1$ m, experiences several oscillations, and runs to plus infinity. The non-homogeneous part of the wire consists of 5 identical sections of a Gaussian form (4.1) (see also Fig. 4.1).

$$x(z) = h - b \sum_{n=-2}^{2} \exp\left(-k_0 (z - nL)^2\right) \tag{4.1}$$
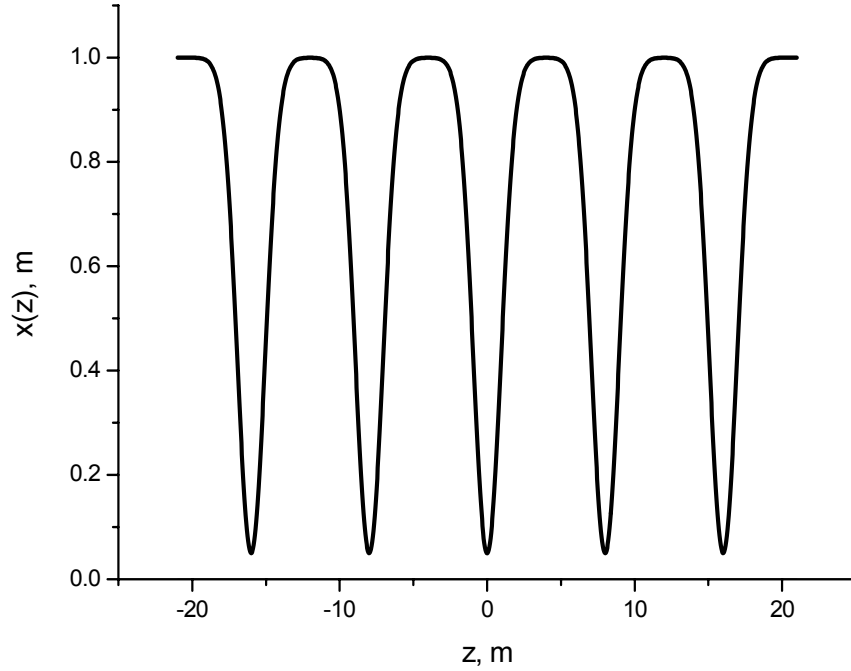
Fig. 4.1: Geometry of the quasi-periodical wiring structure: $h = 1$ m, b=0.75 m, $k_0 = 0.75\, m^{-1}$, $a = 0.01$ m, $L = 8$ m, $N = 5$.

First we consider a perfectly conducting uncoated wire, where only radiation losses are possible. The elements of the matrix of the corresponding global parameters $[P(k,l)]$ of the Full-Wave Transmission Line can be calculated using of our perturbation theory (eqs. (3.2.20)-(3.2.22)). These parameters are complex-valued and coordinate-dependent (see Figs. 4.2 a,b,c,d). To check our calculation, we compare the results for k=0 with the static results for $P_{12}(k,l) \underset{k \to 0}{=} L_0'(l) \underset{k \to 0}{}$ and $P_{21}(k,l) \underset{k \to 0}{=} C_0'(l) \underset{k \to 0}{}$ obtained by eq.(3.23)-(3.24) (see Fig:4.3 a,b). Since the considered wire system is symmetrical around the origin of coordinates, one can observe that the diagonal parameters are symmetrical and the anti-diagonal parameters are anti-symmetrical around the origin of the coordinates (see eq. 3.2.10).

Hereafter we calculate Mei's parameters $U_M(l)$ and $T_M(l)$ in the second order differential equations (see Fig. 4.4 a,b) (3.1.1). We note that for the symmetrical wire $U_M(l)$ is an anti-symmetrical function of l ($U_M(-l) = -U_M(l)$) and the integral $\int_{-\infty}^{\infty} U_M(l)dl = 0$, which confirm the reasoning at the end of the sec. 3.2 (eq. 3.2.11).
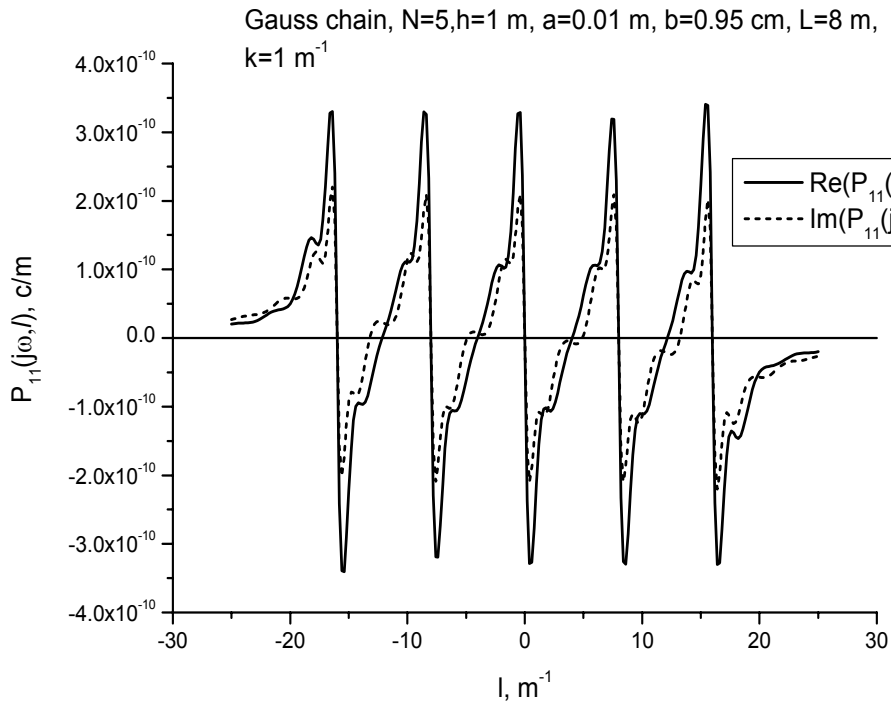
Fig. 4.2 a
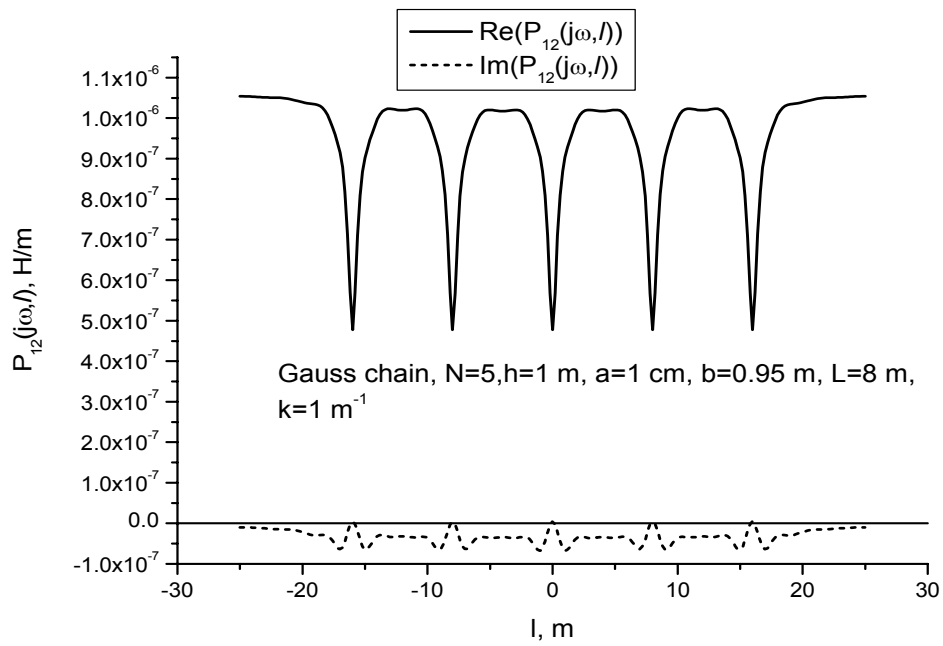
Fig. 4.2 b
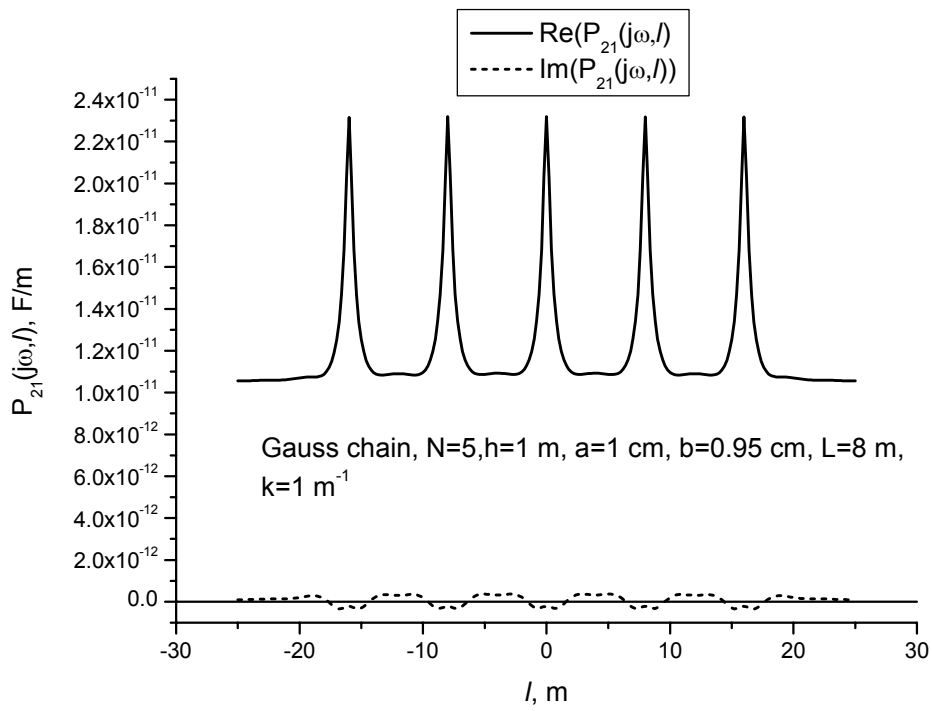
Fig. 4.2 c
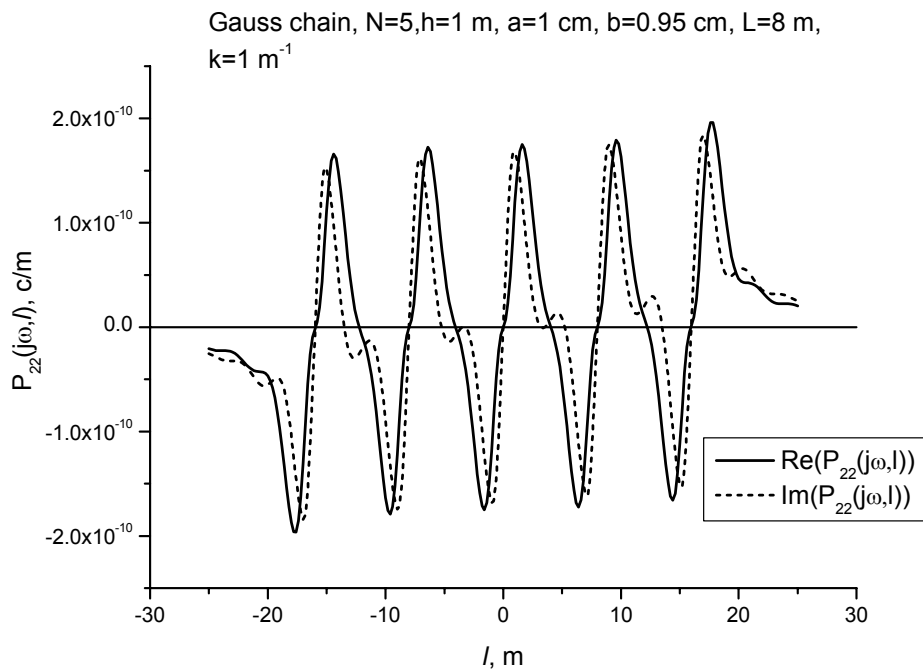
Fig. 4.2 d

Fig. 4.2: Matrix of the global parameters for the Gaussian-chain wiring structure: a - $P_{11}(j\omega,l)$, b - $P_{12}(j\omega,l)$, c - $P_{21}(j\omega,l)$, d - $P_{22}(j\omega,l)$.
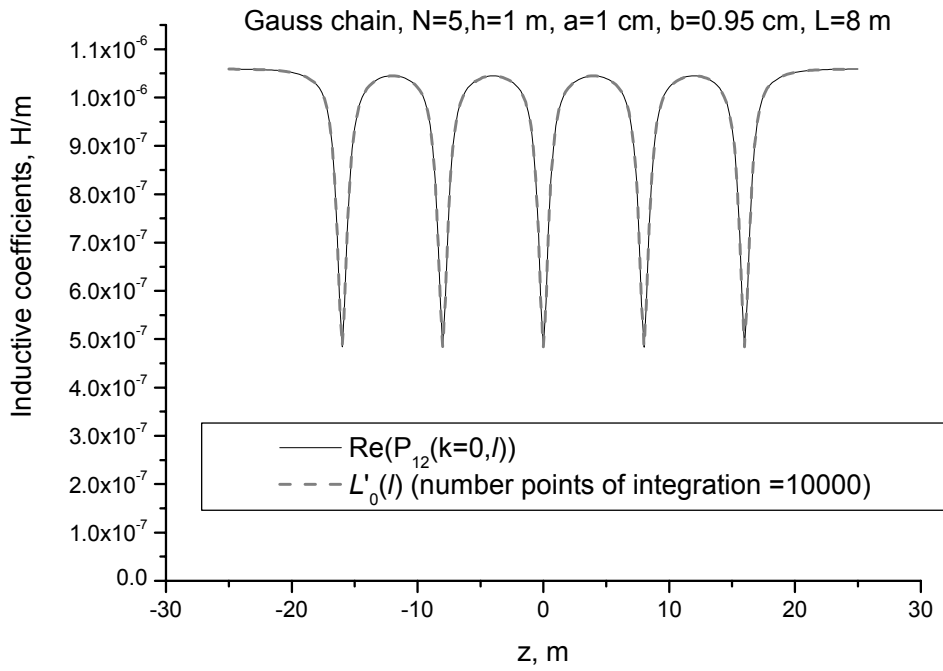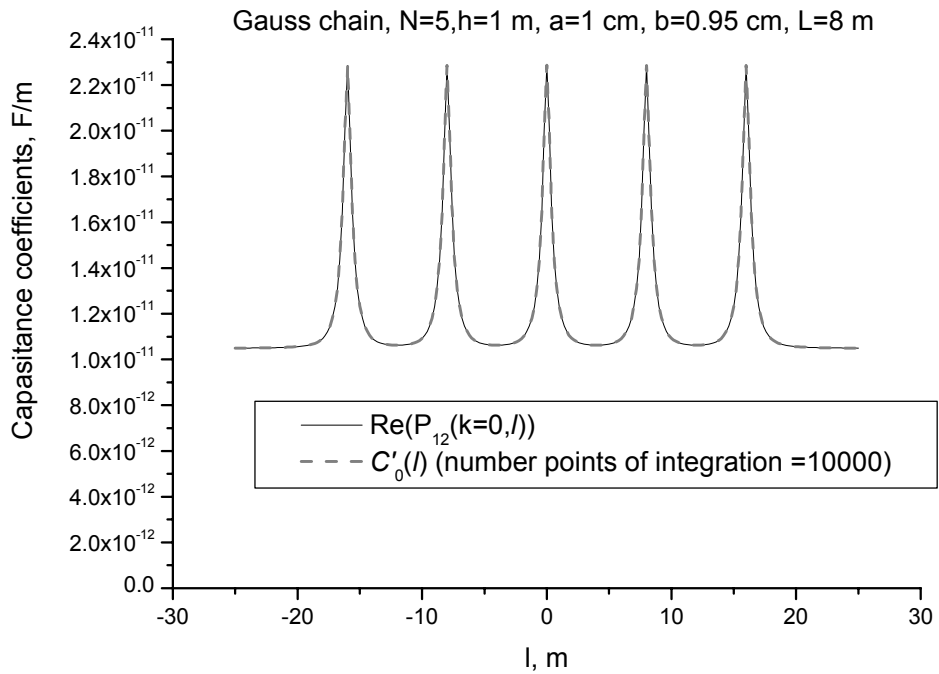
Fig. 4.3 a

Fig. 4.3 b

Fig. 4.3: Comparison of the inductive and capacitive coefficients for $k = 0$ with static inductance and capacitance for the quasi-periodical wire structure:
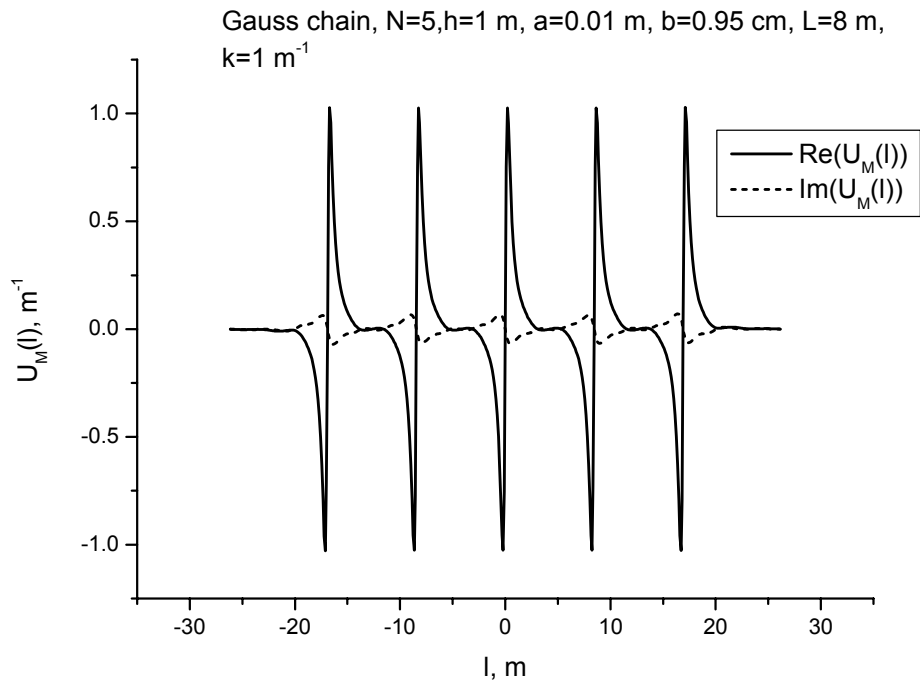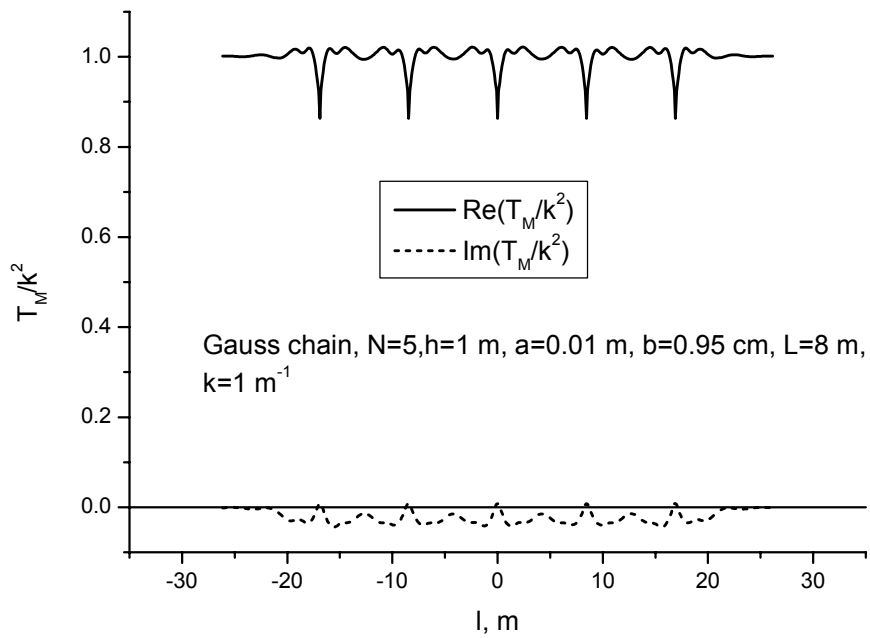a - $P_{12}(k = 0, l)$ and $L'_0(l)$, b - $P_{21}(k = 0, l)$ and $C'_0(l)$.

Gauss chain, N=5,h=1 m, a=0.01 m, b=0.95 cm, L=8 m,
k=1 m^{-1}

Fig. 4.4 a

Fig. 4.4 b

Fig.4.4: Mei's global parameters for the periodical wiring structure: a - $U_M(j\omega,l)$, b - $T_M(j\omega,l)/k^2$.

The total potential $u(k,l)$ for the quasi-periodical wiring system, which is calculated with the help of global parameters $[P(k,l)]$ is presented in Fig. 4.5. Note, that the imaginary part of the potential is not exact periodical. In addition, it is important to note, that the real part of the "potential" practically does not depend on the frequency (at least for $0 \le k \le 1.5$ m^{-1}) (see Fig. 4.6 a), but the imaginary part, which defines radiation, is frequency – dependent (see Fig. 4.6 b). This frequency dependence can be roughly approximated by a quadratic frequency function $u(k,l) \sim k^2$. This approximation for different points (central maximum point of the real part of the potential and a relative minimum point of the real part of the potential) is presented in Fig. 4.7. It is interesting that the imaginary part of potential is positive in one point and is negative in another. The explanation, in our opinion, is that this complex wiring structure in some points radiates energy, but in some other points absorbs energy.

Now, with the knowledge of the potential we can obtain the total transfer matrix of the system and, after that obtain a transmission coefficient of the current wave through the system. To do that, simple numerical methods have been developed (see Appendix).

To check our calculation, we use the well-known MOM code CONCEPT. The configuration of the wire structure, which was used to model the infinite quasi-periodical structure, is shown in Fig. 4.8. The infinite quasi-periodical system was finished at some distance from the periodical part and was supplemented by the vertical risers loaded by the characteristic impedances of the line $Z_C = \eta / 2\pi \ln(2h/a) \approx 317.7\,\Omega$. The system is excited by the unit voltage source $U_0 = 1$ V at the left terminal. Under such condition it is possible to show, that the voltage on the right load is connected with the value of the transfer function. Of course, this connection is valid, if the Transmission Line approximation can be applied to the asymptotical and near – terminal regions of the line.

In the Figure 4.11 there are important results of the present report. The black curve presents the result of the CONCEPT code calculation for the configuration of Fig. 4.8, when the doubled voltage on the matched load with unit voltage excitation is approximately the transmission coefficient of the current wave through the system. On this curve, we can recognize the allowed and forbidden zones. However, in contrast with results of previous modelling with a real potential (red curve) [4] we can see the decrement of the transmission coefficient, which is caused by radiation. The green curve is the result of calculation with the potential from Fig. 4.5 by the matrix method. A quite good agreement with CONCEPT is observed (before the fourth allowed zone). However, because we have to calculate the potential in each frequency point (different from the low-frequency case) the calculation time is very long and we have to use a quite rough division of the interval. (We divided the 50 m distance interval into 1000 subintervals and used 100 frequency points. Moreover, for this rough approximation the calculation time was about 30 hours!). The difference of the two methods can be explained by the strong radiation near the vertical elements.

On the other hand we used our analytical formulae for the propagation coefficient of the chain. First, we define by the matrix method the reflection and transmission coefficients (see Fig. 4.10) through one partial (central) potential of the Gaussian chain (see Fig. 4.9). This calculation is 5 times faster than the transfer matrix calculations for the total system. After that, we used our analytical formulae (with Chebyshev's polynomials). The result is presented by the blue curve. The agreement with the previous curve is quite good. The difference again can be explained by two reasons: in reality, the potential is not periodic and the division is quite rough.
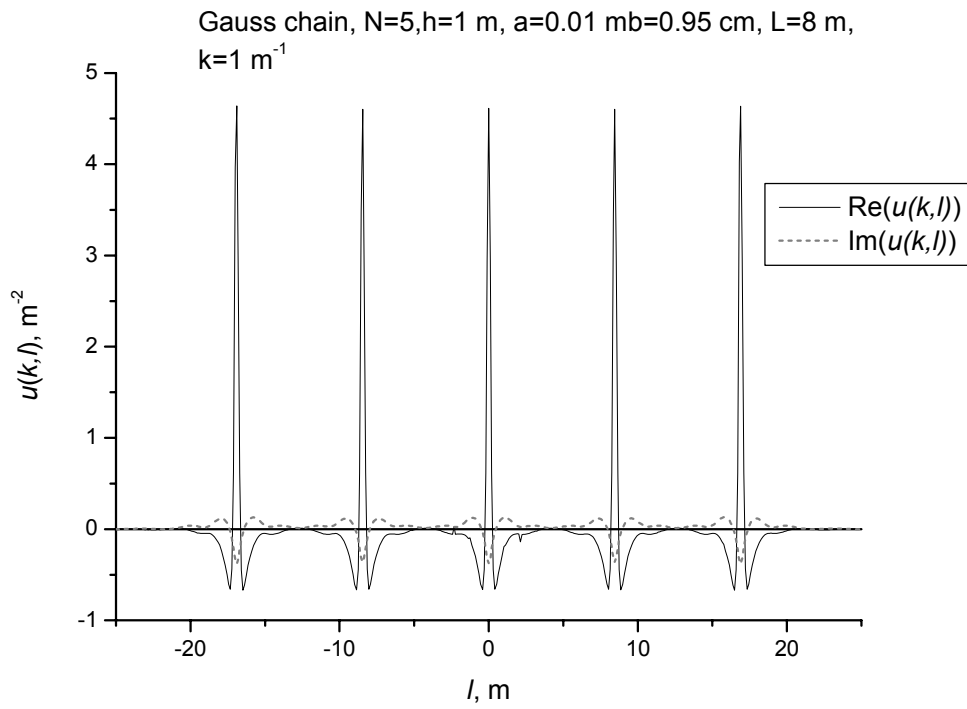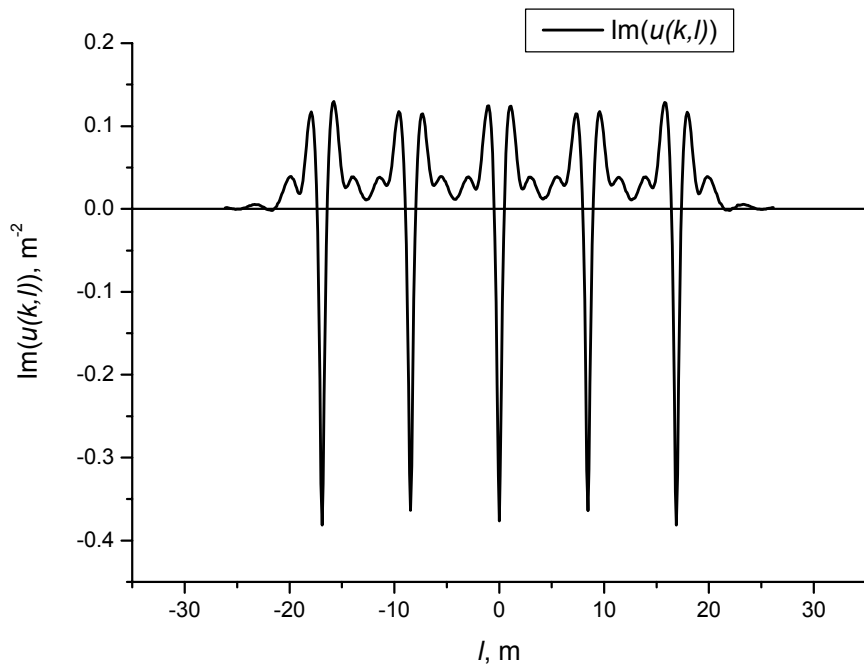
Fig. 4.5 a

Fig. 4.5 b

Fig. 4.5: "Potential" $u(k,l)$ for the periodical wiring structure. a – real and imaginary parts, b – imaginary part.
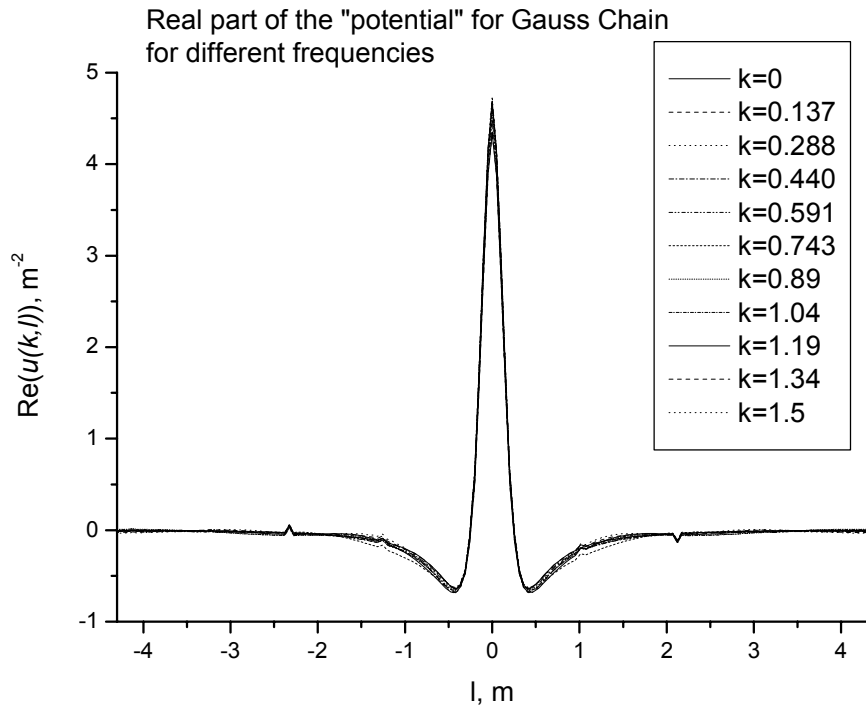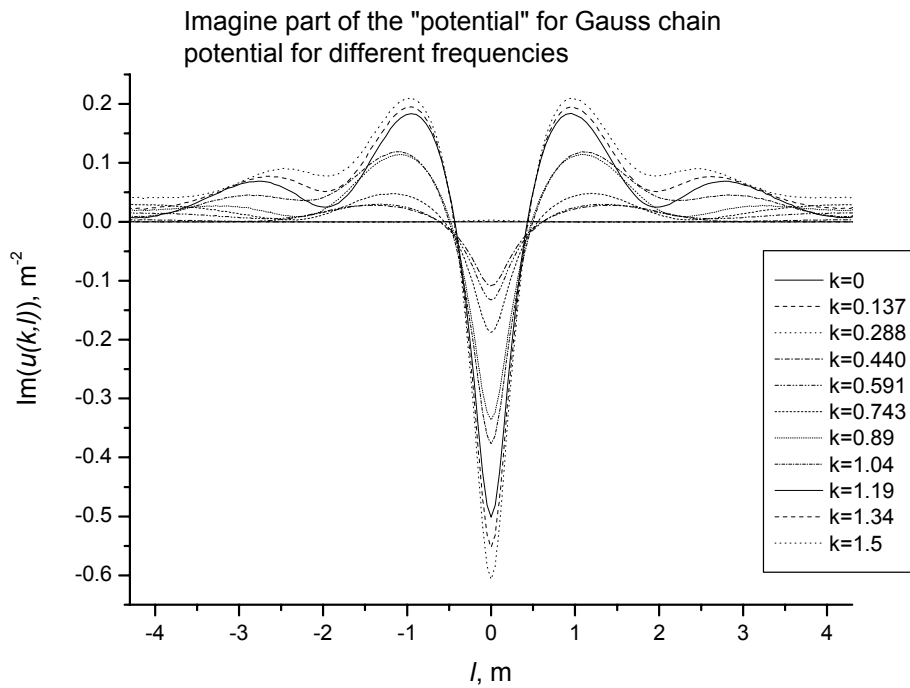
Fig. 4.6 a

Fig. 4.6 b

Fig. 4.6: Spatial dependence of the real and imaginary part of the partial "potential" for the quasi-periodical Gauss system (the central period, $n = 0$, $-4 \leq z \leq 4$) for different frequencies.
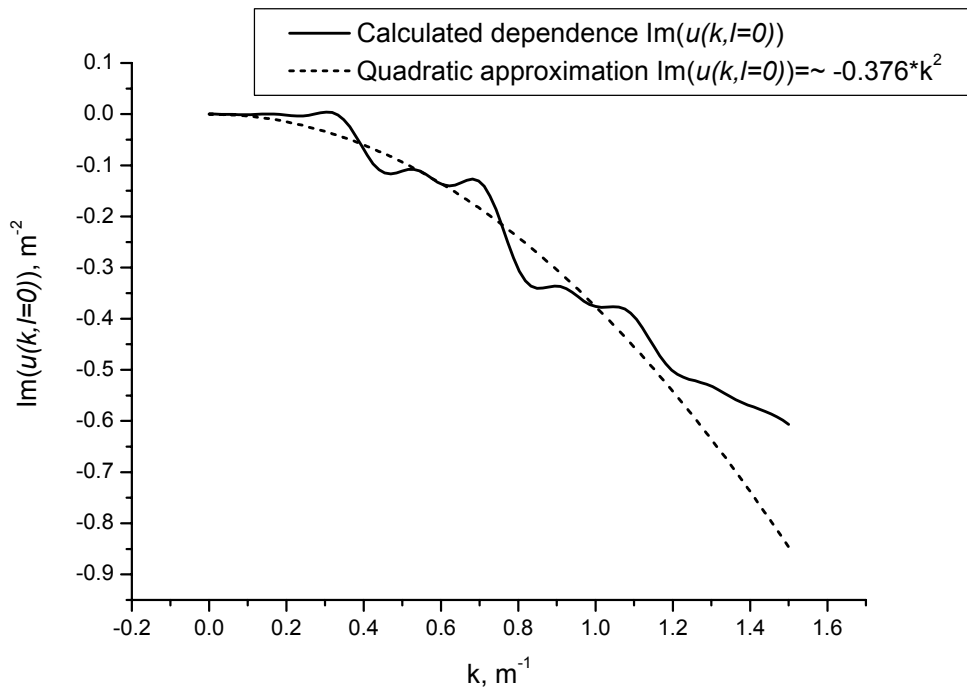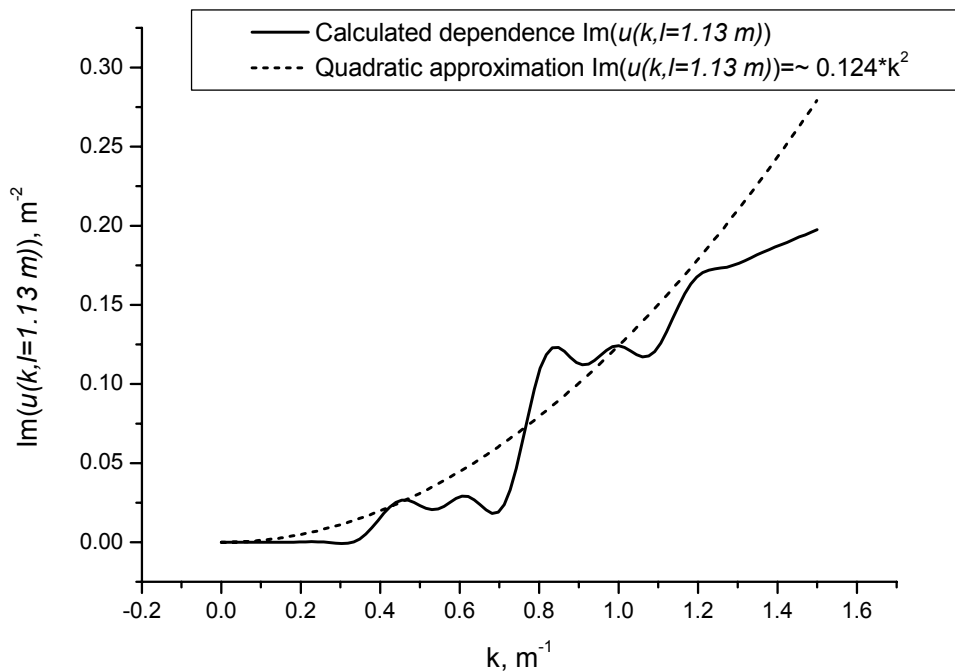
Fig. 4.7 a

Fig. 4.7 b

Fig. 4.7: Frequency dependencies (exact and approximation) of the imaginary part of the partial "potential" for the quasi-periodical Gaussian system (the central period, $n = 0$) for different spatial points: a - $l = 0$; b - $l = 1.13$ m.

The investigated "Gaussian pit" wiring system
h=1 m, Δh=0.05 m
U_0=1 V, Z_c=317,7 Ω
$|D(k)|=\sim 2U_{Z_c}|_{y=50}/U_0$
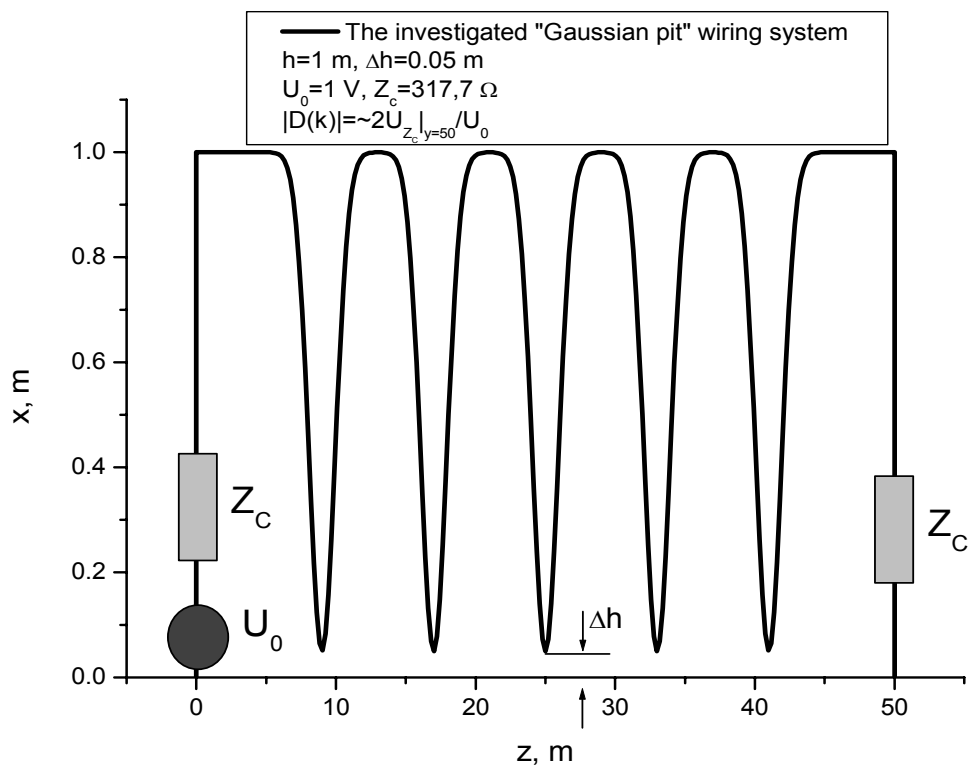
Fig. 4.8: Geometry of the quasi - periodical wiring structure for the CONCEPT simulation.

One "potential" of the Gauss chain,
N=5,h=1 m, b=0.95 cm, a=0.01 m, L=8 m,
k=1 m^{-1}
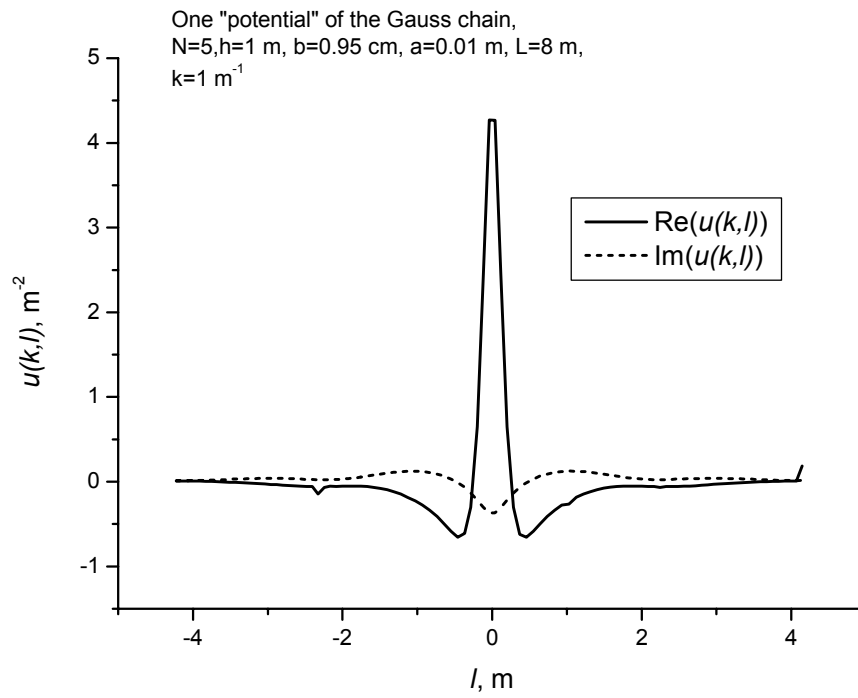
Fig. 4.9: One partial "potential" (central) of the Gaussian wiring chain ($-4 \leq z \leq 4$).
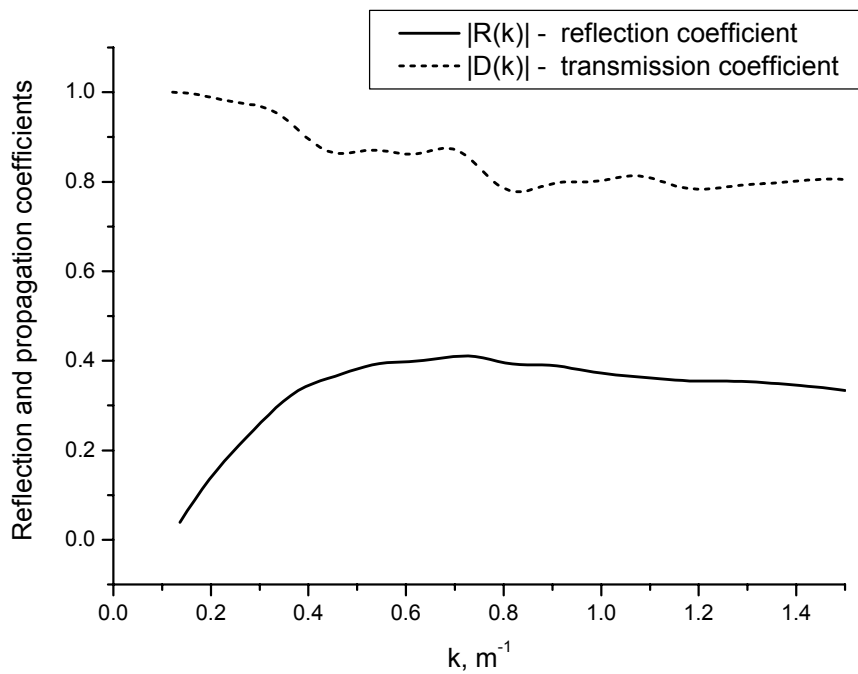
Fig. 4.10: Reflection and transmission coefficients for the one partial "potential" (central) of the Gaussian wiring chain.
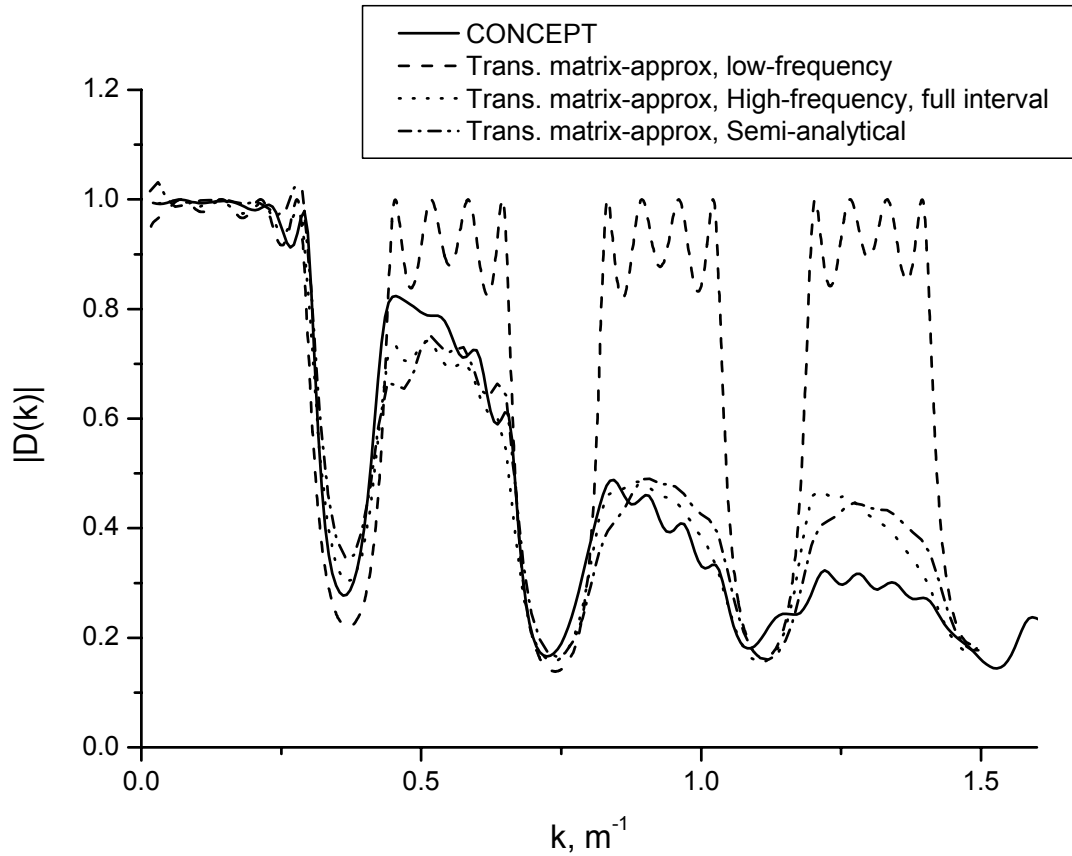
Fig. 4.11: Transmission coefficient for the considered periodical structure calculated by different methods.

Now we consider ohmic losses and dielectric coating on the propagation of current waves through the quasi-periodical wiring structure. First, we consider the ohmic losses without radiation (when the potential is defined by the eq. (3.1.12)). The numerical calculations have shown that the real part of the potential (3.1.12) practically coincides with the lossless case (see Fig. 4.6 for $k = 0$). The imaginary part of the potential for the different conductivities of the wire is shown in Fig. 4.12. The corresponding propagation coefficient is displayed in Fig.4.13. One can observe that the influence of ohmic losses is small in comparison with radiation losses.

Second, we investigate the influence of the dielectric coating of the wire on the transfer coefficient. We consider a perfectly conducting wire with radius $a = 1$ cm coated by a dielectric layer with thickness $\Delta = 0.5$ cm and dielectric permittivity of $\varepsilon = 3$. The corresponding potential for one section calculated by 3.1.12 (in comparison with the uncoated case) is presented in Fig. 4.14, and the propagation coefficient is presented in Fig. 4.15. Again, one can observe that the influence of the dielectric coating on the propagation of the current wave is quite small.
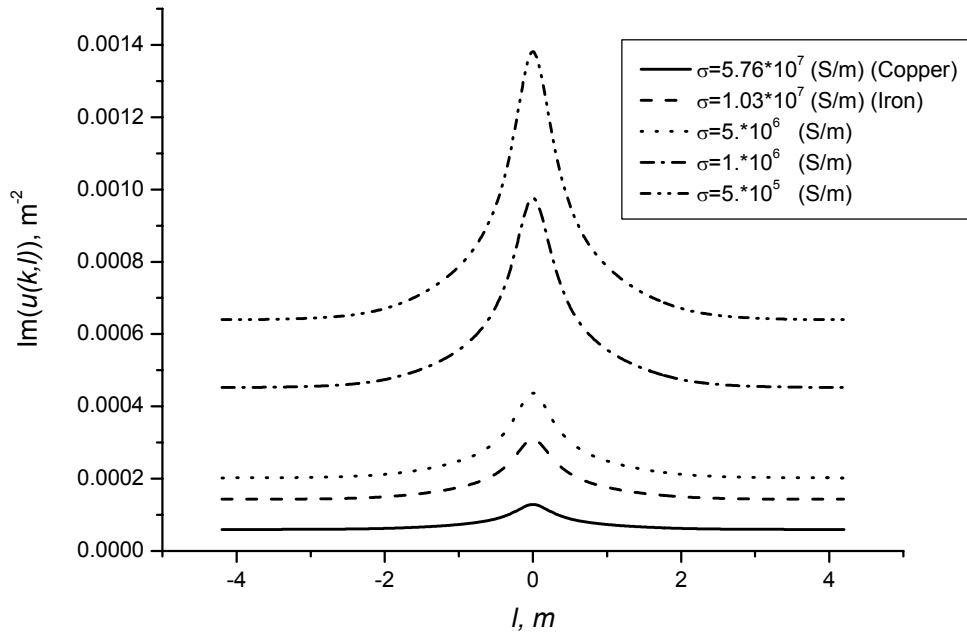
Fig. 4.12: Imaginary part of the "potential " $u(k,l)$ for one "section" ($-4 \leq z \leq 4$) of the periodical structure for different conductivities of the wire.
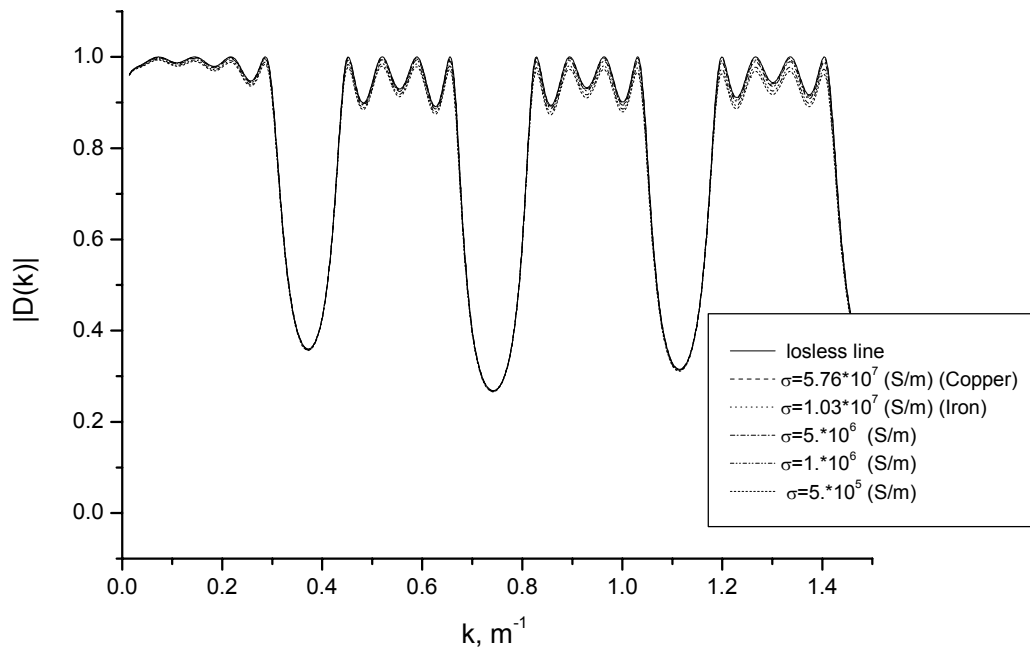

Fig. 4.13: Propagation coefficients through the periodical structure for different conductivities of the wire.
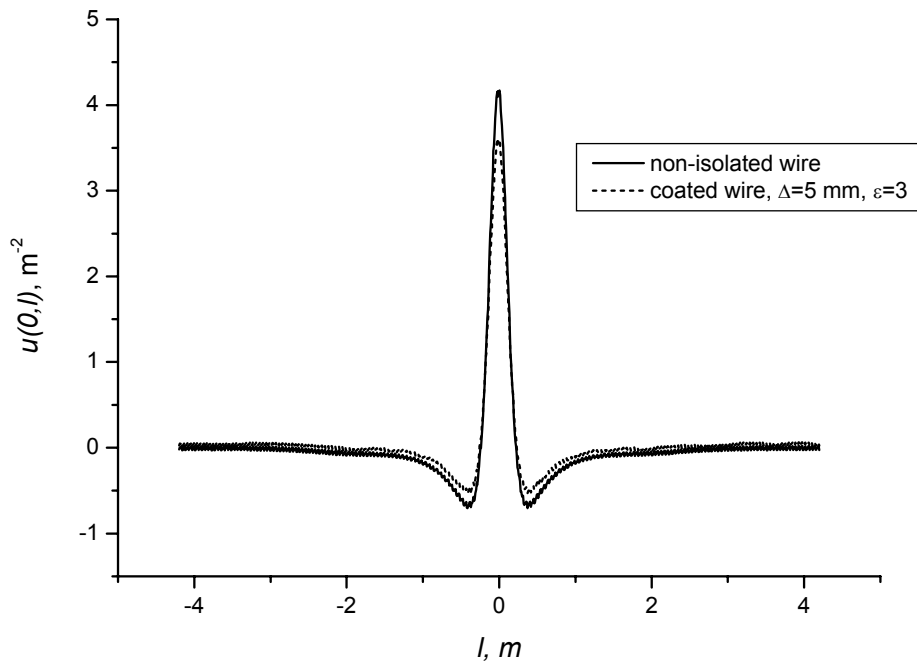
Fig. 4.14: The "potential " $u(0,l)$ for one "section" $(-4 \leq z \leq 4)$ of a perfectly conductive coated and uncoated periodical wiring structure.
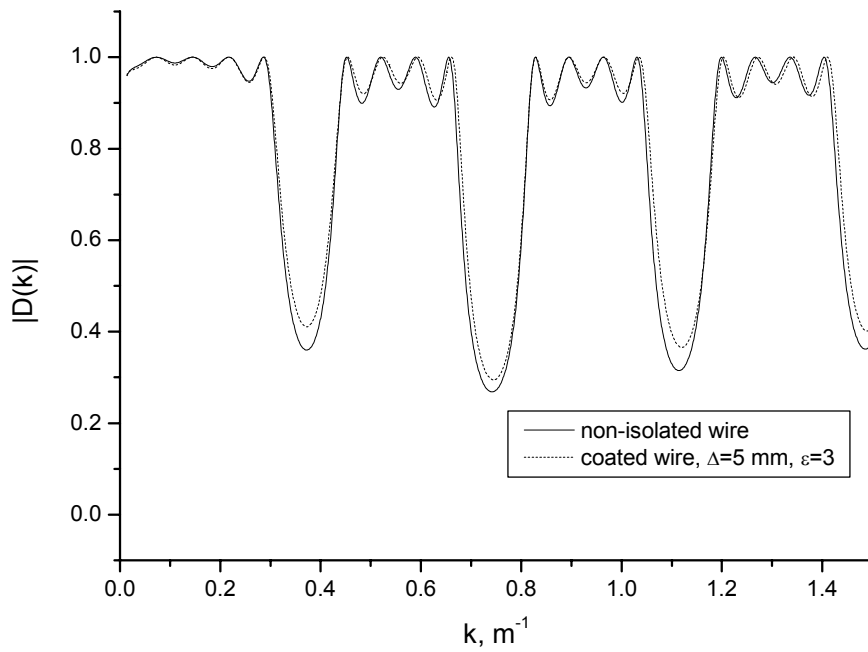

Fig. 4.15: Propagation coefficients through the perfect conductive coated and uncoated periodical wiring structure.

5. Conclusion

We have reduced in an exact way a homogeneous electrodynamical problem for current waves propagating along non-uniform coated wires with finite conductivity to a transmission line – like system of differential equations of first order (Full-Wave Transmission Line equations - FWTL) with global parameters. We applied a simple perturbation theory to define these parameters, which is based on only the thickness of the wire. These FWTL equations, in turn, can be reduced to a second order Schrödinger-like equation with complex "potential", the imaginary part of which defines radiation, ohmic or dielectric losses.

Using the obtained Schrödinger-like equation the propagation of current waves along quasi-periodical thin-wire structures has been considered. Under the hypothesis that the quasi-periodical wiring system corresponds to the quasi-periodical "potential" we have applied the method of transfer matrices, which was generalized to the case of complex potentials, and obtain an explicit form for the reflection and transmission coefficient through such a chain of potentials. The analysis of the formulae obtained has shown that for some frequencies the penetration through the chain is practically free, but for some frequencies it is suppressed. It was established a connection between the frequency regions with allowed and forbidden zones, which are well known from solid-state physics. The accounting for losses leads to the decreasing propagation in the allowed zones, but the effect does not disappear. It was shown that both the ohmic losses and coating effect are negligible in comparison with radiation losses.

Two other effects were noted which are connected to the penetration of the current wave through a quasi - periodical chain. The first effect appears if the potential, which corresponds to one separate non-uniformity, is not easy to penetrate. In this case for allowed zones the current strongly increases inside the structure, what can be a cause, for example, of strong radiation or heating inside the structure. The second effect appears for the infinite periodical chain. In this case it is possible to have opposite directions of phase penetration (direction of the quasi-pulse) and power penetration.

In the future we intend to use group theoretical analogies to consider propagation problems. Also we plan to consider the current propagation along multi-conductor periodical wire structures using one of the methods described in this report for the one-wire case: group theory (transfer matrices) method, standard matrix theory method, or the method of Feynman diagrams. Moreover, we will try to apply the considered method to a stochastically installed wire.

Finally, we note that the developed mathematical apparatus can be applied to other physical and engineering problems, for example, to the penetration of electromagnetic waves through composite materials with lossy sandwiched components.

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Appendix. Numerical solution of the Schrödinger-like equation using the method of transfer matrices.

In this Appendix we shortly describe the procedure of the numerical solution of the Schrödinger-like equation with the aid of the method of transfer matrices. The method has some differences if compared to the method of transfer matrices for potentials (scatterers) with the same asymptotic values at the left and right asymptotes of the wire, which was described in Section 3.

First let us consider the scattering of a quantum mechanical particle at a potential step (see Fig. A.1).
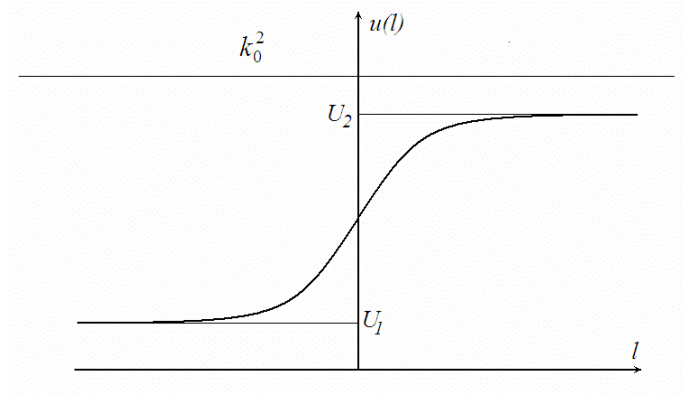

Fig. A.1: Potential step.

The propagation of the particle is described by the one-dimensional Schrödinger equation:

$$\psi'' + (k_0^2 - u(l))\psi = 0 \tag{A.1}$$

This potential can be approximately replaced by the rectangular potential step function (see Fig.A.2) [5], and the solution of the Schrödinger equation can be written in the general form of forward and backward running waves on the left and right sides of a specific point $l = 0$.
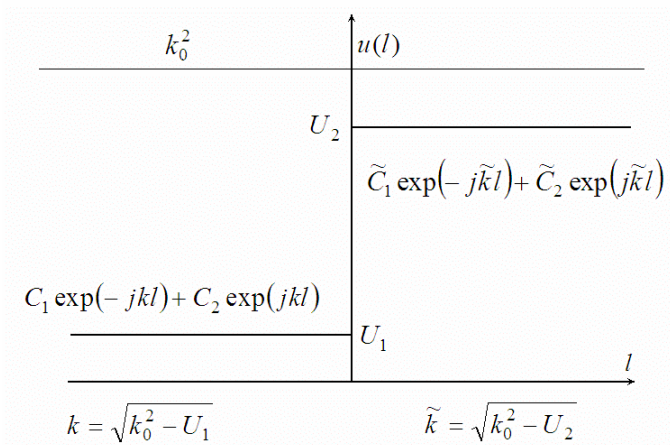

Fig. A.2: Penetration of the particle through the rectangular potential step.

$$\psi(l) = \begin{cases} C_1 \exp(-jkl) + C_2 \exp(jkl) & l \le 0 \\ \widetilde{C}_1 \exp(-j\widetilde{k}l) + \widetilde{C}_2 \exp(j\widetilde{k}l) & l \ge 0 \end{cases} \tag{A.2}$$

where the wave numbers k and $\tilde{k}$ of the potential step function can be written as

$$k = \sqrt{k_0^2 - U_1} \qquad \text{and} \qquad \tilde{k} = \sqrt{k_0^2 - U_2} \qquad\qquad\text{(A.3a,b)}$$

Note that, in contrast to Section 3, these wave numbers are different. Moreover, one or both of them can be complex-valued. Physically this corresponds to the decrease of the wave function of the particle during the penetration through the potential. However, as in Section 3, there is a linear connection between the components of column-vectors before and after the step barrier, which is realized by the transfer matrix $\widehat{S}(\tilde{k}, k)$:

$$\underset{\downarrow}{\tilde{C}} = \widehat{S}(\tilde{k}, k) \cdot \underset{\downarrow}{C} \qquad\qquad\text{(A. 4 a,b)}$$

The components of the transfer matrix can be found in a standard way solving the Schrödinger equation [5] (as the wave functions as well as their derivatives have to be continuous at the point of the step potential $l = 0$). Merging of the solutions yields

$$\widehat{S}(\tilde{k}, k) = \begin{pmatrix} (1 + k/\tilde{k})/2 & (1 - k/\tilde{k})/2 \\ (1 - k/\tilde{k})/2 & (1 + k/\tilde{k})/2 \end{pmatrix} \qquad\qquad\text{(A. 5)}$$

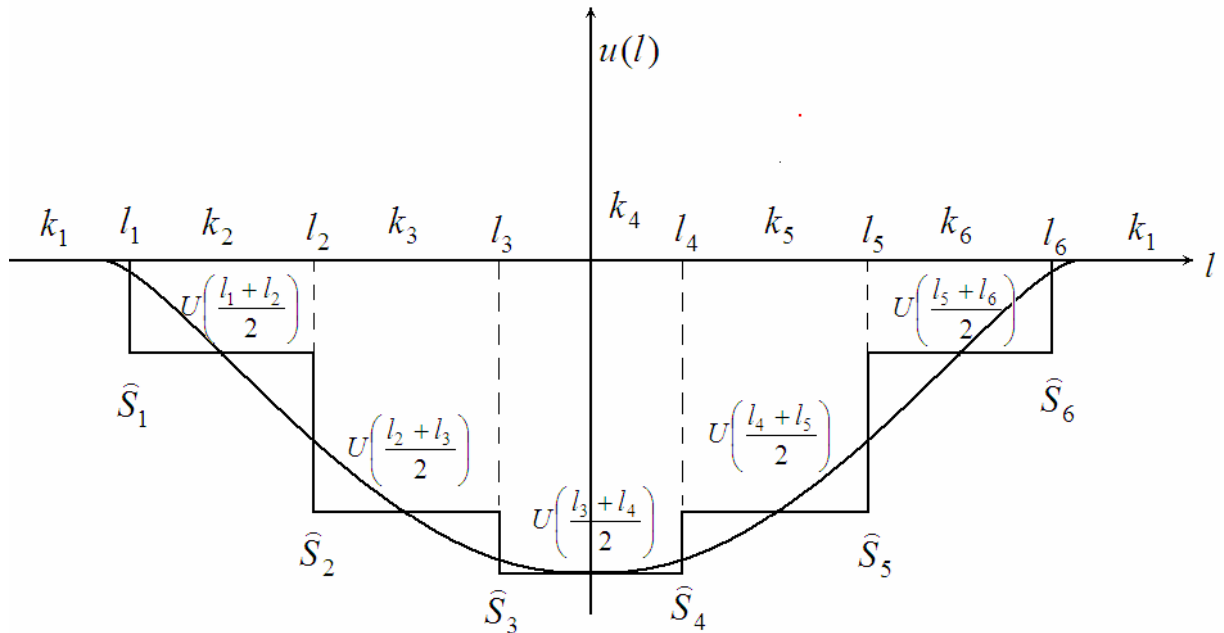Let us consider now a potential of arbitrary form (see Fig. A.3). We can break down

Fig. A.3: A break-down of an arbitrary potential $u(l)$ into rectangular step potentials for a numerical solution.

the potential function $u(l)$ (in the points $l_1, l_{2,} ... l_n ... l_N$) into N rectangular step potentials (in Fig. A.3, $N = 6$). (For practical purposes it is convenient to consider a regular discretization). The value of the potential on the left side of the first barrier and on the right side of the last

barrier is assumed to be zero. The value of constant potential in the interval between l_n and l_{n+1} is taken as a value of the potential function $u(l)$ in the central point of the interval,

$$U_{n+1} = u\left(\frac{l_n + l_{n+1}}{2}\right).$$

(A.6)

This value defines the propagation constant along the interval,

$$k_{n+1} = \sqrt{k_0^2 - U_{n+1}}$$

(A.7)

Then, using the analogue of the consideration in the Section 3, one can now write for the transfer matrix along the interval (l_n, l_{n+1})

$$\widehat{T}_{k_{n+1}}(l_{n+1}, l_n) = \begin{pmatrix} \exp(-jk_{n+1}(l_{n+1} - l_n)) & 0 \\ 0 & \exp(jk_{n+1}(l_{n+1} - l_n)) \end{pmatrix}$$

(A.8)

Formally, this equation looks like eq. (3.3.16) for the transfer matrix for free propagation; however, in contrast to (3.3.16) the constants k_n can be complex-valued for the under-barrier propagation.

Now, using the general approach of the transfer matrix method (as in the Section 3.4), we can write for the total transfer matrix with coordinate origin in the centre of the potential

$$\widehat{S}_{\Sigma(N)} = T_{k_1}((l_1 + l_N)/2, l_N)\,\widehat{S}_N(k_1, k_N)\prod_{n=1}^{N-1}\widehat{T}_{k_{n+1}}(l_{n+1}, l_n)\,\widehat{S}_n(k_{n+1}, k_n)\,T_{k_1}(l_1,(l_1 + l_N)/2)$$

(A.9)

Using the general relations (3.4.5) one can pick up the total reflection and transmission coefficients from the transfer matrix.

The set of formulae (A.5-A.9) and (3.4.5) allows to write a simple algorithm and computer program to calculate the reflection and transmission coefficients for the potential of arbitrary form.