EMP Theoretical Notes

Note V

CLEARED
FOR PUBLIC RELEASE
AFRL/DEO-PA
27 AUG 04

Unsaturated Compton Current and Space-Charge Fields in Evacuated Cavities

Lt C. E. Baum, AFWL

20 January 1965

Abstract:

This note calculates the electric and magnetic fields produced in evacuated cavities by the Compton currents ejected from the cavity walls by the gamma radiation. The energies associated with these fields are also calculated. Three simple geometries are considered: parallel plate, cylindrical and spherical.

I. Introduction

Interaction of γ rays with materials produce Compton electrons which form a current (the Compton current) and in a vacuum next to matter from which this current can flow, due to the lack of neutralizing positive charges, a space charge (which the author chooses to call the Compton space charge). The purpose of this note is to calculate to first order the electric and magnetic fields, the potentials, and the energies associated with this space charge.

The general approach will be to choose cavities with simple geometries (parallel plate, cylindrical, and spherical) in which Laplace's equation has relatively simple forms and in which the boundary conditions can be easily obtained from symmetry considerations. For convenience the voltage from the space charge will be arbitrarily defined as zero on the conducting walls. The formulas developed will be given in two forms, in terms of the constants relating the Compton current and Compton space charge to the radiation intensity and in terms of very approximate numbers for these constants. Thus, as these parameters are empirically determined one can place them in the appropriate formulas.

Since the Compton electrons have a finite average energy (about ½ Mev) they can be affected by voltages approaching this energy. However, it will be assumed throughout this note that the voltages are small compared to the average Compton electron energy and thus have no significant effect on either the Compton current or the Compton space charge. In addition since the high energy Compton electrons produce low energy secondary electrons (less than about 50 e.v.) in the cavity walls, it will be assumed that the voltages are large compared with the secondary electron energy, so that these lower energy electrons never penetrate the cavity interior and thus do not contribute to the space charge. However, the range from about 50 volts to about .25 megavolts is still a large region of application for the formulas which will be developed.

AFRL/DE 04-407

It will be further assumed that the cavities of concern have the dimensions small enough so that the electron transit time across the cavities and the electrical transit time around the cavity walls are short compared to the time in which the radiation flux changes significantly. With these approximations the wave equation need not be solved but instead a solution can be obtained from Poisson's equation, i.e.,

$$\nabla^2 V = -\frac{\rho}{\epsilon_o} \tag{1}$$

where V is the spatially dependent voltage, ρ is the charge density, and ϵ_o is the permittivity of free space.

It is intended that this note will serve as a convenient compilation of the fields and energies associated with the Compton space charge in these simple geometries.

II. Compton Current and Space Charge

In the walls of the cavities to be considered there is an equilibrium relationship between the gamma radiation flux and the Compton electron flux. Since for the cases of interest one is considering undirectional gammas then the magnitude of the gamma current, $|\vec{\gamma}|$, is equal to the gamma flux, γ, There is then a proportionality constant, c_J, between the gamma current and the Compton current density, J_{c_o}, given by the ratio of the mean forward Compton electron range to the gamma mean free path so that

$$J_{c_o} = c_J \gamma \tag{2}$$

where the direction of the gammas is taken as positive. This constant can be taken in any units desired corresponding to the units of J_{c_o} and γ (for a given γ spectrum and material). In this note J_{c_o} will be taken in amps/(meter)^2 and γ will be taken in roentgens/sec (air equivalent dose). Then for fission gammas (and any low atomic number material) c_J is approximately

$$c_J \simeq -2 \times 10^{-8} \ \frac{\text{coulombs}}{\text{meter}^2\text{-roentgen}} \tag{3}$$

or in other units

$$c_J \simeq -10^{-17} \ \frac{\text{coulombs}}{\text{meter}^2} \left(\frac{\gamma\text{-Mev}}{\text{cm}^2}\right)^{-1} \tag{4}$$

the negative sign coming from the fact that the electrons have a negative charge. Neglecting the low energy secondary electrons produced by the Compton electrons this equilibrium Compton current will be the current in an evacuated cavity whose walls (at least the walls plus additional close by, low atomic number matter in the direction of the radiation source) are thicker than the range of the highest energy Compton electrons.

Next one is interested in the space charge associated with this Compton current. If $n(E,\theta)$ is the normalized distribution function of the Compton electrons where $n(E,\theta)dEd\theta$ represents the fraction of the Compton electrons between energies E and $E+dE$ and angles (from the direction of the γ rays) θ and $\theta+d\theta$, then the average value of the reprocal of the forward component of the Compton electron velocity, $\left\langle \frac{1}{v_c} \right\rangle$ or $\frac{1}{v_{c_0}}$, is given by

$$\frac{1}{v_{c_0}} = \left\langle \frac{1}{v_c} \right\rangle = \int_0^{E_{max}} \int_0^{\pi/2} \frac{n(E,\theta)}{v_c(E,\theta)} \, d\theta \, dE \qquad (5)$$

where E_{max} is the maximum Compton electron energy and $v_c(E,\theta)$ is the forward component of the velocity of the Compton electrons. For the case of voltages small compared to the energy (in e.v.) of the Compton electrons then the magnitude of the velocity can be calculated from the energy of the Compton electrons and then multiplied by cos θ to give the forward component. For non-relativistic electrons

$$v_c(E,\theta) = \sqrt{2 e/m}\,\sqrt{E}\,\cos\theta \qquad (6)$$

whre E is in e.V. For relativistic electons

$$v_c(E,\theta) = c\left(1 - \left(\frac{E}{m_0 c^2} + 1\right)^{-2}\right)^{1/2} \cos\theta \qquad (7)$$

where c is the speed of light and $m_0 c^2$ is the rest mass energy of the electron (.51 x 10^6 e.v.).

The Compton space charge desnity, ρ_{c_0} , is then just

$$\rho_{c_0} = J_{c_0} \left\langle \frac{1}{v_c} \right\rangle = \frac{J_{c_0}}{v_{c_0}} \qquad (8)$$

It is planned that in a future note v_{c_0} will be evaluated for realistic distribution functions. For the present this can be approximated by taking the speed of a .25 Mev electron and assuming that this "average Compton electron" is traveling in the forward direction. Thus

$$v_{c_0} \simeq .74c = 2.2 \times 10^8 \text{ meters/sec} \tag{9}$$

From equations (2) and (8) the Compton space charge density can be related to the gamma current by

$$\rho_{c_0} = \frac{J_{c_0}}{v_{c_0}} = \frac{c_J}{v_{c_0}} \gamma \qquad \frac{\text{coulombs}}{\text{meter}^3} \tag{10}$$

so that another constant, c_ρ, can be defined as

$$c_\rho = \frac{c_J}{v_{c_0}} \simeq -.9 \times 10^{-16} \frac{\text{coulombs}}{\text{meter}^3}\left(\frac{\text{roentgen}}{\text{sec}}\right)^{-1} \tag{11}$$

and thus

$$\rho_{c_0} = c_\rho \gamma \qquad \frac{\text{coulombs}}{\text{meter}^3} \tag{12}$$

As mentioned previously the purpose of defining these constants (c_J, v_{c_0}, c_ρ) is to allow their more accurate evaluation in the future at which time these values can be placed in the equations to be developed in this and other notes.

III. Parallel Plate Geometry

The simpliest geometry in which to make these field and energy calculations is parallel plate geometry as shown in figures 1a and 1b where z is taken as the coordinate perpendicular to the parallel plates which are spaced a distance, d, apart. It is assumed that the parallel plates extend for distances in all directions (before they are electrically joined) which are large compa red to d.

In this geometry there are two convenient directions for the γ and Compton currents:

(1) Compton current perpendicular to the plates.

(2) Compton current parallel to the plates.

In either case the electric field and voltage will have the same dependence because these are calculated from the space charge which will be the same in both cases whereas the magnetic field will be different because this quantity depends on the direction of the Compton current.

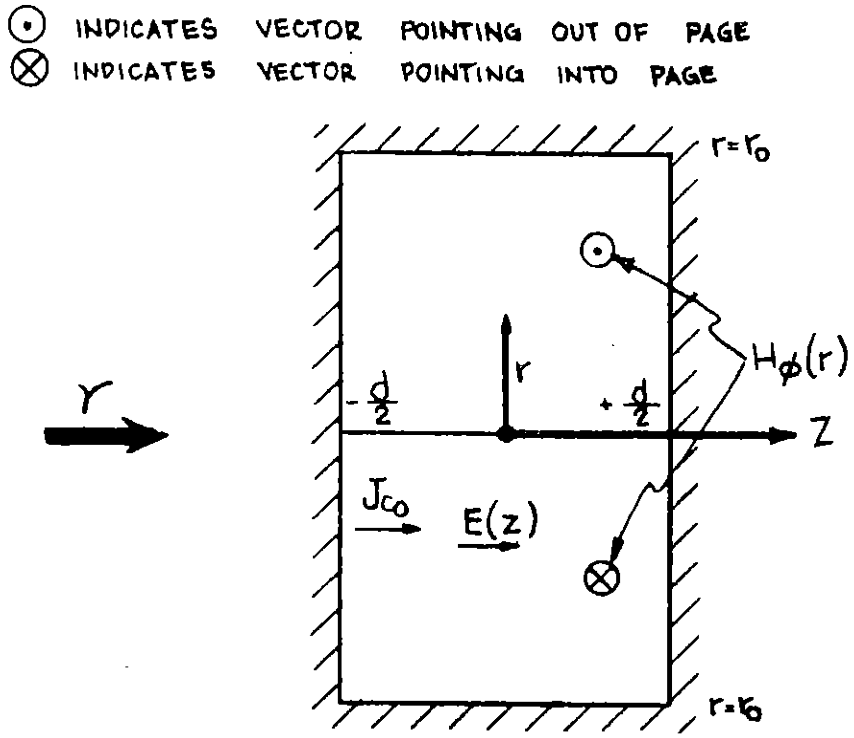

Fig. 1a PARALLEL PLATE GEOMETRY: COMPTON CURRENT PERPENDICULAR TO PARALLEL PLATES.

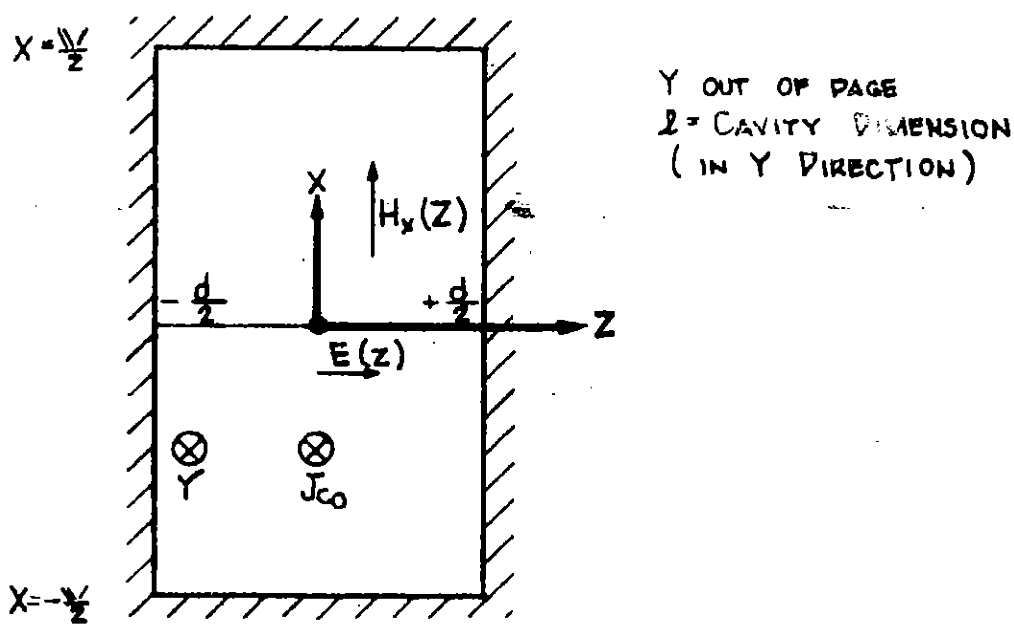

Fig. 1b. PARALLEL PLATE GEOMETRY: COMPTON CURRENT PARRALLEL TO PARALLEL PLATES.

A. Compton Current Perpendicular to Parallel Plates

In parallel plate geometry as defined in figure 1a which represents a right circular cylinder (so that the magnetic field can be later conveniently calculated) there is a constraint placed on the dimensions that the radius, r_0, be much greater than the height, d, i.e.

$$r_0 >> d \tag{13}$$

Then Poisson's equation reduces to

$$\frac{\partial^2 V}{\partial z^2} = -\frac{\rho_{c_0}}{\epsilon_0} \tag{14}$$

where the voltage distribution, V, is now a function of z.

Integrating one with respect to z one has the electric field, E (z),

$$E(z) = -\frac{\partial V}{\partial z} = \frac{\rho_{c_0}}{\epsilon_0}\int_0^z dz = \frac{\rho_{c_0}}{\epsilon_0} z \quad \frac{\text{volts}}{\text{meter}} \tag{15}$$

where the lower limit on the integral comes from the fact that by symmetry the electric field must be zero midway between the parallel plates. Thus

$$E(z) = \frac{c_\rho}{\epsilon_0}\gamma z \qquad \frac{\text{volts}}{\text{meter}} \tag{16}$$

or

$$E(z) \simeq -10^{-5}\gamma z \qquad \frac{\text{volts}}{\text{meter}} \tag{17}$$

where γ is in roentgens/sec and all other quantities are in m.k.s. units. (This procedure will be followed throughout this note.)

The maximum magnitude of the electric field, $|E|_{max}$, occurs at either wall and is

$$|E|_{max} = \frac{|\rho_{c_0}|}{2\epsilon_0} d = \frac{|c_\rho|}{2\epsilon_0}\gamma d \qquad \frac{\text{volts}}{\text{meter}} \tag{18}$$

or

$$|E|_{max} \simeq .5 \times 10^{-5} \gamma d \qquad \frac{\text{volts}}{\text{meter}} \tag{19}$$

Integrating eqn. (15) to obtain the voltage distribution one has

$$V(z) = -\frac{\rho_{c_0}}{\epsilon_0}\int_{-d/2}^{z} z\,dz = \frac{\rho_{c_0}}{8\epsilon_0}(d^2 - 4z^2) \text{ volts} \tag{20}$$

where the lower limit on this integral comes from the arbitrary definition that V be zero on each cavity wall (taken as the wall at $z = -\frac{d}{2}$ in this case.) Thus

$$V(z) = \frac{c_\rho}{8\epsilon_0}\gamma(d^2 - 4z^2) \quad \text{volts} \tag{21}$$

or

$$V(z) \simeq -1.25\times10^{-6}\gamma(d^2 - 4z^2) \text{ volts} \tag{22}$$

The maximum potential difference, $|V|_{max}$, exists between the halfway point between the plates (z=0) and the walls ($z= \pm d/2$) and so

$$|V|_{max} = \frac{|\rho_{c_0}|}{8\epsilon_0}d^2 = \frac{|c_\rho|}{8\epsilon_0}\gamma d^2 \quad \text{volts} \tag{23}$$

or

$$|V|_{max} \simeq 1.25\times10^{-6}\gamma d^2 \qquad \text{volts} \tag{24}$$

It must be noted at this point that the electric field and voltage calculations of eqns. (15) through (24) are not valid near the circular wall ($r = r_0$). However, this is the whole point of the restriction in eqn. (13).

Turning now the the magnetic field in this geometry, this can be easily calculated from Ampere's law using symmetry considerations. Thus the line integral of the magnetic field at a radius, r, is equal to the Compton current flowing through this area. Thus

$$2\pi r H_\phi(r) = \pi r^2 J_{c_0} \tag{25}$$

and therefore

$$H_\phi(r) = \frac{J_{c_0}}{2} r = \frac{c_J \gamma}{2} r \quad \frac{\text{amps}}{\text{meter}} \tag{26}$$

or

$$H_\phi(r) \simeq -10^{-8} \gamma r \quad \frac{\text{amps}}{\text{meter}} \tag{27}$$

In terms of $B_\phi(r)$ this is

$$B_\phi(r) = \frac{\mu_0 J_{c_0}}{2} r = \frac{\mu_0 c_J \gamma}{2} r \quad \frac{\text{webers}}{\text{meter}^2} \tag{28}$$

or

$$B_\phi(r) \simeq -1.3 \times 10^{-14} \gamma r \quad \frac{\text{webers}}{\text{meter}^2} \tag{29}$$

The magnetic field has a maximum value at $r = r_0$ given by

$$|H_\phi|_{max} = \frac{|J_{c_0}|}{2} r_0 = \frac{|c_J| \gamma}{2} r_0 \quad \frac{\text{amps}}{\text{meter}} \tag{30}$$

or

$$|H_\phi|_{max} \simeq 10^{-8} \gamma r_0 \quad \frac{\text{amps}}{\text{meter}} \tag{31}$$

and

$$|B_\phi|_{max} = \frac{\mu_0 |J_{c_0}|}{2} r_0 = \frac{\mu_0 |c_J| \gamma}{2} r_0 \quad \frac{\text{webers}}{\text{meter}^2} \tag{32}$$

or

$$|B_\phi|_{max} \simeq 1.3 \times 10^{-14} \gamma r_0 \quad \frac{\text{webers}}{\text{meter}^2} \tag{33}$$

From the calculations of the field distributions one can calculate the energy stored in the electric and magnetic field. The total energy in the electric field, U_e, is

$$U_e = \pi r_0^2 \int_{-\frac{d}{2}}^{d/2} \frac{\epsilon_0 E^2(z)}{2} \text{ joules} \tag{34}$$

or substituting from eqn. (15)

$$U_e = \frac{\rho_{c_0}^2}{2\epsilon_0} \pi r_0^2 \int_{-d/2}^{d/2} z^2 dz = \frac{\rho_{c_0}^2}{2\epsilon_0} \pi r_0^2 \left. \frac{z^3}{3} \right|_{-d/2}^{d/2} \text{ joules} \tag{35}$$

or

$$U_e = \frac{\rho_{c_0}^2}{24\epsilon_0} \pi r_0^2 d^3 = \frac{c_\rho^2 \pi}{24\epsilon_0} \gamma^2 r_0^2 d^3 \text{ joules} \tag{36}$$

or

$$U_e \simeq 1.2 \times 10^{-22} \gamma^2 r_0^2 d^3 \text{ joules} \tag{37}$$

The average electric energy per unit volume, $\overline{u_e}$, is

$$\overline{u_e} = \frac{\rho_{c_0}^2}{24\epsilon_0} d^2 = \frac{c_\rho^2}{24\epsilon_0} \gamma^2 d^2 \quad \frac{\text{joules}}{\text{meter}^3} \tag{38}$$

or

$$\overline{u_e} \simeq .38 \times 10^{-22} \gamma^2 d^2 \quad \frac{\text{joules}}{\text{meter}^3} \tag{39}$$

Likewise the total energy in the magnetic field, U_m, is

$$U_m = d \int_0^{r_0} \frac{\mu_0 H_\phi^2(r)}{2} (2\pi r) dr \quad \text{joules} \tag{40}$$

or substituting from eqn. (30)

$$U_m = \frac{\pi \mu_0 J_{c_0}^2}{4} d \int_0^{r_0} r^3 dr = \frac{\pi \mu_0 J_{c_0}^2 d}{4} \left. \frac{r^4}{4} \right|_0^{r_0} \tag{41}$$

or

$$U_m = \frac{\pi \mu_0 J_{c_0}^2}{16} r_0^4 d = \frac{\pi \mu_0 c_I^2}{16} \gamma^2 r_0^4 d \text{ joules} \tag{42}$$

or

$$U_m \simeq 10^{-22}\gamma^2 r_o^4 d \quad \text{joules} \tag{43}$$

The average magnetic energy per unit volume, $\bar{u}_m$, is

$$\bar{u}_m = \frac{\mu_o J_{c_o}^2}{16} r_o^2 = \frac{\mu_o c_J^2}{16}\gamma^2 r_o^2 \quad \frac{\text{joules}}{\text{meter}^3} \tag{44}$$

or

$$\bar{u}_m = .31\times 10^{-22}\gamma^2 r_o^2 \quad \frac{\text{joules}}{\text{meter}^3} \tag{45}$$

B. <u>Compton Current Parallel to Parallel Plates</u>

If the Compton current is now assumed to be parallel to the plates as in figure 1b it is more convenient (for the magnetic field calculations) to use a Cartesian (x, y, z) coordinate system rather than a cylindrical (r, ϕ, z) coordinate system. In this case it is also assumed that the cavity is rectangular with the dimensions w in the x direction and ℓ in the y direction and with the constraints

$$\begin{aligned} w &>> d \\ \ell &>> d \end{aligned} \tag{46}$$

As argued previously the calculations for the electric field and voltage will be the same as the previous case as are contained in eqns. (14) through (24). However, the magnetic field will have to be calculated by considering a unit length in the x direction and applying Ampere's Law, recognizing the symmetry of the magnetic field about z=o. Thus

$$-2H_x(z) = 2z\, J_{c_o} \tag{47}$$

and therefore

$$H_x(z) = -z J_{c_o} = -c_J \gamma z \quad \frac{\text{amps}}{\text{meter}} \tag{48}$$

or

$$H_x(z) \simeq 2\times 10^{-8}\gamma z \quad \frac{\text{amps}}{\text{meter}} \tag{49}$$

In terms of $B_x(z)$ this is

$$B_x(z) = -\mu_o z J_{c_o} = -\mu_o c_J \gamma z \quad \frac{\text{webers}}{\text{meter}^2} \tag{50}$$

or

$$B_x(z) \simeq 2.5 \times 10^{-14} \gamma z \quad \frac{\text{webers}}{\text{meter}^2} \tag{51}$$

The magnitude of the magnetic field has a maximum value at $z = \pm d/2$ given by

$$|H_x|_{max} = \frac{d}{2}|J_{c_0}| = \frac{|c_J|}{2}\gamma d \quad \frac{\text{amps}}{\text{meter}} \tag{52}$$

or

$$|H_x|_{max} \simeq 10^{-8}\gamma d \quad \frac{\text{amps}}{\text{meter}} \tag{53}$$

and

$$|B_x|_{max} = \frac{\mu_0 d}{2}|J_{c_0}| = \frac{\mu_0 |c_J|}{2}\gamma d \quad \frac{\text{webers}}{\text{meter}^2} \tag{54}$$

or

$$|B_x|_{max} \simeq 1.3 \times 10^{-14}\gamma d \quad \frac{\text{webers}}{\text{meter}^2} \tag{55}$$

These magnetic field calculations do not hold near $x = \pm \frac{w}{2}$.

Since the electric field is the same as the previous case the energy in the electric field can be calculated from the formulas of eqns. (34) through (39) by replacing the plate area, πr_0^2, by $w\ell$. Thus

$$U_e = \frac{\rho_{c_0}^2}{24\epsilon_0} w\ell d^3 = \frac{c_\rho^2}{24\epsilon_0}\gamma^2 w\ell d^3 \quad \text{joules} \tag{56}$$

or

$$U_e \simeq .38 \times 10^{-22}\gamma^2 w\ell d^3 \quad \text{joules} \tag{57}$$

The average electric energy per unit volume is the same as given in eqns. (38) and (39).

The total energy in the magnetic field is

$$U_m = w\ell \int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{\mu_0 H_x^2(z)}{2}\, dz \quad \text{joules} \tag{58}$$

or substituting from eqn. (48)

$$U_m = \frac{\mu_0 J_{c_0}^2}{2} w\ell \int_{-d/2}^{d/2} z^2 dz = \frac{\mu_0 J_{c_0}^2 w\ell}{2} \left. \frac{z^3}{3} \right|_{-d/2}^{d/2} \text{ joules} \tag{59}$$

or

$$U_m = \frac{\mu_0 J_{c_0}^2}{24} w\ell d^3 = \frac{\mu_0 c_J^2}{24} \gamma^2 w\ell d^3 \text{ joules} \tag{60}$$

or

$$U_m \simeq .21 \times 10^{-22} \gamma^2 w\ell d^3 \text{ joules} \tag{61}$$

The average magnetic energy per unit volume is

$$\bar{u}_m = \frac{\mu_0 J_{c_0}^2 d^2}{24} = \frac{\mu_0 c_J^2 \gamma^2 d^2}{24} \quad \frac{\text{joules}}{\text{meter}^3} \tag{62}$$

or

$$\bar{u}_m \simeq .21 \times 10^{-22} \gamma^2 d^2 \quad \frac{\text{joules}}{\text{meter}^3} \tag{63}$$

IV. Cylindrical Geometry

Another convenient geometry for these calculations, cylindrical geometry is shown in figures 2a and 2b. It is assumed that the length of the cylinder, ℓ, is much greater than its radius, r_0, i.e.

$$\ell >> r_0 \tag{64}$$

Again in this geometry there are two convenient directions for the γ and Compton currents:

(1) Compton current perpendicular to cylinder axis.

(2) Compton current parallel to cylinder axis

As before the electric field and voltage will not be affected by the change in orientation while the magnetic field will be significantly affected.

Z-AXIS NORMAL TO PAGE
ℓ = LENGTH IN Z DIRECTION

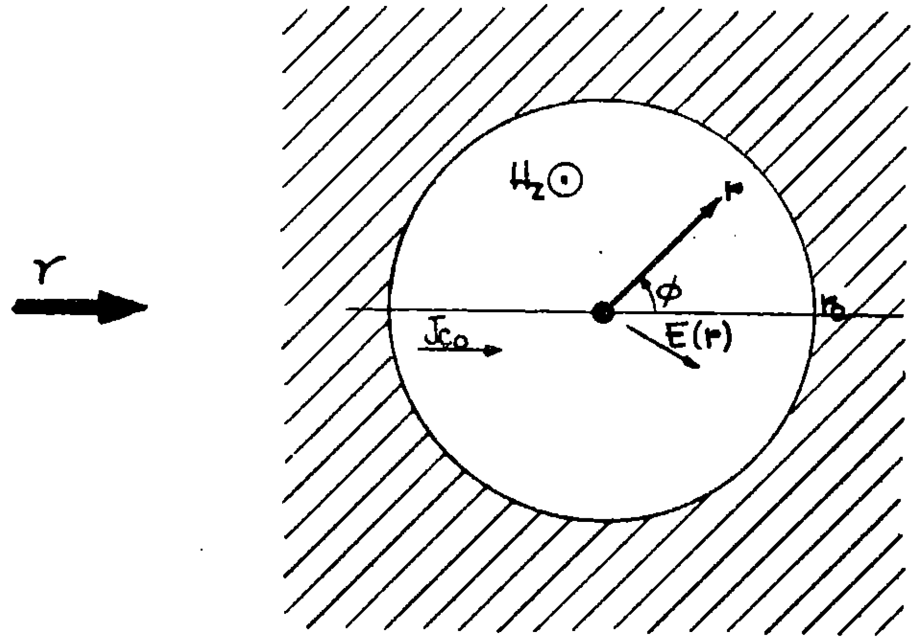

FIG. 2a. CYLINDRICAL GEOMETRY: COMPTON CURRENT PERPENDICULAR TO CYLINDER AXIS.

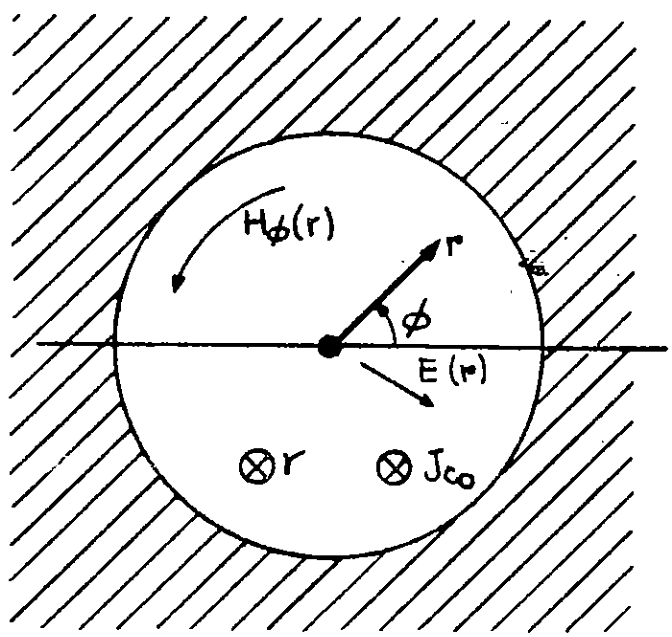

FIG. 2b. CYLINDRICAL GEOMETRY: COMPTON CURRENT PARALLEL TO CYLINDER AXIS.

A. Compton Current Perpendicular to Axis of Cylinder

In cylindrical geometry as defined in figure 2a where the dimensions are restricted as in eqn. (64) and with a uniform charge distribution (thus no ϕ dependence) Poisson's equation reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = -\frac{\rho_{c_o}}{\epsilon_o} \qquad (65)$$

where the voltage distribution, V, is now a function of r.

Multiplying by r and integrating once with respect to r one has

$$r\frac{\partial V}{\partial r} = -\frac{\rho_{c_o}}{\epsilon_o}\int_0^r r\,dr = -\frac{\rho_{c_o}}{2\epsilon_o}r^2 \qquad (66)$$

where the lower limit on the integral comes from the fact that $\partial V/\partial r$ must be zero at r=o. Thus, the electric field, $E(r)$, must be

$$E(r) = -\frac{\partial V}{\partial r} = \frac{\rho_{c_o}}{2\epsilon_o}r = \frac{c_\rho}{2\epsilon_o}\gamma r \quad \frac{\text{volts}}{\text{meter}} \qquad (67)$$

or

$$E(r) \simeq -.5\times 10^{-5}\gamma r \quad \frac{\text{volts}}{\text{meter}} \qquad (68)$$

The maximum magnitude of the electric field occurs at $r=r_o$ and is

$$|E|_{max} = \frac{|\rho_{c_o}|}{2\epsilon_o}r_o = \frac{|c_\rho|}{2\epsilon_o}\gamma r_o \quad \frac{\text{volts}}{\text{meter}} \qquad (69)$$

or

$$|E|_{max} \simeq .5\times 10^{-5}\gamma r_o \quad \frac{\text{volts}}{\text{meter}} \qquad (70)$$

Integrating eqn. (67) to obtain the voltage distribution one has

$$V(r) = -\frac{\rho_{c_o}}{2\epsilon_o}\int_{r_o}^{r} r\,dr = \frac{\rho_{c_o}}{4\epsilon_o}(r_o^2 - r^2) \quad \text{volts} \qquad (71)$$

Thus

$$V(r) = \frac{c_e}{4\epsilon_o}\gamma(r_o^2 - r^2) \quad \text{volts} \tag{72}$$

or

$$V(r) \simeq -.25 \times 10^{-5}\gamma(r_o^2 - r^2) \quad \text{volts} \tag{73}$$

The maximum potential difference exists between r=o and $r=r_o$ and so

$$|V|_{max} = \frac{|\rho_{co}|}{4\epsilon_o} r_o^2 = \frac{|c_e|}{4\epsilon_o}\gamma r_o^2 \quad \text{volts} \tag{74}$$

or

$$|V|_{max} \simeq .25 \times 10^{-5}\gamma r_o^2 \quad \text{volts} \tag{75}$$

These electric field and voltage calculations are not valid near the ends of the cylinder (within a distance of the order of r_o or so).

Since in this geometry it is convenient to define the direction of the Compton current as being in the r direction with $\phi = 0$, the magnetic field can be calculated from symmetry about a plane parallel to the Compton current which contains the axis of the cylinder (the plane defined by $\phi = 0$ and $\phi = \pi$). Thus the magnetic field will be in the z direction and can be calculated using Ampere's Law for a unit length in this direction. Thus

$$2H_z = J_{c_o}(2r\sin\phi) \tag{76}$$

and therefore

$$H_z = J_{c_o} r\sin\phi = c_J \gamma r\sin\phi \quad \frac{\text{amps}}{\text{meter}} \tag{77}$$

or

$$H_z \simeq -2 \times 10^{-8}\gamma r\sin\phi \quad \frac{\text{amps}}{\text{meter}} \tag{78}$$

In terms of B_z this is

$$B_z = \mu_0 J_{c_0} r \sin\phi = \mu_0 c_J \gamma r \sin\phi \quad \frac{\text{webers}}{\text{meter}^2} \tag{79}$$

or

$$B_z \simeq -2.5 \times 10^{-14} \gamma r \sin\phi \quad \frac{\text{webers}}{\text{meter}^2} \tag{80}$$

The magnitude of the magnetic field has a maximum value at $r = r_0$ and $\phi = \pm\frac{\pi}{2}$ given by

$$|H_z|_{max} = |J_{c_0}| r_0 = |c_J| \gamma r_0 \quad \frac{\text{amps}}{\text{meter}} \tag{81}$$

or

$$|H_z|_{max} \simeq 2 \times 10^{-8} \gamma r_0 \quad \frac{\text{amps}}{\text{meter}} \tag{82}$$

and

$$|B_z|_{max} = \mu_0 |J_{c_0}| r_0 = \mu_0 |c_J| \gamma r_0 \quad \frac{\text{webers}}{\text{meter}^2} \tag{83}$$

or

$$|B_z|_{max} \simeq 2.5 \times 10^{-14} \gamma r_0 \quad \frac{\text{webers}}{\text{meter}^2} \tag{84}$$

Again these magnetic field calculations do not apply near the ends of the cylinder.

The energy in the electric field in this geometry is

$$U_e = \ell \int_0^{r_0} \frac{\epsilon_0 E(r)^2}{2} 2\pi r \, dr \quad \text{joules} \tag{85}$$

or substituting from eqn. (67)

$$U_e = \frac{\pi \rho_{c_0}^2 \ell}{4 \epsilon_0} \int_0^{r_0} r^3 dr = \frac{\pi \rho_{c_0}^2}{16 \epsilon_0} r_0^4 \ell \quad \text{joules} \tag{86}$$

or

$$U_e = \frac{\pi c_\rho^2 \gamma^2}{16\epsilon_0} r_0^4 \ell \qquad \text{joules} \tag{87}$$

or

$$U_e \simeq 1.8 \times 10^{-22} \gamma^2 r_0^4 \ell \quad \text{joules} \tag{88}$$

The average electric energy per unit volume is

$$\overline{u}_e = \frac{\rho_{c_0}^2}{16\epsilon_0} r_0^2 = \frac{c_\rho^2 \gamma^2 r_0^2}{16\epsilon_0} \qquad \frac{\text{joules}}{\text{meter}^3} \tag{89}$$

or

$$\overline{u}_e \simeq .57 \times 10^{-22} \gamma^2 r_0^2 \tag{90}$$

The energy in the magnetic field in this geometry is

$$U_m = \ell \int_0^{2\pi}\int_0^{r_0} \frac{\mu_0 H_z^2}{2} r\,dr\,d\phi \qquad \text{joules} \tag{91}$$

or substituting from eqn. (77)

$$U_m = \frac{\mu_0 J_{c_0}^2 \ell}{2} \int_0^{2\pi}\int_0^{r_0} r^3 \sin^2\phi\,dr\,d\phi \quad \text{joules} \tag{92}$$

or

$$U_m = \frac{\mu_0 J_{c_0}^2 r_0^4 \ell}{8} \int_0^{2\pi} \sin^2\phi\,d\phi \quad \text{joules} \tag{93}$$

or

$$U_m = \frac{\pi \mu_0 J_{c_0}^2}{8} r_0^4 \ell = \frac{\pi \mu_0 c_J^2 \gamma^2}{8} r_0^4 \ell \quad \text{joules} \tag{94}$$

or

$$U_m \simeq 2 \times 10^{-22} \gamma^2 r_0^4 \ell \quad \text{joules} \tag{95}$$

The average magnetic energy per unit volume is

$$\overline{u}_m = \frac{\mu_0 J_{c_0}^2 r_0^2}{8} = \frac{\mu_0 c_J^2 \gamma^2 r_0^2}{8} \qquad \frac{\text{joules}}{\text{meter}^3} \tag{96}$$

or

$$U_m \simeq .63 \times 10^{-22} \gamma^2 r_o^2 \quad \frac{\text{joules}}{\text{meter}^3} \tag{97}$$

B. Compton Current Parallel to Axis of Cylinder

As in the case of parallel plate geometry, any shift in direction of the Compton current has no effect on the electric field and voltage distribution so that equations (66) through (75) still hold. In addition, since the geometry has not been altered (as was the case in the parallel plate geometry calculations) the equations for the energy in the electric field (eqns. (85) through (90)) still hold. However, due to the change in direction of the Compton current as shown in figure 2b where this current is now parallel to the axis of the cylinder the magnetic field and magnetic energy calculations will change considerably.

Referring to figure 1a one can notice that the magnetic field calculations are the same as in figure 2b if d is replaced by ℓ, since the magnetic field calculations in this case did not depend at all on the relative size of the dimensions. The equations for the magnetic field do not contain the cylinder length as a dimension, so therefore eqns. (25) through (33) are the solutions for the magnetic field in this geometry.

In the corresponding equations for the energy in the magnetic field (eqns. (40) through (43)) the length of the cylinder is now changed to ℓ so that

$$U_m = \frac{\mu_o \pi J_{c_o}^2}{16} r_o^4 \ell = \frac{\mu_o \pi C_J^2 \gamma^2}{16} r_o^4 \ell \text{ joules} \tag{98}$$

or

$$U_m \simeq 10^{-22} \gamma^2 r_o^4 \ell \quad \text{joules} \tag{99}$$

The average magnetic energy per unit volume is the same as eqns. (44) and (45).

V. Spherical Geometry

Finally these fields and energies can be calculated for spherical geometry as shown in figure 3. In this case there are no approximations to be made regarding the relative dimensions as there is only one dimension, the radius, r_o. Also there is only one set of calculations for a given orientation of Compton current because one direction looks exactly the same as any other direction.

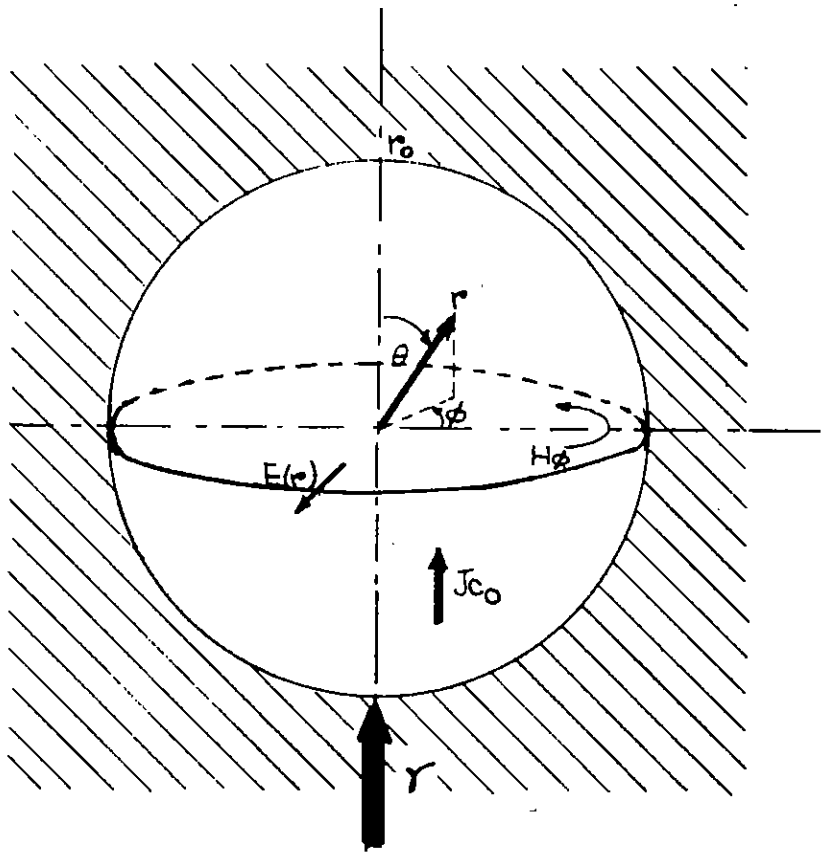

FIG. 3. SPHERICAL GEOMETRY

Given the orientation of the γ and Compton currents a spherical coordinate system (r, θ, ϕ) can be defined as in figure 3. Due to the uniform Compton space charge density and the spherical boundary conditions the electric field and voltage now only depend on r and Poisson's equation reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = -\frac{\rho_{c_o}}{\epsilon_o} \tag{100}$$

Multiplying by r^2 and integrating once with respect to r one has

$$r^2\frac{\partial V}{\partial r} = -\frac{\rho_{c_o}}{\epsilon_o}\int_0^r r^2 dr = -\frac{\rho_{c_o}}{3\epsilon_o} r^3 \tag{101}$$

where again the lower limit on the integral comes from the fact that $\frac{\partial V}{\partial r}$ must be zero at r=o. The electric field is then

$$E(r) = -\frac{\partial V}{\partial r} = \frac{\rho_{c_o}}{3\epsilon_o} r = \frac{c_\rho}{3\epsilon_o}\gamma r \quad \frac{\text{volts}}{\text{meter}} \tag{102}$$

or

$$E(r) \simeq -.34\times 10^{-5}\gamma r \quad \frac{\text{volts}}{\text{meter}} \tag{103}$$

The maximum magnitude of the electric field is at $r=r_o$ and is

$$|E|_{max} = \frac{|\rho_{c_o}|}{3\epsilon_o} r_o = \frac{|c_\rho|}{3\epsilon_o}\gamma r_o \quad \frac{\text{volts}}{\text{meter}} \tag{104}$$

or

$$|E|_{max} \simeq .34\times 10^{-5}\gamma r_o \quad \frac{\text{volts}}{\text{meter}} \tag{105}$$

Integrating eqn. (102) one has the voltage distribution as

$$V(r) = -\frac{\rho_{c_o}}{\epsilon_o}\int_{r_o}^r r\,dr = \frac{\rho_{c_o}}{6\epsilon_o}(r_o^2 - r^2) \text{ volts} \tag{106}$$

Thus

$$V(r) = \frac{c_\rho}{6\epsilon_o}\gamma(r_o^2 - r^2) \quad \text{volts} \tag{107}$$

or

$$V(r) \simeq -.17 \times 10^{-5} \gamma (r_o^2 - r^2) \quad volts \tag{108}$$

The maximum voltage difference (between r=o and $r=r_o$) is

$$|V|_{max} = \frac{|\rho_{c_o}|}{6\epsilon_o} r_o^2 = \frac{|c_\rho|}{6\epsilon_o} \gamma r_o^2 \quad volts \tag{109}$$

or

$$|V|_{max} \simeq .17 \times 10^{-5} \gamma r_o^2 \quad volts \tag{110}$$

These electric field and voltage calculations are valid throughout the spherical volume. There are no "end" or "edge" effects caused by the geometry.

Since the Compton current is parallel to the line defined by $\theta = 0, \pi$ there will only be a ϕ component of the magnetic field which can be calculated from symmetry considerations using Ampere's Law. Taking the line integral of the magnetic field on a circle of radius $r \sin\theta$ around the $\theta = 0, \pi$ line one has the Compton current flowing through this area. Thus

$$2\pi (r \sin\theta) H_\phi = \pi (r \sin\theta)^2 J_{c_o} \tag{111}$$

and therefore

$$H_\phi = \frac{J_{c_o}}{2} r \sin\theta = \frac{c_J}{2} \gamma r \sin\theta \quad \frac{amps}{meter} \tag{112}$$

or

$$H_\phi \simeq -10^{-8} \gamma r \sin\theta \quad \frac{amps}{meter} \tag{113}$$

In terms of B_ϕ this is

$$B_\phi = \frac{\mu_o J_{c_o}}{2} r \sin\theta = \frac{\mu_o c_J}{2} \gamma r \sin\theta \quad \frac{webers}{meter^2} \tag{114}$$

or

$$B_\phi \simeq -1.3 \times 10^{-14} \gamma r \sin\theta \quad \frac{webers}{meter^2} \tag{115}$$

The magnetic field has a maximum value at $r=r_o$ and $\theta = \pi/2$ given by

$$|H_\phi|_{max} = \frac{|J_{c_o}|}{2} r_o = \frac{|c_J|}{2} \gamma r_o \qquad \frac{\text{amps}}{\text{meter}} \tag{116}$$

or

$$|H_\phi|_{max} \simeq 10^{-8} \gamma r_o \qquad \frac{\text{amps}}{\text{meter}} \tag{117}$$

and

$$|B_\phi|_{max} = \frac{\mu_o |J_{c_o}|}{2} r_o = \frac{\mu_o |c_J|}{2} \gamma r_o \qquad \frac{\text{webers}}{\text{meter}^2} \tag{118}$$

or

$$|B_\phi|_{max} \simeq 1.3 \times 10^{-14} \gamma r_o \qquad \frac{\text{webers}}{\text{meter}^2} \tag{119}$$

The energy in the electric field is

$$U_e = \int_0^{r_o} \frac{\epsilon_o E^2(r)}{2} (4\pi r^2)\, dr \qquad \text{joules} \tag{120}$$

or substituting from eqn. (102)

$$U_e = \frac{2\pi \rho_{c_o}^2}{9 \epsilon_o} \int_0^{r_o} r^4 dr = \frac{2\pi \rho_{c_o}^2}{45 \epsilon_o} r_o^5 \quad \text{joules} \tag{121}$$

or

$$U_e = \frac{2\pi c_\rho^2}{45 \epsilon_o} \gamma^2 r_o^5 \qquad \text{joules} \tag{122}$$

or

$$U_e \simeq 1.3 \times 10^{-22} \gamma^2 r_o^5 \quad \text{joules} \tag{123}$$

The average electric energy per unit volume is

$$\bar{u}_e = \frac{\rho_{c_0}^2}{30\,\epsilon_0} r_0^2 = \frac{c_\rho^2}{30\,\epsilon_0}\gamma^2 r_0^2 \quad \frac{\text{joules}}{\text{meter}^3} \tag{124}$$

or

$$\bar{u}_e \simeq .3 \times 10^{-22}\gamma^2 r_0^2 \quad \frac{\text{joules}}{\text{meter}^3} \tag{125}$$

The energy in the magnetic field is

$$U_m = \int_0^{\pi}\int_0^{2\pi}\int_0^{r_0} \frac{\mu_0 H_\phi^2}{2} r^2 \sin\theta \, dr\, d\phi\, d\theta \quad \text{joules} \tag{126}$$

or substituting from eqn. (112)

$$U_m = \frac{\mu_0 J_{c_0}^2}{8}\int_0^{\pi}\int_0^{2\pi}\int_0^{r_0} r^4 \sin^3\theta \, dr\, d\phi\, d\theta \tag{127}$$

$$U_m = \frac{\mu_0 J_{c_0}^2}{40} r_0^5 \int_0^{\pi}\int_0^{2\pi} \sin^3\theta \, d\phi\, d\theta \tag{128}$$

$$U_m = \frac{\pi \mu_0 J_{c_0}^2}{20} r_0^5 \int_0^{\pi} \sin^3\theta \, d\theta \tag{129}$$

$$U_m = \frac{\pi \mu_0 J_{c_0}^2}{20} r_0^5 \left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\pi} \tag{130}$$

and therefore

$$U_m = \frac{\pi \mu_0 J_{c_0}^2}{15} r_0^5 = \frac{\pi \mu_0 c_J^2}{15}\gamma^2 r_0^5 \quad \text{joules} \tag{131}$$

or

$$U_m \simeq 10^{-22}\gamma^2 r_0^5 \tag{132}$$

The average magnetic energy per unit volume is

$$\overline{U}_m = \frac{\mu_0 J_{c_0}^2}{20} r_0^2 = \frac{\mu_0 C_J^2}{20} \gamma^2 r_0^2 \ \frac{\text{joules}}{\text{meter}^3} \tag{133}$$

or

$$\overline{U}_m \simeq .25 \times 10^{-22} \gamma^2 r_0^2 \ \frac{\text{joules}}{\text{meter}^3} \tag{134}$$

As mentioned previously, in spherical geometry there is only one direction of the Compton current for purposes of calculation. Thus, the calculations for these simple geometries are finished.

VI. Summary of the General Characteristics of the Fields and Energies

Comparing the results of the calculations for the fields and energies in these simple geometries one can note some qualitative characteristics of the fields and energies which are generally applicable to maximizing or minimizing the quantities.

The electric field is dependent only on the magnitude of the γ or Compton current, not the direction. This field is proportional to the smallest dimension of the cavity multiplied by the gamma flux. The average electric energy density is proportional to the square of the electric field and thus to the square of the smallest cavity dimension times the square of the gamma flux .

The maximum electric fields and voltages (using the approximate value of C_ρ) can be summarized as follows:

(1) parallel plate geometry

electric field

$$|E|_{max} \simeq .5 \times 10^{-5} \gamma d \ \frac{\text{volts}}{\text{meter}} \tag{19}$$

voltage

$$|V|_{max} \simeq 1.25 \times 10^{-6} \gamma d^2 \ \text{volts} \tag{24}$$

(2) cylindrical geometry

electric field

$$|E|_{max} \simeq .5 \times 10^{-5} \gamma r_0 \ \frac{\text{volts}}{\text{meter}} \tag{70}$$

voltage

$$|V|_{max} \simeq .25 \times 10^{-5} \gamma r_0^2 \ \text{volts} \tag{75}$$

(3) spherical geometry

electric field

$$|E|_{max} \simeq .34 \times 10^{-5} \gamma r_0 \quad \frac{volts}{meter} \qquad (105)$$

voltage

$$|V|_{max} \simeq .17 \times 10^{-5} \gamma r_0^2 \quad volts \qquad (110)$$

When comparing these electric fields and voltages one should remember that in parallel plate geometry d represents the plate spacing and that in cylindrical and spherical geometry r_0 represents the cavity radius.

The magnetic field is also dependent on the direction of the γ or Compton current. This field is proportional to the smallest dimension of the cavity which is perpendicular to the γ or Compton current (i.e. the dimension parallel to the current does not count). The average magnetic energy density is, of course, proportional to the square of this field. The maximum magnetic fields (using the approximate value of c_J) can be summarized as follows:

(1) parallel plate geometry

Compton current perpendicular to plates

$$|B_\phi|_{max} \simeq 1.3 \times 10^{-14} \gamma r_0 \quad \frac{webers}{meter^2} \qquad (33)$$

Compton current parallel to plates

$$|B_x|_{max} \simeq 1.3 \times 10^{-14} \gamma d \quad \frac{webers}{meter^2} \qquad (55)$$

(2) cylindrical geometry

Compton current perpendicular to cylinder axis

$$|B_z|_{max} \simeq 2.5 \times 10^{-14} \gamma r_0 \quad \frac{webers}{meter^2} \qquad (84)$$

Compton current parallel to cylinder axis

$$|B_\phi|_{max} \simeq 1.3 \times 10^{-14} \gamma r_0 \quad \frac{webers}{meter^2} \qquad (33)$$

(3) spherical geometry

$$|B_\phi|_{max} \simeq 1.3 \times 10^{-14} \gamma r_0 \quad \frac{webers}{meter^2} \qquad (119)$$

When comparing these magnetic fields one should remember that the r_o used in parallel plate geometry (first equation of this group) is one half the large dimension of the cavity while in cylindrical geometry it is one half the small dimension. In spherical geometry, of course, it is one half the only dimension of the cavity.

In general the total energy contained in each field component is proportional to the fifth power of the dimensions, i.e., the volume times the square of the dimension applicable to either the electric or magnetic field as defined above. Thus, there are two general considerations in maximizing or minimizing the total energy of any field component.

(1) The total energy in each field component is proportional to the volume of the cavity.

(2) However, for a given cavity volume, the energies can be maximized or minimized by appropriately choosing the dimensions so that the smallest dimension (as considered above) is maximized or minimized.