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# Scattering From a Lossless Electromagnetic Sphere

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**Abstract**

We study fundamental issues in electromagnetic scattering theory, with an emphasis on pole behaviors of a lossless sphere arising from the singularity expansion method (SEM). We use Mie Theory to solve the electromagnetic scattering problems for spheres with lossless boundary conditions and an incident plane wave. We show that for certain lossless impedance boundary conditions there exist second order poles. Our general procedure to directly construct lossless sheet impedance boundary conditions which will produce high order poles is discussed as well as the difficulties to which it leads. Foster's Theorem is imposed on the impedance condition for the electromagnetic scattering to ensure that a lossless scattering problem is obtained.

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# 1 Introduction

The singularity expansion method (SEM) [1] was introduced in 1971 as a way to represent the solution of electromagnetic interaction or scattering problems in terms of the singularities in the complex-frequency ($s$ of two-sided-Laplace-transform) plane. Particularly for the pole terms associated with a scatterer (natural frequencies), their factored form separates the dependencies on various parameters of the incident field, observer location, and scatterer characteristics, with an equally simple form in both frequency (poles) and time (damped sinusoids) domains.

Although in practice, we only encounter the first order scattering poles, an interesting question concerning the SEM concerns the existence of higher order scattering poles. Carl Baum showed that 2nd order poles can be constructed for a transmission line problem [4]. Since the transmission line problem is finitely dimensioned, we can actually use the scattering matrix to find the poles (i.e. the eigenvalues). However, in general the problem is infinitely dimensioned. Thus, we consider a classical model problem, scattering from a sphere with an incident plane wave. We compute the exact solution using Mie Theory. In [1], Carl Baum showed that for a perfectly conductor sphere, there only exist first order poles. Sancer also proved some similar results for a general shape perfectly conductor scatterer in [5]. Those works give a closer simulation of the electromagnetic-scattering case and contribute to the discussion concerning 3-dimensional electromagnetic scattering from lossless, as well as perfectly conducting targets.

In [3] we show that for the acoustic scattering problem high order poles can be constructed for certain impedance boundary conditions, while for hard and soft spheres there only exist first order scattering poles. The general procedure to construct arbitrary order poles is also discussed. In this paper, we show that the electromagnetic scattering from a surface impedance loading sphere can be reduced to the acoustic scattering case (scalar wave equation). Thus, all the results from the acoustics scattering in [3] will follow. We also show that there exist 2nd order poles for electromagnetic scattering from a sheet impedance loading sphere. Foster's Theorem is imposed on the sheet impedance condition to ensure the scatterer is lossless. The notes we mentioned above are available at [7].

# 2 Scattering from a lossless electromagnetic sphere

## 2.1 Introduction

We are considering the problem of a plane wave incident on a sphere (with perfectly conducting surface and lossless sheet impedance loading respectively) as illustrated in Figure 1. An E wave has been chosen as an incident electromagnetic plane wave propagating in $\vec{1}_1$ direction. The scattered solution as well as the surface current density can be written explicitly using vector spherical harmonics. In the case 1, a perfectly conducting surface, we will briefly summarize the work done by Carl Baum, which shows that there exist only first order scattering poles. In the case 2, a lossless surface impedance loading sphere, we can derive the same results as those in the acoustic scattering. In the case 3, a lossless sheet impedance loading sphere, there exist 2nd order scattering poles for some mathematically chosen boundary conditions. Foster's Theorem is enforced on the impedance function $\hat{Z}_s(s)$ to guarantee it is a realizable physical boundary condition.

## 2.2 Formulation of the electromagnetic scattering problem

Define a set of orthogonal (right-handed) unit vectors by

$$\begin{aligned}
\vec{1}_1 &= \sin(\theta_1)\cos(\phi_1)\vec{1}_x + \sin(\theta_1)\sin(\phi_1)\vec{1}_y + \cos(\theta_1)\vec{1}_z \\
\vec{1}_2 &= -\cos(\theta_1)\cos(\phi_1)\vec{1}_x - \cos(\theta_1)\sin(\phi_1)\vec{1}_y + \sin(\theta_1)\vec{1}_z \\
\vec{1}_3 &= \sin(\phi_1)\vec{1}_x - \cos(\phi_1)\vec{1}_y
\end{aligned}$$

As shown in Figure 2, $\vec{1}_1$ is the direction of propagation and $\vec{1}_2$ and $\vec{1}_3$ are mutually orthogonal unit vectors, each orthogonal to $\vec{1}_1$ to indicate the polarization of the electromagnetic fields in the incident plane wave. For convenience $\vec{1}_2$ is chosen in a plane parallel to $\vec{1}_1$ and the $z$ axis (E or TM polarization if the electric field is parallel to $\vec{1}_2$) while $\vec{1}_3$ is parallel to the $x$, $y$ plane (H or TE polarization if the electric field is parallel to $\vec{1}_3$). In free space, electromagnetic plane waves have both electric and magnetic fields orthogonal to $\vec{1}_1$. Thus only $\vec{1}_2$ and $\vec{1}_3$ are concerned. This removes the $\vec{L}$ functions (details

Figure 1: Spherical coordinate system with EM incident wave

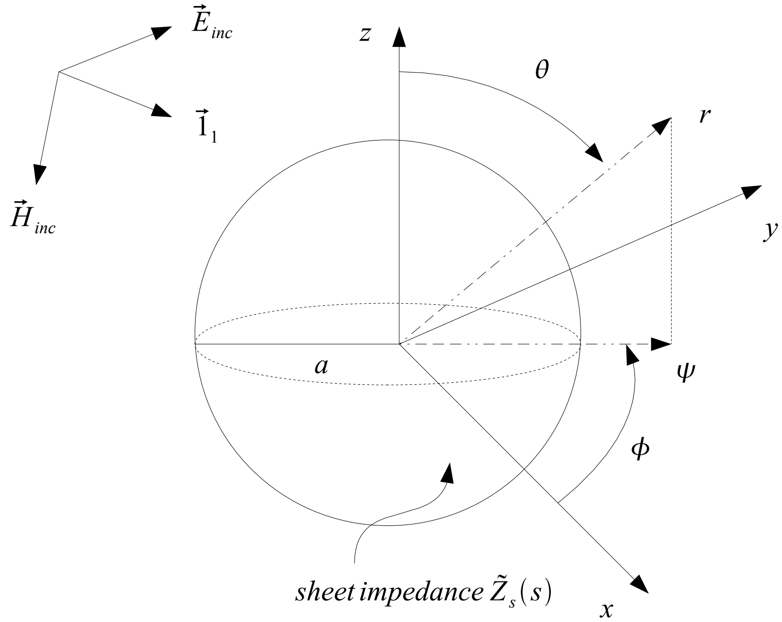
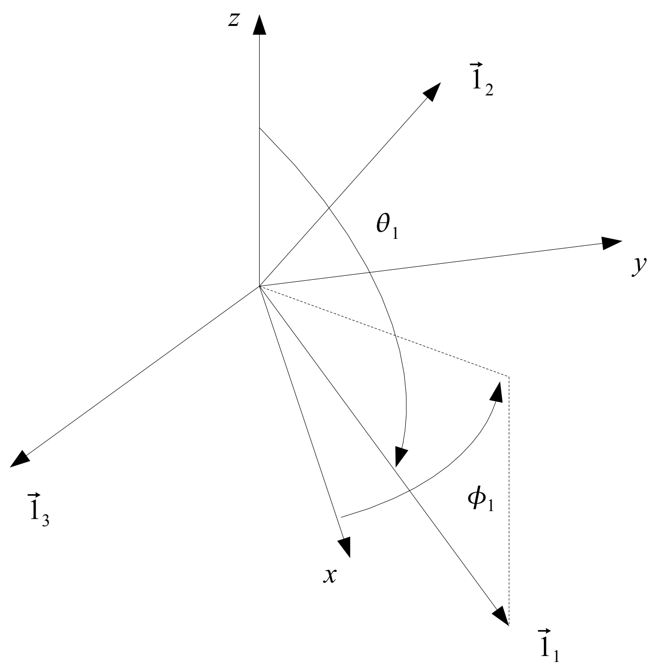


Figure 2: Spherical coordinate system with polarization



are shown later) in the expansion (plane waves have zero-divergence fields). We can use the relations between Cartesian and spherical coordinates

$$
\begin{aligned}
x &= r\sin(\theta)\cos(\phi) \\
y &= r\sin(\theta)\sin(\phi) \\
z &= r\cos(\theta)
\end{aligned}
$$

$$
\begin{aligned}
\vec{1}_x &= \sin(\theta)\cos(\phi)\vec{1}_r + \cos(\theta)\cos(\phi)\vec{1}_\theta - \sin(\phi)\vec{1}_\phi \\
\vec{1}_y &= \sin(\theta)\sin(\phi)\vec{1}_r + \cos(\theta)\sin(\phi)\vec{1}_\theta + \cos(\phi)\vec{1}_\phi \\
\vec{1}_z &= \cos(\theta)\vec{1}_r - \sin(\theta)\vec{1}_\theta
\end{aligned}
$$

to express the incident-wave unit vectors in terms of $(\theta_1, \phi_1)$ and $(\theta, \phi)$ as

$$
\begin{aligned}
\vec{1}_1 &= [\cos(\theta_1)\cos(\theta) + \sin(\theta_1)\sin(\theta)\cos(\phi - \phi_1)]\vec{1}_r \\
&+ [-\cos(\theta_1)\sin(\theta) + \sin(\theta_1)\cos(\theta)\cos(\phi - \phi_1)]\vec{1}_\theta \\
&+ [-\sin(\theta_1)\sin(\phi - \phi_1)]\vec{1}_\phi
\end{aligned}
$$

$$
\begin{aligned}
\vec{1}_2 &= [\sin(\theta_1)\cos(\theta) - \cos(\theta_1)\sin(\theta)\cos(\phi - \phi_1)]\vec{1}_r \\
&- [\sin(\theta_1)\sin(\theta) + \cos(\theta_1)\cos(\theta)\cos(\phi - \phi_1)]\vec{1}_\theta \\
&+ [\cos(\theta_1)\sin(\phi - \phi_1)]\vec{1}_\phi
\end{aligned}
$$

$$
\begin{aligned}
\vec{1}_3 &= -\sin(\theta)\sin(\phi - \phi_1)\vec{1}_r \\
&- \cos(\theta)\cos(\phi - \phi_1)\vec{1}_\theta \\
&- \cos(\phi - \phi_1)]\vec{1}_\phi
\end{aligned}
$$

Having the direction of incidence and two polarizations expressed in spherical coordinates we can go on to express the response to some delta plane wave functions. For an incident delta function plane wave we need spherical harmonics and vector wave function in which to express the expansion in spherical coordinates. In spherical coordinates we have the common differential operators as

$$
\begin{aligned}
\nabla F &= \vec{1}_r \frac{\partial}{\partial r} F + \vec{1}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} F + \vec{1}_\phi \frac{1}{r\sin(\theta)} \frac{\partial}{\partial \phi} F \\
\nabla \cdot \vec{F} &= \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin(\theta)} \frac{\partial}{\partial \theta}(\sin(\theta) F_\theta) + \frac{1}{r\sin(\theta)} \frac{\partial}{\partial \phi} F_\phi \\
\nabla \times \vec{F} &= \vec{1}_r [\frac{1}{r\sin(\theta)} \frac{\partial}{\partial \theta}(\sin(\theta) F_\phi) - \frac{1}{r\sin(\theta)} \frac{\partial}{\partial \phi} F_\theta] \\
&\quad + \vec{1}_\theta [\frac{1}{r\sin(\theta)} \frac{\partial}{\partial \theta}(\sin(\theta) F_r) - \frac{1}{r} \frac{\partial}{\partial r}(r F_\phi)] + \vec{1}_\phi [\frac{1}{r} \frac{\partial}{\partial r}(r F_\theta) - \frac{1}{r} \frac{\partial}{\partial \theta} F_r]
\end{aligned}
$$

$$
\begin{aligned}
\nabla_s F &= \vec{1}_\theta \frac{\partial}{\partial \theta} F + \vec{1}_\phi \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi} F \\
\nabla_s \cdot \vec{F} &= \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta}(\sin(\theta) F_\theta) + \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi} F_\phi \\
\nabla_s \times \vec{F} &= \vec{1}_r [\frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta}(\sin(\theta) F_\phi) - \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi} F_\theta] + \vec{1}_\theta [\frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi} F_r]
\end{aligned}
$$

**Spherical Harmonics**

The scalar spherical harmonics can be written as

$$
Y_{n,m,{}^e_o}(\theta,\phi) = P_n^{(m)}(cos(\theta)) \left\{ \begin{array}{c} cos(m\phi) \\ sin(m\phi) \end{array} \right\}
$$

where $P_n^{(m)}(x)$ is the Legendre function defined as

$$
P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x), \quad P_n(x) = P_n^0(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n
$$

Vector spherical harmonics are defined as follows

$$
\begin{aligned}
\vec{P}_{n,m,p}(\theta,\phi) = &\quad Y_{n,m,p}(\theta,\phi)\vec{1}_r \\
\vec{Q}_{n,m,p}(\theta,\phi) = &\quad \nabla_s Y_{n,m,p}(\theta,\phi) &= \vec{1}_r \times \vec{R}_{n,m,p} \\
\vec{R}_{n,m,p}(\theta,\phi) = &\quad \nabla_s \times \vec{P}_{n,m,p}(\theta,\phi) &= -\vec{1}_r \times \vec{Q}_{n,m,p}
\end{aligned}
$$

They also are mutually orthogonal in an integral sense on the unit sphere. The spherical scalar wave functions are defined as

$$
\Xi^{(l)}_{n,m,p}(\gamma\vec{r}) = f_n^{(l)}(\gamma r) P_n^{(m)}(\theta,\phi)
$$

where $f_n^{(1)}(\gamma r) = i_n(\gamma r)$ , $f_n^{(2)}(\gamma r) = k_n(\gamma r)$ are modified Bessel functions. They satisfy the Wronskian relation

$$W\{\lambda i_n(\lambda), \lambda k_n(\lambda)\} = \lambda i_n(\lambda)[\lambda k_n(\lambda)]' - [\lambda i_n(\lambda)]'\lambda k_n(\lambda) = -1$$

$\gamma = [s\mu(\sigma + s\epsilon)]^{1/2}$ with $\mu, \sigma, \epsilon$ are permeability, conductivity, permittivity, respectively. $s$ is the variable of the two-sided Laplace transformation. Coefficients times the scalar wave function $\Xi_{n,m,p}^{(l)}(\gamma\vec{r})$ when summed over all possible indices, satisfy the scalar wave equation. For each function we can write in operator form as

$$[\nabla^2 - \gamma^2]\Xi_{n,m,p}^{(l)}(\gamma\vec{r}) = 0$$

From the solution of the scalar wave equation one constructs as usual the solutions of the vector wave equation of three kinds.

$$\begin{aligned}
\hat{L}_{n,m,p}^{(l)}(\gamma\vec{r}) &= \frac{1}{\gamma}\nabla\Xi_{n,m,p}^{(l)}(\gamma\vec{r}) \\
\hat{M}_{n,m,p}^{(l)}(\gamma\vec{r}) &= \nabla \times [\vec{r}\Xi_{n,m,p}^{(l)}(\gamma\vec{r})] \\
\hat{N}_{n,m,p}^{(l)}(\gamma\vec{r}) &= \frac{1}{\gamma}\nabla \times \hat{M}_{n,m,p}^{(l)}(\gamma\vec{r})
\end{aligned}$$

Note that all three kinds of vector wave functions satisfy the vector wave equation in Laplacian form which we can summarize as

$$[\nabla^2 - \gamma^2]\left\{\begin{array}{c} \hat{L}_{n,m,p}^{(l)} \\ \hat{M}_{n,m,p}^{(l)} \\ \hat{N}_{n,m,p}^{(l)} \end{array}\right\} = 0$$

We can also write a curl curl wave equation for only the second and third kinds of vector wave functions as

$$[\nabla \times \nabla + \gamma^2]\left\{\begin{array}{c} \hat{M}_{n,m,p}^{(l)} \\ \hat{N}_{n,m,p}^{(l)} \end{array}\right\} = 0$$

The three kinds of vector wave functions have some interrelations as

$$\begin{aligned}
\hat{M}_{n,m,p}^{(l)}(\gamma\vec{r}) &= -\gamma\vec{r} \times \hat{L}_{n,m,p}^{(l)}(\gamma\vec{r}) \\
\hat{M}_{n,m,p}^{(l)}(\gamma\vec{r}) &= -\frac{1}{\gamma}\nabla \times \hat{N}_{n,m,p}^{(l)}(\gamma\vec{r}) \\
\hat{N}_{n,m,p}^{(l)}(\gamma\vec{r}) &= \frac{1}{\gamma}\nabla \times \hat{M}_{n,m,p}^{(l)}(\gamma\vec{r})
\end{aligned}$$

It is also useful to write them as

$$\begin{aligned}
\hat{L}^{(l)}_{n,m,p}(\gamma\vec{r}) &= [f^{(l)}_n(\gamma r)]'\vec{P}_{n,m,p}(\theta,\phi) + [f^{(l)}_n(\gamma r)]\vec{Q}_{n,m,p}(\theta,\phi)/\gamma r \\
\hat{M}^{(l)}_{n,m,p}(\gamma\vec{r}) &= [f^{(l)}_n(\gamma r)]\vec{R}_{n,m,p}(\theta,\phi) \\
\hat{N}^{(l)}_{n,m,p}(\gamma\vec{r}) &= \{n(n+1)[f^{(l)}_n(\gamma r)]\vec{P}_{n,m,p}(\theta,\phi) + [\gamma r f^{(l)}_n(\gamma r)]'\vec{Q}_{n,m,p}(\theta,\phi)\}/\gamma r
\end{aligned}$$

**Plane wave in spherical coordinates**

As shown in Figure 1, the delta function plane waves (transformed) can be written as

$$\begin{aligned}
\vec{1}_2 e^{-\gamma\vec{1}_1\cdot\vec{r}} &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{p=e,o}[a'_{n,m,p}\hat{M}^{(1)}_{n,m,p}(\gamma\vec{r}) + b'_{n,m,p}\hat{N}^{(1)}_{n,m,p}(\gamma\vec{r})] \\
\vec{1}_3 e^{-\gamma\vec{1}_1\cdot\vec{r}} &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{p=e,o}[b'_{n,m,p}\hat{M}^{(1)}_{n,m,p}(\gamma\vec{r}) - a'_{n,m,p}\hat{N}^{(1)}_{n,m,p}(\gamma\vec{r})]
\end{aligned}$$

where

$$\begin{aligned}
a'_{n,m,p} &= [2-1_{0,m}](-1)^{n+1}\tfrac{2n+1}{n(n+1)}\tfrac{(n-m)!}{(n+m)!}m\tfrac{P^{(m)}_n(\cos(\theta_1))}{\sin(\theta_1)}\left\{\begin{array}{c} -\sin(m\phi_1) \\ \cos(m\phi_1)\end{array}\right\} \\
b'_{n,m,p} &= [2-1_{0,m}](-1)^{n}\tfrac{2n+1}{n(n+1)}\tfrac{(n-m)!}{(n+m)!}\tfrac{dP^{(m)}_n(\cos(\theta_1))}{d\theta_1}\left\{\begin{array}{c} \cos(m\phi_1) \\ \sin(m\phi_1)\end{array}\right\}
\end{aligned}$$

Note that we have

$$\begin{aligned}
\frac{1}{\gamma}\nabla\times[\vec{1}_2 e^{-\gamma\vec{1}_1\cdot\vec{r}}] &= \vec{1}_3 e^{-\gamma\vec{1}_1\cdot\vec{r}} \\
\frac{1}{\gamma}\nabla\times[\vec{1}_3 e^{-\gamma\vec{1}_1\cdot\vec{r}}] &= -\vec{1}_2 e^{-\gamma\vec{1}_1\cdot\vec{r}}
\end{aligned}$$

which is associated with the curl relations between the $\hat{M}^{(l)}_{n,m,p}$ and $\hat{N}^{(l)}_{n,m,p}$ functions. Furthermore any divergenceless electric field expansion ($\vec{E}$) can be converted to a magnetic field expansion ($\vec{H}$) by dividing by the wave impedance $Z$ of the medium and changing $\hat{M}^{(l)}_{n,m,p}$ to $-\hat{N}^{(l)}_{n,m,p}$ and $\hat{N}^{(l)}_{n,m,p}$ to $\hat{M}^{(l)}_{n,m,p}$. To go from $\vec{H}$ to $\vec{E}$ multiply by $Z$ and change $\hat{M}^{(l)}_{n,m,p}$ to $\hat{N}^{(l)}_{n,m,p}$ and $\hat{N}^{(l)}_{n,m,p}$ to $-\hat{M}^{(l)}_{n,m,p}$.

### Solution of the scattered field

Define our incident plane wave as an $E$ wave (TM wave)

$$\begin{aligned}\tilde{\vec{E}}_{inc}(\vec{r}, s) &= E_0 \vec{1}_2 e^{-\gamma \vec{1}_1 \cdot \vec{r}} \\ \tilde{\vec{H}}_{inc}(\vec{r}, s) &= \frac{E_0}{Z_0} \vec{1}_3 e^{-\gamma \vec{1}_1 \cdot \vec{r}}\end{aligned}$$

Expand the fields for $r < a$ as

$$\begin{aligned}\tilde{\vec{E}}_{in}(\vec{r}, s) &= E_0 \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{p=e,o} [a''_{n,m,p} \hat{M}^{(1)}_{n,m,p}(\gamma \vec{r}) + b''_{n,m,p} \hat{N}^{(1)}_{n,m,p}(\gamma \vec{r})] \\ \tilde{\vec{H}}_{in}(\vec{r}, s) &= \frac{E_0}{Z_0} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{p=e,o} [b''_{n,m,p} \hat{M}^{(1)}_{n,m,p}(\gamma \vec{r}) - a''_{n,m,p} \hat{N}^{(1)}_{n,m,p}(\gamma \vec{r})]\end{aligned}$$

The solution of the scattered fields for $r > a$ can be written as

$$\begin{aligned}\tilde{\vec{E}}_{sc}(\vec{r}, s) &= E_0 \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{p=e,o} [a'''_{n,m,p} \hat{M}^{(2)}_{n,m,p}(\gamma \vec{r}) + b'''_{n,m,p} \hat{N}^{(2)}_{n,m,p}(\gamma \vec{r})] \\ \tilde{\vec{H}}_{sc}(\vec{r}, s) &= \frac{E_0}{Z_0} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{p=e,o} [b'''_{n,m,p} \hat{M}^{(2)}_{n,m,p}(\gamma \vec{r}) - a'''_{n,m,p} \hat{N}^{(2)}_{n,m,p}(\gamma \vec{r})]\end{aligned}$$

## 2.3 Perfectly conducting sphere

Carl Baum showed in [1] that there only exist simple poles for a perfectly conductor sphere. Here we will just briefly repeat the same argument in our context. Constrain the tangential electric field to be zero on $r = a$, we have $\vec{1}_r \times [\tilde{\vec{E}}_{inc}(\vec{r}, s) + \tilde{\vec{E}}_{sc}(\vec{r}, s)] = 0$. Then we get

$$\begin{aligned}\vec{1}_r \times [a'_{n,m,p} \hat{M}^{(1)}_{n,m,p}(\gamma a \vec{1}_r) + a'''_{n,m,p} \hat{M}^{(2)}_{n,m,p}(\gamma a \vec{1}_r)] &= \vec{0} \\ \vec{1}_r \times [b'_{n,m,p} \hat{N}^{(1)}_{n,m,p}(\gamma a \vec{1}_r) + b'''_{n,m,p} \hat{N}^{(2)}_{n,m,p}(\gamma a \vec{1}_r)] &= \vec{0}\end{aligned}$$

This give equations for the coefficients as

$$\begin{aligned}a'''_{n,m,p} &= - \frac{i_n(\gamma a)}{k_n(\gamma a)} a'_{n,m,p} \\ b'''_{n,m,p} &= - \frac{[\gamma a i_n(\gamma a)]'}{[\gamma a k_n(\gamma a)]'} b'_{n,m,p}\end{aligned}$$

To see that the poles must be simple poles we just need to show that all the zeros of $k_n(s)$ and $sk_n(s)$ are simple zeros. Since $k_n(s)$ is a spherical Bessel function we have

$$s^2 \frac{d^2}{ds^2} k_n(s) + 2s \frac{d}{ds} k_n(s) - [s^2 + n(n+1)] k_n(s) = 0$$

Suppose the zero is higher than first order, say a 2nd order zero at $s_\alpha \neq 0$. Since both $k_n(s)$ and $k_n'(s)$ have to be zero at $s_\alpha$, so does $k_n''(s)$. Thus, the zero must be at least a third order zero. Repeat the same process, we will eventually have all the derivatives at $s_\alpha$ to be zero, thus the function must be identically zero. So there exist only simple poles for $a'''_{n,m,p}$. The argument for $b'''_{n,m,p}$ is similar as $[sk_n(s)]$ satisfies the Riccati-Bessel equation

$$\frac{s^2}{s^2 + n(n+1)} \frac{d^2}{ds^2} [s f_n^{(l)}(s)] - s f_n^{(l)}(s) = 0$$

For more details of the perfectly conducting sphere including surface current and charge densities please see [1].

## 2.4 Surface-impedance-loaded sphere

Assume we choose the following surface impedance boundary

$$\tilde{\vec{E}}_{tan} = \overleftrightarrow{Z}(s) \cdot \tilde{\vec{J}}_s, \quad \overleftrightarrow{Z}(s) = \begin{pmatrix} 0 & \pm \frac{\tilde{Z}_s(s) + 2/a}{s} \\ \pm \frac{\tilde{Z}_s(s)+2/a}{s} & 0 \end{pmatrix}$$

where $\tilde{\vec{E}}_{tan} = \vec{1}_r \times \tilde{\vec{E}}$, $\tilde{\vec{J}}_s = \vec{1}_r \times \tilde{\vec{H}}_{tan}$, $\tilde{Z}_s(s)$ is the scalar impedance function, $a$ is the radius of the sphere, the $\pm$ sign is determined by the choice of the coordinate system. Use the standard spherical coordinate system as illustrated in the Figure 1, the above surface impedance boundary condition is equilvalent to

$$\begin{aligned} \tilde{E}_\theta &= - \frac{\tilde{Z}_s(s) + 2/a}{s} \tilde{H}_\phi \\ \tilde{E}_\phi &= + \frac{\tilde{Z}_s(s) + 2/a}{s} \tilde{H}_\theta \end{aligned} \tag{1}$$

$\nabla \cdot \tilde{\vec{E}} = 0$ can be expressed in the spherical coordinate system as

$$\left( \frac{\partial}{\partial r} + \frac{2}{a} \right) \tilde{E}_r = - \frac{1}{a \sin\theta} \frac{\partial}{\partial \theta} (\tilde{E}_\theta \sin\theta) - \frac{1}{a \sin\theta} \frac{\partial \tilde{E}_\phi}{\partial \phi} \tag{2}$$

Consider the $\vec{1}_r$ component of the the equation $s\tilde{\vec{E}} = \nabla \times \tilde{\vec{H}}$, which is

$$s\tilde{E}_r = \frac{1}{a\sin\theta}\left(\frac{\partial}{\partial\theta}(\tilde{H}_\phi \sin\theta) - \frac{\partial \tilde{H}_\theta}{\partial\phi}\right) \tag{3}$$

Apply the boundary condition (1) to (2)(3), we derive

$$\frac{\partial}{\partial r}\tilde{E}_r = \tilde{Z}_s(s)\tilde{E}_r$$

Thus $\tilde{E}_r$ satisfy the exact impedance condition as appears in acoustic scattering problem in [3]

$$n \cdot \nabla u = \frac{\partial u}{\partial n} = \alpha(s)u$$

Since $\tilde{E}_r$ also satisfy scalar wave equation, the component $\tilde{E}_r$ shares all the results we derived in [3]. Therefore, we are able to construct arbitrary order of scattering poles for the surface impedance loaded sphere.

## 2.5 Sheet-impedance-loaded sphere

### 2.5.1 Lossless sheet impedance

Spherical coordinates $(r, \theta, \phi)$ as in Figure 1 are one of the few coordinate systems in which solutions of Maxwell's equations are separable. In particular let us assume a sheet impedance $\tilde{Z}_s(s)$ (a scalar) which is located on a spherical surface given by $r = a$ and which is independent of $\theta, \phi$. This sheet impedance relates tangential electric field and surface current density as in [2], we have

$$\begin{aligned}
&\overleftrightarrow{1}_t \cdot \tilde{\vec{E}}(a, \theta, \phi, s) = \tilde{Z}_s(s)\tilde{J}_s(\theta, \phi, s) \\
&\overleftrightarrow{1}_t = \overleftrightarrow{1} - \vec{1}_r\vec{1}_r = transverse \;\; dyad \\
&\overleftrightarrow{1} \equiv identity \;\; dyad
\end{aligned}$$

$\sim$ stands for the two-sided Laplace transform. The surface current density is in turn related to the magnetic field via

$$\vec{1}_r \times [\tilde{\vec{H}}(a+, \theta, \phi, s) - \tilde{\vec{H}}(a-, \theta, \phi, s)] = \tilde{J}_s(\theta, \phi, s)$$

The sheet impedance function $\tilde{Z}_s(s)$ also has to satisfy Foster's Theorem to guarantee lossless boundary conditions.

**Foster Theorem**

In [6], a *positive real* function F(s) is an analytic function of the complex variable $s = \sigma + j\omega$, which has the following properties:

1.$F(s)$ is regular for $\sigma > 0$

2.$F(\sigma)$ is real

3.$\sigma \geq 0$ implies $Re[F(s)] \geq 0$

A *reactance function* is a positive real function that maps the imaginary axis into the imaginary axis.

Theorem: A *real rational function* of s is a *reactance function* if and only if all of its poles and zeros are simple, lie on the $j\omega$-axis, and alternate with each other. In other words

$$\psi(s) = K\frac{s(s^2+\omega_1^2)(s^2+\omega_3^2)\cdots(s^2+\omega_{2n-1}^2)}{(s^2+\omega_0^2)(s^2+\omega_2^2)\cdots(s^2+\omega_{2n}^2)}$$

is a reactance function with positive residues, where $k = 2n-2$ or $2n$, $K > 0$, $0 \leq \omega_0 < \omega_1 < \cdots < \omega_{2n-1} < \omega_{2n} < \infty$

Theorem: A *rational function* of $s$ is a *reactance function* if and only if it is the driving-point impedance or admittance of a lossless network.

### 2.5.2 Solving the scattering problem

Matching the boundary condition on $r = a$, with the sheet impedance and continuity of the tangential electric field gives

$$\begin{aligned}
&\overleftrightarrow{1}_t \cdot [\tilde{\vec{E}}_{inc}(a+,\theta,\phi,s) + \tilde{\vec{E}}_{sc}(a+,\theta,\phi,s)] = \overleftrightarrow{1}_t \cdot \tilde{\vec{E}}_{in}(a-,\theta,\phi,s) \\
&\quad = \tilde{Z}_s(s)\tilde{J}_s(\theta,\phi,s) \\
&\quad = \tilde{Z}_s(s) \times [\tilde{\vec{H}}_{inc}(a+,\theta,\phi,s) + \tilde{\vec{H}}_{sc}(a+,\theta,\phi,s) - \tilde{\vec{H}}_{in}(a-,\theta,\phi,s)]
\end{aligned}$$

Plugging in the expansion we derive a system of equations involving $a''_{n,m,p}$, $b''_{n,m,p}$, $a'''_{n,m,p}$, $b'''_{n,m,p}$. Solve for $a''_{n,m,p}$ and $b''_{n,m,p}$ we get

$$a''_{n,m,p} = \frac{a'_{n,m,p}}{1 + \frac{Z_0}{\tilde{Z}_s(s)}(\gamma a)^2 i_n(\gamma a) k_n(\gamma a)}$$

$$b''_{n,m,p} = \frac{b'_{n,m,p}}{1 - \frac{Z_0}{\tilde{Z}_s(s)}[\gamma a i_n(\gamma a)]'[\gamma a k_n(\gamma a)]'}$$

Now the surface current density is

$$
\begin{aligned}
\tilde{J}_s(\theta,\phi,s) &= \frac{1}{\tilde{Z}_s(s)} \overleftrightarrow{1_t} \cdot \tilde{\vec{E}}(a,\theta,\phi,s) \\
&= \frac{E_0}{\tilde{Z}_s(s)} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{p=e,o} [a''_{n,m,p} i_n(\gamma a) \vec{R}_{n,m,p}(\theta,\phi) + b''_{n,m,p} \frac{[\gamma a i_n(\gamma a)]'}{\gamma a} \vec{Q}_{n,m,p}(\theta,\phi)]
\end{aligned}
$$

### 2.5.3 Existence of 2nd order poles

The coefficients we care about concerning the existence of a second-order pole are $c_1 = \frac{i_n(\gamma a)}{\tilde{Z}_s(s)} a''_{n,m,p}$ and $c_2 = \frac{[\gamma a i_n(\gamma a)]'}{\tilde{Z}_s(s)\gamma a} b''_{n,m,p}$. Let's give a simple example showing that a second-order pole does exist for coefficient $c_2$. For simplicity, let's assume $\gamma a = s$. Consider the sheet impedance function

$$
\tilde{Z}_s(s) = \frac{(\frac{1}{2}e^2 + e + \frac{1}{2})s}{s^2 + \frac{1}{2}e + \frac{1}{4}}
$$

Clearly $\tilde{Z}_s(s)$ satisfies Foster's Theorem with $K > 0$ and $\omega_0 > 0$. The expansions of $i_n(s)$ and $k_n(s)$ are

$$
\begin{aligned}
k_n(s) &= \frac{e^{-s}}{s} \sum_{j=0}^{n} \frac{(n+j)!\, 2^{-j} s^{-j}}{j!\ (n-j)!} \\
i_n(s) &= \frac{1}{2}[(-1)^{n+1} k_n(s) - k_n(-s)]
\end{aligned}
$$

For $n = 0$, the denominator of $c_2$ is

$$
De(s) = \left(4\,e^{-2\,s} + 4\right) s^2 + \left(4\,e^2 + 8\,e + 4\right) s + 1 + 2\,e^{1-2\,s} + 2\,e + e^{-2\,s}
$$

It is easy to see that $De(-\frac{1}{2}) = 0$ and $\frac{d}{ds} De(s)|_{s=-\frac{1}{2}} = 0$ or in Taylor expansion around $-\frac{1}{2}$

$$
De(s) = ((16\,e + 4 + 4\,e^2) \left(s + \frac{1}{2}\right)^2 + \left(-\frac{56}{3}\,e - \frac{8}{3}\,e^2\right) \left(s + \frac{1}{2}\right)^3 + O\left(\left(s + \frac{1}{2}\right)^4\right))
$$

Thus we derive a second order pole at $s = -\frac{1}{2}$.

In general, we want to construct a sheet impedance function $\tilde{Z}_s(s) = \frac{Ks}{(s^2+\omega)}$ such that $c_2$ have a second order pole in the left half plane of $s$ mean while $K > 0$ and $\omega > 0$. The denominator of $c_2$ has the following form

$$\begin{aligned} De(s) = Ks &+ \left(-s^2 i_n(s) - s^3 \frac{d}{ds} i_n(s) - \omega\, i_n(s) - \omega\, s \frac{d}{ds} i_n(s)\right) k_n(s) \\ &+ \left(-s^3 i_n(s) - s^4 \frac{d}{ds} i_n(s) - s\omega\, i_n(s) - s^2 \omega \frac{d}{ds} i_n(s)\right) \frac{d}{ds} k_n(s) \end{aligned}$$

We want to solve $De(s) = 0$ and $\frac{d}{ds} De(s) = 0$ for $K$, $\omega$ in terms of $s$. The solution $s_\alpha$ must satisfy $s_\alpha < 0$, $K(s_\alpha) > 0$ and $\omega(s_\alpha) > 0$. For $n = 0$, that is to solve

$$\begin{aligned} 2\,Ks + s^2 e^{-2\,s} + s^2 + \omega\, e^{-2\,s} + \omega = 0 \\ 2\,K + 2\,se^{-2\,s} - 2\,s^2 e^{-2\,s} + 2\,s - 2\,\omega\, e^{-2\,s} = 0 \end{aligned}$$

The solutions are

$$\begin{aligned} K &= -\frac{s\left(e^{-4\,s} + 2\,e^{-2\,s} + 1\right)}{2\,se^{-2\,s} + e^{-2\,s} + 1} \\ \omega &= -\frac{s^2\left(-e^{-2\,s} + 2\,se^{-2\,s} - 1\right)}{2\,se^{-2\,s} + e^{-2\,s} + 1} \end{aligned}$$

From Figures 3,4, approximately when $s$ is chosen from $-0.64$ to 0, both $K$ and $\omega$ will be positive. Pushing the poles to even a higher order is not done here. In order to construct a higher order pole (including the 2nd order case), a transcendental equation has to be solved analytically which in general is not possible. This is different from the perfectly conductor sphere case, where only a system of linear equations need to be solved.

**Remarks**

Due to the symmetry of the expansions of the scattered solutions, there exist 2nd order poles for both E modes and H modes. The above procedure only works for coefficient $c_2$ with $n = 0$. For $n > 0$, We find no region in the left half plane of $s$ where all of our assumptions can be satisfied. A rigorous proof has not been accomplished. We have tested with many different $n$, $K$ and $\omega$. $K$ and $\omega$ will have either different signs or both will be negative. It doesn't work for the coefficient $c_1$ either. We have also tried to include

Figure 3: The plot of the function K in the expansion of $\tilde{Z}_s(s)$ when n=0

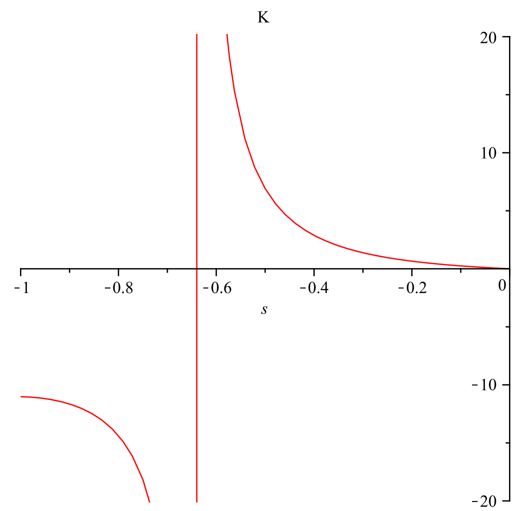
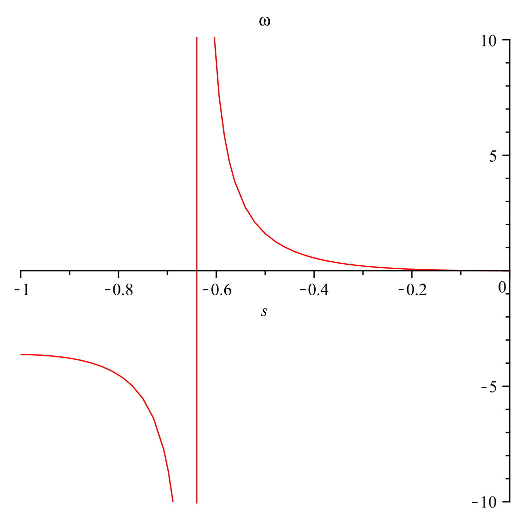



Figure 4: The plot of the function $\omega$ in the expansion of $\tilde{Z}_s(s)$ when n=0



more terms (introduce more freedom) in the $\tilde{Z}_s(s)$ expansion according to Foster's Theorem (i.e. more $\omega_i$) with larger $n$. However, it is not helpful for this case, which is different from the surface-impedance-loaded sphere case (acoustic scattering case). Figures 5, 6 show some results of different cases with different $n$, $K$ and $\omega_i$. There are no regions where both $K$ and $\omega$ are positive simultaneously. It is likely that for lossless sheet-impedance-loaded boundary condition, most scattering poles are first order.

# 3 Conclusions

We show that for the electromagnetic scattering problems for spheres with lossless sheet-impedance-loaded boundary conditions, 2nd order scattering poles can be constructed. For the surface-impedance-loaded sphere, arbitrary order of scattering poles can be derived with less restriction than the sheet impedance loading sphere case. There only exist first order poles for the perfectly conductoring sphere.

# 4 Acknowledgments

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Figure 5: The plots of the function K in the expansion of $\tilde{Z}_s(s)$ with different $n$ for $c_1$ and $c_2$

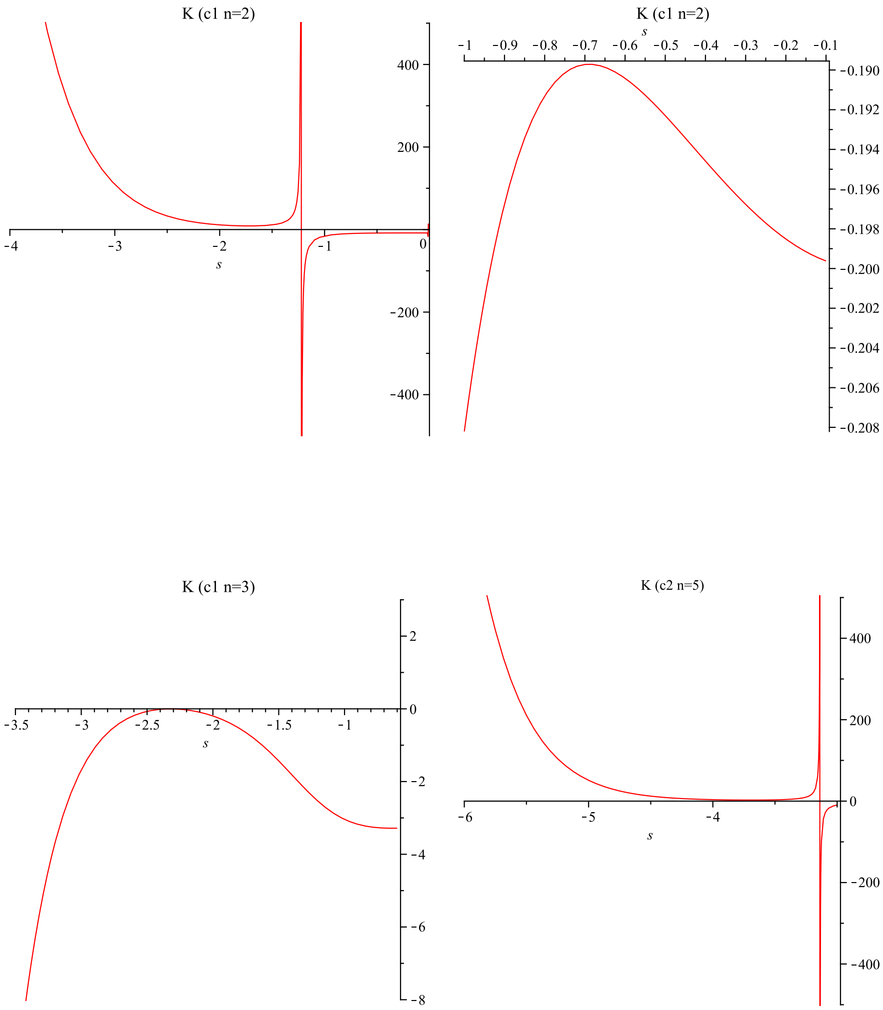
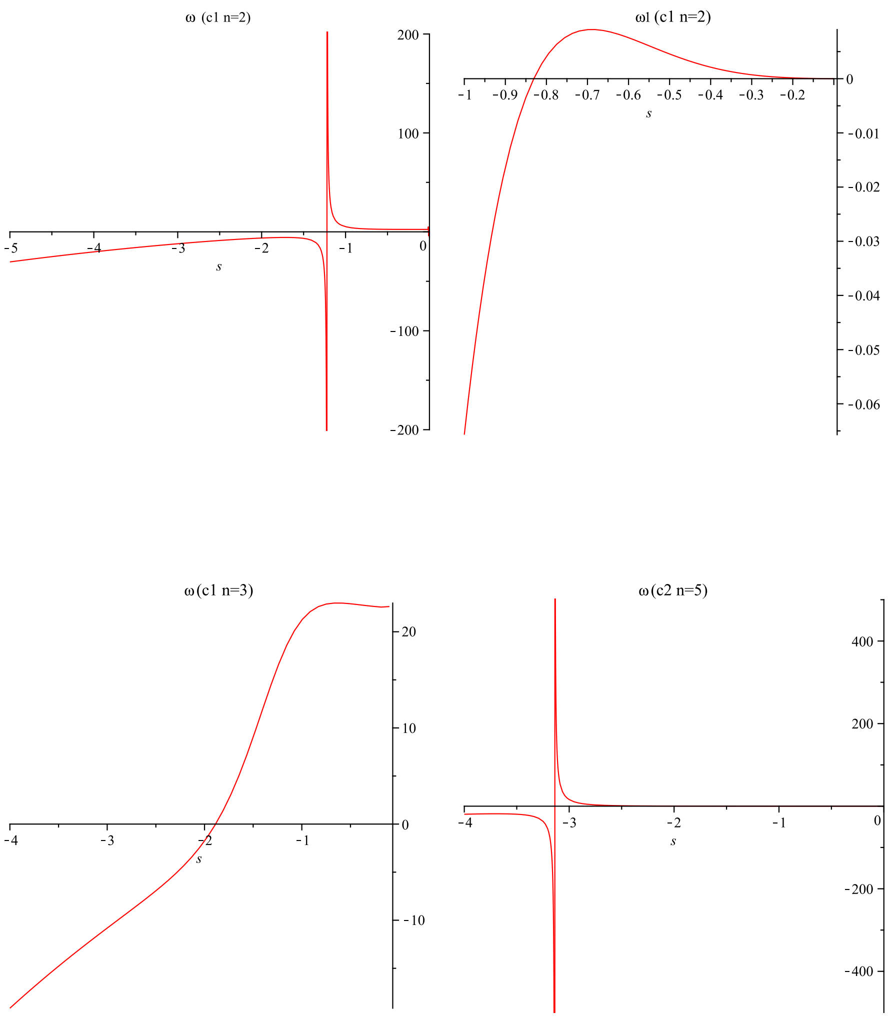


Figure 6: The plots of the function $\omega$ in the expansion of $\tilde{Z}_s(s)$ with different $n$ for $c_1$ and $c_2$



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