Physics Notes

Note 20

11 March 2009

Electromagnetic Stress and Momentum in the Combined-Field Formalism

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Abstract

This paper explores the applicability of the complex combined field (electric and magnetic) to the electromagnetic stress dyadic and momentum vector. This generalization is found to be possible.

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This work was sponsored in part by the Air Force Office of Scientific Research.

1. Introduction

A classical subject in electromagnetic theory is the stress dyadic and momentum vector [3, 4]. As is well known the electromagnetic fields can be recast as a complex vector combining electromagnetic fields, and this can be extended to the other parameters such as current, charge, potentials, etc. [5]. It can also be used for the Lagrangian formulation [2]. Here we look into its applicability to the electromagnetic stress and momentum.

We have the usual Maxwell Equations

$$\begin{aligned}
\nabla \times \vec{E}(\vec{r},t) &= -\frac{\partial \vec{B}(\vec{r},t)}{\partial t} - \vec{J}_m(\vec{r},t) \\
\nabla \times \vec{H}(\vec{r},t) &= \frac{\partial \vec{D}(\vec{r},t)}{\partial t} + \vec{J}(\vec{r},t)
\end{aligned} \tag{1.1}$$

augmented by a magnetic current density. Taking divergences gives

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla \bullet \vec{B}(\vec{r},t) &\equiv -\frac{\partial}{\partial t}\rho_m(\vec{r},t) = -\nabla \bullet \vec{J}_m(\vec{r},t) \\
\frac{\partial}{\partial t}\nabla \bullet \vec{D}(\vec{r},t) &\equiv -\frac{\partial}{\partial t}\rho(\vec{r},t) = -\nabla \bullet \vec{J}(\vec{r},t)
\end{aligned} \tag{1.2}$$

These are augmented by the constitutive relations (free space) as

$$\begin{aligned}
\vec{B}(\vec{r},t) &= \mu_0 \vec{H}(\vec{r},t) \\
\vec{D}(\vec{r},t) &= \varepsilon_0 \vec{E}(\vec{r},t)
\end{aligned} \tag{1.3}$$

For present purposes we summarize the combined field and related quantities. From [1, 5] we have the combined field

$$\begin{aligned}
\vec{E}_q(\vec{r},t) &= \vec{E}(\vec{r},t) + jq Z_0 \vec{H}(\vec{r},t) \\
q &= \pm 1 \equiv \text{separation index} \\
Z_0 &= \left[\frac{\mu_0}{\varepsilon_0}\right]^{1/2} \equiv \text{wave impedance}
\end{aligned} \tag{1.4}$$

and the combined current density

$$\vec{J}_q(\vec{r},t) = \vec{J}(\vec{r},t) + j\frac{q}{Z_0}\vec{J}_m(\vec{r},t)$$

$$\vec{J}(\vec{r},t) \equiv \text{electric current density} \tag{1.5}$$

$$\vec{J}_m(\vec{r},t) \equiv \text{magnetic current density}$$

These give the combined Maxwell equation

$$\left[\nabla \times -j\frac{q}{c}\frac{\partial}{\partial}\right]\vec{E}_q(\vec{r},t) = jq\,Z_0\,\vec{J}_q(\vec{r},t)$$

$$c = \left[\mu_0\varepsilon_0\right]^{-1/2} \equiv \text{speed of light} \tag{1.6}$$

Taking the divergence of (1.6) gives the equation of continuity

$$\nabla \bullet \vec{J}_q(\vec{r},t) = -\frac{1}{\varepsilon_0}\frac{\partial\rho_q(\vec{r},t)}{\partial t}$$

$$\varepsilon_0\,\nabla \bullet \vec{E}_q(\vec{r},t) = \rho_q(\vec{r},t)$$

$$\rho_q(\vec{r},t) = \rho(\vec{r},t) + j\frac{q}{Z_0}\rho_m(\vec{r},t)$$

$$\rho(\vec{r},t) \equiv \text{electric charge density} \tag{1.7}$$

$$\rho_m(\vec{r},t) \equiv \text{magnetic charge density}$$

The magnetic charge and current are included for symmetry and mathematical convenience.

For completeness we include the combined potentials given by

$$\vec{A}_q(\vec{r},t) = \vec{A}(\vec{r},t) + j\,q\,Z_0\,\vec{A}_m(\vec{r},t)$$

$$= \mu_0\int\frac{\vec{J}_q\left(\vec{r},t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi\,|\vec{r}-\vec{r}\,'|}dV'$$

$$\Phi_q(\vec{r},t) \;=\; \Phi(\vec{r},t) \;+\; j\,q\,Z_0\Phi_m(\vec{r},t)$$

$$= \frac{1}{\varepsilon_0}\int \frac{\rho_q\left(\vec{r},t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi|\vec{r}-\vec{r}\,'|}dV' \tag{1.8}$$

From these the combined field is also given by

$$\vec{E}_q(\vec{r},t) \;=\; -\nabla\Phi_q(\vec{r},t)-\frac{\partial}{\partial t}\vec{A}_q(\vec{r},t) \;+\; jqc\nabla\times\vec{A}_q(\vec{r},t) \tag{1.9}$$

2. Electromagnetic Stress Dyadic

There is a lucid development of this in [3 (Ch. 2)]. For our present purposes we write (in free space)

$$\begin{aligned}\overleftrightarrow{S}(\vec{r},t) \;&=\; \varepsilon_0\,\vec{E}(\vec{r},t)\,\vec{E}(\vec{r},t)-\frac{\varepsilon_0}{2}\vec{E}(\vec{r},t)\bullet\vec{E}(\vec{r},t)\,\overleftrightarrow{1}\\ &\quad+\;\mu_0\,\vec{H}(\vec{r},t)\vec{H}(\vec{r},t)-\frac{\mu_0}{2}\vec{H}(\vec{r},t)\bullet\vec{H}(\vec{r},t)\,\overleftrightarrow{1}\\ &\equiv\;\text{electromagnetic stress dyadic}\end{aligned} \tag{2.1}$$

$$\begin{aligned}\overleftrightarrow{1} \;&=\; \vec{1}_x\,\vec{1}_x+\vec{1}_y\,\vec{1}_y+\vec{1}_z\,\vec{1}_z\\ &\equiv\;\text{dyadic identify}\end{aligned}$$

In terms of the combined field we have

$$\begin{aligned}\vec{E}_q(\vec{r},t)\bullet\vec{E}_{-q}(\vec{r},t) \;&=\vec{E}_{-q}(\vec{r},t)\bullet\vec{E}_q(\vec{r},t) \;=\vec{E}(\vec{r},t)\bullet\vec{E}(\vec{r},t)+Z_0^2\,\vec{H}(\vec{r},t)\bullet\vec{H}(\vec{r},t)\\ \vec{E}_q(\vec{r},t)\,\vec{E}_{-q}(\vec{r},t) \;&=\;\left[\vec{E}_{-q}(\vec{r},t)\,\vec{E}_q(\vec{r},t)\right]^{\dagger} \;=\vec{E}(\vec{r},t)\,\vec{E}(\vec{r},t)+Z_0^2\,\vec{H}(\vec{r},t)\vec{H}(\vec{r},t)\\ &\quad+\;jqZ_0\left[\vec{E}(\vec{r},t)\vec{H}(\vec{r},t)-\vec{H}(\vec{r},t)\,\vec{E}(\vec{r},t)\right]\end{aligned} \tag{2.2}$$

$$\dagger \;\equiv\; \text{adjoint} \;\equiv\; \mathrm{T}*$$

So we can write the stress dyadic as

$$\overleftrightarrow{S}(\vec{r},t) = \varepsilon_0 \,\mathrm{Re}\left(\vec{E}_q(\vec{r},t)\,\vec{E}_{-q}(\vec{r},t)\right) - \frac{\varepsilon_0}{2}\vec{E}_q(\vec{r},t)\bullet\vec{E}_{-q}(\vec{r},t)\,\overleftrightarrow{1}$$

$$= \frac{\varepsilon_0}{2}\left[\vec{E}_q(\vec{r},t)\,\vec{E}_{-q}(\vec{r},t) + \vec{E}_{-q}(\vec{r},t)\,\vec{E}_q(\vec{r},t) - \vec{E}_q(\vec{r},t)\bullet\vec{E}_{-q}(\vec{r},t)\,\overleftrightarrow{1}\right] \tag{2.3}$$

$$= \overleftrightarrow{S}^{\mathrm{T}}(\vec{r},t)$$

The next step is to take the divergence as

$$\nabla\bullet\overleftrightarrow{S}(\vec{r},t) = \frac{\varepsilon_0}{2}\left[\nabla\bullet\left[\vec{E}_q(\vec{r},t)\bullet\vec{E}_{-q}(\vec{r},t)\right] + \nabla\bullet\left[\vec{E}_{-q}(\vec{r},t)\bullet\vec{E}_q(\vec{r},t)\right] - \nabla\bullet\left[\vec{E}_q(\vec{r},t)\bullet\vec{E}_{-q}(\vec{r},t)\right]\overleftrightarrow{1}\right]$$

$$\nabla\bullet\left[\vec{E}_q(\vec{r},t)\bullet\vec{E}_{-q}(\vec{r},t)\right] = \left[\nabla\bullet\vec{E}_q(\vec{r},t)\right]\vec{E}_{-q}(\vec{r},t) + \vec{E}_q(\vec{r},t)\bullet\nabla\vec{E}_{-q}(\vec{r},t)$$

$$\nabla\bullet\left[\vec{E}_q(\vec{r},t)\bullet\vec{E}_{-q}(\vec{r},t)\,\overleftrightarrow{1}\right] = \nabla\bullet\left[\vec{E}_q(\vec{r},t)\bullet\vec{E}_{-q}(\vec{r},t)\right] \tag{2.4}$$

$$= \vec{E}_q(\vec{r},t)\times\left[\nabla\times\vec{E}_{-q}(\vec{r},t)\right] + \vec{E}_{-q}(\vec{r},t)\times\left[\nabla\times\vec{E}_q(\vec{r},t)\right]$$

$$+\vec{E}_q(\vec{r},t)\bullet\nabla\vec{E}_{-q}(\vec{r},t) + \vec{E}_{-q}(\vec{r},t)\bullet\nabla\vec{E}_q(\vec{r},t)$$

Collecting terms gives

$$\nabla\bullet\overleftrightarrow{S} = \frac{\varepsilon_0}{2}\left[\left[\nabla\bullet\vec{E}_q(\vec{r},t)\right]\vec{E}_{-q}(\vec{r},t) + \left[\nabla\bullet\vec{E}_{-q}(\vec{r},t)\right]\vec{E}_q(\vec{r},t)\right.$$

$$\left. - \vec{E}_q(\vec{r},t)\times\left[\nabla\times\vec{E}_{-q}(\vec{r},t)\right] - \vec{E}_{-q}(\vec{r},t)\times\left[\nabla\bullet\vec{E}_q(\vec{r},t)\right]\right] \tag{2.5}$$

Substituting from the Maxwell Equations gives

$$\nabla\bullet\overleftrightarrow{S}(\vec{r},t) = \frac{1}{2}\left[\rho_q(\vec{r},t)\,\vec{E}_{-q}(\vec{r},t) + \rho_{-q}(\vec{r},t)\,\vec{E}_q(\vec{r},t)\right.$$

$$+ j\frac{q}{c}\left[\vec{E}_q(\vec{r},t)\times\left[\varepsilon_0\frac{\partial}{\partial t}\vec{E}_{-q}(\vec{r},t) + \vec{J}_{-q}(\vec{r},t))\right]\right.$$

$$\left.\left. - \vec{E}_q(\vec{r},t)\times\left[\varepsilon_0\frac{\partial}{\partial t}\vec{E}_q(\vec{r},t) + \vec{J}_q(\vec{r},t))\right]\right]\right]$$

$$= \frac{1}{2}\left[\rho_q(\vec{r},t)\,\vec{E}_{-q}(\vec{r},t) + \rho_{-q}(\vec{r},t)\,\vec{E}_q(\vec{r},t)\right.$$

$$
\begin{aligned}
&+ \; j\frac{q}{c}\left[\vec{E}_q(\vec{r},t)\times\vec{J}_{-q}(\vec{r},t) \; - \vec{E}_{-q}(\vec{r},t)\times\vec{J}_q(\vec{r},t)\right. \\
&\left.\left.\left. + \varepsilon_0\frac{\partial}{\partial t}\left[\vec{E}_q(\vec{r},t)\times\vec{E}_{-q}(\vec{r},t)\right]\right]\right]\right]
\end{aligned}
\tag{2.6}
$$

Breaking this out into electric and magnetic terms gives

$$
\begin{aligned}
\nabla \bullet \overleftrightarrow{S}(\vec{r},t) \;=\; & \rho(\vec{r},t)\vec{E}(\vec{r},t) + \rho_m(\vec{r},t)\vec{H}(\vec{r},t) \\
& - \mu_0\vec{H}(\vec{r},t) \;\times\; \vec{J}(\vec{r},t) + \varepsilon_0\vec{E}(\vec{r},t) \;\times\; \vec{J}_m(\vec{r},t) \\
& + \; \frac{1}{c^2}\frac{\partial}{\partial t}\left[\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)\right] \\
= \; & \rho(\vec{r},t)\vec{E}(\vec{r},t) \; + \; \rho_m(\vec{r},t)\vec{H}(\vec{r},t) \\
& + \; \vec{J}(\vec{r},t)\times\vec{B}(\vec{r},t) + \vec{D}(\vec{r},t)\times\vec{J}_m(\vec{r},t) \\
& + \; \varepsilon_0\frac{\partial}{\partial t}\left[\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right]
\end{aligned}
\tag{2.7}
$$

This gives the usual result [3] with the additional terms from the magnetic current density. The magnetic current density can be related to the permeability ($\mu-\mu_0$) increase, while the magnetic charge density can be regarded as ficticious, and at times useful in certain equivalence relationships.

3. Combined Stress Dyadic

Having constructed the usual stress dyadic, note in (2.3) the use of the real part of $\vec{E}_q\,\vec{E}_{-q}$. One might naturally ask whether the imaginary part has any significance?

Let us then write a complex stress dyadic (from (2.2)) as

$$
\begin{aligned}
\overleftrightarrow{S}_q(\vec{r},t) \;&=\; \varepsilon_0\,\vec{E}_q(\vec{r},t)\,\vec{E}_{-q}(\vec{r},t) - \frac{\varepsilon_0}{2}\vec{E}_q(\vec{r},t)\bullet\vec{E}_{-q}(\vec{r},t)\,\overleftrightarrow{1} \\
\overleftrightarrow{S}(\vec{r},t) \;&=\; \mathrm{Re}\left(\overleftrightarrow{S}_q(\vec{r},t)\right) \;=\; \overleftrightarrow{S}^{\mathrm{T}}(\vec{r},t) \\
\mathrm{Im}\left(\overleftrightarrow{S}_q(\vec{r},t)\right) \;&=\; \frac{q}{c}\left[\vec{E}(\vec{r},t)\vec{H}(\vec{r},t) - \vec{H}(\vec{r},t)\vec{E}(\vec{r},t)\right]
\end{aligned}
$$

$$= -\mathrm{Im}\left(\overset{\leftrightarrow\mathrm{T}}{S}_q(\vec{r},t)\right) \quad \text{(skew symmetric)}$$

$$\overset{\leftrightarrow}{S}_q(\vec{r},t) = \mathrm{Re}\left(\overset{\leftrightarrow}{S}_q(\vec{r},t)\right) + j\,\mathrm{Im}\left(\overset{\leftrightarrow}{S}_q(\vec{r},t)\right) \tag{3.1}$$

$$= \overset{\leftrightarrow\dagger}{S}_q(\vec{r},t)$$

$$\dagger = \mathrm{T}* = \text{adjoint}$$

Of course, still keeping with the combined field, one could just as easily have reversed the order of $\vec{E}_q$ and $\vec{E}_{-q}$.

There are various ways we might treat this. Taking the divergence of the imaginary part (to compare to the real part as in (2.7)) we have in one form

$$\nabla \bullet \mathrm{Im}\left(\overset{\leftrightarrow}{S}_q(\vec{r},t)\right) = \frac{q}{c}\ \nabla \bullet \left[\vec{E}(\vec{r},t)\vec{H}(\vec{r},t) - \vec{H}(\vec{r},t)\,\vec{E}(\vec{r},t)\right] \tag{3.2}$$

$$= \frac{q}{c}\left[\frac{1}{\varepsilon_0}\rho(\vec{r},t)\vec{H}(\vec{r},t)\ - \frac{1}{\mu_0}\rho_m(\vec{r},t)\,\vec{E}(\vec{r},t) + \vec{E}(\vec{r},t)\bullet\nabla\,\vec{H}(\vec{r},t) - \vec{H}(\vec{r},t)\bullet\nabla\,\vec{E}(\vec{r},t)\right]$$

This shows some cross terms between electric and magnetic parameters.

Another form takes a special dyadic identify [6] as

$$\begin{aligned}\nabla \bullet \mathrm{Im}\left(\overset{\leftrightarrow}{S}_q(\vec{r},t)\right) &= \frac{q}{c}\nabla \times \left[\vec{H}(\vec{r},t)\times\vec{E}(\vec{r},t)\right] \\ &= -\frac{q}{Z_0}\nabla\times\left[\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right]\end{aligned} \tag{3.3}$$

This is a rather compact result. In (2.7) we have a term which is the time derivative of the Poynting vector (describing energy flow). In (3.3) we have the curl of the Poynting vector as the companion imaginary part.

Summarizing, we have

$$\overset{\leftrightarrow}{S}_q(\vec{r},t) = \varepsilon_0 \vec{E}_q(\vec{r},t)\vec{E}_{-q}(\vec{r},t) - \frac{\varepsilon_0}{2}\vec{E}_q(\vec{r},t)\bullet\vec{E}_{-q}(\vec{r},t)\overset{\leftrightarrow}{1}$$

$$\begin{aligned}\nabla\bullet\overset{\leftrightarrow}{S}_q(\vec{r},t) &= \rho(\vec{r},t)\vec{E}(\vec{r},t) + \rho_m(\vec{r},t)\vec{H}(\vec{r},t) \\ &\quad + \vec{J}(\vec{r},t)\times\vec{B}(\vec{r},t) + \vec{D}(\vec{r},t)\times\vec{J}_m(\vec{r},t) \\ &\quad + \varepsilon_0\left[\frac{\partial}{\partial t} - qjc\nabla\times\right]\left[\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right]\end{aligned} \tag{3.4}$$

4. Combined Momentum

Now integrate the divergence of the stress dyadic over a volume V, bounded by a surface Σ, giving

$$\begin{aligned}\int_V \nabla\bullet\overset{\leftrightarrow}{S}_q(\vec{r},t)dV &= \int_\Sigma \vec{1}_\Sigma\bullet\overset{\leftrightarrow}{S}_q(\vec{r},t)d\Sigma \\ &= \varepsilon_0\int_\Sigma \vec{1}_\Sigma\bullet\left[\vec{E}_q(\vec{r},t)\vec{E}_{-q}(\vec{r},t) - \frac{1}{2}\vec{E}_q(\vec{r},t)\bullet\vec{E}_{-q}(\vec{r},t)\overset{\leftrightarrow}{1}\right]d\Sigma\end{aligned}$$

$$\vec{1}_\Sigma \equiv \text{outward pointing normal (from V) on the surface } S. \tag{4.1}$$

As in [3] let the charges and currents in V be zero. Then (3.4) gives

$$\begin{aligned}\int_V \nabla\bullet\overset{\leftrightarrow}{S}_q(\vec{r},t)dV &= \varepsilon_0\int_V\left[\frac{\partial}{\partial t} - qjc\nabla\times\right]\left[\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right]dV \\ &= \varepsilon_0\frac{\partial}{\partial t}\int_V \vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\,dV - \frac{qj}{Z_0}\int_\Sigma \vec{1}_\Sigma\times\left[\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right]d\Sigma\end{aligned} \tag{4.2}$$

One might interpret this as the force (net) across a volume equal to the time rate of change of momentum inside the volume.

In [3] there is defined (dimentionally)

$$\begin{aligned}\vec{g}(\vec{r},t) &\equiv \varepsilon_0\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t) \\ &\equiv \text{momentum per unit volume}\end{aligned} \tag{4.3}$$

$$\int_\Sigma \vec{1}_\Sigma\bullet\vec{S}(\vec{r},t)d\Sigma = \int_V \vec{g}(\vec{r},t)dV$$

This leads us to define a complex quantity via

$$\nabla \bullet \overleftrightarrow{S}_q(\vec{r},t) \equiv \frac{\partial}{\partial t}\vec{g}_q(\vec{r},t) \;=\; \frac{\partial}{\partial t}\vec{g}(\vec{r},t) + j\,\mathrm{Im}\left(\frac{\partial}{\partial t}\vec{g}_q(\vec{r},t)\right)$$

$$\equiv \varepsilon_0\left[\frac{\partial}{\partial t} - qjc\nabla\times\right]\left[\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right]$$

$$\vec{g}_q(\vec{r},t) \;\equiv\; \text{combined momentum per unit volume} \tag{4.4}$$

The imaginary part gives

$$\mathrm{Im}\left(\vec{g}_q(\vec{r},t)\right) \;=\; -\frac{q}{Z_0}\int_{-\infty}^{t}\nabla\times\left[\vec{E}(\vec{r},t')\;\times\;\vec{B}(\vec{r},t')\right]dt' \tag{4.5}$$

with assumed zero parameters for initial conditions (t = -∞). In integral form this is

$$\int_V \nabla\bullet\overleftrightarrow{S}_q(\vec{r},t)dV \;=\; \int_\Sigma \vec{1}_\Sigma\bullet\overleftrightarrow{S}_q(\vec{r},t)d\Sigma$$

$$= \frac{\partial}{\partial t}\int_V \vec{g}_q(\vec{r},t)dV \tag{4.6}$$

$$= \varepsilon_0\frac{\partial}{\partial t}\int_V \vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)dV - j\frac{q}{Z_0}\int_\Sigma \vec{1}_\Sigma\times\left[\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right]d\Sigma$$

The first term is dimensionally a time rate of change of momentum, and, therefore, so is the second. How should one interpret this?

Traditionally, $\vec{E}\times\vec{H}$ is interpreted as power flow through a boundary, but using $\vec{1}_\Sigma\bullet$, not $\vec{1}_\Sigma\times$. This latter form might be interpreted as power running along (parallel to) a boundary. From (4.5) one can also look at the volume integral of a curl. Again $\vec{E}\times\vec{H}$ can be interpreted as power flow.

These two terms come from symmetric and skew symmetric parts of $\overleftrightarrow{S}_q$ as in (3.1). It is the skew symmetric part as in (3.3) where the curl is introduced. Note that, while the symmetric $\overleftrightarrow{S} = \mathrm{Re}(\overleftrightarrow{S}_q)$ has 6 "independent" elements, the skew symmetric ($\overleftrightarrow{S}_q$) has only 3.

In the spirit of [3], let us do some dimensional analysis. As has been already established

$$\nabla \bullet \overleftrightarrow{S}(\vec{r}, t) \text{ has dimensions } \frac{kgm}{m^3} \frac{m}{s^2}$$

$$= \text{time derivative of momentum per unit volume} \tag{4.7}$$

$$= \text{force per unit volume}$$

$$= \text{force } \mathrm{m}^{-3}$$

Looking at the imaginary part

$$\nabla \bullet \operatorname{Im}\left(\overleftrightarrow{S}_q (\vec{r}, t) \right) \text{ has dimensions } \frac{kgm}{m^2} \frac{m}{s^2}$$

$$= \frac{\text{force}}{m^3} = \frac{\text{torque}}{m^4}$$

$$= \text{torque } m^{-4} \tag{4.8}$$

$$= \frac{1}{c} \text{ torque } sm^{-3}$$

So, another possible interpretation of this term is as the time derivative of a torque per unit volume. This is interpreting this term as a kind of twisting action associated with the curl or cross product.

5. Concluding Remarks

This has shown at least one generalization of the electromagnetic stress dyadic (tensor). Consistent with the definition of the combined field, we can write the usual stress dyadic and generalize it to a combined stress dyadic. This may not be the only possible generalization.

Taking the divergence we have a classical relation to the electromagnetic momentum. There is also a new term, the imaginary part, which can be interpreted in various ways.

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