Physics Notes

Note 21

3 June 2013

Extending Classical Physics into the Quantum Domain

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Abstract: An equation of motion of point charged particles including radiation is developed. It is first applied to the electron and shown that the solutions conserve energy and causality. The addition of one hypothesis allows the extension of classical physics into the quantum domain.. The modified equation is applied to the electron, resulting in a model of the spinning electron. This model is then applied to the hydrogen atom, the results agreeing with the standard approach. The principal result is a formula for the fine structure constant, giving a value within 1 σ of the latest QED calculations.

Extending Classical Physics into the Quantum Domain

1. Introduction.

1.1 The starting point for the derivation of a theory extending classical physics into the quantum domain is the equation of motion of a classical charged particle. The presently accepted equation is that of Abraham and Lorentz [1][2][3][4][5], a third order linear differential equation. This is presented as the inevitable result of classical physics despite violating the conservation laws and causality. The conclusion is that classical physics is inconsistent or that the equation is incorrect. It is assumed here that the A-L equation is incorrect.

1.2 An equation of motion is developed that is consistent with the conservation laws and causality. It transpires that the resulting equation, a version of which was originally found by Planck[5], is a non-linear second order equation, which makes for difficulties. However some analytical solutions are available and even the relativistic version has an important analytic solution as well as yielding readily obtained approximations.

1.3 Having established that the new equation of motion at least conforms to the requirements of classical physics, it is then necessary to introduce the concept of a stationary state into electron physics, a concept well understood in circuit theory. This allows a modification of the equation of motion to develop a model of a rotating electron and the hydrogen atom.

1.4 The singularity of the point charge is removed by assuming a radius consistent with the observed mass of the electron. This introduces the necessity of considering the internal stresses so as to obtain covariant relativistic equations[7]. *The assumption made here is that the effect of these stresses is ignorable in the first approximation.*

1.5 Applied to the hydrogen atom, the formula for the energy levels are obtained, the Zeeman splitting of the levels is demonstrated and the transition radiation is shown to occur in two stages, a relatively slow move from one level to the next, radiating a small amount of energy, followed by a rapid radiation of the remaining energy on entering the lower energy level.

1.6 The determination of the energy levels requires a certain function of the fine structure constant and the electron –proton mass ratio to be equal to a prime number. Using current values of α gives a prime number plus a small excess. Introducing a relativistic correction the discrepancy is greatly reduced. Accepting the correct number to be the prime, inversion of the function gives a new value of α which is ~σ/4 below the latest QED calculation, which means that the calculated value agrees with the QED calculation to 11 places of decimals. The success of the proposed equation together with the stationary state hypothesis in reproducing quantum mechanical results and providing the first accurate theoretical formula for evaluating the fine structure constant justifies the assumption of §*1.4* and gives credibility to the correctness of the equation.

2. The Proposed Equation for Non-relativistic Motion

2.1 The required equation of motion of a charged particle is obtained by equating the rate of change of momentum acquired plus the rate of change of momentum lost by radiation, to the applied force. This results in non-linear second order differential equations for both the non-relativistic and the special relativistic forms. Planck obtained such an equation for the harmonic oscillator, but thought it too complicated[5].

2.2. Carrying out the above scheme, the rate of loss of energy by radiation from an accelerating charge is $m\tau\dot{\mathbf{v}}^2$ [8] where $\tau = q^2/6\pi\varepsilon_0 m_0 c^3 = 2e^2/3m_0c^3$. Multiplying by unity in the form $\mathbf{v.v}/v^2$ the rate that energy is being radiated can be expressed as

$$\mathbf{F_{rad}.v} = m\tau\left(\frac{\dot{\mathbf{v}}^2}{v^2}\right)\mathbf{v.v} \tag{2.1}$$

$\mathbf{F_{rad}}$ may then be defined as

$$\mathbf{F_{rad}} = m\tau\left(\frac{\dot{\mathbf{v}}^2}{v^2}\right)\mathbf{v} \tag{2.2}$$

The equation of motion can then be written

$$\dot{\mathbf{v}} + \tau\left(\frac{\dot{\mathbf{v}}^2}{v^2}\right)\mathbf{v} = \frac{\mathbf{F}}{m_0} \tag{2.3}$$

which is the equation which was to be developed. Note that this reduces to the Newtonian form for uncharged particles.

2.3. Consider the equation of motion with zero applied force, and make the assumption that the acceleration is non-zero,

$$\dot{\mathbf{v}} + \tau(\dot{v}^2/v^2)\mathbf{v} = 0 \tag{2.4}$$

This implies that the acceleration is parallel to the velocity, and so the equation reduces to the scalar equation

$$1 + \tau(\dot{v}/v) = 0 \tag{2.5}$$

which has a solution, which is the only solution

$$v = v_0\exp(-t/\tau) \tag{2.6}$$

This non-physical result implies that the initial assumptions are false. Specifically, if the applied force is zero, the acceleration cannot be non-zero

3. Linear Motion under a Constant Force

3.1. Analytical solutions can be found for a few problems, in particular for rectilinear motion. For linear motion the three vectors in the equation of motion become co-linear, and the equation reduces to the scalar equation

$$\dot{v} + \tau\dot{v}^2 / v = \frac{F}{m_0} \tag{3.1}$$

Treating this equation as a quadratic in the acceleration and noting that positive forces give positive acceleration

$$\dot{v} = \frac{v}{2\tau}\left[\sqrt{1+\frac{4\tau F}{m_0 v}} - 1\right] \tag{3.2}$$

3.2 Consider a constant force. Rearranging the above and integrating

$$\int_0^v \left[\sqrt{1+\frac{4\tau F}{m_0 v}} - 1\right]^{-1} \frac{dv}{v} = t/2\tau \tag{3.3}$$

Observing that $\dot{v} = v dv/dx$

$$\int \left[\sqrt{1+\frac{4\tau F}{m_0 v}} - 1\right]^{-1} dv = x/2\tau \tag{3.4}$$

The integrands in these two equations are reduced to rational algebraic functions by the substitution $w = \sqrt{1+\frac{4\tau F}{m_0 v}}$ yielding

$$\int w(w+1)^{-1}(w-1)^{-2}dw = -t/4\tau \qquad \int w(w+1)^{-2}(w-1)^{-3}dw = -mx/16\tau^2 F \tag{3.5}$$

Carrying out the indicated integrations and reverting to the original variables

$$t/\tau = 2\left[\sqrt{1+v_0/v} - 1\right]^{-1} + \ln\left\{\left(\sqrt{1+v_0/v}+1\right)\left(\sqrt{1+v_0/v}-1\right)^{-1}\right\} \tag{3.6}$$

$$mx/\tau^2 F = 2\left(1+v_0/v\right)^{-1/2} + 2\left[\sqrt{1+v_0/v} - 1\right]^{-2} - \ln\left\{\left(\sqrt{1+v_0/v}+1\right)\left(\sqrt{1+v_0/v}-1\right)^{-1}\right\} \tag{3.6}$$

where

$$4F\tau/m = v_0 \tag{3.7}$$

Introducing the dimension-less variables $T = t/\tau, \quad X = mx/\tau^2 F$ the graph obtained is presented in Fig2.

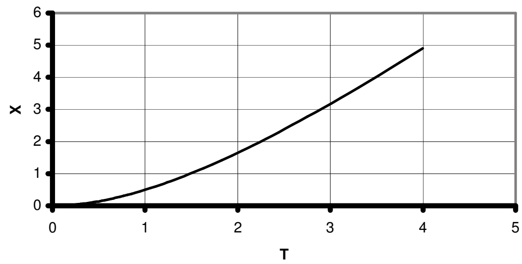

Figure 1 Displacement *vs* time

4 The Relativistic Equation of Motion

4.1. The relativistic equation of motion is now derived in the covariant 4-tensor notation of relativity. Ensuring that the terms of the equation are 4-tensors guarantees that the equation is invariant under a Lorentz transformation. All 4-tensors are denoted by capitals, and 3-vectors now by bold face lower case letters. Introducing T as the proper time, the equation of motion for a non-radiating particle is

$$dP^\mu / dT = F^\mu \tag{4.1}$$

where P^μ is the 4-momentum and F^μ the 4-force, and this may be written displaying the space-like and time-like components

$$(d/dT)(\mathbf{p}, mc) = \left(1-(v/c)^2\right)^{-1/2}(\mathbf{f}, (\mathbf{f.v})/c) = \gamma(\mathbf{f}, (\mathbf{f.v})/c) \tag{4.2}$$

where **p is** the 3-momentum

$$\mathbf{p} = m\,\mathbf{v} = \gamma m_0\,\mathbf{v} \tag{4.3}$$

and it is to be noted that the time-like component of the 4-force is the rate at which work is being done by the force on the particle. To modify the equation of motion a 4-vector is introduced that represents the momentum not acquired by the particle by virtue of the particle radiating,

$$dP^{\mu} / dT + dP^{\mu}_{rad} / dT = F^{\mu} \tag{4.4}$$

Note that the 4-velocity and 4-acceleration are given by

$$V^{\mu} = dx^{\mu} / dT = \gamma dx^{\mu} / dt = (\gamma v, \gamma c) \tag{4.5}$$

$$A^{\mu} = dV^{\mu} / dT = \gamma dV^{\mu} / dt = \gamma d(\gamma v, \gamma c) / dt \tag{4.6}$$

The magnitude of the squared acceleration is

$$-g_{\mu\nu} A^{\mu} A^{\nu} = -A_{\mu} A^{\nu} = \gamma^4 \dot{v}^2 + \gamma^6 v^2 \dot{v}^2 / c^2 \tag{4.7}$$

The 3-force then becomes

$$\mathbf{f} = \left(\gamma^4 \, \dot{\mathbf{v}}^2 + \gamma^6 (v\dot{v})^2 / c^2\right) \; \mathbf{v} / v^2 \tag{4.8}$$

The 4-force is

$$F^{\mu} = \gamma(\mathbf{f}, \mathbf{f} \,.\, \mathbf{v} / c) \tag{4.9}$$

and ultimately

$$\gamma \frac{d}{dt}(\gamma \mathbf{v}, \gamma c) + \tau_0 \left(\gamma^4 \dot{\mathbf{v}}^2 + \gamma^6 (\mathbf{v}/c)^2\right) \gamma \left(\mathbf{v}/v^2, 1/c\right) = (\gamma / m_0)(\mathbf{f}, \mathbf{f.v}/c) \tag{4.10}$$

Extracting the space-like components and making the velocity explicit

$$(d/dt)\mathbf{v}\left(1-(v/c)^2\right)^{-1/2} + \tau_0 \left(1-(v/c)^2\right)^{-2} \left\{ \dot{\mathbf{v}}.\dot{\mathbf{v}} + \left(1-(v/c)^2\right)^{-1} (\mathbf{v}.\dot{\mathbf{v}})^2 / c^2 \right\} \mathbf{v} / v^2 = \mathbf{f} / m_0 \tag{4.11}$$

5 The Stationary State Hypothesis

5.1 The classical equations of motion for charged particles imply that all accelerated motion results in radiation and steady states can only be maintained by an input of energy, stationary states cannot exist. But classical physics has examples of stationary states, the iconic model being the L-C circuit. The relevant parameter in this case is imaginary, and this suggests that introduction of the imaginary unit into the equation of motion for a charged particle will result in stationary state solutions. This is the Stationary State Hypothesis, and the idea is now pursued with some interesting results.

6 The N-R Stationary State Equation of Motion

6.1 The stationary state equation is

$$\dot{\mathbf{v}} + i\tau \frac{\dot{v}^2}{v^2} \mathbf{v} = \frac{\mathbf{f}}{m} \tag{6.1}$$

The simplest stationary state is that of the electron with no forces acting, and accordingly we look for solutions to

$$\dot{\mathbf{v}} + i\tau \frac{\dot{v}^2}{v^2} \mathbf{v} = 0 \tag{6.2}$$

The simplest form of this equation is for rectilinear motion, when it reduces to

$$\frac{\dot{v}}{v} = -\frac{i}{\tau} \tag{6.3}$$

and the solution is

$$v = v_0 \, \mathbf{exp}\left\{-i\frac{t}{\tau}\right\} \tag{6.4}$$

A complex solution to a 1-D problem indicates a 2-D motion, and in this case it is motion in a circle. The Stationary State Hypothesis has led to a model of a rotating electron! Note that this is in contrast to the standard model that insists that the angular momentum of an electron is a 'purely quantum effect'.

6.2 To confirm this really is the case we now consider two-dimensional motion and we write

$$z = x + iy \tag{6.5}$$

Substituting in the equation of motion, the solution ultimately reduces to

$$z = iz_0 \, \mathbf{exp}\left\{-\frac{it}{\tau}\right\} \tag{6.6}$$

and so

$$r = r_0 \, \mathbf{exp}\left\{-\frac{it}{\tau}\right\} \tag{6.7}$$

$$v = \frac{r_0}{\tau} = v_0 \, \mathbf{exp}\left\{-\frac{it}{\tau}\right\} \tag{6.8}$$

The motion is circular with constant speed v_0 and radius $v_0\tau$, and we observe that this motion is consistent with the 'zitterbewegung' of quantum mechanics. If this solution is to represent an electron, the associated parameters must agree, and so we attempt to match the angular momentum

$$m_0 v_0^2 \tau = \frac{\hbar}{2} \tag{6.9}$$

Assuming m_0 to be the rest mass of the electron, the velocity is

$$v_0 = \sqrt{\frac{\hbar}{2m_0\tau}} \cong 3.10^9 \text{ m/s} \tag{6.10}$$

and we see that a relativistic treatment is necessary.

7 The Relativistic Rotating Electron

7.1 The electron is now seen, on this model, as a point charge moving at relativistic speeds. Accordingly the observed rest mass must be written as

$$m_0 = \frac{m_i}{\left[1 - \frac{v_0^2}{c^2}\right]^{1/2}} \tag{7.1}$$

where m_i is an intrinsic mass, assumed to be made up of a mechanical mass m_m, together with an electromagnetic mass, m_{em}. Heitler [2] discusses the need to consider mechanical mass for an electron, '*we must ---attribute to the electron a mechanical inert mass----*'.

7.2 The SS equation of motion with zero applied force is

$$\frac{d}{dt}\frac{m_i \mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{m_i i\tau_i}{\left[1 - \frac{v^2}{c^2}\right]^2}\left\{\dot{\mathbf{v}}^2 + \frac{(\mathbf{v}.\dot{\mathbf{v}})^2}{c^2\left[1 - \frac{v^2}{c^2}\right]}\right\}\frac{\mathbf{v}}{v^2} = 0 \tag{7.2}$$

Assuming a circular motion as indicated by the non-relativistic solution, the equation reduces to

$$\dot{\mathbf{v}} + \frac{i\tau_i \dot{v}^2}{\left[1 - \frac{v^2}{c^2}\right]^{3/2}}\frac{\mathbf{v}}{v^2} = 0 \tag{7.3}$$

Making the replacement $\mathbf{r} \to z$ the equation has the solution

$$z = \frac{i v_0 \tau_i}{\left[1 - \frac{v_0^2}{c^2}\right]^{3/2}} \mathbf{exp}\left\{-i\left[1 - \frac{v_0^2}{c^2}\right]^{3/2}\frac{t}{\tau_i}\right\} \tag{7.4}$$

$$\dot{z} = v_0 \,\mathbf{exp}\left\{-i\left[1 - \frac{v_0^2}{c^2}\right]^{3/2}\frac{t}{\tau_i}\right\} \tag{7.5}$$

and so

$$r = \frac{v_0 \tau_i}{\left[1 - \frac{v_0^2}{c^2}\right]^{3/2}} \tag{7.6}$$

8 The Electron Rotational Velocity

8.1 The angular momentum of the electron is

$$\mathbf{\Omega} = m_0 \mathbf{v} \times \mathbf{r} = \frac{m_i \tau_i v_0^2}{\left[1 - \frac{v_0^2}{c^2}\right]^2}\hat{\mathbf{n}} \tag{8.1}$$

Equating the magnitude to the known value of the electron angular momentum

$$m_i \tau_i \frac{v_0^2}{\left[1 - \frac{v_0^2}{c^2}\right]^2} = \frac{\hbar}{2} \tag{8.2}$$

or

$$\frac{v_0}{\left[1-\frac{v_0^2}{c^2}\right]} = \pm\sqrt{\frac{\hbar}{2m_0\tau_0}} \tag{8.3}$$

on noting that $m_i\tau_i = m_0\tau_0$. Introducing the fine structure constant

$$\alpha = \frac{3}{2}\frac{m_0 c^2\tau_0}{\hbar} \tag{8.4}$$

we have

$$\frac{v_0}{\left[1-\frac{v_0^2}{c^2}\right]} = \pm c\sqrt{\frac{3}{4\alpha}} \tag{8.5}$$

Solving the quadratic for v_0, for $v_0<c$

$$\frac{v_0}{c} = \eta_e' = \frac{\sqrt{1+\frac{3}{\alpha}}-1}{\sqrt{\frac{3}{\alpha}}} = 0.9518956037 \tag{8.6}$$

where α has been taken as given by $\alpha^{-1}=137.035999070=1/7.297352570.10^{-3}$ [5]

9. The Electron Angular Velocity

9.1 The electron angular velocity is given by

$$\omega = \frac{\left[1-\frac{v_0^2}{c^2}\right]^{3/2}}{\tau_i} = \frac{\left[1-\frac{v_0^2}{c^2}\right]^2}{\tau_0} \tag{9.1}$$

From 8.5

$$\left[1-\frac{v_0^2}{c^2}\right]^2 = \frac{v_0^2}{c^2}\frac{4\alpha}{3} \tag{9.2}$$

and

$$\omega = \frac{v_0^2}{c^2}\frac{4\alpha}{3\tau_0} = \eta_e^2\frac{4\alpha}{3\tau_0} \tag{9.3}$$

The quantum mechanical result for the frequency of the 'Zitterbewegung' is

$$\omega = \frac{4\alpha}{3\tau_0} \tag{9.4}$$

implying that the Dirac equation assumes that the mass of the electron is all electromagnetic.

10 The Electron Rotation Radius

10.1 The rotation radius for the electron is given by

$$r_s = \frac{v_0\tau_0}{\left[1-\frac{v_0^2}{c^2}\right]^2} \tag{10.1}$$

This reduces to

$$r_s = \frac{3}{4}\frac{c^2\tau_0}{\alpha v_0} = \frac{3}{4}\frac{c\tau_0}{\alpha\eta_e} \tag{10.2}$$

Introducing the Compton Wavelength via the relation

$$\lambda\!\!\!^{-} = \frac{3c\tau_0}{2\alpha} \tag{10.3}$$

the radius is given by

$$2r_s = \left(\frac{c}{v_0}\right)\lambda\!\!\!^{-} = \frac{\lambda\!\!\!^{-}}{\eta_e} = 1.05053537\lambda\!\!\!^{-} \tag{10.4}$$

This result is consistent with the uncertainty principle, the uncertainty of the momentum being $\sqrt{2}mr\omega$ and the uncertainty in position $\sqrt{2}r$ giving

$$\Delta p\Delta x = 2mr^2\omega = \hbar \tag{10.5}$$

11 The Electron Magnetic Moment

11.1 The magnetic moment is the turning force per unit magnetic flux density

$$\boldsymbol{\mu} = q\mathbf{v}\times\mathbf{r} = \frac{q\hbar}{2m_0}\hat{\mathbf{n}} = \boldsymbol{\mu}_\mathbf{B} \tag{11.1}$$

the Bohr Magneton.

12 The Electric Circuit Analogue

12.1 The Steady State hypothesis was supported by the analogy with an LC circuit. If we consider a capacitor discharging into a parallel LR circuit, the voltage across the inductor has a time constant

$$\lambda = \frac{\dfrac{-1}{RC} \pm \sqrt{\dfrac{1}{(RC)^2} - \dfrac{4}{LC}}}{2} \tag{12.1}$$

Allowing the resistance to go to ∞ reduces this to

$$\lambda_\infty = \frac{\pm i}{\sqrt{LC}} = \pm i\omega \tag{12.2}$$

which is saying that in the stationary state radiation is inhibited. The analogy can be pursued further by analysing the current in a single turn loop and taking for the capacitance the only possibility, the capacitance of the electron considered as a sphere of radius $\kappa.r_e$, where κ is a constant to be determined. We have [3]

$$L = \mu_0(a-b)\left\{\ln\left(\frac{8a}{b}\right) - 2 + \frac{a}{4(a-b)}\right\} \tag{12.3}$$

where a is the loop radius and b the conductor radius, here taken to be the electron radius. The capacitance is taken to be

$$C = \frac{4\pi\varepsilon_0\kappa\alpha^2 a_0}{\tan^{-1}\left\{\dfrac{\eta_e}{\sqrt{1-\eta_e^2}}\right\}} \tag{12.4}$$

where a_0 is the Bohr radius and $\eta_e c$ is the electron velocity. Equating the two expressions for the angular frequency

$$\left[\frac{2\pi\ a_0^2\kappa\alpha^3}{c^2}\frac{\{\eta_e - 2\kappa\alpha\}}{\tan^{-1}\left\{\dfrac{\eta_e}{\sqrt{1-\eta_e^2}}\right\}}\left\{\ln\left(\frac{4}{\kappa\alpha\eta_e}\right) - 2 + \frac{\frac{c}{v}}{4(\eta_e - 2\kappa\alpha)}\right\}\right]^{-1/2} = \eta_e^2\frac{4\alpha}{3\tau} = \eta_e^2\frac{2c}{\alpha a_0} \tag{12.5}$$

Squaring, inverting and cancelling common factors

$$8\pi\kappa\alpha\Gamma^3\frac{(1-2\kappa\alpha\eta_e)}{\tan^{-1}\left\{\dfrac{\eta_e}{\sqrt{1-\eta_e^2}}\right\}}\left\{\ln\left(\frac{4}{\kappa\alpha\eta_e}\right) - 2 + \frac{1}{4(1-2\kappa\alpha\eta_e)}\right\} = 1 \tag{12.6}$$

where

$$\eta_e = \frac{\sqrt{1+\dfrac{3}{\alpha}} - 1}{\sqrt{\dfrac{3}{\alpha}}} = 0.9518956038 \tag{12.7}$$

A numerical solution gives

$$\kappa = 2.125570591 \tag{12.8}$$

where the value of α is as given in §8.

13 The Steady State Hypothesis Applied to the Hydrogen Atom

13.1 The velocity of electrons within atoms is of order αc suggesting that a non-relativistic treatment would be adequate. However the precision of the prediction of physical constants involved requires a relativistic treatment. The complexity of the relativistic equation of motion is such that an approximate procedure is necessary. This can be achieved in two ways: a non-relativistic solution can be corrected for relativistic effects and the relativistic equation can be simplified by introducing the non-relativistic solution. Both procedures are presented below to show consistency between the two approaches.

14. The Non-Relativistic Stationary State Equation

14.1 The non-relativistic stationary state vector equation of motion for an electron in the field of a proton is

$$\dot{\mathbf{v}} + i\tau\frac{\dot{v}^2}{v^2}\mathbf{v} = -\frac{e^2}{m_0}\frac{\mathbf{r}}{r^3} \tag{14.1}$$

Looking for a circular orbit, put

$$\mathbf{r} = \ell r_0 \exp(i\omega t) \tag{14.2}$$

where ℓ is an integer. Setting ℓ =1 and substituting into 2.1

$$\tau\omega^3 + \omega^2 - \frac{e^2}{m_0}\frac{1}{r_0^3} = 0 \tag{14.3}$$

The discriminant is<0 and so the equation has three real roots. Noting that the constant is <<1 and setting r=sxa$_0$, a_0 being the Bohr radius, we have an approximate solution

$$\omega\tau \sim -1 \text{ or } \pm\sqrt{\frac{\tau^2 e^2}{m_0 s^3 a_0^3}} = -1 \text{ or } \pm\frac{2}{3}s^{-3/2}\alpha^3 \tag{14.4}$$

where α is the fine structure constant and s is a constant. Discarding the –1 solution, this gives

$$\omega_0 = \pm s^{-3/2}\frac{\alpha c}{a_0} \tag{14.5}$$

An improved approximation can be obtained by putting $\omega = \omega_0 + \delta$ into 2.3 retaining only first order terms in δ, the result being

$$\omega \approx \pm\frac{s^{-3/2}\alpha c}{a_0}\left(1 \pm \frac{s^{-3/2}\alpha^3}{3}\right) = \omega_0\left(1 \pm \frac{\omega_0\tau}{2}\right) \tag{14.6}$$

This then gives for the ground state

$$\mathbf{r}_0 = sa_0\exp(i\omega t) \quad = sa_0\exp\left(i\omega_0\left[1 \pm \frac{\omega_0\tau}{2}\right]t\right) \tag{14.7}$$

and the orbital speed is

$$v = \pm\frac{2s^{-1/2}\alpha^3 a_0}{3\tau}\left\{1 \pm \frac{s^{-3/2}\alpha^3}{3}\right\} = s^{-1/2}\alpha c\left\{1 \pm \frac{s^{-3/2}\alpha^3}{3}\right\} \tag{14.8}$$

Imposing the requirement that the orbital angular momentum of the ground state is $\hbar$

$$m_0 s^2 a_0^2 . s^{-3/2}\frac{\alpha}{a_0}c\left\{1 \pm \frac{s^{-3/2}\alpha^3}{3}\right\} = \hbar \tag{14.9}$$

which reduces to

$$s^{1/2}\left\{1 \pm \frac{s^{-3/2}\alpha^3}{3}\right\} = 1 \tag{14.10}$$

This has the approximate solution

$$s \cong \left(1 \pm \frac{2\alpha^3}{3}\right) \tag{14.11}$$

The velocity in the ground state is then, retaining terms to α^3 in the individual expansions

$$v = \pm\left\{1 \pm \frac{2\alpha^3}{3}\right\}^{-1/2}\alpha c\left\{1 \pm \frac{\left[1 \pm \frac{2\alpha^{3/2}}{3}\right]^{-3/2}\alpha^3}{3}\right\} \cong \pm\left(1 - \frac{\alpha^3}{3}\right)\left(1 + \frac{\alpha^3}{3}\right)\alpha c \cong \pm\alpha c \tag{14.12}$$

and the angular frequency is

$$\omega_{orb} \cong \frac{\alpha c}{a_0} \tag{14.13}$$

14.2 For a true stationary state the trajectory should be stationary, and this requires that the relative rotation period should be commensurate with the orbital period, that is, there should be an integral number of revolutions in a period of the orbit. Imposing this condition on the non-relativistic approximation gives

$$N = \frac{\omega_e}{\omega_{orb}} = \left(\frac{2\eta_e{}^2 c}{\alpha a_0}\frac{a_0}{\eta_{orb}c}\right) == \frac{2\eta_e{}^2}{\alpha\eta_{orb}} = \frac{2\eta_e^2}{\alpha^2} = 34031.2562\,1068 \tag{14.14}$$

14.3 The orbital speed will modify the rotation velocity, the two needing to be combined relativistically. The sum is given by [2]

$$\mathbf{w} = \mathbf{v_1} \oplus \mathbf{v_2} = \frac{\mathbf{\Phi_1} \cdot \mathbf{v_2} + \beta_1 \mathbf{v_1}}{\beta_1 \left(1 + \frac{\mathbf{v_1} \cdot \mathbf{v_2}}{c^2}\right)} \tag{14.15}$$

where **Φ** is the dyadic

$$\mathbf{\Phi} = \mathbf{I} + \frac{(\beta - 1)}{v^2} \mathbf{vv} \tag{14.16}$$

The change to the spin period will be given by the $\hat{\varphi}$ component, and so we put

$$\mathbf{v}_1 = \eta_e c \hat{\varphi} \quad \mathbf{v}_2 = \alpha c \sin\varphi \hat{\varphi} \tag{14.17}$$

The rotation velocity and frequency are given by

$$\frac{v_e}{c} = \eta_e = \frac{\sqrt{1 + \frac{3}{\alpha}} - 1}{\sqrt{\frac{3}{\alpha}}} \qquad \omega_e = \frac{v_0^2}{c^2} \frac{4\alpha}{3\tau_0} = \frac{2c\eta_e^2}{\alpha a_0} \tag{14.18}$$

and we have on making use of the above dyadic

$$\frac{\eta'_e}{\eta_e} = \frac{1 - \frac{\alpha}{\eta_e} \sin\varphi}{1 - \alpha\eta_e \sin\varphi} \tag{14.19}$$

This correction varies as the electron rotates, and so an average is taken.

$$\phi : 0 \to 2\pi, \tag{14.20}$$

$$\frac{\overline{\eta}'_e}{\eta_e} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\left\{1 - \frac{\alpha}{\eta_e} \sin\phi\right\}}{(1 - \alpha\eta_e \sin\phi)} d\phi \tag{14.21}$$

We have chosen to subtract the orbital velocity. However the integral depends only on the even powers of the expansion and the same even powers are obtained for the sum of the velocities. The result to 4th order in α is

$$\frac{\overline{\eta}'_e}{\eta_e} = \left[1 - \frac{\alpha^2}{2}\left(1 - \eta_e^2\right)\left(1 - \frac{3\alpha^2}{4\eta_e^2}\right)\right] \tag{14.22}$$

Conservation of angular velocity requires that the rotation radius be increased by the same factor.

14.4 Imposing the stationary state condition

$$N = \frac{\omega_e}{\omega_{orb}} = \frac{2\eta_e^2}{\alpha\eta_{orb}} \left(\frac{\overline{\eta}'_e}{\eta_e}\right)^3 = \frac{2\eta_e^2}{\alpha^2} \left(\frac{\overline{\eta}'_e}{\eta_e}\right)^3 \tag{14.23}$$

i.e.

$$N = \frac{\omega_e}{\omega_{orb}} = \frac{2\eta_e^2}{\alpha^2} \left[1 - \frac{1}{2}\alpha^2\left(1 - \eta_e^2\right)\left(1 - \frac{3\alpha^2}{4\eta_e^2}\right)\right]^3 \tag{14.24}$$

Evaluating using $\alpha = 7.297352570.10^{-3}$ [4], N=34031.000823.

14.5 The calculations have so far ignored the magnetic fields of the electron and proton spins. The energy of the electron due to the magnetic field of the proton is

$$U = -\frac{\mu_e \mu_p \mu_0}{2\pi a_0^3} = \delta E \tag{14.25}$$

where μ_e , μ_p are the magnetic moments of the electron and proton respectively. Dividing by the energy of the ground state and simplifying

$$\frac{\delta E}{E} = \alpha^2 \frac{m_e}{m_p} = 2.90015941.10^{-8} \tag{14.26}$$

It is readily shown that the fractional reduction in orbit radius is equal to δE/E, resulting in an increased orbital angular velocity, the increase being given by 1+δE/E. The formula for N is now

$$N = \frac{\omega_e}{\omega_{orb}} = \frac{2\eta_e^2}{\alpha^2} \left[1 - \frac{1}{2}\alpha^2\left(1 - \eta_e^2\right)\left(1 - \frac{3\alpha^2}{4\eta_e^2}\right)\right]^3 \left\{1 - \alpha^2 \frac{m_e}{m_p}\right\} \tag{14.27}$$

The corrected value for N is then

$$N = 34031.000823x(1-2.90015941.10^{-8}) = 34031 - 0.00016 \quad (14.28)$$

It seems reasonable to accept this as an integer and we note that it is a prime. In general where the orbit radius is ℓr_0

$$N_\ell = \ell^{3/2} 34031 \quad (14.29)$$

We have that p_0 is an integer and so ℓ must be a perfect square and moreover it must also be an integer, say n^2, as fractional values would require 34031 to have factors.

14.6 In general

$$\omega_{orb} = \frac{\alpha c}{n^3 a_0}\left(1 - \frac{n^3\alpha^3}{3}\right) \quad (14.30)$$

The value of r_0 is modified to maintain the angular momentum

$$r_0 = n^2 a_0 \left(1 - \frac{s^{-3/2}\alpha^3}{3}\right)^{-1} \quad (14.31)$$

The orbital velocity is then

$$v_{orb} = \frac{\alpha c}{n} \quad (14.32)$$

The potential energy is

$$E_{pot} = -\frac{m_0\alpha^2 c^2}{n^2}\left(1 - \frac{\alpha^3}{3n^3}\right) \quad (14.33)$$

and the kinetic energy is

$$E_{kin} = \frac{m_0 c^2}{\sqrt{1 - \frac{\alpha^2}{n^2}}} - m_0 c^2 = \frac{m_0 c^2 \alpha^2}{2n^2}\left(1 + \frac{3}{4}\frac{\alpha^2}{n^2}\right) \quad (14.34)$$

To 4th order in α, the total energy is

$$E_{tot} = -\frac{m_0 c^2 \alpha^2}{2n^2}\left(1 - \frac{3\alpha^2}{4n^2}\right) \quad (14.35)$$

15 The Zeeman Effect

15.1 The equation of motion for a stationary state with an applied magnetic field becomes

$$\dot{\mathbf{v}} + i\tau\frac{\dot{v}^2}{v^2}\mathbf{v} = -\frac{k}{r^3}\mathbf{r} + \frac{q}{m_0}\mathbf{v}\times\mathbf{B} \quad 15.1$$

For 2-D circular motion we may write

$$\mathbf{r} = r_0 e^{i\omega t} \quad 15.2$$

With this substitution the equation becomes

$$\tau\omega + \omega^2 - \frac{qB}{m_0}\omega = \frac{k}{r_0^3} \quad 15.3$$

This reduces to

$$\xi^3 + \xi^2 - \nu\xi = \xi_0^2 \quad 15.4$$

where

$$\xi = \omega\tau \qquad \text{and} \qquad \nu = \frac{qB\tau_0}{m_0} \quad 15.5$$

We may again make the approximation

$$\xi^3 \sim 0 \quad 15.6$$

and the equation becomes

$$\xi^2 - \nu\xi - \xi_0^2 \sim 0 \quad 15.7$$

Solving this equation

$$\xi = \frac{\nu \pm \sqrt{\nu^2 + 4\xi_0^2}}{2} \quad 15.8$$

For weak fields we may assume

$$\nu \gg 2\xi_0 \quad 15.9$$

and we have

$$\xi = \pm\xi_0 + \frac{\nu}{2} \quad 15.10$$

Taking the modulus and replacing ν and ξ_0,

$$\omega = \omega_0 \pm \frac{qB}{2m_0} \qquad 15.11$$

This is the correct quantum mechanical result for the two new levels introduced by the magnetic field.

16. Perturbed Orbits

16.1 More general orbits may be considered by allowing the radial component to be a function of time,

$$\mathbf{r} = r\exp\{i\theta\} \qquad 16.1$$

Repeated differentiation then gives

$$\dot{\mathbf{r}} = \{\dot{r} + i r\dot{\theta}\}e^{i\theta} \quad \ddot{\mathbf{r}} = \{\ddot{r} - r\dot{\theta}^2 + i(r\ddot{\theta} + \dot{r}\dot{\theta})\}e^{i\theta} \qquad 16.2$$

It is assumed that the perturbation removes the electron from the stationary state. Inserting these results into the equation of motion

$$(\ddot{r} - r\dot{\theta}^2) + i(r\ddot{\theta} + 2\dot{r}\dot{\theta}) + \tau A(\dot{r} + i\ r\dot{\theta}) = -\frac{e^2}{m_0 r^2} \qquad 16.3$$

where

$$A = \frac{\left[(\ddot{r} - r\dot{\theta}^2)^2 + (\dot{r}\dot{\theta} + r\ddot{\theta})^2\right]}{\left[\dot{r}^2 + r^2\dot{\theta}^2\right]} \qquad 16.4$$

Separating the real and imaginary parts

$$\ddot{r} - r\dot{\theta}^2 + \tau A\dot{r} = -\frac{e^2}{m_0 r^2} \qquad 16.5$$

$$r\ddot{\theta} - \dot{r}\dot{\theta} + \tau A r\dot{\theta} = 0 \qquad 16.6$$

Eliminating τ

$$\ddot{r} - r\dot{\theta}^2 - \frac{r\ddot{\theta} + 2\dot{r}\dot{\theta}}{r\dot{\theta}}\dot{r} = -\frac{e^2}{m_0 r^2} \qquad 16.7$$

We now look for a decaying solution and put $r = r_a\exp(-\theta)$, where r_a is an arbitrary initial radius, with the result that

$$\dot{\theta} == \omega_o \frac{1}{\sqrt{2}}\left(\frac{a_o}{r_a}e^{\theta}\right)^{3/2} \qquad 16.8$$

and we ultimately have

$$r = r_a\left[1 - \frac{3\omega_o}{2\sqrt{2}}\left(\frac{a_o}{r_a}\right)^{3/2} t\right]^{2/3} \qquad 16.9$$

where $\omega_o^2 = e^2/m_0 r_0^3$.

16.2 If we now consider a small positive displacement from $r_a = a_o$ we have

$$r = a_o(1+\delta)\left[1 - \frac{3}{2\sqrt{2}}\omega_o(1+\delta)^{-3/2} t\right]^{2/3} \qquad 16.10$$

The time to return to the stationary state is then

$$t_g \approx \sqrt{2}\frac{\delta}{\omega_0} \qquad 16.11$$

and the stationary state is stable to small positive displacements.

16.3 For negative displacements, consider an infinitesimal displacement from the state with n=2. We then have $r_a = 4a_0$ which, substituting in 16.9 leads to

$$1 = 4\left\{1 - \frac{3}{16\sqrt{2}}\omega_0 t_g\right\}^{2/3} \qquad 16.12$$

Solving for the time to reach the ground state

$$t_g = \left[1 - 4^{-3/2}\right]\frac{16\sqrt{2}}{3\omega_0} \approx 1.05T_0 \qquad 16.13$$

16.4 The rate of energy emission during the transition from the n=2 state to the ground state is given by

$$\dot{\mathcal{E}} = m_0\tau\left[(\ddot{r} - r\dot{\theta}^2)^2 + (2\dot{r}\dot{\theta}^2)\right] \qquad 16.14$$

Substituting from the above solution and integrating

$$\mathcal{E} = \frac{m_0\,\tau a_0^2\omega_0^2}{64^2}\int_0^{t_g}\left\{\frac{20 + 4u^{2/3} + u^{4/3}}{u^{8/3}}\right\}dt \qquad 16.15$$

where

$$u = \left\{1 - \frac{3}{16\sqrt{2}}\omega_0 t\right\} \qquad 16.16$$

Carrying out the integration

$$\mathcal{E} = 47.494 m_0 \tau a_0^2 \omega_0^3 = \frac{\alpha^5}{15.83} m_0 c^2 \qquad 16.17$$

Comparing this with the difference in energy between the levels

$$\frac{\varepsilon}{\Delta E} = 0.0168\alpha^3 \approx 7.10^{-9} \qquad 16.18$$

This low value of the fraction of energy radiated during the transition indicates that the majority of the energy is radiated on entering the ground state. If this energy is emitted as one photon, it would correspond to a frequency of~$1.6.10^8$ Hz, or a wavelength of ~1.9 m. This could be the basis of an experimental check on the theory.

17. The Relativistic Stationary State Equation

17.1 The relativistic stationary state equation of motion for the electron under the central field of a proton is

$$\frac{d}{dt}\frac{m_0\mathbf{v}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} + \frac{m_0 i\tau}{\left(1-\frac{v^2}{c^2}\right)^2}\left\{\dot{\mathbf{v}}.\dot{\mathbf{v}} + \frac{(\mathbf{v}.\dot{\mathbf{v}})^2}{c^2\left(1-\frac{v^2}{c^2}\right)}\right\}\frac{\mathbf{v}}{v^2} = -\frac{e^2\mathbf{r}}{m_0 r^3} \qquad 17.1$$

Inserting the trial solution for circular motion

$$\mathbf{r} = r_0 \exp\{i\omega t\} \qquad 17.2$$

and setting $r_0=sa_0$ reduces the equation to

$$\frac{\omega^2}{\left(1-\frac{s^2a_0^2\omega^2}{c^2}\right)^{1/2}} + \frac{\omega^3\tau}{\left(1-\frac{s^2a_0^2\omega^2}{c^2}\right)^2} = \frac{e^2}{m_0 s^3 a_0^3} \qquad 17.3$$

Multiplying by τ^2

$$\frac{(\omega\tau)^3}{\left[1-\frac{\omega^2s^2a_0^2}{c^2}\right]^2} + \frac{(\omega\tau)^2}{\left[1-\frac{\omega^2s^2a_0^2}{c^2}\right]^{1/2}} = \frac{4}{9}\frac{\alpha^6}{s^3} \qquad 17.4$$

This has the approximate solution

$$\frac{(\omega\tau)^2}{\left[1-\frac{\omega^2s^2a_0^2}{c^2}\right]^{1/2}} = \frac{4}{9}\frac{\alpha^6}{s^3} \qquad 17.5$$

yielding

$$\omega \cong \pm\left(1-\frac{\omega^2s^2a_0^2}{c^2}\right)^{1/4}\frac{2}{3}\frac{\alpha^3}{\tau s^{3/2}} \cong \pm\left(1-\frac{\alpha^2s^2}{4}\right)\frac{\alpha c}{s^{3/2}a_0} \qquad 17.6$$

This in turn gives

$$v = s^{-1/2}\left(1-\frac{\alpha^2s^2}{4}\right)\alpha c \qquad 17.7$$

Noting that the orbital angular momentum is $\hbar$

$$m_0 s a_0 s^{-1/2}\alpha c\left(1-\frac{s^2\alpha^2}{4}\right) = \hbar \qquad 17.8$$

Noting that $\hbar = m_0\alpha a_0 c$

$$s = \left(1-\frac{\alpha^2}{4}\right)^{-2} \cong 1+\frac{\alpha^2}{2} \qquad 17.9$$

The velocity is then

$$v = \left(1-\frac{\alpha^2}{4}\right)\left(1+\frac{\alpha^2}{4}\right)\alpha c \cong \alpha c\left(1-\frac{\alpha^4}{16}\right) \qquad 17.10$$

agreeing to first order in α with the NR approximation. Writing $v/c = \eta'_{orb}$ and evaluating, η'_{orb} =7.297352564.10^{-3}. Replacing α by η_{orb} in the appropriate position in 14.22

$$\overline{\eta}'_e = \eta'_e\left[1-\frac{\eta'^2_{orb}}{2}\left(1-\eta'^2_e\right)\left(1+\frac{3\eta'^2_{orb}\eta'^2_e}{4}\right)\right] \qquad 17.8$$

where $\overline{\eta}'_e$ is the relativistic average. The formula for N is then

$$N = 2\frac{\overline{\eta}'^3_e}{\eta'_e\alpha\eta'_{orb}}\left\{1-\alpha^2\frac{m_e}{m_p}\right\} = 34031\text{-}0.00016 \qquad 17.9$$

as before,

17.2 The agreement to 10 significant figures is indicative of a consistent approach to the problem. Regarding the equation for N as an equation for α, given N=34031, leads to a value of α of 7.2973525528.10^{-3} from the NR calculation) with R correction, and 7.2973525529.10^{-3} from the approximate R calculation. The QED result being 7.297352570(72) 10^{-3}, with a 1-σ value of 7.2973525498.10^{-3}, makes the above prediction ~σ from the Gabrielse result.

17.3 The formula for α implies that for it to have changed over the lifetime of the universe, the ratio of electron to proton mass would also have to have changed.

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