Physics Notes

Note 6

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ON THE FIELDS OF A TORUS AND THE OBSERVABILITY OF THE VECTOR POTENTIAL

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ABSTRACT

To study the observability and properties of the magnetic vector potential $\boldsymbol{A}$ when $\boldsymbol{B}=\nabla\times\boldsymbol{A}=0$, we compute the fields and potential outside a torus carrying a toroidal current sheet. An immediate general result of magnetostatics is used to show that the exterior static $\boldsymbol{A}$ in a common gauge is exactly the same as the static magnetic *field* $\boldsymbol{B}$ of an ordinary current loop with the same dimensions as the torus; $\boldsymbol{A}$ falls off as $1/r^3$, and its form becomes obvious.

When the current I varies in time, non-zero quasi-static fields $\boldsymbol{E}(t)$ and $\boldsymbol{B}(t)$ are produced ($\boldsymbol{E}\sim\omega I/r^3$ and $\boldsymbol{B}\sim\omega^2 I/r^2$). Radiation is also produced. The radiation pattern is that of an *electric dipole*. The torus provides a counterexample to the common erroneous notion that if all multipole moments of a current distribution vanish then quasi-static fields and radiation must also vanish.

A general result of radiation theory is that if a radiated vector potential is not zero then the radiated fields are also not zero, meaning that the radiated $\boldsymbol{A}$ cannot be separated from its fields as the static $\boldsymbol{A}$ can; there is no such thing as a "radiated curl-free vector potential" referred to in some literature.

Maxwell's Equations are formulated in a way that obviates the role of gauge transformations, clarifying the relation of potentials to current sources. We also discuss shielding of fields and potentials by a conducting enclosure.

Attempts to generalize the result of magnetostatics to time varying sources reveals a seldom recognized symmetry of Maxwell's Equations.

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SECTION 1

INTRODUCTION

In classical electromagnetic theory, the magnetic vector potential A is generally considered only a mathematical aid to solving for the field $B = \nabla \times A$. It is the field that is real.

However, in classical physics energy is quite real, and the electrostatic potential ϕ is usually considered as real as the electric field $E = -\nabla\phi$. Since relativity requires ϕ and A be components of a four-vector, one should attribute just as much reality to A.

Moreover, in quantum mechanics, the canonical quantization procedure *requires* the use of potentials. A problem would seem to arise in those situations in which $B = 0$, but $A \neq 0$, for there should be no classical difference, but there may be a QM difference.

Aharonov and Bohm [Ah59] first pointed out experiments to demonstrate the reality and importance of the potentials in quantum mechanics. There are observable differences when, say, an electron is passed through a region of zero fields but non-zero potentials. The differences show up in the phase of the wave function, requiring an electron interference experiment to detect.

Ever since the reality of A in quantum mechanics was emphasized, and various experiments confirmed it [Bo60, Ch60, We60, We85, Ch85], interest has attached to measuring A directly, especially where $B = 0$. In fact, a recent proposal suggests a non-destructive photon detector by "passively" measuring only A of the photon [Le92].

The possibility then arises of a class of electromagnetic sensors that would detect the vector potential rather than the fields. The sensitivity of such detectors, and their merits relative to usual field sensors, would have to be determined.

To investigate this phenomenon it is useful to have a ready current distribution that provides a working volume in which $B = 0$, but $A \neq 0$. This is the case for the static A outside a torus on which flows a steady current only in the "toroidal" direction, around the thin limb. (But it is not the case for time varying currents.)

In addition to the static case, the further question arises whether a propagating time varying A can similarly be separated from E and B, and so be measured on its own. It is easily shown that for radiation fields A cannot be so separated.

Two suggested detectors of A both operate via variants of the Aharonov-Bohm effect. In place of a coherent electron beam split into two sub-beams enclosing magnetic flux, one can use a superconductor with a Josephson junction. The superconducting state is a macroscopic coherent wave function that plays the role of the electron beam. The Josephson junction detects A since the tunneling current depends on the phase of the wave function, which shifts when $A \neq 0$.

A second detector [Le92] would employ the conventional split electron beam along the surface of a crystal, detecting the evanescent A of a light beam undergoing total internal reflection.

These sensors work best on a time varying $\boldsymbol{A}$. Therefore for time dependent laboratory current sources one needs to know the full environment at the detector, both fields and potentials.

A torus provides a simple source of non-zero $\boldsymbol{A}$ but zero $\boldsymbol{B}$ in a workable volume of space. In this note we first obtain a very useful expression for the static $\boldsymbol{A}$ outside a torus, and for the quasi-static and radiated fields and potential when the current varies in time. An Appendix calculates the exact fields.

In the process we make useful observations in magnetostatics and electrodynamics concerning the calculation of the vector potential, and its relationship to sources in the light of gauge invariance. Maxwell's Equations are formulated in a way that explicitly incorporates gauge invariance; once in that form the questions of gauge transformations and gauge invariance never arise.

SECTION 2

MAGNETOSTATICS

The equations of magnetostatics,

$$\nabla \cdot \boldsymbol{B} = 0 , \qquad \nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J} , \tag{2.1}$$

are conveniently formulated with

$$\boldsymbol{B} \equiv \nabla \times \boldsymbol{A} , \qquad \nabla \cdot \boldsymbol{A} \text{ arbitrary} , \tag{2.2}$$

so that the Cartesian components of $\boldsymbol{A}$ obey

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_o \boldsymbol{J} . \tag{2.3}$$

The arbitrariness in $\nabla \cdot \boldsymbol{A}$ means the gradient of any scalar may be added to $\boldsymbol{A}$ without changing $\boldsymbol{B}$ (gauge transformation); it is sufficient to compute $\boldsymbol{A}$ in any gauge. Noting that when $\partial/\partial t = 0$, the Lorentz gauge ($\nabla \cdot \boldsymbol{A} + \partial\phi/\partial t = 0$) and Coulomb gauge ($\nabla \cdot \boldsymbol{A} = 0$) are the same, we choose

$$\nabla \cdot \boldsymbol{A} = 0 . \tag{2.4}$$

Then $\boldsymbol{A}$ obeys Poisson's equation $\nabla^2 \boldsymbol{A} = -\mu_o \boldsymbol{J}$, with the solution

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_o}{4\pi} \int d^3r' \, \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} . \tag{2.5}$$

However the basic equations of $\boldsymbol{A}$,

$$\nabla \cdot \boldsymbol{A} = 0 , \qquad \nabla \times \boldsymbol{A} = \boldsymbol{B} , \tag{2.6}$$

are the same as obeyed by $\boldsymbol{B}$, Equations(2.1), with $\boldsymbol{A}$ replacing $\boldsymbol{B}$ and $\boldsymbol{B}$ replacing $\mu_o \boldsymbol{J}$. We can therefore introduce a vector Λ by

$$\boldsymbol{A} = \nabla \times \Lambda \tag{2.7}$$

and choose $\nabla \cdot \Lambda = 0$, so that

$$\Lambda(\boldsymbol{r}) = \frac{\mu_o}{4\pi} \int d^3r' \, \frac{\boldsymbol{B}(\boldsymbol{r}')}{\mu_o \, |\boldsymbol{r} - \boldsymbol{r}'|} . \tag{2.8}$$

Taking the curl of this shows $\boldsymbol{A}$ is given by a Biot-Savart law in terms of $\boldsymbol{B}$,

$$\boldsymbol{A}(\boldsymbol{r}) = \nabla \times \Lambda(\boldsymbol{r}) = \frac{\mu_o}{4\pi} \int d^3\boldsymbol{r}' \, \frac{\boldsymbol{r}' - \boldsymbol{r}}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \times \frac{\boldsymbol{B}(\boldsymbol{r}')}{\mu_o} . \tag{2.9}$$

The differential relations (2.1) and (2.6), or (2.9) itself, show that, as an immediate result of magnetostatics, $\boldsymbol{B}$ is the source of $\boldsymbol{A}$ just as $\boldsymbol{J}$ is the source of $\boldsymbol{B}$. This can be used to visualize $\boldsymbol{A}$ in many situations, for $\boldsymbol{A}$ is constructed from $\boldsymbol{B}$ in the same way $\boldsymbol{B}$ is constructed from $\boldsymbol{J}$. One can, for example, apply the "right-hand rule" to get $\boldsymbol{A}$'s direction from $\boldsymbol{B}$.

Magnetostatics admits an endless hierarchy of potentials,

$$\begin{array}{ll} \cdots & \\ \nabla\times\boldsymbol{J} = (\text{prescribed}) & \nabla\cdot\boldsymbol{J} = 0\ , \\ \nabla\times\boldsymbol{B} = \boldsymbol{J} & \nabla\cdot\boldsymbol{B} = 0\ , \\ \nabla\times\boldsymbol{A} = \boldsymbol{B} & \nabla\cdot\boldsymbol{A} = 0\ , \\ \nabla\times\boldsymbol{\Lambda} = \boldsymbol{A} & \nabla\cdot\boldsymbol{\Lambda} = 0\ , \\ \cdots & \end{array} \tag{2.10}$$

each determined from the next one by the curl operation. $\boldsymbol{B}$, for example, is the "potential" of the "field" $\boldsymbol{J}$, and $\boldsymbol{A}$ is the field of the current $\boldsymbol{B}$. This can be very useful in computing the vector potential.

The hierarchy shows that the vector potential $\boldsymbol{A}_1$ of a current distribution $\boldsymbol{J}_1$ with field $\boldsymbol{B}_1$ is the same as the magnetic field $\boldsymbol{B}_2$ of a current distribution $\boldsymbol{J}_2$ equal to $\boldsymbol{B}_1$. Then for any given $\boldsymbol{J}$ the solution for $\boldsymbol{B}$ immediately provides the solution for an infinite set of problems, and the hierarchy may be summarized as

$$\begin{bmatrix} \boldsymbol{J} \\ \boldsymbol{B} \\ \boldsymbol{A} \end{bmatrix}_{(n)} = \nabla\times\begin{bmatrix} \boldsymbol{J} \\ \boldsymbol{B} \\ \boldsymbol{A} \end{bmatrix}_{(n+1)} . \tag{2.11}$$

This is a general symmetry of magnetostatics. Use will be made of these observations in Section 9 to construct a hierarchy of current distributions based on that of a torus. Equ (2.11) also allows easy construction of many current distributions which produce vanishing $\boldsymbol{B}$ but non-vanishing $\boldsymbol{A}$. In Section 10 it is extended to time varying sources and fields.

SECTION 3

STATIC VECTOR POTENTIAL OF TORUS

Direct evaluation of (2.5) for a torus runs into integrals of elliptic integrals. While these cannot be avoided for an exact solution by quadratures, the observations of Section 2 permit a very useful approximate expression.

The torus lies in the x,y plane, has minor radius a and major radius b (Figure 1). To keep

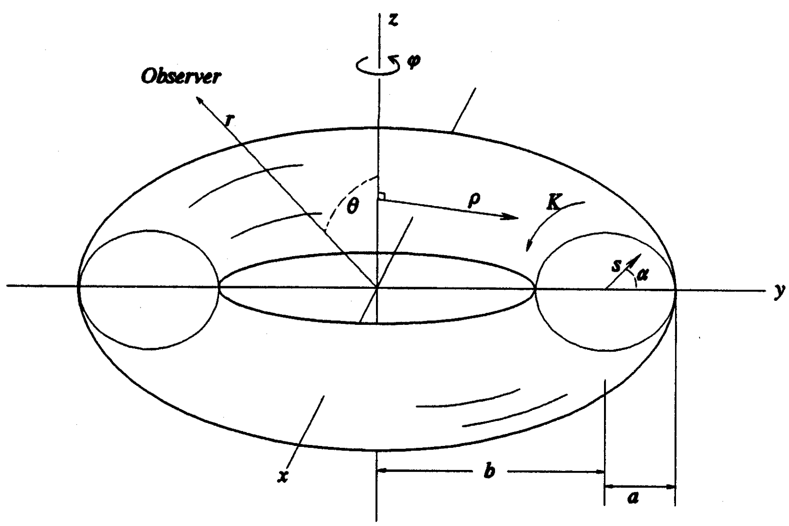

Figure 1. Torus geometry definitions.

$\boldsymbol{B} = 0$ outside, we require no circumferential current in the azimuthal direction about the symmetry axis z. The only allowed current is in the direction of increasing α, as would be created by a tightly spaced toroidal wire coil with an even number of counter-rotating layers. If i_w is the wire current, and the total number of turns is N, then the total current is $I = Ni_w$.

We employ the usual spherical coordinate system (r,θ,φ), a Cartesian set of axes (x,y,z) the usual cylindrical system (ρ,φ,z), and occasionally the internal polar coordinates (s,α).

The field inside is

$$B = B_\varphi = \frac{\mu_o I}{2\pi\rho} . \tag{3.1}$$

B decreases across the interior. The surface current density K is

$$K = \frac{I}{2\pi\rho} = \frac{I}{2\pi(b + a\cos\alpha)} , \tag{3.2}$$

and the current density $\boldsymbol{J}$ is $K\delta(s\text{-}a)\,\hat{\alpha}$. The flux in the torus is

$$\Phi = \int dS \cdot B = \pi a^2 \frac{\mu_o I}{2\pi b} g(\tfrac{a}{b}) \ , \tag{3.3}$$

where $dS = sdsd\alpha$, and, with $u = s/a$, $\xi = a/b$,

$$g(\xi) = 2\int_0^1 du\, u \int_0^{\pi} \frac{d\alpha}{\pi} \frac{1}{1+u\xi\cos\alpha} = 2\,\frac{1-\sqrt{1-\xi^2}}{\xi^2} \tag{3.4}$$

is the shape factor, $1 \leq g \leq 2$. $g \to 1$ as $a/b \to 0$ (bicycle tire), and $g \to 2$ as $a/b \to 1$ (hole-less donut). Torus self inductance is $L = \mu_o N^2(a^2/2b)g$.

In the gauge in which $\nabla \cdot A = 0$, the vector potential is given by the curl of (2.8),

$$A = \nabla \times \frac{\mu_o}{4\pi} \int d^3r' \, \frac{\hat{\varphi}' B(r')}{\mu_o \, |r - r'|} \ , \tag{3.5}$$

the integral being taken over the torus volume. A has no φ component. With B given by (3.1), this equation shows A is exactly the same as the magnetic field of a single fat wire loop coinciding with the torus with current distribution inside the wire proportional to $1/\rho$. When $b \gg a$, or when the observer is at $r \gg (b+a)$, this current variation within the wire is not important, and A looks like the dipole field of an ordinary current loop.

The integral in (3.5) behaves as $1/r^2$ for $r \gg (b+a)$, and there is easily evaluated to obtain

$$A_r = \frac{\mu_o}{4\pi} \frac{VI}{2\pi r^3} \cos\theta \ , \qquad A_\theta = \frac{\mu_o}{4\pi} \frac{VI}{4\pi r^3} \sin\theta \ , \qquad [r \gg (b+a)] \tag{3.6}$$

proportional to torus volume $V = 2\pi^2 a^2 b$. Lines of A are sketched in Figure 2. The exact solution in Appendix A shows that A contains only odd powers of $1/r$. The next correction to the static potential (3.6) is $\sim 1/r^5$, smaller by a factor $\sim (b/r)^2$.

Outside the torus A is locally the gradient of a scalar, $A = -\nabla\Psi$ [for the $1/r^3$ terms, (3.6), Ψ is

$$\Psi = \frac{\mu_o}{4\pi} \frac{VI}{4\pi r^2} \cos\theta\,] . \tag{3.7}$$

Thus A can be locally transformed away with the gauge transformation

$$A \to A' = A + \nabla\Psi = 0 \ . \tag{3.8}$$

However A cannot be transformed to zero everywhere in the doubly connected space outside the torus. Due to (2.2), we have around any closed path encircling the limb,

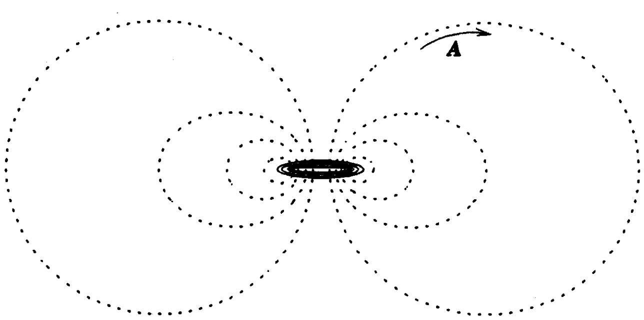

Figure 2. Lines of A.

$$\oint d\ell \cdot A = \Phi \; . \tag{3.9}$$

Therefore, on the surface, the average A around the limb is

$$\bar{A} = \frac{\Phi}{2\pi a} = \frac{\mu_o}{4\pi}\frac{aI}{b}g \; , \tag{3.10}$$

in any gauge. Largest $\bar{A}$ occurs for $a = b$, $\bar{A}_{max} = (\mu_o/4\pi)2I$.

In general, since $A \propto \Phi$, but $B \propto I$, A outside can be increased relative to any leakage B by keeping I constant and increasing a (so long as the gap between wires is not also increased).

The equivalence of A of a torus to B of a current loop becomes clear by dividing (3.5) by a, and using (3.1).

$$\frac{1}{a}A = \frac{\mu_o}{4\pi}\nabla\times\int d^3r' \frac{I\hat{\varphi}}{2\pi a\rho|r-r'|} \; . \tag{3.11}$$

A of a torus is a times the magnetic field of an ordinary loop with current density

$$J' = \frac{I}{2\pi a\rho}\hat{\varphi} \tag{3.12}$$

and current

$$I' = \int dS\, J' = \left(\frac{a}{2b}\right)Ig \; . \tag{3.13}$$

Potential On Axis

On the z axis, A is simple when $a \ll b$. Then the magnetic field of a ring current $(a/2b)I$ is [Smythe, 1989]

$$B_z = \frac{\mu_o}{4\pi}\frac{2\pi b^2(a/2b)I}{(b^2+z^2)^{3/2}} \; , \tag{3.14}$$

so that A_z for a torus with $a \ll b$ and current I is

$$A_z = \frac{\mu_o}{4\pi}\frac{\pi a^2 bI}{(b^2+z^2)^{3/2}} \; . \tag{3.15}$$

A_z is largest at $z = 0$, and there is quite comparable to the distant components (3.6) extrapolated back to $r \approx b$, but smaller by about a factor $\pi a/b$ than A on the surface.

The Aharonov-Bohm Effect

Equation (3.9) contains the essence of the Aharonov-Bohm effect. The phase shift of an electron moving from point P to point Q (Figure 3) is $-(e/\hbar)\int_P^Q dl \cdot A$ [Ah59]. An electron

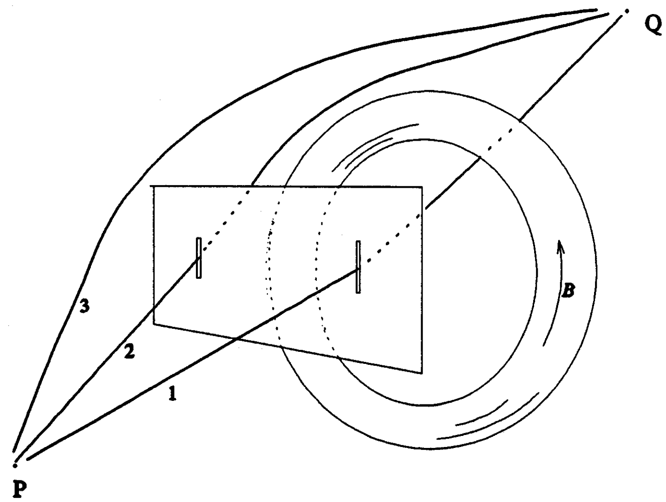

Figure 3. The Aharonov Bohm effect arises when magnetic flux passes through two paths, as between paths *1* and *2*, while the magnetic field vanishes along each path.

traversing path 1, which passes through the torus, suffers a different phase shift $\delta\varphi$ from one traversing path 2, which does not, by a gauge invariant amount

$$\delta\varphi = -(e/\hbar)\oint_{(1,2)} dl \cdot A = -(e/\hbar)\Phi, \tag{3.16}$$

even though fields are zero along both paths. Electrons taking paths 2 or 3 undergo the same phase shift since no flux cuts the 2,3 loop.

Therefore the interference pattern at Q of a coherent electron source emanating from P, passing through the two slits, differs according as the magnetic flux between paths 1 and 2 is zero or non-zero, even though no force acts on an electron taking either path.

SECTION 4

ALTERNATE CONFIGURATION FOR EXPERIMENTS

Before proceeding to time varying currents, we note that the observations of Section 2 aid in designing other static configurations to produce a region of $\boldsymbol{B}=0$, but $A \neq 0$.

Any localized current distribution $\boldsymbol{J}_1$ that produces a field $\boldsymbol{B}_1$ in a region where $J_1=0$ can be transformed into a configuration that produces an $A \neq 0$ where $\boldsymbol{B}=0$ by inventing a different current distribution that will produce a field equal to $\boldsymbol{J}_1$. According to the hierarchy of Eq(2.11), the needed current distribution is proportional to $\nabla \times \boldsymbol{J}_1$. Since practical current sources are localized and produce a field outside themselves, this shows that, by constructing the new current distribution $\propto \nabla \times \boldsymbol{J}_1$, there are many ways to produce a field-free region of space with $A \neq 0$.

The torus provides an example. The required current distribution is the curl of the torus current of Figure 1, and is an azimuthally flowing double layer on the torus surface. The inner surface layer, say, flows in the direction $\hat{\varphi}$, the outer surface layer in the direction $-\hat{\varphi}$. $\boldsymbol{B}$ is in a thin sheet confined between the layers, and is in the toroidal direction, being a vector parallel to the current layer K of Figure 1. As shown in Figure 4 the vector potential circulates inside the torus, like the $\boldsymbol{B}$ of Figure 1. This would provide a volume of slowly varying A. Unfortunately, this volume is physically inaccessible.

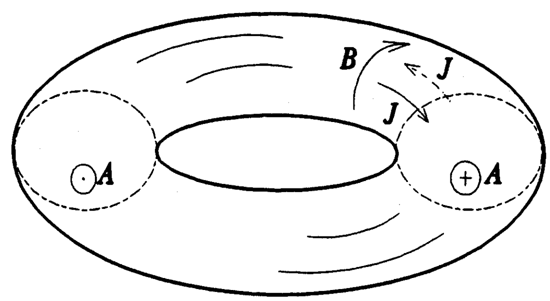

Figure 4. Torus with counter-rotating surface current layers. The layer flowing in the $-\varphi$ direction is on the outer suface of the tube; the layer flowing in the $+\varphi$ direction is on the inner surface of the tube.

However, if this torus is now cut through its limb at one place and then straightened out, we have a long narrow cylinder, of length $2\pi b$ and radius a, with current flowing up its length on the outside cylinder surface, and back down its length on the inside. The magnetic field is circumferential about the axis, confined between the two current sheets. The vector potential is the same as the usual magnetic field of a solenoid.

This configuration is itself topologically equivalent to the torus. Instead of changing the current direction and cutting the torus, we can stretch the torus of Figure 1 in the z direction, its cross sectional circle of radius a being elongated into an ellipse, as in Figure 5. After stretching to a length ℓ, and taking $a \to 0$, the resulting geometry is a "solenoid" of radius b and length ℓ, with azimuthal magnetic field confined between the two axial surface current sheets. $\boldsymbol{B} = 0$ inside and outside this "solenoid". Inside, at cylindrical radii $\rho < b$, $A \neq 0$; outside ($\rho > b$), A is only the "fringing field", significantly different from zero only near the ends. Inside the solenoid is a readily accessible volume for experiments.

A inside is proportional to the flux of $\boldsymbol{B}$, and so can be increased by thickening the walls, as in the cylindrical torus of Figure 6. A inside, on or off axis, is axial and is, for $\ell \gtrsim 2\rho_i$,

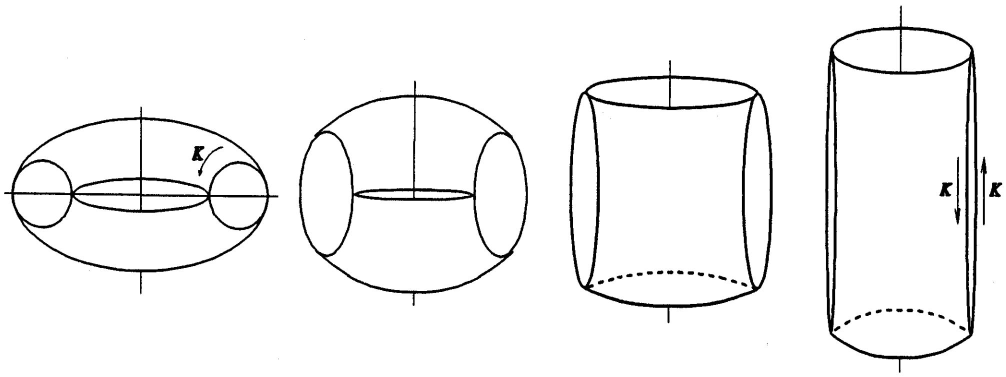

Figure 5. Showing topological equivalence of torus and cylindrical "solenoid".

$$A = \frac{\mu_o I}{2\pi} \ln\left[\frac{\rho_o}{\rho_i}\right] . \qquad (4.1)$$

A is maximized by increasing the ratio of outer to inner radius. For practical construction the logarithm is likely to be $\sim$ 1, providing a useable volume with radius ρ_i of nearly constant A with magnitude as large as the largest occurring anywhere for any torus carrying the same current. If I varies in time, B always stays zero on axis, but $E_z = -\partial A_z / \partial t$ is non-zero, providing a (smaller) region near the axis of vanishing B, but non-zero $A(t)$ and $E(t)$.

The value for A in (4.1) is, of course, gauge dependent, here being in the gauge $\nabla \cdot A = 0$. As written this constant z component of A is $\nabla(zA)$, and so could be transformed away. But Equ.(3.9) requires that some A would then appear outside this cylindrical torus.

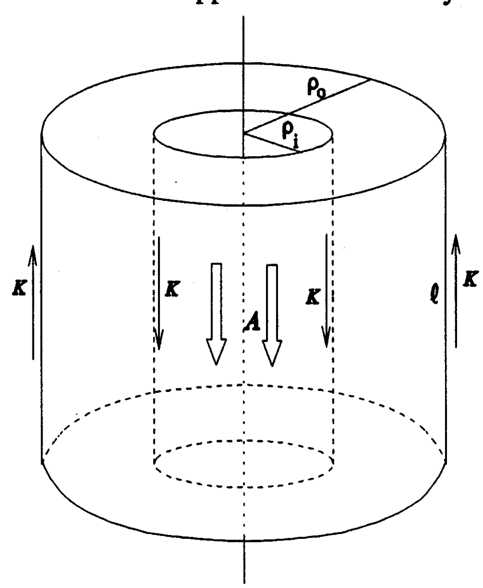

Figure 6. Cylindrical torus with accessible region of large, constant A inside (at $\rho < \rho_i$).

SECTION 5

QUASI-STATIC FIELDS

We return now to time dependent fields of a torus with circular cross section.

If J and I vary in time, an observer in the near zone will see the previously static A vary as $I(t)/r^3$ and give rise to an electric field. If, further, $\nabla \cdot J$ remains zero, the scalar potential ϕ vanishes, and the field is $E = -\partial A/\partial t$. The quasi-static electric field pattern is the same as the static A, and, for harmonic variation with frequency ω, is proportional to $\omega I/r^3$. A displacement current $\epsilon_0 \partial E/\partial t \sim \omega^2 I/r^3$ appears through, say, the area enclosed in the circular path drawn above the torus in Figure 7. Then Ampere's law

$$\nabla \times B = \mu_o J + \frac{1}{c^2}\frac{\partial E}{\partial t} , \tag{5.1}$$

requires there be a quasi-static azimuthal magnetic field to balance this displacement current. Integrating Equation (5.1) over the area of radius ρ, the required magnetic field is given by

$$\begin{aligned} 2\pi\rho B &= \frac{1}{c^2}\frac{\partial}{\partial t}\int dS \cdot E \\ &= \frac{2\pi}{c^2}\frac{\partial}{\partial t}\int d\rho\, \rho\, E_z . \end{aligned} \tag{5.2}$$

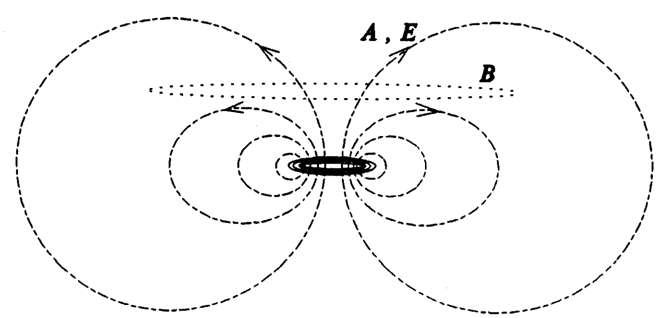

Figure 7. A quasi-static azimuthal magnetic field is required on the indicated loop due to the displacement current through it.

Since $E_z \sim \omega I/\rho^3$ for large ρ, this quasi-static B varies as $\omega^2 I/r^2$ for large r.

This B must also be $\nabla \times A$. But the just determined quasi-static B is not the curl of the quasi-static A, which vanishes. The field discussion thus far therefore cannot be complete, and we need to look closer at time varying fields and potentials.

In the Lorentz gauge, Maxwell's Equations reduce to the wave equation for the vector potential, whose Cartesian components are given exactly by the full retarded solution

$$A(r,t) = \frac{\mu_o}{4\pi}\int d^3r' \, \frac{J(r',t')}{|r-r'|} , \tag{5.3}$$

where

$$t' = t - \frac{|r - r'|}{c} \tag{5.4}$$

is retarded time. When A is expanded in powers of b/r, and each coefficient further developed in the low frequency expansion in powers of $kb \equiv (\omega/c)b$, the lowest order surviving terms are given in Appendix A, Equation (A-17). The quasi-static vector potential is

$$A_{QS} = e^{-i\omega t} e^{ikr} A_{\text{stat}}(r) \tag{5.5}$$

where A_{stat} is the static vector potential of Equation (3.6).

The quasi-static fields (those to lowest order in frequency) are

$$E_{QS} = \frac{\mu_o}{4\pi} \frac{VI}{4\pi} \frac{ikc}{r^3} (2\cos\theta\hat{r} + \sin\theta\hat{\theta}) \ ,$$

$$B_{QS} = \frac{\mu_o}{4\pi} \frac{VI}{4\pi} \frac{k^2}{r^2} \sin\theta\hat{\varphi} \ , \tag{5.6}$$

also proportional to torus volume. The quasi-static B does balance the displacement current as required by Ampere's Law, $\partial E_{QS}/\partial t = c^2 \nabla \times B_{QS}$. As seen in the Appendix, E_{QS} arises from the time derivative of the quasi-static potential A_{QS}, but B_{QS} arises as the curl of the "inductive" A which is $-ikr$ times A_{QS}. However these quasi-static fields do not separately satisfy Faraday's law, $\partial B_{QS}/\partial t \neq -\nabla \times E_{QS}$. Rather, $\partial B_{QS}/\partial t$ is balanced by the curl of the part of E arising from the inductive A, which is kr times smaller that E_{QS}.

These fields, valid for $kr \ll 1$, are ordered according to

$$B_{QS} \sim \frac{VIk^2}{r^2} \sim kr\frac{E_{QS}}{c} \ll \frac{E_{QS}}{c} \ . \tag{5.7}$$

SECTION 6

RADIATED FIELDS AND POTENTIAL

Before computing the radiation fields from a torus we make some general observations.

General Comments on Radiation

In any gauge, the scalar potential is not needed for the radiated fields. Both $\boldsymbol{E}_{\text{rad}}$ and $\boldsymbol{B}_{\text{rad}}$ are derivable solely from $\boldsymbol{A}_{\text{rad}}$, since $\boldsymbol{B} = \nabla\times\boldsymbol{A}$, and

$$\boldsymbol{E}_{\text{rad}} = -c\boldsymbol{n}\times\boldsymbol{B}_{\text{rad}} , \tag{6.1}$$

where $\boldsymbol{n}$ is the outgoing unit vector. In the Lorentz gauge, $\boldsymbol{A}$ is given by (5.3). The radiated vector potential is the part of (5.3) that falls as $1/r$, obtained by expanding $1/|\boldsymbol{r}-\boldsymbol{r}'|$ and keeping only the $1/r$ term,

$$\boldsymbol{A}_{\text{rad}} = \frac{\mu_o}{4\pi r}\int d^3r'\,\boldsymbol{J}(\boldsymbol{r}',t') . \tag{6.2}$$

If the time argument of $\boldsymbol{J}$ were t, the integral of $\boldsymbol{J}$ over all space at one instant of time would vanish when $\nabla\cdot\boldsymbol{J}=0$. Retarded time means wavelets emanating from different parts of the source do not quite cancel, allowing radiation. Retardation, of course, is the physical reason any divergenceless time dependent current distribution of finite extent radiates.

$\boldsymbol{B}_{\text{rad}}$ is the $1/r$ part of $\nabla\times\boldsymbol{A}_{\text{rad}}$, or

$$\boldsymbol{B}_{\text{rad}} = \frac{\mu_o}{4\pi r}\nabla\times\int d^3r'\,\boldsymbol{J}(\boldsymbol{r}',t') . \tag{6.3}$$

This does not vanish, of course, even for a torus. The integral depends on $\boldsymbol{r}$ only through t',

$$\nabla\times\boldsymbol{J}(\boldsymbol{r}',t') = \nabla t'\times\frac{\partial\boldsymbol{J}}{\partial t'} = -\frac{1}{c}\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\times\frac{\partial\boldsymbol{J}}{\partial t'} \tag{6.4}$$

where we have used the gradient of (5.4). Therefore,

$$\boldsymbol{B}_{\text{rad}} = -\frac{\mu_o}{4\pi c r}\int d^3r'\,\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\times\frac{\partial\boldsymbol{J}(\boldsymbol{r}',t')}{\partial t'} . \tag{6.5}$$

The first factor in this integrand is $(\boldsymbol{r}-\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'| = \boldsymbol{n} - O(r'/r)$, so that

$$\boldsymbol{B}_{\text{rad}} = -\frac{\mu_o}{4\pi c r}\boldsymbol{n}\times\int d^3r'\,\frac{\partial\boldsymbol{J}}{\partial t'} . \tag{6.6}$$

Now noting $\partial t'/\partial t = 1$, the derivative may be pulled out,

$$B_{\rm rad} = -\frac{\mu_o}{4\pi cr}\, n \times \frac{\partial}{\partial t}\int d^3r'\, J(r',t') = -\frac{1}{c}\, n \times \frac{\partial A_{\rm rad}}{\partial t} \, . \qquad (6.7)$$

Since J and $A_{\rm rad}$ are not parallel to n, $B_{\rm rad}$ is non-zero. Equations (6.1) and (6.7) also show $E_{\rm rad} = -\partial A_{\rm rad}/\partial t$ as it should.

$A_{\rm rad}$ varies much more rapidly in the radial direction (rate $\sim kA_{\rm rad}$) than in the φ or θ directions ($\sim A_{\rm rad}/r$). Therefore the radial derivative of a transverse component is large, producing a non-zero component of $\nabla \times A_{\rm rad}$ perpendicular to n. In any gauge, if $A_{\rm rad}$ is non-zero, $B_{\rm rad}$ and $E_{\rm rad}$ are also non-zero. Unlike the static A, the radiated vector potential cannot be separated from its fields. That is, there is no such thing as a "radiated curl-free vector potential" referred to in some literature. It is easily shown that $\nabla \cdot A_{\rm rad} = 0$, so that $A_{\rm rad}$ is a transverse vector.

The discussion so far is completely general, applying to any current distribution.

As a rule, if a current increases in time from zero to a non-zero steady value, radiation is produced that leaves behind the static field of the non-zero current. When I was turned on for the torus, radiated A, E and B propagated out, leaving behind a non-zero static A, and a static B which happens to have value 0. The static A outside a torus is only *accidentally* curl-free because of high symmetry geometry. As shown in Section 7 it is more fundamentally to be considered a transverse vector, with the defining property of being divergence-free everywhere.

Physical Argument for Radiation

That a torus, or any system that encloses magnetic flux in an area that can be "looped", must radiate can be seen physically as follows. Consider a resistive test wire that encircles the torus limb in a closed loop as in Figure 8. Let R be the full resistance of the wire. When the torus current I varies, so does the magnetic flux Φ in the torus interior, and an emf

$$\mathcal{E} = \oint_{\rm wire} dl \cdot E = -\frac{\partial \Phi}{\partial t} \qquad (6.8)$$

is produced, according to Faraday's law. A current $\mathcal{E}/R$ flows in the test wire. The electrons in the wire know to move because the induced electric field drives them.

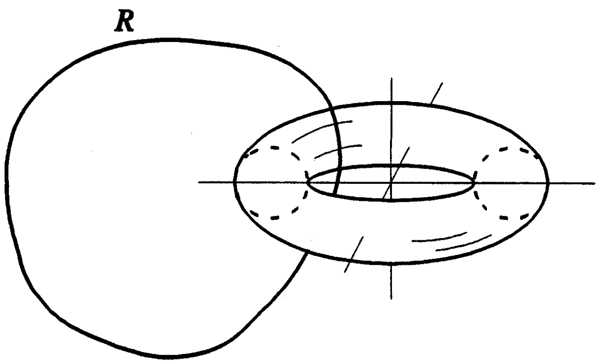

Figure 8. Resistive wire loop through a torus.

All electromagnetic disturbances start where $\nabla \times J \neq 0$, in this case on the torus windings. The only way E can get to the wire is by propagating from the torus to the wire. This propagation, carried to larger distances, is radiation.

If the torus and test wire are of very great radius L, then a propagation time of order L/c is necessary before E gets to the wire. But since (6.8) holds instantaneously, the radiated magnetic flux through the big wire loop is equal and opposite to the quasi-static flux inside the torus limb until the radiation front gets to the wire. It is only after the radiated fields pass that Φ in (6.8) can be taken to be the usual interior quasi-static flux. Only $1/r$ fields can account for the necessary emf. Radiation from a torus has been discussed by Baum [Ba91].

It is interesting to note that, from (6.8), and from (3.9), which holds for time varying conditions as well, the wire current is

$$\frac{\mathscr{E}}{R} = \frac{1}{R}\oint dl \cdot E = -\frac{1}{R}\oint_{\text{wire}} dl \cdot \frac{\partial A}{\partial t} , \tag{6.9}$$

so the total charge moved through the wire is

$$Q(t) = \int \frac{\mathscr{E}(t)}{R}\, dt = -\frac{1}{R}\oint dl \cdot A . \tag{6.10}$$

That is, displaced charge is as much a *direct* measure of $\oint dl \cdot A$ as current is of $\oint dl \cdot E$. From this point of view A could be considered just as real a physical variable as E. Feynman [Fe64] stresses how A may be considered physically real in spite of the arbitrariness in its divergence.

Parallel to the fact that only $\nabla \times A$ enters Maxwell's Equations and $\nabla \cdot A$ is arbitrary, it is worth noting that only the divergence of the energy flux $S = E \times H$ (Poynting vector) enters the energy conservation law that follows from Maxwell's Equations; $\nabla \times S$ is arbitrary. That is, S itself is arbitrary to the extent that the curl of any vector may be added to it. Yet we are quite accustomed to thinking of S as physically real in spite of the arbitrariness in its curl. An arbitrary additive curl or gradient need not prevent a vector from being considered real. Konopinski [Ko78] discusses how in classical Electromagnetism A may be considered the potential momentum per unit charge of a particle in external fields, just as ϕ is the potential energy per unit charge.

Explicit Radiated Fields

To compute the radiated fields, we directly evaluate (6.2) for a harmonic toroidal current. For frequency ω we write for J,

$$J(r,t') = \frac{I}{2\pi\rho}\,\delta(s-a)\, e^{-i\omega t'}\, \hat{\alpha} \tag{6.11}$$

where $\hat{\alpha}$ is the unit vector in the direction of increasing α (Figure 1). Then in (6.2) the s integral in $d^3r' = \rho'\, d\varphi'\, ds\, s\, d\alpha$ is trivial, leaving

$$A_{\text{rad}}(r',t) = \frac{\mu_o}{4\pi}\frac{Ia}{2\pi r}\int_0^{2\pi} d\varphi' \int_0^{2\pi} d\alpha\ \hat{\alpha}\ e^{-i\omega t'} . \tag{6.12}$$

In t', $|r - r'|$ is expanded,

$$t' = t - \frac{r}{c} + \frac{1}{c} n \cdot r' + O(r'^2/cr) \tag{6.13}$$

so the integral in (6.12) becomes

$$e^{-i\omega\tau}\int d\varphi' \int d\alpha\ \hat{\alpha}\ e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'}\ , \tag{6.14}$$

where $\tau = t\text{-}r/c$ is the observer's retarded time, and $\boldsymbol{k} = (\omega/c)\boldsymbol{n}$. We have

$$\hat{\alpha} = -(\hat{x}\cos\varphi' + \hat{y}\sin\varphi')\sin\alpha + \hat{z}\cos\alpha \tag{6.15}$$

and

$$\boldsymbol{k}\cdot\boldsymbol{r}' = k[(b + a\cos\alpha)\cos(\varphi-\varphi')\sin\theta + a\sin\alpha\cos\theta]\ , \tag{6.16}$$

where θ,φ are those of the observer.

Equations (6.15) and (6.16) are to be inserted in (6.14). Rather than grapple with the resultant integrals we proceed with a low frequency approximation $kb \ll 1$. Then the exponential in the integrand in (6.14) is expanded,

$$e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'} = 1 - i\boldsymbol{k}\cdot\boldsymbol{r}' - \tfrac{1}{2}(\boldsymbol{k}\cdot\boldsymbol{r}')^2 \pm \ldots \tag{6.17}$$

It is easy to show the first two terms do not contribute to (6.14), and the remaining integral is trivial. One finds,

$$\begin{aligned}\int_0^{2\pi} d\varphi' \int_0^{2\pi} d\alpha\,\hat{\alpha}\,e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'} &= \pi^2 k^2 ab\sin\theta\,(\hat{x}\cos\theta\cos\varphi + \hat{y}\cos\theta\sin\varphi - \hat{z}\sin\theta) \\ &= \pi^2 k^2 ab\sin\theta\,\hat{\theta}\ .\end{aligned} \tag{6.18}$$

Then, if we call the time dependent current I_{AC}, to lowest non-vanishing order in kb the radiated $\boldsymbol{A}$ of a torus is, except for phase,

$$\boldsymbol{A}_{\text{rad}} = \frac{\mu_o}{4\pi}\frac{k^2 I_{AC} V}{4\pi r}\sin\theta\,\hat{\theta}\ , \tag{6.19}$$

a factor of order $(I_{AC}/I)(kr)^2$ times the static $\boldsymbol{A}$ (3.6) in the same (Lorentz) gauge. Fields are

$$\begin{aligned}\boldsymbol{B}_{\text{rad}} &= \frac{\mu_o}{4\pi}\frac{k^3 I_{AC} V}{4\pi r}\sin\theta\,\hat{\varphi}\ , \\ \boldsymbol{E}_{\text{rad}} &= \frac{\mu_o}{4\pi}\frac{c k^3 I_{AC} V}{4\pi r}\sin\theta\,\hat{\theta} = -c\hat{r}\times\boldsymbol{B}_{\text{rad}}\ .\end{aligned} \tag{6.20}$$

Radiated field amplitude is proportional to torus volume. The field pattern is that of an *electric dipole*. However the fields are proportional to ω^3 rather than ω^2. Physically, the origin of the fields is as follows. The torus of Figure 1 has a current I flowing in the $+z$ direction at $\rho=b+a$, and I flowing in the $-z$ direction at $\rho=b\text{-}a$, with connecting paths above and below. Exaggerating the radii, Figure 9 sketches a ring of upward flowing current balanced by a downward

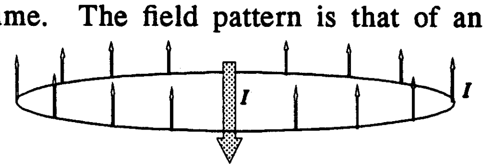

Figure 9. Exaggerated model of vertical torus currents.

flowing current on axis. If I varies in time, both the center and the outer ring currents will radiate like individual electric dipoles. The electric dipole moment of the entire configuration is zero, since the charge displaced by one current is taken by the other. Since the sources are of different dimensions, their radiated fields do not cancel. Rather, the net field is proportional to ka, the separation relative to a wavelength. This explains the extra power of ω and of a in the radiated fields relative to an ordinary electric dipole.

Even two simple dipoles, separated by d as in Figure 10, together having zero dipole moment, will radiate an electric dipole field at frequencies $\omega \gtrsim c/d$. At lower frequencies the quadrupole field dominates, the electric dipole part being smaller by a factor of order $\omega d/c$.

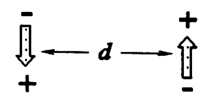

Figure 10.

Multipole Coefficients

It is curious that all electric and magnetic multipole moments of a toroidal current distribution vanish, and yet quasi-static fields and radiation do not.

The general theory of multipole radiation [e.g., Ja75; Ch.16] relates radiated fields to sources. The relevant source parameters are the electric and magnetic *multipole coefficients*, different from the multipole moments. The electric multipole coefficients are[1] [Ja75, §16.5]

$$a_E(l,m) = \frac{4\pi k^2}{i\sqrt{l(l+1)}} \int Y^*_{lm}(\theta,\varphi)\left[\rho\frac{\partial[rj_l(kr)]}{\partial r} + i\frac{k}{c}(\mathbf{r}\cdot\mathbf{J})\,j_l(kr) - ik\nabla\cdot(\mathbf{r}\times\mathbf{M})\,j_l(kr)\right]d^3r \,, \tag{6.21}$$

where $\mathbf{M}$ is the magnetization of the source. The term in the charge density ρ corresponds to the usual electric multipole moment when $kr \ll 1$ (it is generalized in (6.21) to include the radial radiation function $rj_l(kr)$), and vanishes for a torus, as does the term in $\mathbf{M}$. The term in $\mathbf{r}\cdot\mathbf{J}$ is not zero. The extra factor kr/c in this term accounts for the destructive interference from opposite sides of closed loop currents. The lowest surviving one is $l=1$, $m=0$, and corresponds precisely to the previous explanation in terms of equal up and down currents at different radii. Working it out for the case $kb \ll 1$, we have

$$a_E(1,0) = \frac{4\pi k^2}{i\sqrt{2}}\,\frac{ik}{c}\,\frac{k}{3}\int Y^*_{10}\,\mathbf{r}\cdot\mathbf{J}\,r\,d^3r \,, \tag{6.22}$$

having used $j_1(x) \to x/3$ for $x \ll 1$. Since $rY_{10} = (3/4\pi)^{1/2}z$, and

$$\mathbf{r}\cdot\mathbf{J} = (z\cos\alpha - \rho\sin\alpha)\,K(\rho)\,\delta(s-a) \,, \tag{6.23}$$

this reduces to

[1] Equations (6.21) through (6.26) are in Gaussian units.

$$a_E(1,0) = \frac{4\pi k^2}{i\sqrt{2}} \frac{ik^2}{3c}\left(\frac{3}{4\pi}\right)^{1/2} \frac{I}{2\pi} \int z(z\cos\alpha - \rho\sin\alpha)\, a\, d\varphi\, d\alpha$$

$$= \left(\frac{2\pi}{3}\right)^{1/2} \frac{\pi a^2 b I k^4}{c} . \tag{6.24}$$

The radiated magnetic field for the $l=1$, $m=0$ coefficient is

$$\boldsymbol{B}_{\text{rad}} = a_E(1,0) h_1^{(1)}(kr) \frac{1}{\sqrt{2}} \boldsymbol{L} Y_{10} , \tag{6.25}$$

where $h_1^{(1)}$ is the outgoing spherical Hankel function of order 1, and $\boldsymbol{L} = -i\boldsymbol{r}\times\nabla$. Doing the algebra one finds

$$\boldsymbol{B}_{\text{rad}} = i \frac{\pi a^2 b}{2} \frac{k^3}{r} \frac{I}{c} \sin\theta\, \hat{\varphi} , \tag{6.26}$$

which reproduces our (6.20) upon converting units and replacing I by I_{AC}.

All magnetic multipole coefficients vanish for the torus, but the electric multipole coefficients are non-zero. The exact radiated fields are given by a sum over terms proportional to $a_E(l,m)$.

Multipole *moments* form a complete set only for static charge-current distributions. For the time varying case, it is the multipole coefficients that are complete. These depend on k as well as ρ and $\boldsymbol{J}$, reflecting the role of retardation in radiation.

For a system with intrinsic magnetization, the $\boldsymbol{M}$ term in the square brackets in (6.21) shows that even a source with zero charge density and zero current density, $\rho=\boldsymbol{J}=0$, but with a time varying magnetization, can produce electric dipole radiation.

SECTION 7

GAUGE TRANSFORMATIONS AND SOURCES

Due to the arbitrariness arising from a gauge transformation, only certain parts of A are measurable. One needs to inquire how A is related to current sources. It turns out the longitudinal part of A can be combined with the scalar potential ϕ, while the transverse part retains its identity and is measurable.

When potentials A and ϕ are used to formulate Maxwell's equations,

$$\nabla \cdot \boldsymbol{B} = 0 , \qquad \nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_o} ,$$
$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 , \qquad \nabla \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \mu_o \boldsymbol{J} \tag{7.1}$$

the source-free ones serve to define the fields in terms of the potentials,

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} , \qquad \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \varphi . \tag{7.2}$$

The other two become dynamical equations determining the potentials:

$$\nabla^2 \phi + \nabla \cdot \dot{\boldsymbol{A}} = -\frac{\rho}{\epsilon_o} ,$$
$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\ddot{\boldsymbol{A}} - \nabla\left(\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right) = -\mu_o \boldsymbol{J} . \tag{7.3}$$

Any potentials A, ϕ having the same fields (7.2) obey Equations (7.3).

The sole *raison d'etre* of A is to have its curl equal to B. If we have one set of potentials A_1, ϕ_1, we get another with the gauge transformation

$$\boldsymbol{A}_1 \rightarrow \boldsymbol{A}_2 = \boldsymbol{A}_1 + \nabla \chi ,$$
$$\phi_1 \rightarrow \phi_2 = \phi_1 - \dot{\chi} , \tag{7.4}$$

where χ is an arbitrary function. Since A_2, ϕ_2 have the same fields they also obey (7.3). In addition to the arbitrary constant always accompanying ϕ, gauge invariance adds the additional arbitrariness of adding $-\dot{\chi}$ to ϕ when $\nabla\chi$ is added to A.

Although gauge invariance assures the irrelevance of χ, it will be helpful to make more explicit some features of this invariance. Writing the equations for A_2, ϕ_2 in terms of χ and potentials 1, they are

$$\nabla^2(\phi_1 - \dot{\chi}) + \nabla\cdot(\dot{A}_1 + \nabla\dot{\chi}) = -\frac{\rho}{\epsilon_o} ,$$
$$\Box(A_1 + \nabla\chi) - \nabla[\nabla\cdot A_1 + \nabla^2\chi + \frac{1}{c^2}\dot{\phi}_1 - \frac{1}{c^2}\ddot{\chi}] = -\mu_o J . \tag{7.5}$$

where $\Box \equiv \nabla^2 - (1/c^2)\partial^2/\partial t^2$ is the D'Alembertian. Collecting terms, these become

$$\nabla^2\phi_1 + \nabla\cdot\dot{A}_1 - [\![\nabla^2\dot{\chi} - \nabla^2\dot{\chi}]\!] = -\frac{\rho}{\epsilon_o} ,$$
$$\Box A_1 - \nabla(\nabla\cdot A_1 + \frac{1}{c^2}\dot{\phi}_1) + [\![\Box\nabla\chi - \nabla^2\nabla\chi + \frac{1}{c^2}\nabla\ddot{\chi}]\!] = -\mu_o J . \tag{7.6}$$

It is immediately seen that χ drops out altogether, the quantities in $[\![\ldots]\!]$ vanishing identically. This demonstrates explicitly that the arbitrary gauge function χ is, of course, physically irrelevant.

Transverse and Longitudinal Fields

The relation $\nabla\times A = B$ and the arbitrariness in $\nabla\cdot A$ suggest writing all fields in their transverse (solenoidal) and longitudinal (irrotational) parts[2]. Thus, for A:

$$A = A_T + A_L ,$$
$$\nabla\times A_T = \nabla\times A = B , \qquad \nabla\times A_L \equiv 0 , \tag{7.7}$$
$$\nabla\cdot A_T \equiv 0 , \qquad \nabla\cdot A_L = \nabla\cdot A = \text{arbitrary} .$$

If two vectors are equal, their transverse and longitudinal parts are separately equal. Then, decomposing E, B, and J into their parts, Maxwell's Equations can be written

[2] For any vector field V we have $V = V_T + V_L$, where

$$V_T = \frac{1}{4\pi}\nabla\times\int d^3r' \frac{\nabla'\times V(r')}{|r-r'|} , \qquad V_L = -\frac{1}{4\pi}\nabla\int d^3r' \frac{\nabla'\cdot V(r')}{|r-r'|} .$$

That is, *any* vector field has its own vector and scalar potentials. V_T and V_L may each be non-local even if V is local. By Gauss' theorem the integrals involve only the normal component of the vector at infinity, and for transverse radiation fields these vanish faster than $1/r$, assuring convergence.

$$B_L = 0\ , \qquad \nabla\times E_T + \frac{\partial B_T}{\partial t} = 0\ ;$$

$$\nabla\cdot E_L = \frac{\rho}{\epsilon_o}\ , \tag{7.8}$$

$$\nabla\times B_T - \frac{1}{c^2}\frac{\partial E_T}{\partial t} = \mu_o J_T\ , \qquad -\frac{1}{c^2}\frac{\partial E_L}{\partial t} = \mu_o J_L\ ;$$

with charge conservation

$$\nabla\cdot J_L + \frac{\partial\rho}{\partial t} = 0\ . \tag{7.9}$$

Maxwell's Equations without Gauge Transformations

Now introduce the usual potentials A and ϕ:

$$B_T = \nabla\times A_T\ ;$$

$$E_T = -\frac{\partial A_T}{\partial t}\ , \qquad E_L = -\frac{\partial A_L}{\partial t} - \nabla\phi\ . \tag{7.10}$$

Only A_L and ϕ are affected by a gauge transformation, with E_L left unchanged. A_L can always be written in terms of a scalar χ_o, $A_L = \nabla\chi_o$, so that E_L can be expressed as

$$E_L = -\nabla\psi\ , \tag{7.11}$$

where

$$\psi = \phi + \dot{\chi}_o \tag{7.12}$$

is a *gauge invariant scalar potential* with only an additive constant arbitrary. By itself A_L is physically irrelevant, being a matter of gauge choice and having no relation to sources. It can be fully subsumed in the definition of the scalar potential in a gauge invariant way, and only ψ and A_T survive. In terms of them Maxwell's Equations are

$$B_T = \nabla\times A_T\ , \qquad B_L = 0\ ;$$

$$E_T = -\frac{\partial A_T}{\partial t}\ , \qquad E_L = -\nabla\psi\ ;$$

$$\nabla^2\psi = -\frac{\rho}{\epsilon_o}\ ; \tag{7.13}$$

$$\Box A_T = -\mu_o J_T\ , \qquad \frac{\partial}{\partial t}\nabla\psi = \frac{J_L}{\epsilon_o}\ .$$

Since A_L has been combined with ϕ, the last equation for ψ is redundant and may be dropped. It is equivalent to the remaining Poisson equation and the continuity equation (7.9). The Poisson equation for ψ, and wave equation for A_T, are the same as the usual equations for ϕ and A in the Coulomb gauge, with the significant difference that now there has been no specification of gauge. Every quantity in Equ (7.13) is gauge-independent.

With Maxwell's Equations in this form there is no longer any arbitrariness in the potentials, and the questions of gauge transformations or gauge invariance do not arise. Longitudinal sources, fields, and potentials completely decouple from transverse ones.

Maxwell's Equations are usually not formulated this way. Most theories insist on, and build in, relativistic invariance, and separately inquire as to gauge invariance. In contrast, the above equations build in gauge invariance, but are not manifestly Lorentz invariant. The very division into transverse and longitudinal parts is not Lorentz invariant. However the formulation can be convenient for analysis of a given experiment in one reference frame.

J_L gives rise only to ψ. J_T gives rise only to A_T. Closed current loops, for example, which are commonly well approximated as divergenceless, especially at low frequencies, are transverse currents, and produce only A_T and transverse fields. In this same approximation, the complete fields of a torus with time-dependent current are purely transverse.

The discussion following (6.7), that A_{rad} cannot be separated from its fields, left open the possibility that the longitudinal part of A might be separable from its fields. But A_L, and the field associated with it, is a gauge artifice that is eliminated in Equations (7.13), and has no physical meaning.

The static A of a torus, Equation (3.6), is related to the source current. This A is a transverse vector for which $\nabla \cdot A = 0$ everywhere and which *happens* to have $\nabla \times A = 0$ outside the torus (but $\nabla \times A \neq 0$ inside). It is *locally* curl-free due to high symmetry.

These equations make it clear that it is only the transverse part of the current source that produces radiation. This same result is apparent from the usual wave equation for $\boldsymbol{B}$

$$\Box \boldsymbol{B} = -\mu_o \nabla \times \boldsymbol{J} = -\mu_o \nabla \times \boldsymbol{J}_T \tag{7.14}$$

that follows from Maxwell's Equations.

SECTION 8

CONDUCTORS AS SHIELDS

In this section we inquire as to the effectiveness of conducting enclosures in shielding the vector potential.

Boundary Conditions on A

Boundary conditions on A can be obtained from those on B by using the symmetry observed in Section 2. Since the normal component $B_\perp$ of B is continuous at an interface, and since this B is the A of a different problem, the normal component $A_\perp$ of A is continuous also. Actually, for the general case, the discontinuity in $A_\perp$ is formally gauge-dependent, since $\nabla \cdot A$ need not vanish at the surface. But so long as the gauge is such that the volume integral of $\nabla \cdot A$ across the surface remains zero, then $A_\perp$ is continuous.

Similarly, that the tangential component $B_\parallel$ is discontinuous by the normal integral of J means the tangential component $A_\parallel$ is continuous unless there is a delta function sheet of B on the surface. Excluding this unphysical condition, we have that all three components of A are continuous at an interface between two media with different ϵ, μ, and σ, even across a dielectric-conductor interface on which there may be a single-layer current sheet. These boundary conditions are conventionally derived [St41] by applying the usual Gaussian pill box or Stokes' loop over the interface.

The normal derivative of $A_\parallel$ is not continuous if there is a skin current, and is determined by the usual boundary conditions on $B_\parallel$. The discontinuity of $E_\perp$ at a surface charge density shows up in the potentials as a discontinuity in the normal component of $\nabla\phi$.

Conductor Moving Through Field-free Region

If a conductor is moved through a region of space where $E=B=0$, but $A\neq 0$, there are no physical effects, for there are no forces to produce any. Since A is continuous across the conductor surface, A effectively penetrates freely through the conductor.

To an observer at rest on the conductor, A is changing in time at a rate $dA/dt = \mathbf{v}\cdot\nabla A$, where $\mathbf{v}$ is the conductor velocity. By a Lorentz transformation, the observer sees a scalar potential $\phi = -\mathbf{v}\cdot A$, just such as to keep $E = -\partial A/\partial t - \nabla\phi$ equal to zero.

Shielding by a Perfectly Conducting Enclosure

Consider a localized current distribution with its attendant vector potential with $B=0$. We enclose it in a conducting shield. Two "experiments" are considered:

1. The current and vector potential are pre-established and the shield box is erected around the source in the space where $A \neq 0$.

2. The shield box is constructed while the source is off, then the source is turned on.

We will choose a torus for the source.

1. Pre-established DC field.

With a steady current flowing, **A** is non-zero outside. As the conductors are assembled, they move in the space through **A**. **A** passes freely through the metal pieces; there is no interaction between **A** and the conductors. We end up with the torus with its unperturbed static **A**, enclosed by a conducting box, with $A \neq 0$ inside and outside the box. **A** has no way of knowing that the shield box was built.

2. DC field turned on after shield is built.

In this case we erect the shield box around the dead torus, then turn the current on. The shield interrupts **A**, **E**, and **B** that are propagating out. So that concepts are not clouded by non-compatible geometries, choose for the box a concentric torus.

E, **B**, and **A** remain zero inside the shield metal. Therefore, on the inner surface of the shield, **E**, **A**, and $B_{\perp}$ also remain zero. Surface currents which terminate $B_{\parallel}$ are generated on the inner shield surface. The space between the driving torus and the shield torus is filled with **B** and **A**.

If **I** is suddenly turned on and then held constant, the final static configuration is as shown in Figure 11. A skin current flows on the inner surface of the shield, in the opposite direction of the driving current. An azimuthal **B** field persists within the shield box, opposite the main **B** in the driving torus, so that the total flux through the large torus arm stays zero. The static **A** is in the toroidal direction between the two tori, but is 0 at the shield. **A**, **E**, and **B** all vanish outside the shield. The box shields **A** as well as the fields.

There is thus a significant difference according as the shielding enclosure is constructed after or before the driving current is activated. In the former case, the static **A** is non-zero outside the shield, but in the latter case it stays zero.

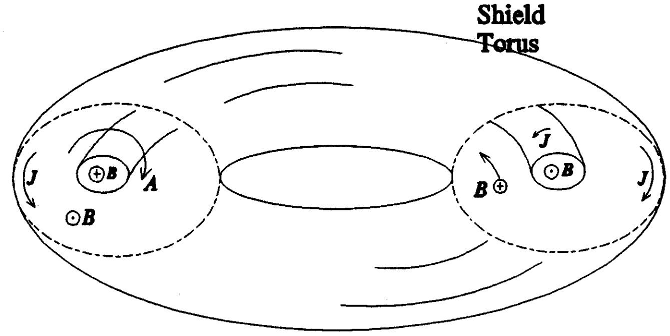

Figure 11. Torus shielded by a larger torus.

Shielding by a Finitely Conducting Enclosure

If the enclosing box is a finite

conductor, and the current is a step function turned on at $t=0$, the currents and fields diffuse into the metal and penetrate the shield of thickness d after a time of order

$$t_d = \mu_o \sigma d^2 \tag{8.1}$$

($t_d \approx 180$ μsec for 2 mm thick Al, using $\sigma(\mathrm{Al})=3.6\times10^7$ mho/m). After this time, A will appear outside the "shield box", and the exterior DC vector potential will have been established.

If there is an AC torus current in addition to the DC component, the AC fields and potential will diffuse with a skin depth

$$\delta = \sqrt{2/\mu_o \omega \sigma} \tag{8.2}$$

($\approx$ 0.084 cm in Al at $\omega/2\pi=10$ kHz). A shielding calculation then shows low frequency fields and vector potential, with $\omega \ll 1/t_d$ and $\delta > d$, penetrate the shield attenuated by the thin sheet amplitude transmission factor

$$T_o = \frac{\omega \delta^2}{cd} = \frac{2}{Z_o \sigma d} , \tag{8.3}$$

where $Z_0=377\Omega$ is free space impedance. For 2 mm thick Al, $T_o = 7.4\times10^{-8}$ (=-143 dB).

High frequency fields and potential, $\omega \gg 1/t_d$, $d>\delta$, are attenuated by the amplitude shielding factor

$$T_\omega = \frac{4\omega\delta}{\sqrt{2}\,c} e^{-d/\delta} = \frac{4}{\sqrt{2}} \frac{d}{\delta} e^{-d/\delta} T_o \tag{8.4}$$

relative to the unshielded torus.

The division between low and high frequencies occurs at $d=\delta$, and in 2 mm Al is at

$$\frac{\omega}{2\pi} = \frac{1}{\pi t_d} = 1.6 \text{ kHz} . \tag{8.5}$$

The shielding calculation is standard, however it means no conducting enclosure can shield out the fields yet permit a time dependent A to pass. The fields and potentials are shielded together.

Thus, regarding the full vector potential, A_L is a mathematical artifact totally disconnected from the source current. Time varying A_T is shielded along with the fields, and, upon penetrating at reduced amplitude, is accompanied by its usual transverse fields. A DC A_T can be established outside a finite conducting enclosure in a time of order t_d.

SECTION 9

VARIATIONS ON A THEME

The observation in Section 2 that A bears the same relation to $\boldsymbol{B}$ as $\boldsymbol{B}$ does to $\mu_o \boldsymbol{J}$ can be used to develop a physically meaningful hierarchy of static current configurations.

Hierarchy of J, B, A

Consider the current distribution of the torus, Figure 1, redrawn in Figure 12-(2). Call its current $\boldsymbol{J}_2$, field $\boldsymbol{B}_2$, and vector potential A_2. (In this section subscripts have no relation to gauge choice).

Construct a new current distribution $\boldsymbol{J}_3 = \boldsymbol{B}_2$ having only an azimuthal ($\hat{\varphi}$) component. This is the current of an ordinary current loop (except that $\boldsymbol{J}_3$ drops off as $1/\rho$ within the loop). Its field $\boldsymbol{B}_3$ will be $\boldsymbol{B}_3 = A_2$. Its vector potential must be computed anew. A_3 is also azimuthal, and lines of A_3 form circles about the z axis in space, as sketched in Figure 12-(3).

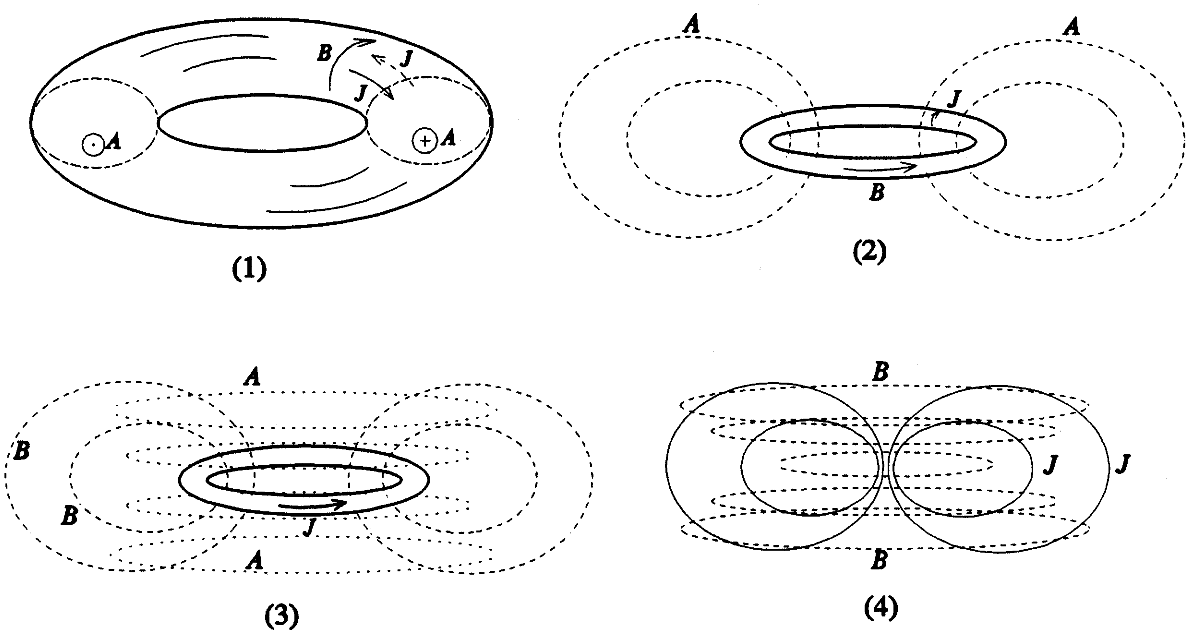

Figure 12. Hierarchy of current distributions based on torus.

Continuing, construct $\boldsymbol{J}_4 = \boldsymbol{B}_3$. This current distribution would be that of the discharge current of a battery immersed in a partially conducting fluid. The associated magnetic field is $\boldsymbol{B}_4 = A_3$, being circles about the axis, encircling lines of $\boldsymbol{J}_4$. The vector potential A_4 would have a field line pattern close to that of $\boldsymbol{J}_4$.

Proceeding in the opposite direction, from configuration (2) invent a current distribution $\boldsymbol{J}_1 \propto \nabla\times\boldsymbol{J}_2$ whose field $\boldsymbol{B}_1 = \boldsymbol{J}_2$, and vector potential $\boldsymbol{A}_1 = \boldsymbol{B}_2$. This is the azimuthal double-layer current sheet discussed in Section 4, whose field and vector potential vanish outside the torus. It is sketched in Figure 12-(1).

One could go one step further and construct four alternating layers of toroidal currents, $\boldsymbol{J}_0$, confining $\boldsymbol{B}_0$ to be the same double layer as $\boldsymbol{J}_1$. $\boldsymbol{A}_0$ would be like $\boldsymbol{B}_1$, confined to a thin toroidal sheet, the same as $\boldsymbol{J}_2$. $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{J}$ vanish inside and outside the torus.

Developments similar to this can be based on a common cylindrical solenoid or any other current distribution. The hierarchy is summarized by:

$$\begin{array}{lclclclcl}
\boldsymbol{J}_m .. & = & \nabla\times\boldsymbol{J}_1 & & & & & & \\
\boldsymbol{B}_m .. & = & \boldsymbol{J}_1 & = & \nabla\times\boldsymbol{J}_2 & & & & \\
\boldsymbol{A}_m .. & = & \boldsymbol{B}_1 & = & \boldsymbol{J}_2 & = & \nabla\times\boldsymbol{J}_3 & & \\
 & & \boldsymbol{A}_1 & = & \boldsymbol{B}_2 & = & \boldsymbol{J}_3 & = & ..\nabla\times\boldsymbol{J}_n \\
 & & & & \boldsymbol{A}_2 & = & \boldsymbol{B}_3 & = & ..\ \boldsymbol{J}_n \\
 & & & & & & \boldsymbol{A}_3 & = & ..\ \boldsymbol{B}_n \\
 & & & & & & & & ..\ \boldsymbol{A}_n
\end{array}$$

which is written more succinctly in (2.11).

Toroidal Coax

In the configuration of Figure 12-(1), one can separate the inner current sheet from the outer one, collapsing it to an inner wire ring concentric with the torus limb, as in Figure 13. One has a toroidal coax line. $\boldsymbol{J}$ and $\boldsymbol{A}$ are azimuthal, $\boldsymbol{A}$ being non-zero both inside the inner conductor, and between the inner and outer one. $\boldsymbol{B}$ is in the "toroidal" direction, confined to between the inner conductor and outer "shield", being the magnetic field of an ordinary coax line. $\boldsymbol{A}$ vanishes outside the entire configuration and inside at the outer conductor.

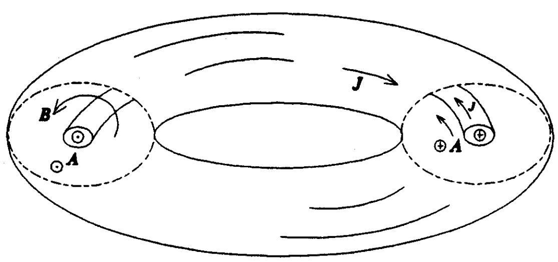

Figure 13. Toroidal coax line.

As the inner wire is an ordinary current loop, this configuration is a "shielded current loop". The shield is an ordinary (perfect) conductor and shields both $\boldsymbol{A}$ and $\boldsymbol{B}$. The same is true for ordinary straight coax cables.

SECTION 10

A SYMMETRY OF MAXWELL'S EQUATIONS

As a separate question, it is interesting to inquire whether the magnetostatic observations of Section 2 on the $\boldsymbol{J}$, $\boldsymbol{B}$, $\boldsymbol{A}$ hierarchy can be extended to time varying currents and fields.

Due to the general relations $\boldsymbol{B}=\nabla\times\boldsymbol{A}$, $\nabla\cdot\boldsymbol{A}=0$, and the static relation $\boldsymbol{J}=\nabla\times\boldsymbol{B}$, as well as $\nabla\cdot\boldsymbol{B}=0$, we have for the arbitrary time dependent case,

> The vector potential $\boldsymbol{A}_2(\boldsymbol{r},t)$ of any current distribution $\boldsymbol{J}_2(\boldsymbol{r},t)$ with field $\boldsymbol{B}_2(\boldsymbol{r},t)$, is the same as the instantaneous static magnetic *field* $\boldsymbol{B}_1(\boldsymbol{r},t)$ of a current $\boldsymbol{J}_1(\boldsymbol{r},t)$ instantaneously equal to $\boldsymbol{B}_2(\boldsymbol{r},t)$.

Here $\boldsymbol{A}_2$ is in the Coulomb gauge. It appears this rule cannot be extended to develop a hierarchy of $\boldsymbol{J}$, $\boldsymbol{B}$, $\boldsymbol{A}$ configurations in parallel with the static case since the displacement current destroys the analogy. But a closely related hierarchy still holds.

In the Lorentz gauge Maxwell's Equations are

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} \ , \qquad \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi \ ,$$
$$\Box\boldsymbol{A} = -\mu_o\boldsymbol{J} \ , \qquad \Box\phi = -\frac{\rho}{\epsilon_o} \ . \tag{10.1}$$

Let these hold for a current $\boldsymbol{J}_1$ with fields $\boldsymbol{E}_1$, $\boldsymbol{B}_1$, and potentials $\boldsymbol{A}_1$, ϕ_1.

Form the new current

$$\boldsymbol{J}_2 = \nabla\times\boldsymbol{J}_1 \ , \qquad \nabla\cdot\boldsymbol{J}_2 = 0 \ . \tag{10.2}$$

Clearly, its potentials are

$$\boldsymbol{A}_2 = \nabla\times\boldsymbol{A}_1 = \boldsymbol{B}_1 \ ,$$
$$\phi_2 = 0 \ . \tag{10.3}$$

The associated fields are

$$\boldsymbol{E}_2 = -\frac{\partial \boldsymbol{A}_2}{\partial t} = -\frac{\partial \boldsymbol{B}_1}{\partial t} = \nabla\times\boldsymbol{E}_1 \ ,$$
$$\boldsymbol{B}_2 = \nabla\times\boldsymbol{A}_2 = \nabla\times\boldsymbol{B}_1 \tag{10.4}$$
$$= \mu_o\left(\boldsymbol{J}_1 + \epsilon_o\frac{\partial \boldsymbol{E}_1}{\partial t}\right) \ .$$

In summary,

$$
\begin{bmatrix} \boldsymbol{J} \\ \boldsymbol{B} \\ \boldsymbol{E} \\ \boldsymbol{A} \end{bmatrix}_{(2)} = \nabla\times \begin{bmatrix} \boldsymbol{J} \\ \boldsymbol{B} \\ \boldsymbol{E} \\ \boldsymbol{A} \end{bmatrix}_{(1)} , \qquad (10.5)
$$

a complete parallel to the magnetostatic case (2.11). Even for full time dependence the curl of one set of currents, fields, and potentials is another set, so that a solution for one $\boldsymbol{J}$ provides the solution for other problems. That this should be true for Maxwell's Equations is less obvious than for magnetostatics, although if follows as well simply by taking the curl of Maxwell's field equations (7.1). It differs from magnetostatics in that $\boldsymbol{E}\neq 0$, and $\nabla\times\boldsymbol{B}$ is no longer $\boldsymbol{J}$, so the hierarchy is not as tight. This is more explicit by writing (10.5) as

$$
\begin{bmatrix} \boldsymbol{J}_2 \\ \boldsymbol{B}_2 \\ \boldsymbol{E}_2 \\ \boldsymbol{A}_2 \end{bmatrix} = \begin{bmatrix} \nabla\times\boldsymbol{J}_1 \\ \boldsymbol{J}_1 \\ 0 \\ \boldsymbol{B}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \partial\boldsymbol{E}_1/\partial t \\ -\partial\boldsymbol{B}_1/\partial t \\ 0 \end{bmatrix} , \qquad (10.6)
$$

in which we have dropped factors of μ_o and ϵ_o in the second line for simplicity. A problem specific numerical coefficient with dimension of length should also appear multiplying the right hand sides of (10.5) and (10.6). Based on any given $\boldsymbol{J}(\boldsymbol{r},t)$ one can develop a hierarchy of physical current and field distributions in parallel with the magnetostatics case.

Fields of a Rotating Torus

One instance in which this symmetry is useful is in computing the fields of a rotating torus.

Let a DC current flow in the torus windings of Figure 1. Suppose the current is delivered to the windings by slip rings so the torus is free to rotate about, say, the y axis, maintaining its DC current. (Or we could imagine a superconducting arrangement, with no need to drive the currents once established). When stationary, there is a vector potential, but no fields, outside. When the torus is spinning, are there external fields?

Clearly there are, for to an external observer the static $\boldsymbol{A}$ also rotates, producing non-zero $\boldsymbol{E}=-\partial\boldsymbol{A}/\partial t$, without a cancelling $-\nabla\phi$. Since $\partial\boldsymbol{E}/\partial t$ is also non-zero, $\nabla\times\boldsymbol{B}=(1/c^2)\partial\boldsymbol{E}/\partial t\neq 0$, and therefore $\boldsymbol{B}$ does not vanish outside. An oscillating EMF is generated around a fixed loop in, say, the $x=0$ plane that encircles the torus limb in its original position. A rotating torus carrying a DC current possesses non-zero near zone fields. It also radiates. The question is how to calculate the fields. The symmetry discussed above is quite powerful in this regard, and makes the task easy.

The surface current density on the torus is the curl of the current density of an ordinary current loop with the same dimensions. According to (10.5) then, the exact fields and potential of the rotating torus are precisely the curl of the corresponding fields and potential of a spinning current loop. These latter fields are not difficult to compute. Here we sketch only the radiation fields.

More precisely, a torus wound with N turns carrying total current $I=i_w N$, where i_w is the wire current, has a current density J proportional to the curl of the current density J' of a loop carrying current $I'=\pi a^2 J'$:

$$J = \kappa \nabla \times J' \ , \qquad \kappa = \frac{a^2}{2b}\frac{I}{I'} \ . \tag{10.7}$$

The coefficient κ should appear multiplying the right-hand-side of (10.5) and (10.6).

A loop of radius b, of thin wire with radius $a \ll b$, carrying current I' has a magnetic moment $m=\pi b^2 I'$. When rotating with angular velocity ω in the positive sense about $\hat{y}$ its components are

$$m_z = m\cos\omega t \ , \qquad m_x = m\sin\omega t \ . \tag{10.8}$$

Each of these is in turn the magnetic moment of a stationary loop with oscillating current. Each radiated electric field is of the standard form, and the total is the superposition of the two:

$$\begin{aligned} E_{rad}^{loop} &= E_{rad}^{(x)} + E_{rad}^{(z)} \ , \\ E_{rad}^{(x)} &= -i\frac{\mu_o}{4\pi}\frac{\omega^2 m}{cr}\, e^{i(kr-\omega t)} \sin\theta_x\, \hat{\varphi}_x \ , \\ E_{rad}^{(z)} &= -\frac{\mu_o}{4\pi}\frac{\omega^2 m}{cr}\, e^{i(kr-\omega t)} \sin\theta_z\, \hat{\varphi}_z \ , \end{aligned} \tag{10.9}$$

where the real part is understood. Here θ_z and φ_z are the usual polar and azimuthal angles in spherical coordinates defined on the z axis, and θ_x and φ_x are those of a spherical system defined on the x axis. The magnetic field is

$$B_{rad}^{loop} = \frac{1}{c}\hat{r} \times E_{rad}^{loop} \ . \tag{10.10}$$

For these, as for any radiation fields,

$$\nabla \times E_{rad} = -\frac{\partial B_{rad}}{\partial t} = i\omega B_{rad} = i\frac{\omega}{c}\hat{r}\times E_{rad} \ . \tag{10.11}$$

Then, using $\hat{r}\times\hat{\varphi}_x = -\hat{\theta}_x$, etc., the radiated field of a spinning torus comes out

$$\begin{aligned} E_{rad}^{torus} &= \kappa\nabla\times E_{rad}^{loop} = i\kappa\omega B_{rad}^{loop} \\ &= \frac{\mu_o c}{4\pi}\frac{Vk^3 I}{4\pi}\frac{e^{i(kr-\omega t)}}{r}\left[-\sin\theta_x\hat{\theta}_x + i\sin\theta_z\hat{\theta}_z\right] \ . \end{aligned} \tag{10.12}$$

κ has been replaced using $\kappa m=(\pi a^2 b/2)I=(V/4\pi)I$. The full radiation pattern can be mapped from

Equation (10.12). As for a loop, this is the field of two stationary perpendicular toruses carrying oscillating currents, each with fields of the form (6.20).

For an observer on the rotation axis (+y), for example, $\sin\theta_x = \sin\theta_z = 1$, $\hat{\theta}_x = -\hat{x}$, $\hat{\theta}_z = -\hat{z}$, and

$$E^{torus}_{rad}(x=0, y=r, z=0) = \frac{\mu_o c}{4\pi} \frac{V k^3 I}{4\pi} \frac{e^{i(kr-\omega t)}}{r} [\hat{x} - i\hat{z}] \ . \tag{10.13}$$

Radiation in the +y direction is right-circularly polarized.

One can similarly compute the quasi-static fields of a spinning torus from those of a spinning loop.

The energy of the torus consists of its rotational kinetic energy plus the magnetic field energy inside. These supply the energy radiated. The kinetic energy is an artifact of the mass of material chosen for fabrication, and can in principle be made as small as desired. Therefore the energy radiated comes from the enclosed magnetostatic field energy. Due to radiation the DC current of a spinning torus will decay.

This example has been illustrative only. One can employ the symmetry noted in this Section to problems less academic than a rotating torus.

APPENDIX A.

EXACT FIELDS OF A TORUS

The complete vector potential and fields of a harmonically driven torus may be computed to arbitrary accuracy by expanding in small parameters.

The full expression for $A(\boldsymbol{r},t)$ in the Lorentz gauge is Equation (5.3). Taking

$$J(\boldsymbol{r}',t') = e^{-i\omega t'} J(\boldsymbol{r}') = e^{-i\omega t} e^{ik|\boldsymbol{r}-\boldsymbol{r}'|} J(\boldsymbol{r}') \tag{A-1}$$

and $A = A_\omega(\boldsymbol{r}) e^{-i\omega t}$, then

$$A_\omega(\boldsymbol{r}) = \frac{\mu_o}{4\pi} \int d^3r' \, \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} J(\boldsymbol{r}') \ . \tag{A-2}$$

$\nabla \cdot \boldsymbol{A} = 0$, and since $\nabla \cdot \boldsymbol{J} = 0$, the scalar potential vanishes. We consider only $r > b$ and $kb < 1$, but expressions will be valid in the near zone $kr < 1$, or far zone $kr > 1$. When $kb \gtrsim 1$, azimuthal asymmetries arise due to propagation time delays around the torus when driven at one point. We do not take into account these asymmetries.

The dependence on r and r' is separated using the standard expansion

$$\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} = ik \sum_{\ell=0}^{\infty} (2\ell+1) j_\ell(kr') \, h_\ell^{(1)}(kr) \, P_\ell(\mu) \ , \qquad r > r' \tag{A-3}$$

where j_ℓ is the spherical Bessel function, P_ℓ is the Legendre polynomial, and $\mu = \cos(\boldsymbol{r},\boldsymbol{r}')$. Powers of kr are explicit because the Hankel function

$$h_\ell^{(1)}(kr) = \frac{1}{i^\ell} \frac{e^{ikr}}{ikr} \sum_{q=0}^{\ell} \frac{(\ell+q)!}{q!(\ell-q)!} \left(\frac{i}{2kr}\right)^q \ . \tag{A-4}$$

Then

$$A_\omega = \frac{\mu_o}{4\pi} \frac{e^{ikr}}{r} \sum_{\ell=0}^{\infty} S_\ell(k) \sum_{q=0}^{\ell} f_{\ell q} \left(\frac{1}{kr}\right)^q \ , \tag{A-5}$$

where

$$S_\ell(k) = \int d^3r' \, j_\ell(kr') \, P_\ell(\mu) \, J(\boldsymbol{r}') \tag{A-6}$$

characterizes the source and observer angle, and

$$f_{\ell q} = \frac{(2\ell+1)}{i^{\ell-q}} \frac{(\ell+q)!}{2^q q!(\ell-q)!} \ . \tag{A-7}$$

Since for small argument $j_\ell(kr') = (kr')^\ell/(2\ell+1)!!$, Equation (A-6) shows

$$S_\ell(k) = (kb)^\ell C_\ell \ , \qquad \ell > 0 \tag{A-8}$$

with C_ℓ independent of k to lowest order. The only exception is $S_0 \propto (kb)^2$. Using this in (A-5) and interchanging summations gives

$$A_\omega = \frac{\mu_o}{4\pi}\frac{e^{ikr}}{r}\left\{f_{00}\,S_0 + \sum_{q=0}^{\infty}\sum_{\ell=q}^{\infty}{}'\,C_\ell f_{\ell q}\frac{(kb)^\ell}{(kr)^q}\right\}, \tag{A-9}$$

where the prime means the $\ell=0$ term is deleted from the sum. Now setting $\ell=q+j$,

$$A_\omega = \frac{\mu_o}{4\pi}\frac{e^{ikr}}{r}\left\{f_{00}\,S_0 + \sum_{q=0}^{\infty}\left[\sum_{j=0}^{\infty}{}'\,C_{q+j}f_{q+j,q}\,(kb)^j\right]\left(\frac{b}{r}\right)^q\right\}. \tag{A-10}$$

This is an explicit series in powers of b/r with coefficients that are rapidly converging series in kb.

It is not difficult to show that the combination of azimuthal symmetry and reflection symmetry in the $z=0$ plane implies

$$S_\ell(k) = C_\ell = 0, \qquad \ell \text{ odd} \tag{A-11}$$

so that only even indices survive in (A-10):

$$\begin{aligned} A_\omega = \frac{\mu_o}{4\pi}\frac{e^{ikr}}{r}\Bigg\{ & f_{00}\,S_0 + \sum_{\ell=1}^{\infty} C_{2\ell}f_{2\ell,0}\,(kb)^{2\ell} + \frac{b}{r}\sum_{\ell=1}^{\infty} C_{2\ell}f_{2\ell,1}\,(kb)^{2\ell-1} \\ & + \left(\frac{b}{r}\right)^2\sum_{\ell=1}^{\infty} C_{2\ell}f_{2\ell,2}\,(kb)^{2\ell-2} + \left(\frac{b}{r}\right)^3\sum_{\ell=2}^{\infty} C_{2\ell}f_{2\ell,3}\,(kb)^{2\ell-3} \\ & + \left(\frac{b}{r}\right)^4\sum_{\ell=2}^{\infty} C_{2\ell}f_{2\ell,4}\,(kb)^{2\ell-4} + \left(\frac{b}{r}\right)^5\sum_{\ell=3}^{\infty} C_{2\ell}f_{2\ell,5}\,(kb)^{2\ell-5} + \ldots \Bigg\}. \end{aligned} \tag{A-12}$$

A_ω contains all powers of $1/r$, but as $k \to 0$ the static vector potential

$$A_0 = \frac{\mu_o}{4\pi}\frac{1}{r}\left\{\left(\frac{b}{r}\right)^2 C_2 f_{22} + \left(\frac{b}{r}\right)^4 C_4 f_{44} + \ldots\right\} \tag{A-13}$$

contains only odd inverse powers of r.

The first few terms of (A-12) are

$$\begin{aligned} A_\omega = \frac{\mu_o}{4\pi}\frac{e^{ikr}}{r}\Bigg\{ & f_{00}\,S_0 + f_{20}\,C_2\,(kb)^2 + \frac{b}{r}\left[f_{21}\,C_2(kb) + f_{41}\,C_4\,(kb)^3\right] \\ & + \left(\frac{b}{r}\right)^2\left[f_{22}\,C_2 + f_{42}\,C_4\,(kb)^2\right] + \left(\frac{b}{r}\right)^3\left[f_{43}\,C_4\,(kb) + f_{63}\,C_6\,(kb)^3\right] \\ & + O((kb)^4) + O\left(\left(\frac{b}{r}\right)^4\right)\Bigg\}. \end{aligned} \tag{A-14}$$

C_2 has already been evaluated, since according to (A-13) it is determined by the static $1/r^3$ potential, which was obtained in Section 3. Comparing the first term of (A-13) with (3.6) gives

$$f_{22}C_2 = \frac{\pi a^2 I}{2b}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) \ , \qquad \text{(A-15)}$$

and direct evaluation shows

$$S_0 = (kb)^2 C_0 = (kb)^2 \frac{\pi a^2 I}{3b}(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) \ . \qquad \text{(A-16)}$$

Then, taking the f's from (A-7), one gets

$$A(t) = \frac{\mu_o}{4\pi}\frac{VI}{4\pi r^3} e^{-i\omega(t-r/c)}\left[(1-ikr)(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) - k^2r^2\sin\theta\,\hat{\theta}\right] , \qquad \text{(A-17)}$$

plus smaller terms of higher order in b/r and kb. Here $V = 2\pi^2 a^2 b$ = torus volume.

The first term (1) in square brackets is the lowest order static vector potential, separately obtained in (3.6). The ikr term is the "inductive" potential. The term in k^2r^2 is the radiated potential, previously obtained in (6.19).

The fields to the same order are:

$$E = -\frac{\partial A}{\partial t} = i\omega A \ , \qquad \text{(A-18)}$$

having terms behaving as $\omega I/r^3$, $\omega^2 I/r^2$, and $\omega^3 I/r$; and

$$B = \nabla\times A = \frac{\mu_o}{4\pi}\frac{VI}{4\pi} e^{-i\omega(t-r/c)}\frac{k^2}{r^2}[1 - ikr]\sin\theta\,\hat{\varphi} \ , \qquad \text{(A-19)}$$

behaving as $\omega^2 I/r^2$ and $\omega^3 I/r$.

In the near zone, the quasistatic magnetic and electric fields, given in (5.6), are related by (5.7).

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