EMP Theoretical Notes

Note I

The EMP Fields at the Surface of the Ground and Below the Ground

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Abstract:

This note describes the calculation of the EMP magnetic field below the surface of the ground and the radial electric field at the surface and below the ground. These calculations proceed from a knowledge of the magnetic field at the surface of the ground. We shall consider the propagation into the ground of electromagnetic fields typical of those generated by a nuclear weapon detonation.[1,2] From the onset, we will work with the Laplace transforms of the electromagnetic fields, and develop the necessary inverse transforms which will allow us to calculate the radial electric field and magnetic field below the ground by performing a single integration. The ground is assumed to be a flat half-sphere with the electrical parameters ϵ, σ and μ.

I. The Magnetic Field Below the Ground: Maxwell's equations for the fields in the ground,

$$\nabla \times \nabla \times \mathcal{H} = -s\mu(\sigma + s\epsilon)\mathcal{H} \qquad (1)$$

may be expanded in spherical coordinates into the form (with the assumption that H_ϕ is the only magnetic field component):

$$\frac{\hat{e}_\phi}{r}\left[2\frac{\partial \mathcal{H}_\phi}{\partial r} + r\frac{\partial^2 \mathcal{H}_\phi}{\partial r^2} + \frac{1}{r}\cot\theta\frac{\partial \mathcal{H}_\phi}{\partial \theta} - \frac{\csc^2\theta}{r}\mathcal{H}_\phi + \frac{1}{r}\frac{\partial^2 \mathcal{H}_\phi}{\partial \theta^2}\right]$$

$$= +s\mu(\sigma + s\epsilon)\mathcal{H}_\phi \qquad (2)$$

[1]The problems we shall consider have been treated in the diffusion approximation by Dr. J. Malik in Los Alamos Document J-13-459, "E.M. Pulse Fields in Dissipative Media".

[2]The diffusion approximation will not be made in the following development.

The assumption will now be made that the fields change much more rapidly with depth into the ground than they do in the horizontal directions. The mathematical requirements for this assumption will be stated later in this note. Then equation (2) reduces to

$$\frac{1}{r^2}\frac{\partial^2 \mathcal{H}_\phi}{\partial \theta^2} = s\mu(\sigma + s\epsilon)\mathcal{H}_\phi \tag{3}$$

Define a new variable, z, such that

$$z = r\cos\theta \tag{4}$$

$$\frac{d}{d\theta} = -r\sin\theta\frac{d}{dz} \tag{5}$$

and

$$\frac{1}{r^2}\frac{d^2}{d\theta^2} = -\frac{1}{r}\cos\theta\frac{d}{dz} + \sin^2\theta\frac{d^2}{dz^2} \tag{6}$$

By the previous assumption,

$$\frac{1}{r^2}\frac{d^2}{d\theta^2} \approx \sin^2\theta\frac{d^2}{dz^2} \approx \frac{d^2}{dz^2} \tag{7}$$

Then equation 3 becomes

$$\frac{d^2\mathcal{H}_\phi}{dz^2} = s\mu(\sigma + s\epsilon)\mathcal{H}_\phi$$

which has the solution

$$\mathcal{H}_\phi(z) = A\,e^{\sqrt{s\mu(\sigma+s\epsilon)}\,z} + B\,e^{-\sqrt{s\mu(\sigma+s\epsilon)}\,z} \tag{9}$$

As z, which is negative below the surface of the ground, becomes large in magnitude, this solution must tend to 0, hence

$$\mathcal{B} = 0 \tag{10}$$

If we know H_ϕ at the surface of the ground, $H_\phi(0)$, then

$$\mathcal{H}_\phi(z) = \mathcal{H}_\phi(0)\, e^{\sqrt{s\mu(\sigma+s\epsilon)}\, z} \quad . \tag{11}$$

II. <u>The Radial Electric Field Below the Ground</u>

From the radial component of Maxwell's equation

$$\nabla \times \mathcal{H} = (\sigma + s\epsilon)\mathcal{E} \quad , \tag{12}$$

we find that

$$\frac{1}{r}\frac{\partial \mathcal{H}_\phi}{\partial \theta} + \frac{\cot\theta}{r}\mathcal{H}_\phi = (\sigma + s\epsilon)\mathcal{E}_r \quad . \tag{13}$$

Again, by assumption, this is approximately

$$\frac{1}{r}\frac{d\mathcal{H}_\phi}{d\theta} = (\sigma + s\epsilon)\mathcal{E}_r \tag{14}$$

or, using the z coordinate,

$$\frac{d\mathcal{H}_\phi}{dz} = -(\sigma + s\epsilon)\mathcal{E}_r \tag{15}$$

Using equation 11, this just states that

$$\mathcal{E}_r(z) = -\mathcal{H}_\phi(0)\,\frac{\sqrt{s\mu}}{\sqrt{\sigma+s\epsilon}}\, e^{\sqrt{s\mu(\sigma+s\epsilon)}\, z} \tag{16}$$

In particular, at the surface of the ground,

$$\mathcal{E}_r(0) = -\mathcal{H}_\phi(0)\,\frac{\sqrt{s\mu}}{\sqrt{\sigma+s\epsilon}} \quad . \tag{17}$$

III. <u>Mathematical Form of the Assumption Made on Sections I and II</u>:

The requirements for self-consistency of the assumptions may now be stated: For equation (3), the requirement is

$$\left\{ 2\left|\frac{\partial \mathcal{H}_\phi}{\partial r}\right|,\ r\left|\frac{\partial^2 \mathcal{H}_\phi}{\partial r^2}\right|,\ \left|\cot\theta\sqrt{s\mu(\sigma+s\epsilon)}\,\mathcal{H}_\phi\right|,\ \frac{\csc^2\theta}{r}\left|\mathcal{H}_\phi\right| \right\} \ll \left| r s\mu(\sigma+s\epsilon)\mathcal{H}_\phi \right| \tag{18}$$

for equation (7) and equation (15) the requirement is

$$\frac{1}{r}\left|\cos\theta\sqrt{s\mu(\sigma+s\epsilon)}\right| \ll \left|s\mu(\sigma+s\epsilon)\right|\sin^2\theta \tag{19}$$

and

$$\sin^2\theta \approx 1 \quad . \tag{20}$$

The approximation leading to equation (14) is

$$\left|\frac{\cot\theta}{r}\right| \ll \left|\sqrt{s\mu(\sigma+s\epsilon)}\right| \quad . \tag{21}$$

Several of these approximations are valid by virtue of the fact that $\theta \approx \pi/2$. The rest of the approximations will certainly be valid when $\epsilon \gg \epsilon_o$ or $\sigma \gg |s\epsilon|$.

IV. <u>Fields Below the Ground in Terms of Convolution Integrals</u>: If the equation relating two variables in terms of their Laplace transforms is known, for example

$$\mathcal{E}(z) = \eta(z) \mathcal{H}_\phi(0) \tag{22}$$

then the dependent quantity may be expressed as a function of time in terms of a convolution integral:

$$E_r(t,z) = \int_0^t N(\tau,z) H_\phi(t-\tau,0)\, d\tau \ , \tag{23}$$

where $E_r(t,z)$, $N(t,z)$ and $H_\phi(t,0)$ are the inverse Laplace transforms of the original quantities. Thus, these inverse transforms must be found, and then the convolution may proceed. We shall assume here that $H_\phi(t,0) \equiv H_\phi(t)$ is known:

A. <u>E_r (t,o) as a Convolution of H_ϕ (t)</u>

From equation (17),

$$\mathcal{E}_r(s,0) = \mathcal{H}_\phi(s)\eta(s) \ ,$$

$$\text{where} \quad \eta(s) = \frac{\sqrt{s\mu}}{\sqrt{\sigma + s\epsilon}} . \tag{24}$$

Hence, to express $E_r(t,0)$ as a convolution integral of $H_\phi(t)$ one needs the inverse transform of $\eta(s)$.

$$\eta(s) = \sqrt{\frac{\mu}{\epsilon}}\,\frac{s}{\sqrt{s^2+\frac{\sigma}{\epsilon}s}} = \sqrt{\frac{\mu}{\epsilon}}\,\frac{s}{\sqrt{(s+\sigma/2\epsilon)^2-(\sigma/2\epsilon)^2}}$$

$$\eta(s) = \sqrt{\frac{\mu}{\epsilon}}\left[\frac{(s+\sigma/2\epsilon)}{\sqrt{(s+\sigma/2\epsilon)^2-(\sigma/2\epsilon)^2}} - \frac{\sigma/2\epsilon}{\sqrt{(s+\sigma/2\epsilon)^2-(\sigma/2\epsilon)^2}}\right]$$

$$\eta(s) = \sqrt{\frac{\mu}{\epsilon}}\left[\frac{(s+\sigma/2\epsilon)-\sqrt{(s+\sigma/2\epsilon)^2-(\sigma/2\epsilon)^2}}{\sqrt{(s+\sigma/2\epsilon)^2-(\sigma/2\epsilon)^2}}\right.$$

$$\left. - \frac{\sigma/2\epsilon}{\sqrt{(s+\sigma/2\epsilon)^2-(\sigma/2\epsilon)^2}} + 1\right] \tag{25}$$

From the Handbook of Chemistry and Physics, thirty-ninth edition (1957), p. 288, transform no. 59, one finds

$$\mathcal{L}^{-1}\left\{\frac{\left(s-\sqrt{s^2-a^2}\right)^{\nu}}{\sqrt{s^2-a^2}}\right\} = a^{\nu}\, I_{\nu}(at)\ , \quad \nu > -1\ . \tag{26}$$

Thus from Equation (26) and the fact that $\mathcal{L}^{-1}\{f(s+a)\} = e^{-at}F(t)$, where $\mathcal{L}^{-1}\{f(s)\} = F(t)$, one finds that $\mathcal{L}^{-1}\{\eta(s)\}$ can be written

$$\mathcal{L}^{-1}\{\eta(s)\} = \sqrt{\frac{\mu}{\epsilon}}\, e^{-\frac{\sigma}{2\epsilon}t}\left[\frac{\sigma}{2\epsilon} I_1\left(\frac{\sigma}{2\epsilon}t\right) - \frac{\sigma}{2\epsilon} I_0\left(\frac{\sigma}{2\epsilon}t\right) + \delta(t)\right] \tag{27}$$

The convolution integral can then be written

$$E_r(t,o) = -\sqrt{\frac{\mu}{\epsilon}}\left\{H_\phi(t) + \frac{\sigma}{2\epsilon}\int_0^t e^{-\frac{\sigma}{2\epsilon}(t-t')}\left[I_1\left(\frac{\sigma}{2\epsilon}(t-t')\right) - I_0\left(\frac{\sigma}{2\epsilon}(t-t')\right)\right] H_\phi(t')\,dt'\right\} \tag{28}$$

It is interesting to note that at early times, $E_r(t,o)$ is directly proportional to $H_\phi(t,o)$.

B. $H_\phi(t,z)$ as a Convolution Integral of $H_\phi(t,o)$.

From equation 11,

$$H_\phi(s,z) = H_\phi(s,0)\, e^{\gamma z}, \quad z < 0,$$

where

$$\gamma = \sqrt{s\mu(\sigma + s\epsilon)} \tag{29}$$

Thus, the inverse Laplace transform of $e^{\gamma z}$ is required. Again, from the Handbook of Chemistry and Physics (ibid.), p. 290, transform no. 93, one finds

$$\mathcal{L}^{-1}\left\{e^{-k\sqrt{s^2-a^2}} - e^{-ks}\right\} = \frac{ak}{\sqrt{t^2-k^2}}\, I_1\left(a\sqrt{t^2-k^2}\right) u(t-k),$$

where u is the unit step function. The inverse transform of interest can be written

$$\mathcal{L}^{-1}\left\{e^{\gamma z}\right\} = \mathcal{L}^{-1}\left\{\left[e^{-k\sqrt{(s+a)^2-a^2}} - e^{-k(s+a)}\right] + e^{k(s+a)}\right\},$$

where

$$k = -\frac{z}{c}, \quad a = \frac{\sigma}{2\epsilon} \quad \text{and} \quad c = \frac{1}{\sqrt{\mu\epsilon}}.$$

The inverse transform of $e^{\gamma z}$ is

$$\mathcal{L}^{-1}\left\{e^{\gamma z}\right\} = e^{-at}\left\{\frac{ak}{\sqrt{t^2-k^2}} I_1\left(a\sqrt{t^2-k^2}\right)u(t-k) + \delta(t-k)\right\}. \tag{30}$$

The appropriate convolution integral is then

$$H_\phi(t,z) = \left\{ e^{\frac{\sigma z}{2\epsilon c}} H_\phi(t+z/c, 0) - \frac{\sigma}{2\epsilon}\int_0^{t+z/c} e^{-\frac{\sigma}{2\epsilon}(t-t')} \frac{z/c}{\sqrt{(t-t')^2+(z/c)^2}} I_1\left[\frac{\sigma}{2\epsilon}\sqrt{(t-t')^2-(z/c)^2}\right] H_\phi(t',0)\,dt' \right\} u(t+z/c). \tag{31}$$

C. $E_r(t,z)$ as a Convolution Integral of $H_\phi(t)$.

From equation 16,

$$\mathcal{E}_r(s,z) = -\mathcal{H}_\phi(s)\,\eta(s)\,e^{\gamma z} \qquad . \tag{32}$$

Hence the kernel of the convolution integral is given by $-\mathcal{L}^{-1}\{\eta(s)\,e^{\gamma z}\}$. Following the approach of the preceding sections one can write

$$\eta(s)\,e^{\gamma z} = \sqrt{\frac{\mu}{\epsilon}}\,\frac{s}{\sqrt{(s+a^2)-a^2}}\,e^{-k\sqrt{(s+a)^2-a^2}}$$

$$= \sqrt{\frac{\mu}{\epsilon}}\left\{\frac{(s+a)}{\sqrt{(s+a)^2-a^2}}\,e^{-k\sqrt{(s+a)^2-a^2}} - \frac{a}{\sqrt{(s+a)^2-a^2}}\,e^{-k\sqrt{(s+a)^2-a^2}}\right\} \tag{33}$$

where a and k are defined in the preceding seqtion. From the Handbook of Chemistry and Physics (ibid.), p. 290, no. 90, one finds

$$\mathcal{L}^{-1}\left\{\frac{e^{-k\sqrt{s^2-a^2}}}{\sqrt{s^2-a^2}}\right\} = I_0\left(a\sqrt{t^2-k^2}\right)u(t-k)\ . \tag{34}$$

Equation (34) is the transform needed for the second term on the right of Equation (33). To find the inverse of the first term, use the relations

$$\mathcal{L}\{F'(t)\} = sf(s) - F(0)$$

$$\mathcal{L}^{-1}\{sf(s)\} = F'(t) + F(0)\delta(t)$$

(35)

If F(t) = $I_0(a\sqrt{t^2-k^2})u(t-k)$ Equation (35) becomes

$$\mathcal{L}^{-1}\left\{\frac{s\, e^{-k\sqrt{s^2-a^2}}}{\sqrt{s^2-a^2}}\right\} = \frac{at}{\sqrt{t^2-k^2}} I_0'(a\sqrt{t^2-k^2})u(t-k) + I_0(0)\delta(t-k)$$

$$= \frac{at}{\sqrt{t^2-k^2}} I_1(a\sqrt{t^2-k^2})u(t-k) + \delta(t-k).$$

(36)

Using Equations (34) and (36), the inverse transform of $\eta(s)\, e^{\gamma z}$ may be written

$$\mathcal{L}^{-1}\{\eta(s)e^{\gamma z}\} = \sqrt{\frac{\mu}{\epsilon}}\; e^{-at}\left\{\frac{at}{\sqrt{t^2-k^2}} I_1(a\sqrt{t^2-k^2})u(t-k) + \delta(t-k) - aI_0(a\sqrt{t^2-k^2})u(t-k)\right\}$$

(37)

Finally, the convolution integral is

$$E_r(t,z) = -\sqrt{\frac{\mu}{\epsilon}}\left\{ e^{\frac{\sigma z}{2\epsilon c}} H_\phi(t+z/c) + \frac{\sigma}{2\epsilon}\int_0^{t+z/c}\left[\frac{(t-t')}{\sqrt{(t-t')^2-(z/c)^2}}\, I_1\left(\frac{\sigma}{2\epsilon}\sqrt{(t-t')^2-(z/c)^2}\right) - I_0\left(\frac{\sigma}{2\epsilon}\sqrt{(t-t')^2-(z/c)^2}\right)\right] e^{-\frac{\sigma}{2\epsilon}(t-t')} H_\phi(t')\,dt'\right\} * u(t+z/c) \,, \tag{38}$$

where of course $z < 0$.

Even if an analytic form for $H_\phi(t)$ is not known, it requires only a simple machine computation to determine $E_r(t, z)$.