Physics Notes

Note 5

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The Equation of Motion for a Classical Charged Particle

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Abstract

The accepted equation of motion for a classical charged particle is second order in velocity and has solutions that violate conservation of energy and causality. This note derives a more satisfactory equation that is a first order differential equation in velocity, conserves energy and preserves causality. It has the additional feature of being non-linear. The result is extended into the relativistic domain by formulating the procedure in 4-vector notation.

The Equation of Motion for a Classical Charged Particle

1 Introduction

The inclusion of the effect of radiation on the motion of an accelerating charged particle has a long and unsatisfactory history. The generally accepted equation of motion is the Abraham - Lorentz equation, which may be written.

$$m(\dot{v} - \tau \ddot{v}) = f \qquad (1)$$

where $\tau = 6.24 \cdot 10^{-24}$ seconds, and the term involving τ supposedly accounts for the radiation. The derivation of this equation hinges on equating integrands of definite integrals and so is intrinsically unsound. The resulting equation is of third order, that is it depends upon the rate of change of acceleration, and consequently is not of a form generally accepted as an equation of motion. A consequence of this last observation is the generation of solutions involving 'pre-acceleration', that is the particle is predicted to move just before the force is applied! In an attempt to avoid this violation of causality, Dirac introduced an ad hoc boundary condition, which merely transformed the difficulty into making the motion of the particle dependent on all future forces! The form of equation (1) is also curious in that the radiation correction term is negative which implies that it contributes a force in the same sense as f, and this result is vigorously defended by Dirac. However, ignoring these arguments and changing the sign does lead to solutions that look more hopeful. Pre-acceleration disappears, but we find that particles subjected to a particular history of forces end up with the same velocity regardless of whether the radiation term is included or not, and so violates the conservation of energy. There is a further factor worthy of comment. The A - L equation is linear, and yet the rate of energy loss from an accelerating particle is

$$\frac{dE}{dt} = \frac{q^2 \dot{v}^2}{6\pi\varepsilon_o c^3} = m\tau\dot{v}^2 \qquad (2)$$

that is, it depends on the square of the acceleration and hence non-linearly on the applied force. Summarising the objections to the A - L equation we have

(1) Third order equation
(2) Derivation unsound
(3) Solutions violate causality
(4) Solutions violate conservation of energy
(5) The equation is expected to be non-linear

A full discussion of the shortcomings of A - L equation can be found in [1], the standard derivation in [2] and Diracs defence of the equation in [3].

2 The Non-Relativistic Equation of Motion

Consider a charged particle released in a force field. The force causes the charge to move with a resultant change in momentum. In the absence of radiation we would write

$$\frac{dp}{dt} = f \tag{3}$$

as the equation of motion. We wish to find how to modify this equation to take into account the fact that accelerating particles radiate. We can write for the kinetic energy of a particle

$$E_{kin} = \frac{1}{2} m\, v\,.\,v = \frac{1}{2}\,\frac{p.p}{m} \tag{4}$$

Differentiating

$$\frac{dE_{kin}}{dt} = \frac{1}{m}\, p.\, \frac{dp}{dt} = \frac{1}{m}\, p\frac{dp}{dt}\cos\theta \tag{5}$$

However, if the change in momentum is entirely due to a loss of kinetic energy and not to some force that does no work we must have

$$\cos\theta = 1 \tag{6}$$

In other words, it is only the component of the change in momentum that is parallel to the momentum that can cause a loss in kinetic energy. A loss of kinetic energy implies a change of momentum parallel to the momentum.

Denoting the loss of kinetic energy due to radiation as E_{rad}, we can write

$$\frac{dE_{rad}}{dt} = \frac{1}{m}\, p\, \frac{dp_{rad}}{dt} \tag{7}$$

and accordingly the momentum not acquired by the charge is given by

$$\frac{d\,p_{rad}}{dt} = \frac{m}{p}\,\frac{dE_{rad}}{dt} \tag{8}$$

This loss of momentum is in the direction of the momentum, and so we have

$$\frac{dp_{rad}}{dt} = \frac{m}{p}\,\frac{dE_{rad}}{dt}\,\frac{p}{p} \tag{9}$$

and the equation of motion becomes

$$\frac{dp}{dt} + \frac{m}{p^2}\frac{dE_{rad}}{dt} p = f \qquad (10)$$

Expressing this equation in terms of velocity

$$\dot{v} + \tau\frac{\dot{v}^2}{v^2} v = \frac{f}{m} \qquad (11)$$

as the non-relativistic equation of motion for a radiating charged particle.

3 Discussion of the Equation

We first note that for $\tau = o$, i.e., no radiation occurring, the equation reduces to the Newtonian form. Inclusion of the radiation term leads to a non-linear term as expected, and the equation remains of second order, i.e., first order in velocity.

If we form the scalar product of this equation with the velocity, we obtain

$$\dot{v}.\ v + \tau\frac{\dot{v}^2}{v^2} v.\ v = \frac{1}{m} f.\ v \qquad (12)$$

which reduces to

$$m\left[\dot{v}.\ v + \tau\ \dot{v}^2\right] = f.\ v \qquad (13)$$

and these terms are immediately identifiable,

$$\frac{dE_{kin}}{dt} + \frac{dE_{rad}}{dt} = \frac{dW}{dt} \qquad (14)$$

where dW/dt is the rate at which the force is doing work on the particle.

If we consider the equation of motion with zero applied force and make the assumption that the acceleration is not zero we have

$$\dot{v} + \tau\frac{\dot{v}^2}{v^2} v = 0 \qquad (15)$$

This imples that

$$\dot{v} \parallel v \tag{16}$$

and the equation reduces to

$$1 + \tau \frac{\dot{v}}{v} = 0 \tag{17}$$

which has a solution, which is the only solution

$$v = V_o e^{-\frac{t}{\tau}} \tag{18}$$

This states that any velocity will decrease rapidly to zero in the absence of an applied force. This non-physical result implies that the assumptions are false. Specifically, if the applied force is zero, the accelration cannot be non-zero.

We have now demonstrated that the new equation of motion conserves energy and preserves causality, two of the fundamental requirements for a classical equation of motion and so would appear to be satisfactory as the equation of motion for a classical charged particle.

4 Linear Motion Under a Constant Force

Considering now linear motion, the three vectors in equation (11) become co-linear and the equation reduces to

$$\dot{v} + \tau \frac{\dot{v}^2}{v} = \frac{f}{m} \tag{19}$$

Treating this equation as a quadratic in acceleration we obtain

$$\dot{v} = -\frac{v}{2\tau}\left(1 \pm \sqrt{1 + \frac{4\tau f}{mv}}\right) \tag{20}$$

Inspection of equation (16) shows that we must take the negative sign, the positive sign giving negative acceleration for positive forces. The equation to be solved is then

$$\dot{v} = \frac{v}{2\tau}\left(\sqrt{1+\frac{4\tau f}{mv}} - 1\right) \tag{21}$$

Rearranging we have

$$\int_o^v \frac{dv}{v\left(\sqrt{1+\frac{4\tau f}{mv}} - 1\right)} = \frac{t}{2\tau} \tag{22}$$

Observing that

$$\dot{v} = v\,\frac{dv}{dx} \tag{23}$$

we obtain from (17)

$$\int_o^v \frac{dv}{\left(\sqrt{1+\frac{4\tau f}{mv}} - 1\right)} = \frac{x}{2\tau} \tag{24}$$

The integrands in equations (18) and (20) are reduced to rational algebraic functions by the substitution

$$w = \sqrt{1+\frac{4\tau f}{mv}} \tag{25}$$

yielding

$$\int \frac{w\,dw}{(w+1)\,(w-1)^2} = -\frac{t}{4\tau} \tag{26}$$

$$\int \frac{wdw}{(w+1)^2(w-1)^3} = -\frac{mx}{16\tau^2 f} \tag{27}$$

Carrying out the integration and reverting to the original variables, we have the parametric solution, with velocity as the parameter

$$\frac{t}{4\tau} = \frac{1}{2\left(\sqrt{1+\frac{v_o}{v}} - 1\right)} + \frac{1}{4}\,\ell n \left\{ \frac{\sqrt{1+\frac{v_o}{v}} + 1}{\sqrt{1+\frac{v_o}{v}} - 1} \right\} \tag{28}$$

$$\frac{mx}{\tau^2 f} = \frac{2}{\sqrt{1+\frac{v_o}{v}}+1} + \frac{2}{\left(\sqrt{1+\frac{v_o}{v}}-1\right)^2} - \ell n \left\{ \frac{\sqrt{1+\frac{v_o}{v}}+1}{\sqrt{1+\frac{v_o}{v}}-1} \right\} \tag{29}$$

where we have written

$$\frac{4\tau f}{m} = v_o \tag{30}$$

Introducing the dimensionless variables

$$T = \frac{t}{4\tau} \tag{31}$$

$$X = \frac{mx}{\tau^2 f} \tag{32}$$

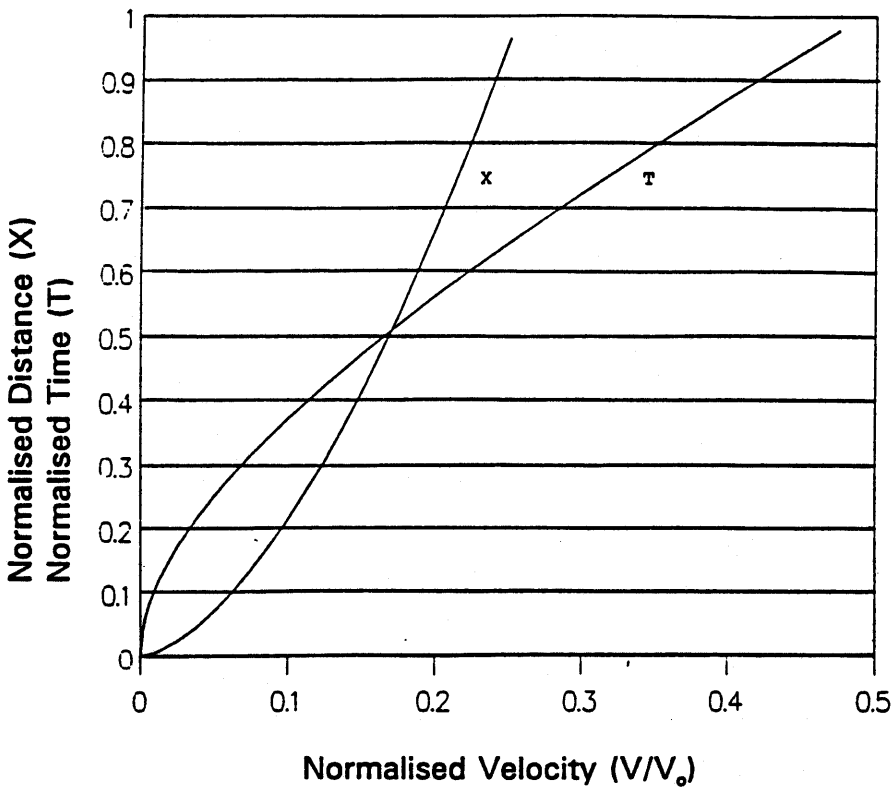
we obtain the graphs presented in fig 1.

5 Radiated Energy for Linear Motion under a Constant Force

The standard approach to calculating the energy radiated by an accelerating electron is to calculate the motion ignoring radiation and to follow this by integrating the radiation loss over the acceleration history. If we carry out this process for linear motion, we have, ignoring radiation,

$$v = \frac{ft}{m} \tag{33}$$

where we have assumed a constant force. The kinetic energy of the electron is then



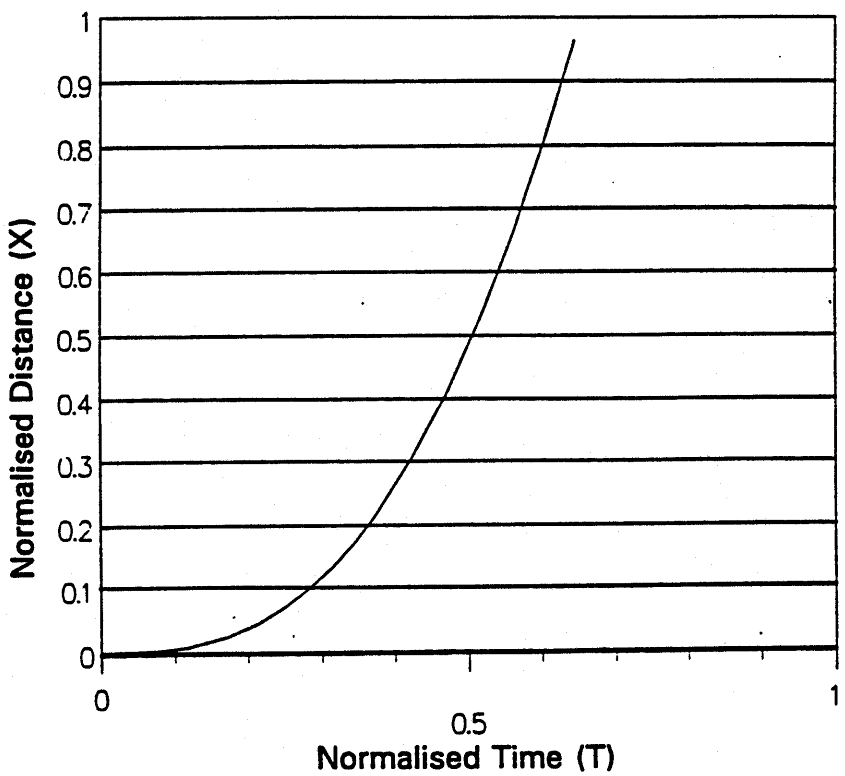


Fig 1 Non-Relativisit Solution for Linear Motion Under a Constant Force

$$E_k = \frac{1}{2} mv^2 = \frac{1}{2} m \left(\frac{ft}{m}\right)^2 \tag{34}$$

The radiation rate is

$$\frac{dE_{rad}}{dt} = m\tau\dot{v}^2 = m\tau \left(\frac{f}{m}\right)^2 \tag{35}$$

Integrating

$$E_{rad} = m\tau \left(\frac{f}{m}\right)^2 t \tag{36}$$

The fractional loss of energy due to radiation is then

$$\left(\frac{E_{rad}}{E_k}\right)_{standard} = \frac{2\tau}{t} \tag{37}$$

To compare this result with deductions based on the new equation of motion, it is necessary to obtain an approximate explicit solution for the velocity. With the assumption

$$\frac{v_o}{v} < 1 \tag{38}$$

equation (28) reduces to

$$t = \frac{1 + \frac{\tau f}{mv} \ell n\,[1 + \frac{mv}{\tau f}]}{\frac{f}{mv}} \tag{39}$$

solving for v

$$v = \frac{\frac{ft}{m}}{1 + \frac{\tau f}{mv} \ln\,[1 + \frac{mv}{\tau f}]} \tag{40}$$

The first iterative solution is obtained by noting that

$$v \approx \frac{ft}{m} \tag{41}$$

yielding

$$v \approx \frac{\frac{ft}{m}}{1 + \frac{\tau}{t} \ell n \left(1 + \frac{t}{t}\right)} \tag{42}$$

Making use of the binomial expansion

$$v = \frac{ft}{m} \left(1 - \frac{\tau}{t} \ell n \left(1 + \frac{t}{\tau}\right)\right) \tag{43}$$

The kinetic energy of the particle is then

$$E_k = \frac{1}{2} \left(\frac{ft}{m}\right)^2 \left(1 - \frac{\tau}{t} \ell n \left(1 + \frac{t}{\tau}\right)\right)^2 \tag{44}$$

Again using the binomial therein

$$E_k = \frac{1}{2} \left(\frac{ft}{m}\right)^2 \left(1 - \frac{2\tau}{t} \ell n \left(1 + \frac{t}{\tau}\right)\right) \tag{45}$$

The logarithmic term represents the radiated energy and so we have immediately

$$\frac{E_{rad}}{E_k} = \frac{2\tau}{t} \ell n \left(1 + \frac{t}{\tau}\right) \tag{46}$$

Comparing the two calculations for the radiated energy

$$\frac{E_{rad}}{E_{rad\ st}} = \ell n \left(1 + \frac{t}{\tau}\right) \tag{47}$$

As an example, consider an electron gun with 10^4 volts applied over a distance of 0.1m. The force is

$$f = Eq = \frac{Vq}{d} \tag{48}$$

The transit time is

$$t = \sqrt{\frac{2md^2}{Vq}} = 3.37.10^{-9}s \tag{49}$$

Noting that $\tau \sim 6.266\ .10^{-24}$s, the ratio is

$$\frac{E_{rad}}{E_{rad\ st}} = \ell n\ (1 + 5.382.10^{14}) = 33.92 \tag{50}$$

6 Non-relativistic Motion in a Magnetic Field

The force on an electron in a uniform magnetic field is

$$\boldsymbol{f} = q\ \boldsymbol{v} \times \boldsymbol{B} \tag{51}$$

The equation of motion then becomes

$$\dot{\boldsymbol{v}} + \tau \frac{\dot{v}^2}{v^2} \boldsymbol{v} = \frac{q}{m} \boldsymbol{v} \times \boldsymbol{B} \tag{52}$$

Consider an electron injected into a field with its velocity vector perpendicular to the field. Expressing the equation of motion in cartesian coordinates

$$\dot{v}_x \hat{i} + \dot{v}_y \hat{j} + \tau \frac{\dot{v}^2}{v^2} (v_x \hat{i} + v_y \hat{j}) = \frac{q}{m} (v_x \hat{i} + v_y \hat{j}) \times B \hat{k} \tag{53}$$

Separating this equation into its two component equations

$$\dot{v}_x + \tau \frac{\dot{v}^2}{v^2} v_x = \frac{q}{m} v_y \tag{54}$$

$$\dot{v}_y + \tau \frac{\dot{v}^2}{v^2} v_y = -\frac{q}{m} B v_x \tag{55}$$

where

$$v^2 = v_x^2 + v_y^2 \tag{56}$$

$$\dot{v}^2 = \dot{v}_x^2 + \dot{v}_y^2 \tag{57}$$

As a first approximation

$$\dot{v}_x = \frac{q}{m} B v_y \quad \dot{v}_y = \frac{-q}{m} B v_x \tag{58}$$

Introducing these approximations into the radiation terms, the equations of motion become

$$\dot{v}_x + \tau \frac{q^2}{m^2} B^2 v_x \approx \frac{q}{m} B v_y \tag{59}$$

$$\dot{v}_y + \tau \frac{q^2}{m^2} B^2 v_y \approx - \frac{q}{m} B v_x \tag{60}$$

Differentiating the first of these and substituting for v_y and v_y in the second and collecting terms the resulting equation is

$$\ddot{v}_x + 2\,\tau \left(\frac{qB}{m}\right)^2 \dot{v}_x + \left(\frac{qB}{m}\right)^2 \left\{1 + \left(\frac{\tau qB}{m}\right)^2\right\} v_x = o \tag{61}$$

This is the equation for damped harmonic motion with a damping factor

$$\tau \frac{q^2 B^2}{m^2} \tag{62}$$

The equation for the y-component is identical and so we have a quasi spiral motion. We can assume without loss of generality that the initial motion is along the x axis, we then have the initial conditions

$$\begin{aligned} v_x &= V_o & v_y &= 0 \\ \dot{v}_x &= 0 & \dot{v}_y &= - \frac{q}{m} B V_o \end{aligned} \tag{63}$$

Introducing $\alpha\, \tau$ for the damping factor the x-equation becomes

$$\ddot{v} + 2\,\alpha\tau\, \dot{v}_x + \alpha\,\{1 + \alpha\tau^2\}\, v_x = 0 \tag{64}$$

and similarly for the y-equation.

The solutions to the two equations can be set in the form

$$\begin{aligned} v_x &= e^{-\alpha\tau t}\{A_1 \cos \omega t + B_2 \sin\omega t\} \\ v_y &= e^{-\alpha\tau t}\{A_3 \cos \omega t + A_4 \sin\omega t\} \end{aligned} \tag{65}$$

where

$$\omega = \sqrt{\alpha} = \frac{qB}{m} \tag{66}$$

Imposing the initial conditions the solutions become

$$\begin{aligned} v_x &= V_o\, e^{-\alpha\tau t} \left\{ \cos\omega t + \frac{\alpha\tau}{\omega} \sin\omega t \right\} \\ v_y &= - V_o\, e^{-\alpha\tau t} \sin\omega t \end{aligned} \tag{67}$$

The magnitude of the velocity is then given by

$$v^2 = V_o^2\, e^{-2\omega^2\tau t} \left(1 + 2\omega\tau \sin \omega t \cos \omega t + \omega^2 \tau^2 \sin^2 \omega t\right) \tag{68}$$

For fields such that

$$\omega < \frac{1}{\tau} \tag{69}$$

we may write this as

$$v^2 = V_o^2\, e^{-2\omega^2\tau t} \{1 + 2\, \omega\tau \cos \omega t \sin \omega t\} \tag{70}$$

or

$$v^2 \approx V_o^2\, e^{-2\omega^2\tau t} \{1 + \omega\tau\, \sin 2\omega t\} \tag{71}$$

To the same order of approximation

$$v \approx V_o\, e^{-\omega^2\tau t} \left\{1 + \frac{\omega\tau}{2} \sin 2\, \omega t\right\} \tag{72}$$

The traditional approach is as follows. The approximate equation of motion is

$$\dot{v} = \frac{q}{m} B\, v \tag{73}$$

and this gives directly

$$\dot{E}_{rad} = m\tau \frac{q^2 B^2}{m^2} v^2 \tag{74}$$

Now the kinetic energy is

$$E_k = \frac{1}{2} mv^2 \tag{75}$$

On differentiating

$$\dot{E}_k = m\, v\, \dot{v} \tag{76}$$

We must have

$$\dot{E}_r = - \dot{E}_{rad} \tag{77}$$

This yields

$$\dot{v} = - \tau \frac{q^2B^2}{m^2} v \tag{78}$$

or

$$v = V_o\, e^{-\omega^2 \tau t} \tag{79}$$

Comparing the two solutions we see that the amplitude of the velocity decreases at the same average rate, but the new equation indicates a modulation at twice the angular frequency. This effect will cause the rate of emission to be modulated and change the spectral content of the radiation.

Differentiating the velocity components and retaining only first order terms in $\omega\tau$

$$\begin{aligned} \dot{v}_x &\approx - \omega V_o\, e^{-\omega^2\tau t} \sin \omega t \\ \dot{v}_y &\approx - \omega V_o\, e^{-\omega^2\tau t} \{\cos\omega t - \omega\tau\sin\omega t\} \end{aligned} \tag{80}$$

Squaring and adding

$$\dot{v}^2 = \omega^2\, V_o^2\, e^{-2\omega^2\tau t} \{1 - \omega\tau \sin 2\omega t\} \tag{81}$$

giving for the radiation rate

$$\dot{E}_{rad} = m\omega^2\tau\, V_o^2\, e^{-2\omega^2\tau t} \{1 - \omega\tau \sin 2\omega t\} \tag{82}$$

These results were determined under the assumption that

$$\omega\tau < 1 \tag{83}$$

To understand the implication of this restriction we may assume the approximation to be reasonably accurate up to values of $\omega\tau \sim 0.1$ This gives accurate estimates for magnetic flux densities up to

$$B = \frac{.1m}{q\tau} \approx 10^{11}\ Wb/m^2 \tag{84}$$

For all practical magnetic fields the approximations that have been developed are adequate

7 The Relativistic Equation of Motion

The equation of motion for a relativistic particle is now derived in the covariant 4-vector notation of relativity. By ensuring that the terms of the equation are 4-vectors we ensure that the equation is invariant to a Lorentz transformation. We denote 4-vectors by bold face capitals and 3-vectors by lower case as before. Introducing T as the proper time, the equation of motion for a non-radiating particle is

$$\frac{d\mathbf{P}}{dT} = \mathbf{F} \tag{85}$$

where P is the 4 momentum and F is the 4-force, and we may write this equation displaying their space-like and time-like components

$$\frac{d}{dT}(\mathbf{p}, imc,) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \left(\mathbf{f}, \frac{i}{c}\ \mathbf{f}.\mathbf{v}\right) \tag{86}$$

where **p** is the momentum

$$\mathbf{p} = m\ \mathbf{v} = \frac{m_o\ \mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{87}$$

and we note that the time like component of the 4-force is the rate at which work is done by the force on the particle. To modify the equation we introduce a 4-vector that represents the 4-momentum not acquired by the particle by virtue of the particle radiating,

$$\frac{d\mathbf{P}}{dT} + \frac{d\mathbf{P}_{rad}}{dT} = \mathbf{F} \tag{88}$$

The relativistic equation for the rate of loss of energy by an accelerating particle is

$$\dot{E}_{rad} = \frac{m_o \tau}{\left(1 - \frac{v^2}{c^2}\right)^2} \left\{ \dot{\mathbf{v}}.\dot{\mathbf{v}} + \frac{(\dot{\mathbf{v}}.\mathbf{v})^2}{c^2\left(1 - \frac{v^2}{c^2}\right)} \right\} \tag{89}$$

and we wish to relate this loss to the components of $\mathbf{P}_{rad}$. Writing the energy as

$$E = mc^2 = \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{90}$$

we differentiate, obtaining

$$\dot{E} = \frac{m_o c^2}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \frac{v\,\dot{v}}{c^2} = \frac{m_o v \dot{v}}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \tag{91}$$

and we may then write

$$\dot{E}_{rad} = \frac{m_o v\, \dot{v}_{rad}}{(1 - \frac{v^2}{c\tau^2})^{3/2}} \tag{92}$$

where $\dot{v}_{rad}$ is the rate of change of velocity due to the radiation of energy. This radiation is acting to retard the particle and so the rate of change of momentum due to the radiation emission is in the direction of the momentum. Accordingly we must obtain an expression for the rate of change of momentum under the condition that the direction of the particle does not change, that is

$$\dot{\mathbf{p}}_{rad} = \frac{\partial}{\partial t} (\mathbf{p})_{(\frac{\mathbf{v}}{v})} \tag{93}$$

where we have borrowed the notation from thermodynamics. To obtain this derivative we write

$$\mathbf{p} = \frac{m_o v}{(1 - \frac{v^2}{c^2})^{1/2}} \frac{\mathbf{v}}{v} \tag{94}$$

making the unit vector in the direction of the velocity explicit. Carrying out the differentiation

$$p_{rad} = \frac{m_o \dot{v}_{rad}}{(1 - \frac{v^2}{c^2})^{3/2}} \cdot \frac{\mathbf{v}}{v} \tag{95}$$

Making use of equation (96)

$$p_{rad} = \frac{\dot{E}_{rad}}{v^2}\, \mathbf{v} \tag{96}$$

Noting that

$$\frac{dt}{dT} = (1 - \frac{v^2}{c^2})^{-1/2} \tag{97}$$

we have

$$\dot{m} = \frac{m_o}{(1 - \frac{v^2}{c^2})^{3/2}} \frac{v\,\dot{v}}{c^2} \tag{98}$$

and similarly

$$\dot{m}_{rad} = \frac{m_o}{(1 - \frac{v^2}{c^2})^{3/2}} \frac{v\dot{v}_{rad}}{c^2} = \frac{\dot{E}_{rad}}{c^2} \tag{99}$$

Combining these results, the relativistic equation of motion becomes

$$\left\{\frac{d}{dt} \frac{m_o\,\mathbf{v}}{(1 - \frac{v^2}{c^2})^{1/2}}\, ,\, ic\, \frac{d}{dt} \frac{m_o}{(1 - \frac{v^2}{c^2})^{1/2}}\right\} + \left\{\dot{E}_{rad}\frac{\mathbf{v}}{v^2}\, ,\, i\frac{\dot{E}_{rad}}{c}\right\} = \left\{\mathbf{f}, \frac{i}{c}\mathbf{f}.\mathbf{v}\right\} \tag{100}$$

Separating this equation into its space-like and time-like components

$$\frac{d}{dt} \frac{m_o\mathbf{v}}{(1 - \frac{v^2}{c^2})^{1/2}} + \dot{E}_{rad}\frac{\mathbf{v}}{v^2} = \mathbf{f} \tag{101}$$

$$c\frac{d}{dt}\frac{m_o}{(1-\frac{v^2}{c^2})^{1/2}} + \frac{\dot{E}_{rad}}{c} = \frac{1}{c}\,f.v \qquad (102)$$

Rearranging this latter equation, it becomes

$$\frac{d}{dt}\frac{m_o c^2}{(1-\frac{v^2}{c^2})^{1/2}} + \dot{E}_{rad} = f.v \qquad (103)$$

and this is simply a statement of the conservation of energy. Substituting for $\dot{E}_{rad}$ in equation (103), we have the three-vector equation of motion.

$$\frac{d}{dt}\frac{m_o\,v}{(1-\frac{v^2}{c^2})^{1/2}} + \frac{m_o\tau}{(1-\frac{v^2}{c^2})^{2}}\left\{\dot{v}.\dot{v} + \frac{(v.\dot{v})^2}{c^2(1-\frac{v^2}{c^2})}\right\}\frac{v}{v^2} = f \qquad (104)$$

8 Relativistic Linear Motion under a Constant Force

We simplify the equation of motion as before obtaining the scalar equation

$$\frac{\dot{v}}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} + \frac{\tau\dot{v}^2}{v\left(1-\frac{v^2}{c^2}\right)^{3}} = \frac{f}{m_o} \qquad (105)$$

Solving this equation for v and discarding the negative root of the radical occurring in the solution, we have

$$\dot{v} = \frac{v}{2\tau}(1-\frac{v^2}{c^2})^{3/2}\left\{\sqrt{1+\frac{4f\tau}{m_o v}} - 1\right\} \qquad (106)$$

The solution to this equation is

$$\int_o^v \frac{dv}{v\,(1-\frac{v^2}{c^2})^{3/2}\left\{\sqrt{1+\frac{4f\tau}{m_o v}} - 1\right\}} = \frac{t}{2\tau} \qquad (107)$$

For all practical fields

$$v_o = \frac{4f\tau}{m_o} \ll c \qquad (108)$$

for example, with a field as high as 10^8 v/m

$$\frac{4f\tau}{m_o} \approx 4.4.10^{-4}\ m/s \qquad (109)$$

We may accordingly make a non-relativistic calculation for velocities up to a more or less arbitrary large multiple of v_o such that

$$nv_o \ll c \qquad (110)$$

A suitable multiple would be of the order of 10^{10}. To evaluate the integral, we now split it into two ranges

$$\int_o^{nv_o} \text{----}\ dv + \int_{nv_o}^{v} \text{----}\ dv = \frac{t}{2\tau} \qquad (111)$$

The first of these has been evaluated for the general case and so we have, with an obvious notation

$$\frac{1}{\left(\sqrt{1+\frac{1}{n}}-1\right)} + \frac{1}{2}\ln\left\{\frac{\sqrt{1+\frac{1}{n}}+1}{\sqrt{1+\frac{1}{n}}-1}\right\} = \frac{t_1}{2\tau} \qquad (112)$$

To a high degree of approximation, this reduces to

$$t_1 \approx 4\ n\tau \qquad (113)$$

The radical in the second integrand may be approximated

$$\sqrt{1+\frac{4f\tau}{m_o v}} - 1 \approx \frac{2f\tau}{m_o v} \qquad (114)$$

and the integral simplifies to

$$\int_{nv_o}^{v} \frac{dv}{\frac{2f\tau}{m_o}\left(1-\frac{v^2}{c^2}\right)^{3/2}} = \frac{t_2}{2\tau} \qquad (115)$$

The substitution

$$v = c \sin \theta \tag{116}$$

reduces the integral to

$$\frac{m_o}{f} \int \sec^2\theta \, d\theta = \frac{m_o}{f} \tan\theta \Bigg|_{\theta_1}^{\theta_2} = \frac{m_o v}{f(1 - \frac{v^2}{c^2})^{1/2}} \Bigg|_{nv_o}^{v} = t_2 \tag{117}$$

Inserting the limits and noting

$$nv_o \ll c \tag{118}$$

we obtain

$$\frac{m_o v}{f(1 - \frac{v^2}{c^2})^{1/2}} - 4n\tau = t_2 \tag{119}$$

and we have the result that to first order in $4f\tau/m_o$

$$t = t_1 + t_2 = \frac{m_o v}{f(1 - \frac{v^2}{c^2})^{1/2}} \tag{120}$$

which is of course, the result for no radiation! This simply means that electrons that are accelerated from rest by macroscopic electric fields to relativistic velocities radiate a very small fraction of their energy. This can be understood by observing that as the particle approaches c the energy goes into mass rather than changing the velocity. To estimate the loss from relativisitic electrons we must refine the approximations to retain terms involving τ^2.

We start again with the radical in the second integral in eq (115) and we write

$$\sqrt{1 - \frac{4f\tau}{m_o v}} - 1 \approx \frac{2f\tau}{m_o v} - 2\left(\frac{f\tau}{m_o v}\right)^2 \tag{121}$$

and the integral becomes

$$\frac{m_o}{f}\int_{nv_o}^{v}\frac{v\,dv}{\left(v-\frac{f\tau}{m_o}\right)\left(1-\frac{v^2}{c^2}\right)^{3/2}} = t_2 \tag{122}$$

Noting that

$$nv_o > \frac{f\tau}{m_o} \tag{123}$$

we make use of the Binomial theorem and expand the bracket involving τ, yielding

$$\frac{m_o}{f}\int_{nv_o}^{v}\frac{dv}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} + \tau\int_{nv_o}^{v}\frac{dv}{v\left(1-\frac{v^2}{c^2}\right)^{3/2}} = t_2 \tag{124}$$

The first of these has already been evaluated, while the second, with the same subsitution becomes

$$-\int\frac{d\theta}{\sin\theta\cos^2\theta} = \tau\left\{\sec\theta + \frac{1}{2}\ell n\left(\frac{1-\cos\theta}{1+\cos\theta}\right)\right\} \tag{125}$$

Reverting to the original variables and inserting the limits we have

$$\frac{m_o v}{f\left(1-\frac{v^2}{c^2}\right)^{1/2}} - 4n\tau + \tau\left\{\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} - \frac{1}{\left(1-\frac{(nv_o)^2}{c^2}\right)^{1/2}} + \frac{1}{2}\ell n\left(\frac{1-\sqrt{1-\frac{v^2}{c^2}}\;1+\sqrt{\frac{1-(nv_o)^2}{c^2}}}{1+\sqrt{1-\frac{v^2}{c^2}}\;1-\sqrt{1-\frac{(nv_o)^2}{c^2}}}\right)\right\} = t_2 \tag{126}$$

Again making use of the Binomial theorem and (127)

$$\frac{m_o v}{f\left(1-\frac{v^2}{c^2}\right)^{1/2}} 4n\tau + \tau \left\{ \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} - 1 - \frac{(nv_o)^2}{2c^2} + \frac{1}{2}\ell n \left(\frac{1-\sqrt{1-\frac{v^2}{c^2}}}{1+\sqrt{1-\frac{v^2}{c^2}}} \frac{1+\sqrt{1-\frac{(nv_o)^2}{c^2}}}{1-\sqrt{1-\frac{(nv_o)^2}{c^2}}} \right) \right\} = t_2$$

(127)

Without making significant error for relativistic particles we may simplify further to give

$$\frac{m_{ov}}{f\left(1-\frac{v^2}{c^2}\right)^{1/2}} + \tau \left\{ \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} - 1 + \ell n 4n + \frac{1}{2}\ell n \left(\frac{1-\sqrt{1-\frac{v^2}{c^2}}}{1+\sqrt{1-\frac{v^2}{c^2}}} \cdot \frac{4c^2}{n^2 v_o^2} \right) \right\} = t \qquad (128)$$

where (116) has been extended to give

$$t_1 \approx 4n\,\tau + \tau\,\ell n\, 4n \qquad (129)$$

The first term is the solution under the assumption of no radiation, leaving the second term to account for the delay in reaching a spepcified final velocity, or equivalently the delay in attaining a specified energy due to the emission of radiation. We may write this delay as

$$\Delta t = \tau \left\{ \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} - 1 + \ell n\, 4 + \frac{1}{2} \ell n \left(\frac{1 - \sqrt{1 - \frac{v^2}{c^2}}}{1 + \sqrt{1 - \frac{v^2}{c^2}}} \cdot \frac{4c^2}{v_o^2} \right) \right\} \tag{130}$$

At high energies the E - t curves for both the radiating particle and the non-radiating particle will be approximately parallel.

The total energy is

$$E = \frac{m_o c^2}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} \tag{131}$$

Expressing v in terms of E and substituting into (123) we have

$$t = \frac{E}{fc} \sqrt{1 - \left(\frac{m_o c^2}{E}\right)^2} \tag{132}$$

Differentiating with respect to E

$$\frac{dt}{dE} = \frac{1}{fc \left\{1 - \left(\frac{m_o c^2}{E}\right)^2\right\}^{1/2}} \tag{133}$$

which gives

$$\Delta t = \frac{\Delta E}{cf\left\{1 - \left(\frac{m_o c^2}{E}\right)^2\right\}^{1/2}} \tag{134}$$

Equating the two expressions for Δt we obtain an estimate of the energy radiated after first expressing (74) in terms of energy

$$\Delta t = \tau \left\{\frac{E}{m_o c^2} - 1 + \ln 4 + \frac{1}{2} \ln\left(\frac{1 - \frac{m_o c^2}{E}}{1 + \frac{m_o c^2}{E}} \cdot \frac{4c^2}{v_o^2}\right)\right\} \tag{135}$$

Comparing (137) and 138)

$$\Delta E \approx fc\,\tau \left(1 - \frac{(m_o c^2)^2}{E}\right)^{1/2} \left\{\frac{E}{m_o c^2} - 1 + \ln 4 - \ln\left(\frac{v_o}{2c}\right) + \frac{1}{2} \ln\left(\frac{1 - \frac{m_o c^2}{E}}{1 + \frac{m_o c^2}{E}}\right)\right\} \tag{136}$$

The conventional approach to obtaining an estimate is to calculate the acceleration assuming no radiation and then to use this value in the radiation rate relation, followed by integration. Following this process we w rite the acceleration as

$$v = \left(1 - \frac{v^2}{c^2}\right)^{3/2} \frac{f}{m_o} \tag{137}$$

The radiation rate reduces to

$$\dot{E}_{rad} = \frac{m_o \tau \ v^2}{(1 - v^2)^{3/2}} \qquad (138)$$

Combining these two results

$$\dot{E}_{rad} = \frac{f^2 \tau}{m_o} \qquad (139)$$

The total energy radiated in time t is then

$$E_{rad} = \frac{f^2 \tau t}{m_o} = \Delta E \qquad (140)$$

t is the time to attain an energy E, and so we have that the radiated energy is

$$\Delta E = \frac{f^2 \tau}{m_o} \ . \ \frac{m_o c}{f} \sqrt{\alpha^2 - 1} = f \tau c \sqrt{\alpha^2 - 1} \qquad (141)$$

where

$$\alpha = \frac{E}{m_o c^2} \qquad (142)$$

$$\Delta E = f \tau c \sqrt{\alpha^2 - 1} \ . \sqrt{1 - \frac{v^2}{c^2}} = \frac{f \tau \sqrt{\alpha^2 - 1}}{a} \qquad (143)$$

Comparing this estimate with the first estimate based on the new equation of motion, and which we designate as ΔE_1, we have

$$\frac{\Delta E_1}{\Delta E_2} = \left\{\alpha - 1 + \ell n4 - \ell n\left(\frac{V_o}{2c}\right) + \frac{1}{2}\,\ell n\left(\frac{\alpha - 1}{\alpha + 1}\right)\right\} \tag{144}$$

If we consider a highly relativistic electron, say 10Mev, that has been accelerated over 1m we find

$$\frac{\Delta E_1}{\Delta E_2} = 34.9 \tag{145}$$

Conclusion

A new equation of motion for charged particles has been developed which takes into account the radiated energy in a consistent manner, consistent that is, with classical physics. The equation has been deduced using the conservation of energy, and the one analytical solution found preserves causality. Approximate solutions for motion in a magnetic field have indicated the modulation of the motion at twice the rotational frequency. The same processes, expressed in the 4-vector notation of relativity theory, yield the relativistic equation of motion. Here it was found possible to find an approximate solution for linear motion expressing time as a function of velocity. Changing the dependent variable to energy it has been shown that a conventional approach leads to errors of factors of ~ 34 in the estimate of the radiated energy for linearly accelerated electrons in a constant electric field for both relativistic and non-relativistic electrons. The complexity of the relativistic equation indicates that a numerical approach to obtaining solutions may be necessary.

References

[1] A Critical Examination of the Abraham - Lorentz Equation for a Radiating Charged Particle.

J.L.Jimenez and I.Campus.
Am Journal of Physics 55(11) Nov 1987.

[2] Classical Electrodynamics.

J.D. Jackson.
John Wiley & Son.

[3] Classical Theory of Radiating Electrons.

P.A.M. Dirac.
Proceedings of the Royal Society, 1938.