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A Technique for the Approximate Solution of EMP Fields from a Surface Burst in the Vicinity of an Air-Ground or an Air-Water Interface

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Abstract

Under suitable restrictions Maxwell's equations assume a simpler form for the EMP fields near the air-ground or air-water interface. These equations can then be used to include various effects in a computer solution for the early-time fields. An analytic solution for the fields for simplified forms of the conductivities and the Compton current density is presented. This illustrates the influence of some of the parameters on the field components and gives rough estimates of the field components for certain conditions.

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I. Introduction

In attempting to calculate the time histories of the electromagnetic fields that comprise the nuclear EMP from a surface burst there are many physical parameters to include in the calculation with the result that a full solution requires a complex digital computer program. (This, of course, presupposes that all these physical parameters are sufficiently well understood, which is not obvious.) However, some parts of this problem can be approximately calculated if we make simplifying assumptions regarding the various pertinent physical parameters. In this note we consider a technique for approximately calculating the initial or early part of the time histories of these fields.

First, the time span of application is restricted to the transit time for a γ-ray mean free path in air. Then it is assumed that the air conductivity is large enough so all fields are the result of the local Compton current density and local electromagnetic parameters of the air and soil or water. By local is meant that the positions of these parameters are within a γ-ray mean free path (computed in air) of the position being considered. We can then define a local Cartesian coordinate system in which Maxwell's equations are somewhat simpler. Then for analytic calculations linear, time-independent parameters (ε, μ, and σ) and a step function Compton current density to illustrate the contribution of the various parameters are assumed. Specifically, the ground or water parameters are included by letting the ground or water conductivity be finite in the calculations. Thus, we can see the effect of the ground or water conductivity on the various field components. We also have some simple approximate formulas for the fields during a square pulse of γ rays, which might be used for a rough approximation for the fields from a γ-ray pulse of about the same pulse width.

II. Local Field-Generation Model

Consider a source of γ rays at the center of a spherical coordinate system as in figure 1A where positive z' is air and negative z' is soil or water. We assume a Compton current density, $\vec{J}_c$, for positive z' with only a radial component, J_{c_r}, of the form

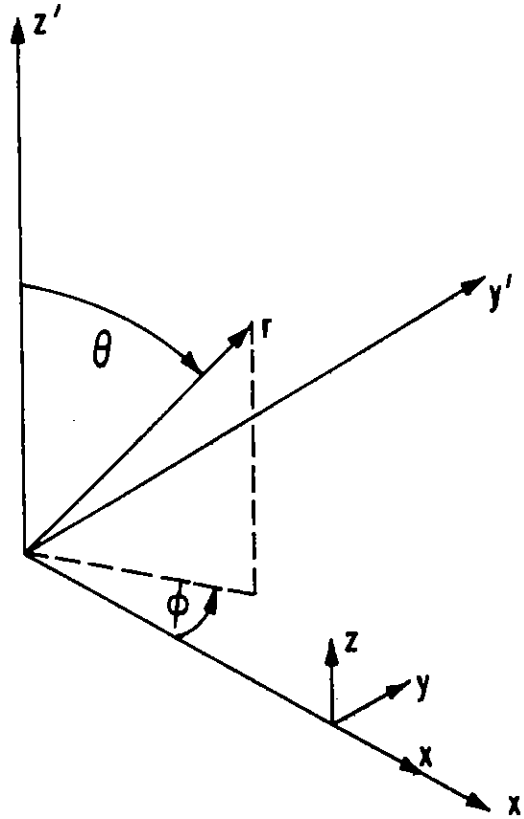
$$J_{c_r} = f_1\left(t - \frac{r}{c}\right) \frac{e^{-\frac{r}{r_\gamma}}}{4\pi r^2} \tag{1}$$

where

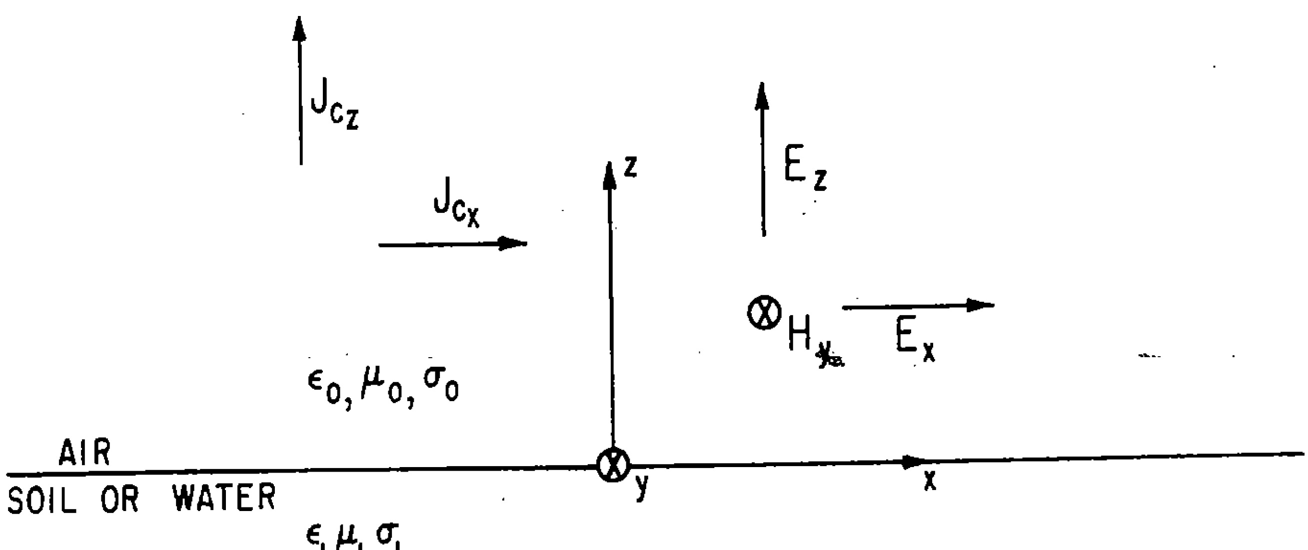
$$c \equiv \frac{1}{\sqrt{\mu_o \varepsilon_o}} \tag{2}$$

and is the speed of light in vacuum. We have an arbitrary function, $f_1(t - \frac{r}{c})$, of retarded time for the time variation and an attenuation due to the finite γ-ray mean free path, r_γ. Equation (1) does not include multiple scattering of the γ rays which spreads out the Compton current density in time, varies its distribution with θ and introduces a θ component of the Compton current density near the ground or water surface.[1] Similarly, for the air conductivity, σ_o, considering only electron attachment to oxygen, so that σ_o is linear in the γ-ray dose rate,

1. Lt Richard R. Schaefer, EMP Theoretical Note X, Prompt Gamma Effects in the Vicinity of a Ground-Air Interface, May 1965.



A. SPHERICAL COORDINATES WITH LOCAL CARTESIAN COORDINATES



y AND H_y ARE POINTING INTO THE PAGE.

B. LOCAL CARTESIAN COORDINATES WITH ELECTROMAGNETIC QUANTITIES

FIGURE 1 COORDINATE SYSTEMS

$$\sigma_o = f_2(t-\frac{r}{c})\ \frac{e^{-\frac{r}{r_\gamma}}}{4\pi r^2} \qquad (3)$$

where effects due to multiple scattering of the γ rays and the influence of the electric field on the conductivity have been ignored.

By choosing a particular point on the x' axis, as illustrated in figure 1A, for the origin of a local Cartesian (x,y,z) coordinate system, we have a new reference frame for our problem as illustrated in figure 1B. By local is meant that the origin of the coordinates is defined as a position on the x' axis near which the electromagnetic fields are calculated. Since the electromagnetic quantities are assumed independent of ϕ we choose $\phi = 0$, without loss of generality, to base these local Cartesian coordinates. It is convenient, however, to place the origin at the air-ground or air-water interface because of the discontinuity there, even if positions off this plane are of interest. In the spherical coordinates we have, by symmetry, only r and θ components of the electric field and a ϕ component of the magnetic field. Only consider positions which are much closer to the center of the local Cartesian coordinates than are the centers of the two coordinate systems with respect to each other. Then we only have x and z components of the electric field, E_x and E_z, a y component of the magnetic field, H_y, and x and z components of the Compton current density, J_{c_x} and J_{c_z}, where in general effects due to multiple γ-ray scattering are now included.

Further assume that the attenuation of the magnitudes of the Compton current density components and the air conductivity with distance can be neglected. Specifically, assume that these quantities are of the forms $J_{c_x}(t-\frac{x}{c})$, $J_{c_z}(t-\frac{x}{c})$, and $\sigma_o(t-\frac{x}{c})$ so that the only variation with x is in the retarded time variable, $t-\frac{x}{c}$. These three quantities are assumed independent of y but are allowed to vary with z. However, there are certain restrictions associated with these assumptions.

Consider the charge density associated with the Compton current density. In spherical coordinates the equation of continuity (considering only J_{c_r}) is given by

$$\nabla \cdot \vec{J}_c = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 J_{c_r}\right) = -\frac{\partial \rho_c}{\partial t} \qquad (4)$$

where ρ_c can be called the Compton charge density. From equation (1) then

$$\frac{\partial \rho_c}{\partial t} = \frac{1}{4\pi r^2}\left[\frac{1}{c}\,\frac{\partial f_1(t-\frac{r}{c})e^{-\frac{r}{r_\gamma}}}{\partial(t-\frac{r}{c})} + \frac{1}{r_\gamma} f_1(t-\frac{r}{c})e^{-\frac{r}{r_\gamma}}\right] \qquad (5)$$

We have assumed that the exponential attenuation, $e^{-\frac{r}{r_\gamma}}$, with distance can be ignored. Equation (5) shows that for a Compton current density which changes significantly in times much less than the transit time, $\frac{r_\gamma}{c}$, for one γ-ray mean free path this exponential term can be ignored. At a given location, then, if we consider $t = 0$ as the start of the Compton current density at that location, then for times much less than $\frac{r_\gamma}{c}$ the exponential attenuation, $e^{-\frac{r}{r_\gamma}}$, can be ignored. For a total scattering mean free path, r_γ, of about 200 meters for sea level air

$$\frac{r_\gamma}{c} \approx 700 \text{ ns} \qquad (6)$$

Thus, in our local Cartesian coordinate system we take J_{c_x} and J_{c_z} as zero for $t - \frac{x}{c} < 0$ and only consider times of

$$t - \frac{x}{c} << \frac{r_\gamma}{c} \qquad (7)$$

Next an important restriction is that the local electromagnetic fields (E and H) are determined only by the other local electromagnetic parameters (ϵ, μ, σ, J_{c_x}, and J_{c_z}). Specifically, the electromagnetic fields at the position of interest are assumed to be insignificantly affected by electromagnetic parameters from some other position where they might be significantly different (for a given retarded time). Since J_{c_x}, J_{c_z}, and σ_o depend on the γ-ray mean free path in a manner similar to equations (1) and (3), r_γ can be taken as the distance at which the electromagnetic parameters are significantly different in retarded time, provided the source is far enough away so that the $1/r^2$ dependence is not as significant as the $e^{-\frac{r}{r_\gamma}}$ dependence.

For $\sigma_o = 0$ fields generated at one radius can propagate to another radius, larger than the first by r_γ, arriving at the larger radius at the same time as the Compton current density. To inhibit this propagation effect we can consider only those values of σ_o or σ_1 which are large enough to significantly attenuate fields which might propagate a γ-ray mean free path. For the moment, considering a time independent σ, we have a characteristic form of $e^{-\sqrt{s\mu\,(\sigma+s\epsilon)}\,r_\gamma}$ for plane wave propagation for a distance, r_γ, where s is the Laplace transform variable (with respect to time). The transit time is just $\sqrt{\mu\epsilon}\;r_\gamma$ while a characteristic diffusion time constant is $\mu\sigma r_\gamma^2$. If this diffusion time constant is made much larger than the transit time, then in propagating a distance, r_γ, and for times of the order of the transit time after the initial arrival of the wave (one free space transit time, $\frac{r_\gamma}{c}$), the wave is severely attenuated, as desired. Thus, we consider only those cases in which

$$\mu\sigma r_\gamma^2 >> \sqrt{\mu\epsilon}\;\; r_\gamma \qquad (8)$$

or

$$\frac{\epsilon}{\sigma} << \sqrt{\mu\epsilon}\;\; r_\gamma \qquad (9)$$

where ϵ/σ is the relaxation time for the medium. For the air medium ($z > 0$) we have

$$\frac{\varepsilon_o}{\sigma_o} << \frac{r_\gamma}{c} \tag{10}$$

which is approximated in equation (6). This implies a lower limit for the air conductivity consistent with

$$\sigma_o >> 10^{-5} \text{ mhos/m} \tag{11}$$

Of course, in a real case σ_o starts at a value much lower than this (zero for EMP purposes) so that this limitation does not strictly apply to the practical case. There is some retarded time at the beginning of a γ-ray pulse for which there is a Compton current density generating fields while σ_o is lower than indicated above. However, in many cases this may not be practically important, introducing only a small error into the results due to this early time propagation.

In the local Cartesian coordinate system two of Maxwell's equations are

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{12}$$

and

$$\nabla \times \vec{H} = \vec{J}_c + \left(\sigma + \varepsilon \frac{\partial}{\partial t}\right) \vec{E} \tag{13}$$

Reducing these to the scalar equations in the field components we have

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t} \tag{14}$$

$$-\frac{\partial H_y}{\partial z} = J_{c_x} + \left(\sigma + \varepsilon \frac{\partial}{\partial t}\right) E_x \tag{15}$$

and

$$\frac{\partial H_y}{\partial x} = J_{c_z} + \left(\sigma + \varepsilon \frac{\partial}{\partial t}\right) E_z \tag{16}$$

The various electrical parameters, ε, μ, and σ, are subscripted with a zero for the upper medium ($z > 0$) and with a one for the lower medium ($z < 0$) as in figure 1B. For greater generality we have included a z component of the Compton current density, J_{c_z}, on the right side of equation (16), but this is neglected at some points in this note. The x component of the Compton current density, J_{c_x}, is later assumed zero for $z < 0$ and independent of z for $z > 0$, but in general we can let it vary with z.

We assume that locally J_{c_x}, J_{c_z}, and σ are functions of x only as contained in the retarded time expression, $t - \frac{x}{c}$. Then whatever fields are at $x = x_o$ are at $x = x_1$ at a time later by $(x_1 - x_o)/c$ (for the same z). The conditions relative to x_1 at the later time are the same as those at x_o at the earlier time. The solution of Maxwell's equations is the same for these two position-time combinations. Thus, the three field components are each functions of x, only as contained in retarded time. Note that σ_o is a function of the magnitude of the electric field.[2] However, since σ_o and the magnitude of the electric field are both functions of x only in the retarded time formulation, the fact that σ_o is a function of the magnitude of the electric field does not alter its dependence only on retarded time for its x variation. Since t and x variations are combined in retarded time, derivatives with respect to both t and x are redundant. Define retarded time

$$\tau = t - \frac{x}{c} \tag{17}$$

Since all quantities of interest are dependent on τ and not on t or x separately, we can replace derivatives with respect to both t and x by derivatives with respect to τ. We then have for these differential operators

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} \tag{18}$$

and

$$\frac{\partial}{\partial x} = -\frac{1}{c}\frac{\partial}{\partial \tau} \tag{19}$$

Thus, equations (18) and (19) can be used to rewrite equations (14) through (16) as

$$\mu \frac{\partial H_y}{\partial \tau} = -\frac{\partial E_x}{\partial z} - \frac{1}{c}\frac{\partial E_z}{\partial \tau} \tag{20}$$

$$\frac{\partial H_y}{\partial z} = -J_{c_x} - \left(\sigma + \epsilon\frac{\partial}{\partial \tau}\right) E_x \tag{21}$$

and

$$\frac{1}{c}\frac{\partial H_y}{\partial \tau} = -J_{c_z} - \left(\sigma + \epsilon\frac{\partial}{\partial \tau}\right) E_z \tag{22}$$

The number of independent variables are now reduced to two, z and τ , simplifying the equations.

Combining equations (20) and (22) to remove H_y

$$-\frac{1}{\mu}\frac{\partial E_x}{\partial z} - \frac{1}{\mu c}\frac{\partial E_z}{\partial \tau} = -cJ_{c_z} - c\left(\sigma + \epsilon\frac{\partial}{\partial \tau}\right) E_z \tag{23}$$

or

$$\frac{\partial E_x}{\partial z} = \mu c J_{c_z} + \mu\left[c\sigma + \left(\epsilon c - \frac{1}{\mu c}\right)\frac{\partial}{\partial \tau}\right] E_z \tag{24}$$

2. Lt Carl E. Baum, EMP Theoretical Note XII, Electron Thermalization and Mobility in Air, July 1965.

Differentiating equation (21) with respect to τ gives

$$\frac{\partial^2 H_y}{\partial\tau\partial z} = -\frac{\partial J_{c_x}}{\partial\tau} - \frac{\partial}{\partial\tau}\left(\sigma + \varepsilon\frac{\partial}{\partial\tau}\right) E_x \tag{25}$$

Rearranging equation (22) and differentiating with respect to z gives

$$\frac{\partial^2 H_y}{\partial z\partial\tau} = -c\frac{\partial J_{c_z}}{\partial z} - c\frac{\partial}{\partial z}\left(\sigma+\varepsilon\frac{\partial}{\partial\tau}\right) E_z \tag{26}$$

Equating the two second derivations of H_y in equations (25) and (26) then gives

$$\frac{\partial J_{c_x}}{\partial\tau} + \frac{\partial}{\partial\tau}\left(\sigma+\varepsilon\frac{\partial}{\partial\tau}\right)E_x = c\frac{\partial J_{c_z}}{\partial z} + c\frac{\partial}{\partial z}\left(\sigma+\varepsilon\frac{\partial}{\partial\tau}\right) E_z \tag{27}$$

Equations (24) and (27) are two equations for E_x and E_z, having removed H_y. Note that at this point J_{c_x}, J_{c_z}, E_x, E_z, and σ can all be functions of both independent variables, z and τ. In this form we can, for example, have σ as a function of E_x and E_z besides the γ-ray source. Likewise, J_{c_x} and J_{c_z} can be functions of z and in this formulation can even be affected by the field components, E_x, E_z, and H_y, although in this latter instance we would use the three equations (20) through (22). The essential condition is that the various electromagnetic quantities can be approximated as functions of τ, instead of t and x separately.

Associated with these differential equations there are boundary conditions for the field components. Across the boundary plane, $z = 0$, we have continuity of

E_x (tangential E)

$J_{c_z} + \left(\sigma+\varepsilon\frac{\partial}{\partial\tau}\right) E_z$ (total normal current density)

and

H_y (tangential H)

In the limit of large positive z the presence of ground or water has no influence on the solution leaving an E_x (uniform with z), for J_{c_x} independent of z and no J_{c_z}, as the only field component. For large negative z there is no Compton current density and thus no fields. Then, for large positive and negative z, if J_{c_x} and σ_o are uniform with z and J_{c_z} is zero, $\frac{\partial E_x}{\partial z}$, E_z, and H_y are zero. Also E_x is zero for large negative z. However, for more general forms of J_{c_x}, J_{c_z}, and σ_o, for large positive z, we can have other forms for the field components.

At this point equations (20) through (22) could be used as the basis for a detailed computer calculation of the early-time fields near the ground. Since we have but two independent variables and can base our coordinates near our location of interest, we can grid z and τ rather finely, if desired. With this simpler form for Maxwell's equations we can also more easily include other effects in the calculation, such as the dependence of σ_o on the electric field which in turn is a function of z and τ, the distortion of the Compton current density within an electron range of the ground or water due to the presence of the denser medium, and the distortion of the Compton current density due to the generated electric and magnetic fields. We could even let the ground or water conductivity depend on various parameters such as the γ-ray dose rate. In all these cases we must, of course, first know the dependence of these parameters on one another.

There are a few special cases of interest which further simplify the equations. Consider the upper medium (air) where μ and ε are the free space values, then equation (24) reduces to

$$\frac{\partial E_x}{\partial z} = \mu_o c J_{c_z} + \mu_o c \sigma_o E_z \tag{28}$$

such that the retarded-time derivative and the dependence on ε have been removed. This can be substituted into equation (27) giving

$$\frac{\partial J_{c_x}}{\partial \tau} + \frac{\partial}{\partial \tau}\left(\sigma_o + \varepsilon_o \frac{\partial}{\partial \tau}\right) E_x = -c\varepsilon_o \frac{\partial^2}{\partial z \partial \tau}\left(\frac{J_{c_z}}{\sigma_o}\right) + \frac{1}{\mu_o}\frac{\partial}{\partial z}\left(\sigma_o + \varepsilon_o \frac{\partial}{\partial \tau}\right)\frac{1}{\sigma_o}\frac{\partial E_x}{\partial z} \tag{29}$$

which is an equation containing only one field component, E_x. Another simplification which might be made is to ignore the displacement current with respect to the conduction current. In this case, equations (24) and (27) reduce to

$$\frac{\partial E_x}{\partial z} = \mu c J_{c_z} + \mu c \sigma E_z \tag{30}$$

and

$$\frac{\partial J_{c_x}}{\partial \tau} + \frac{\partial}{\partial \tau}\left(\sigma E_x\right) = c\frac{\partial J_{c_z}}{\partial z} + c\frac{\partial}{\partial z}\left(\sigma E_z\right) \tag{31}$$

These can be combined to give

$$\frac{\partial^2 E_x}{\partial z^2} - \mu \frac{\partial}{\partial \tau}\left(\sigma E_x\right) = \mu \frac{\partial J_{c_x}}{\partial \tau} \tag{32}$$

which again has only one field component.

III. Laplace Transformed Solutions

Now let us use this formulation of Maxwell's equations to obtain some approximate analytic solutions for the fields. For this purpose we make some simplifying assumptions. First, assume that the electromagnetic parameters, ε, μ, and σ, are independent of retarded time so that we can conveniently Laplace transform Maxwell's equations. Actually the air conductivity is not independent of retarded time, but for a step function (in retarded time) of γ rays the air conductivity will approach a constant value within a few attachment times. (Electrons attach to O_2 molecules in about 10 ns.) For our problem we can think of σ_o as a step function commencing at $\tau = 0$, the same retarded time that the step function of γ rays begins. Then we can still Laplace transform the equations because σ_o is a constant for $\tau > 0$. We must also neglect the dependence of σ_o on the electric field (which varies with retarded time) to make it independent of retarded time. These approximations, and others to be made, reduce the quantitative value of the solutions. However, we can perhaps qualitatively see the influence of the various parameters.

The assumption of a step function γ-ray pulse is perhaps a little more plausible if we note that the solution for a square pulse, of width, Δt, and starting at $\tau = 0$, is identical to the solution for a step pulse, also starting at $\tau = 0$, for retarded times such that $\tau < \Delta t$. Thus, the solutions apply during a square γ-ray pulse. We might think of approximating an initial γ-ray pulse by a square pulse, and taking J_{c_x} and σ_o as average values over this Δt. However, we should be cautious in assigning quantitative value to such an approximation.

In the following table are listed the variables for the components of the fields and the Compton current density, in both retarded-time domain and Laplace domain. The Laplace transform is performed over τ, giving a Laplace variable, s. A tilde, $\sim$, over a field variable indicates the Laplace transform of the variable.

Retarded-Time Domain	Laplace Domain
$J_{c_x}(z,\tau)$	$\tilde{J}_{c_x}(z,s)$
$J_{c_z}(z,\tau)$	$\tilde{J}_{c_z}(z,s)$
$E_x(z,\tau)$	$\tilde{E}_x(z,s)$
$E_z(z,\tau)$	$\tilde{E}_z(z,s)$
$H_y(z,\tau)$	$\tilde{H}_y(z,s)$

Table I. Listing of Variables

If these variables are further subscripted with a zero they apply to the air medium ($z > 0$), or if subscripted with a one they apply to the soil or water medium ($z < 0$). With neither of these subscripts the variables apply to both media.

Then for ε, μ, and σ independent of τ(for $\tau>0$) we can Laplace transform equations (20) through (22). Taking the fields and the Compton current density as initially zero (in retarded time) we have

$$\mu s\, \tilde{H}_y = -\frac{\partial \tilde{E}_x}{\partial z} - \frac{s}{c} \tilde{E}_z \tag{33}$$

$$\frac{\partial \tilde{H}_y}{\partial z} = -\tilde{J}_{c_x} - (\sigma+\varepsilon s)\, \tilde{E}_x \tag{34}$$

and

$$\frac{s}{c} \tilde{H}_y = -\tilde{J}_{c_z} - (\sigma+\varepsilon s)\, \tilde{E}_z \tag{35}$$

Associated with these equations are Laplace-transformed boundary conditions. At the boundary plane, $z = 0$,

$$\tilde{E}_{x_o}(0,s) = \tilde{E}_{x_1}(0,s) \tag{36}$$

$$\tilde{J}_{c_{z_o}}(0,s) + (\sigma_o+\varepsilon_o s)\, \tilde{E}_{z_o}(0,s) = \tilde{J}_{c_{z_1}}(0,s)+(\sigma_1+\varepsilon_1 s)\tilde{E}_{z_1}(0,s) \tag{37}$$

and

$$\tilde{H}_{y_o}(0,s) = \tilde{H}_{y_1}(0,s) \tag{38}$$

Also for large positive and negative z, if $\tilde{J}_{c_x}$ and σ_o are independent of z and if $\tilde{J}_{c_z}$ is zero, then $\frac{\partial \tilde{E}_x}{\partial z}$, $\tilde{E}_z$, and $\tilde{H}_y$ are zero.

For convenience assume that the vertical component of the Compton current density is zero and that the horizontal component is zero for $z<0$ and a step function, independent of z, for $z>0$. Then

$$J_{c_{x_o}}(z,\tau) = J_o\, u(\tau) \tag{39}$$

$$J_{c_{x_1}}(z,\tau) = 0 \tag{40}$$

and

$$J_{c_z}(z,\tau) = 0 \tag{41}$$

where $u(\tau)$ is zero for $\tau<0$ and one for $\tau>0$ and J_o is a constant. This is roughly consistent with a step function of γ rays travelling in the x direction for positive z, ignoring the scattered γ rays. In Laplace domain then

$$\tilde{J}_{c_{x_o}}(z,s) = \frac{J_o}{s} \tag{42}$$

$$\tilde{J}_{c_{x_1}}(z,s) = 0 \tag{43}$$

and

$$\tilde{J}_{c_z}(z,s) = 0 \tag{44}$$

Combining Maxwell's equations to remove $\tilde{H}_y$ gives, as before (equations (24) and (27)),

$$\frac{\partial \tilde{E}_x}{\partial z} = \mu \left[c\sigma + \left(\varepsilon c - \frac{1}{\mu c} \right) s \right] \tilde{E}_z \tag{45}$$

and

$$s\tilde{J}_{c_x} + s\,(\sigma+\varepsilon s)\tilde{E}_x = c\,\frac{\partial}{\partial z}\,(\sigma+\varepsilon s)\,\tilde{E}_z \tag{46}$$

where $\tilde{J}_{c_z}$ has been dropped. Further assuming that σ is independent of z, equation (46) becomes

$$s\tilde{J}_{c_x} + s(\sigma+\varepsilon s)\tilde{E}_x = c(\sigma+\varepsilon s)\,\frac{\partial \tilde{E}_z}{\partial z} \tag{47}$$

At this point define some convenient parameters

$$t_{r_o} \equiv \frac{\varepsilon_o}{\sigma_o} \tag{48}$$

$$t_{r_1} \equiv \frac{\varepsilon_1}{\sigma_1} \tag{49}$$

$$t'_{r_1} \equiv \frac{\varepsilon_1}{\sigma_1}\left(1 - \frac{1}{\mu_1 \varepsilon_1 c^2}\right) \equiv t_{r_1}\left(1 - \frac{\mu_o \varepsilon_o}{\mu_1 \varepsilon_1}\right) \tag{50}$$

$$E_o \equiv \frac{J_o}{\sigma_o} \tag{51}$$

$$t_{z_o} \equiv \frac{\mu_o \sigma_o z^2}{4} \tag{52}$$

and

$$t_{z_1} \equiv \frac{\mu_1 \sigma_1 z^2}{4} \tag{53}$$

The relaxation times for the two media are given in equations (48) and (49) while equation (50) defines a modified relaxation time for the lower medium. Equation (51) gives a parameter, E_o, which is convenient for expressing the electric field solutions. The final two parameters are characteristic diffusion times for the two media, which are used later in the note.

Rewriting equations (45) and (47) for the two media

$$\frac{\partial \tilde{E}_{x_o}}{\partial z} = \mu_o c\sigma_o \ \tilde{E}_{z_o} \tag{54}$$

$$\sigma_o + s(1+t_{r_o} s)\ \tilde{E}_{x_o} = c\ (1+t_{r_o} s)\ \frac{\partial \tilde{E}_{z_o}}{\partial z} \tag{55}$$

$$\frac{\partial \tilde{E}_{x_1}}{\partial z} = \mu_1 c\sigma_1 (1+t'_{r_1} s)\tilde{E}_{z_1} \tag{56}$$

and

$$s\tilde{E}_{x_1} = c\ \frac{\partial \tilde{E}_{z_1}}{\partial z} \tag{57}$$

These can be further refined, removing $\tilde{E}_z$, giving

$$\frac{\partial^2 E_{x_o}}{\partial z^2} - s\mu_o\sigma_o\tilde{E}_{x_o} = \frac{\mu_o\sigma_o}{1+t_{r_o} s}\ E_o \tag{58}$$

and

$$\frac{\partial^2 E_{x_1}}{\partial z^2} - s\mu_1\sigma_1(1+t'_{r_1} s)\tilde{E}_{x_1} = 0 \tag{59}$$

Rewrite one of the boundary conditions (equation (37)) as

$$\sigma_o\ (1+t_{r_o} s)\tilde{E}_{z_o}(0,s) = \sigma_1(1+t_{r_1} s)\tilde{E}_{z_1}(0,s) \tag{60}$$

We can solve for $\tilde{E}_x$ from equations (58) and (59) and $\tilde{E}_z$ from equations (54) and (56), applying the boundary conditions as in equations (36) and (60). Finally, we can obtain $\tilde{H}_y$ from equation (35) which can be rewritten as

$$\tilde{H}_{y_o} = -\frac{c\sigma_o}{s}\ (1+t_{r_o} s)\ \tilde{E}_{z_o} \tag{61}$$

and

$$\tilde{H}_{y_1} = -\frac{c\sigma_1}{s}\ (1+t_{r_1} s)\ \tilde{E}_{z_1} \tag{62}$$

Solving for $\tilde{E}_x$ gives

$$\tilde{E}_{x_o} = -\frac{E_o}{s(1+t_{r_o}s)}\left[1 - C_o e^{-\sqrt{s\mu_o\sigma_o}\,z}\right] = -\frac{E_o}{s(1+t_{r_o}s)}\left[1-C_o e^{-2\sqrt{t_{z_o}s}}\right] \quad (63)$$

and

$$\tilde{E}_{x_1} = -\frac{E_o C_1}{s(1+t_{r_o}s)} e^{\sqrt{s\mu_1\sigma_1(1+t'_{r_1}s)}\,z} = -\frac{E_o C_1}{s(1+t_{r_o}s)} e^{-2\sqrt{t_{z_1}s(1+t'_{r_1}s)}} \quad (64)$$

where C_o and C_1 are independent of z. These solutions match the boundary conditions for large positive and negative z, but for z = 0 these must be equal (equation (36)) giving

$$C_o + C_1 = 1 \quad (65)$$

For $\tilde{E}_z$ equations (54), (56), (63), and (64) give

$$\tilde{E}_{z_o} = -\frac{E_o C_o e^{-\sqrt{s\mu_o\sigma_o}z}}{c\sqrt{s\mu_o\sigma_o}\,(1+t_{r_o}s)} = -\frac{E_o C_o e^{-2\sqrt{t_{z_o}s}}}{\sqrt{\frac{s}{t_{r_o}}}\,(1+t_{r_o}s)} \quad (66)$$

and

$$\tilde{E}_{z_1} = -\frac{E_o C_1\, e^{\sqrt{s\mu_1\sigma_1(1+t'_{r_1}s)}z}}{c\sqrt{s\mu_1\sigma_1(1+t'_{r_1}s)}(1+t_{r_o}s)} = -\sqrt{\frac{\mu_o\sigma_o}{\mu_1\sigma_1}}\,\frac{E_o C_1 e^{-2\sqrt{t_{z_1}s(1+t'_{r_1}s)}}}{\sqrt{\frac{s}{t_{r_o}}(1+t'_{r_1}s)}\,(1+t_{r_o}s)} \quad (67)$$

Applying the boundary condition at z = 0 (equation (60)) gives

$$\frac{\sigma_o E_o C_o}{\sqrt{\frac{s}{t_{r_o}}}} = \sigma_1\sqrt{\frac{\mu_o\sigma_o}{\mu_1\sigma_1}}\,E_o C_1\,\frac{1+t_{r_1}s}{\sqrt{\frac{s}{t_{r_o}}(1+t'_{r_1}s)}(1+t_{r_o}s)} \quad (68)$$

or

$$C_1 = C_o\sqrt{\frac{\mu_1\sigma_o}{\mu_o\sigma_1}}\,\frac{(1+t_{t_o}s)\sqrt{1+t'_{r_1}s}}{1+t_{r_1}s} \quad (69)$$

Combining with equation (65)

$$C_o = \left\{1 + \sqrt{\frac{\mu_1\sigma_o}{\mu_o\sigma_1}}\,\frac{(1+t_{r_o}s)\sqrt{1+t'_{r_1}s}}{1+t_{r_1}s}\right\}^{-1} \quad (70)$$

and

$$C_1 = \left\{1 + \sqrt{\frac{\mu_o\sigma_1}{\mu_1\sigma_o}}\,\frac{1+t_{r_1}s}{(1+t_{r_o}s)\sqrt{1+t'_{r_1}s}}\right\}^{-1} \quad (71)$$

By substituting these into equations (63), (64), (66), and (67) we have the electric field components in the Laplace domain.

For $\tilde{H}_y$ equations (61), (62), (66), and (67) give

$$\tilde{H}_{y_o} = \frac{J_o C_o e^{-\sqrt{s\mu_o\sigma_o}z}}{s\sqrt{s\mu_o\sigma_o}} = \frac{J_o c t_{r_o} C_o e^{-2\sqrt{t_{z_o} s}}}{s^{3/2}\sqrt{t_{r_o}}} \tag{72}$$

and

$$\tilde{H}_{y_1} = \frac{\frac{\sigma_1}{\sigma_o} J_o C_1 e^{\sqrt{s\mu_1\sigma_1(1+t'_{r_1}s)}z}}{s\sqrt{s\mu_1\sigma_1(1+t'_{r_1}s)}} \; \frac{1+t_{r_1}s}{1+t_{r_o}s} \tag{73}$$

or in terms of C_o (from equation (69))

$$\tilde{H}_{y_1} = \frac{J_o C_o e^{\sqrt{s\mu_1\sigma_1(1+t'_{r_1}s)}z}}{s\sqrt{s\mu_o\sigma_o}} = \frac{J_o c t_{r_o} C_o e^{-2\sqrt{t_{z_1} s(1+t'_{r_1}s)}}}{s^{3/2}\sqrt{t_{r_o}}} \tag{74}$$

Substitution of C_o from equation (70) into equations (72) and (74) yields the magnetic field in the Laplace domain.

IV. Retarded-Time Domain Solutions.

As the Laplace-transformed field components stand, they are rather complex, particularly from the point of view of performing an inverse Laplace transform. However, we can consider various simplifying cases which bring out different aspects of the solutions. First, let the ground or water conductivity be infinite and solve for the fields above the ground, including the effects of ε_o, μ_o, and σ_o. Second, look at retarded times, large compared to the relaxation times, which is equivalent to neglecting displacement currents compared to conduction currents, but include the effects of μ_1 and σ_1 in the lower medium. Third, consider the initial behavior of the fields at the interface between the two media with a finite σ_1 and finally, calculate some approximate numbers for the time constants and appropriate coefficients for some typical conditions.

A. Infinite Soil or Water Conductivity.

Letting $\sigma_1 = \infty$ we have fields only for positive z. Equations (70) and (71) reduce to

$$C_o = 1 \tag{75}$$

and

$$C_1 = 0 \tag{76}$$

Then we have in Laplace domain, from equations (63), (66), and (72),

$$\tilde{E}_{x_o} = -\frac{E_o}{s(1+t_{r_o} s)} \left[1 - e^{-2\sqrt{t_{z_o} s}}\right] \tag{77}$$

$$\tilde{E}_{z_o} = -\frac{E_o \, e^{-2\sqrt{t_{z_o} s}}}{\sqrt{\frac{s}{t_{r_o}}}\,(1+t_{r_o} s)} \tag{78}$$

and

$$\tilde{H}_{y_o} = \frac{J_o c t_{r_o} \, e^{-2\sqrt{t_{z_o} s}}}{s^{3/2}\sqrt{t_{r_o}}} \tag{79}$$

We can invert these Laplace transforms, in some cases using the convolution integral if we separately invert parts of the transforms which are multiplied together.

First considering E_x we have the inverse Laplace transforms[3]

$$L^{-1}\left[\frac{1-e^{-2\sqrt{st_{z_o}}}}{s}\right] = \operatorname{erf}\left(\sqrt{\frac{t_{z_o}}{\tau}}\right) \tag{80}$$

$$L^{-1}\left[\frac{1}{1+t_{r_o} s}\right] = \frac{1}{t_{r_o}} e^{-\frac{\tau}{t_{r_o}}} \tag{81}$$

and

$$L^{-1}\left[\frac{1}{s(1+t_{r_o} s)}\right] = 1 - e^{-\frac{\tau}{t_{r_o}}} \tag{82}$$

where L^{-1} is the inverse Laplace transform operator. Then

$$E_{x_o} = -E_o \int_0^{\tau} \frac{1}{t_{r_o}} e^{-\frac{\tau-\tau'}{t_{r_o}}} \operatorname{erf}\left(\sqrt{\frac{t_{z_o}}{\tau'}}\right) d\tau' \tag{83}$$

which applies for all positive z. Simpler forms exist for limiting cases of z as

$$E_{x_o}(\infty,\tau) = -E_o\left[1-e^{-\frac{\tau}{t_{r_o}}}\right] \tag{84}$$

3. For these transforms and various functions see AMS 55, Handbook of Mathematical Functions, National Bureau of Standards, 1964.

and

$$E_{x_o}(0,\tau) = 0 \qquad (85)$$

For the limiting case of small t_{r_o}, which should apply for $\tau >> t_{r_o}$, we have the simpler result, including the z variation,

$$E_{x_o} \approx -E_o \operatorname{erf}\left(\sqrt{\frac{t_{z_o}}{\tau}}\right) \qquad (86)$$

Second, considering E_z we have additional inverse Laplace transforms

$$L^{-1}\left[\frac{e^{-2\sqrt{st_{z_o}}}}{\sqrt{\frac{s}{t_{r_o}}}}\right] = \sqrt{\frac{t_{r_o}}{\pi\tau}}\; e^{-\frac{t_{z_o}}{\tau}} \qquad (87)$$

$$L^{-1}\left[\frac{1}{s^{3/2}\sqrt{t_{r_o}}}\right] = 2\sqrt{\frac{\tau}{\pi t_{r_o}}} \qquad (88)$$

and

$$L^{-1}\left[\frac{1}{\sqrt{\frac{s}{t_{r_o}}}\,(1+t_{r_o}s)}\right] = \frac{2}{\sqrt{\pi}} \quad F\left(\sqrt{\frac{\tau}{t_{r_o}}}\right) \qquad (89)$$

where

$$F(\eta) = e^{-\eta^2}\int_o^{\eta} e^{\nu^2}\, d\nu \qquad (90)$$

and is called Dawson's integral and has a maximum value given by

$$F(.924) = .541 \qquad (91)$$

Then for all positive z

$$E_{z_o} = -E_o\int_o^{\tau}\frac{1}{t_{r_o}}\; e^{-\frac{\tau-\tau'}{t_{r_o}}}\sqrt{\frac{t_{r_o}}{\pi\tau'}}\; e^{-\frac{t_{z_o}}{\tau'}}\, d\tau' \qquad (92)$$

For large z

$$E_{z_o}(\infty,\tau) = 0 \qquad (93)$$

while for z = 0

$$E_{z_o}(0,\tau) = -E_o \frac{2}{\sqrt{\pi}} F\left(\sqrt{\frac{\tau}{t_{r_o}}}\right) \tag{94}$$

which for $\tau << t_{r_o}$ is

$$E_{z_o}(0,\tau) \approx -E_o\, 2\sqrt{\frac{\tau}{\pi t_{r_o}}} \tag{95}$$

and for $\tau >> t_{r_o}$ is

$$E_{z_o}(0,\tau) \approx -E_o\sqrt{\frac{t_{r_o}}{\pi\tau}} \tag{96}$$

For $\tau >> t_{r_o}$ (including the z variation)

$$E_{z_o} \approx -E_o\sqrt{\frac{t_{r_o}}{\pi\tau}}\; e^{-\frac{t_{z_o}}{\tau}} \tag{97}$$

Finally, considering H_y we have another inverse Laplace transform

$$L^{-1}\left[\frac{e^{-2\sqrt{st_{z_o}}}}{s^{3/2}\sqrt{t_{r_o}}}\right] = 2\sqrt{\frac{\tau}{\pi t_{r_o}}}\; e^{-\frac{t_{z_o}}{\tau}} - 2\sqrt{\frac{t_{z_o}}{t_{r_o}}}\; \mathrm{erfc}\left(\sqrt{\frac{t_{z_o}}{\tau}}\right) \tag{98}$$

Then for all positive z

$$H_{y_o} = J_o c t_{r_o}\left\{2\sqrt{\frac{\tau}{\pi t_{r_o}}}\; e^{-\frac{t_{z_o}}{\tau}} - 2\sqrt{\frac{t_{z_o}}{t_{r_o}}}\; \mathrm{erfc}\left(\sqrt{\frac{t_{z_o}}{\tau}}\right)\right\} \tag{99}$$

For large z

$$H_{y_o}(\infty,\tau) = 0 \tag{100}$$

and for z = 0

$$H_{y_o}(0,\tau) = J_o c t_{r_o}\, 2\sqrt{\frac{\tau}{\pi t_{r_o}}} \tag{101}$$

Conveniently H_y does not depend on t_{r_o} as a characteristic time for the waveform. It enters as an amplitude scale factor. If we rewrite c and t_{r_o} in terms of ε_o, μ_o, and σ_o, ε_o cancels out of the equations for H_y.

B. Retarded Times Much Greater Than Relaxation Times.

Now let σ_1 be finite. However, only consider the case in which $\tau >> t_{r_o}$, t_{r_1}, and t'_{r_1}. This last assumption considerably simplifies the Laplace transforms of the field quantities which take the limiting form for small t_{r_o}, t_{r_1}, and t'_{r_1}. The coefficients from equations (70) and (71) reduce to

$$C_o \simeq \left[1 + \sqrt{\frac{\mu_1\sigma_o}{\mu_o\sigma_1}}\,\right]^{-1} \tag{102}$$

and

$$C_1 \simeq \left[1 + \sqrt{\frac{\mu_o\sigma_1}{\mu_1\sigma_o}}\right]^{-1} = \sqrt{\frac{\mu_1\sigma_o}{\mu_o\sigma_1}} \quad C_o \tag{103}$$

which are independent of s. The Laplace-transformed field quantities (equations (63),(64), (66), (67), (72), and (74)) become

$$\tilde{E}_{x_o} \simeq -E_o \frac{1-C_o e^{-2\sqrt{t_{z_o} s}}}{s} \tag{104}$$

$$\tilde{E}_{x_1} \simeq -E_o \frac{C_1 e^{-2\sqrt{t'_{z_1} s}}}{s} \tag{105}$$

$$\tilde{E}_{z_o} \simeq -E_o \frac{C_o e^{-2\sqrt{t_{z_o} s}}}{\sqrt{\frac{s}{t_{r_o}}}} \tag{106}$$

$$\tilde{E}_{z_1} \simeq -E_o \sqrt{\frac{\mu_o\sigma_o}{\mu_1\sigma_1}} \frac{C_1 e^{-2\sqrt{t_{z_1} s}}}{\sqrt{\frac{s}{t_{r_o}}}} \tag{107}$$

$$\tilde{H}_{y_o} \simeq J_o c t_{r_o} \frac{C_o e^{-2\sqrt{t_{z_o} s}}}{s^{3/2}\sqrt{t_{r_o}}} \tag{108}$$

and

$$\tilde{H}_{y_1} = J_o c t_{r_o} \frac{C_o e^{-2\sqrt{t_{z_1} s}}}{s^{3/2}\sqrt{t_{r_o}}} \tag{109}$$

In the retarded-time domain, for τ much longer than the relaxation times,

$$E_{x_o} \simeq -E_o \left\{ 1 - \left[1+\sqrt{\frac{\mu_1\sigma_o}{\mu_o\sigma_1}}\,\right]^{-1} \operatorname{erfc}\left(\sqrt{\frac{t_{z_o}}{\tau}}\right)\right\} \tag{110}$$

$$E_{x_1} \simeq -E_o \left[1+\sqrt{\frac{\mu_o\sigma_1}{\mu_1\sigma_o}}\,\right]^{-1} \operatorname{erfc}\left(\sqrt{\frac{t_{z_1}}{\tau}}\right) \tag{111}$$

$$E_{z_o} \simeq -E_o \left[1+\sqrt{\frac{\mu_1\sigma_o}{\mu_o\sigma_1}}\,\right]^{-1} \sqrt{\frac{t_{r_o}}{\pi\tau}}\; e^{-\frac{t_{z_o}}{\tau}} \tag{112}$$

$$E_{z_1} \simeq -E_o \frac{\sigma_o}{\sigma_1} \left[1+\sqrt{\frac{\mu_1\sigma_o}{\mu_o\sigma_1}}\right]^{-1} \sqrt{\frac{t_{r_o}}{\pi\tau}}\; e^{-\frac{t_{z_1}}{\tau}} \tag{113}$$

$$H_{y_o} \simeq J_o c t_{r_o} \left[1+\sqrt{\frac{\mu_1\sigma_o}{\mu_o\sigma_1}}\right]^{-1} \left\{2\sqrt{\frac{\tau}{\pi t_{r_o}}}\; e^{-\frac{t_{z_o}}{\tau}} - 2\sqrt{\frac{t_{z_o}}{t_{r_o}}}\operatorname{erfc}\left(\sqrt{\frac{t_{z_o}}{\tau}}\right)\right\} \tag{114}$$

and

$$H_{y_1} \simeq J_o c t_{r_o} \left[1+\sqrt{\frac{\mu_1\sigma_o}{\mu_o\sigma_1}}\right]^{-1} \left\{2\sqrt{\frac{\tau}{\pi t_{r_o}}}\; e^{-\frac{t_{z_1}}{\tau}} - 2\sqrt{\frac{t_{z_1}}{t_{r_o}}}\operatorname{erfc}\left(\sqrt{\frac{t_{z_1}}{\tau}}\right)\right\} \tag{115}$$

For the special case of $z = 0$

$$E_{x_o}(0,\tau) = E_{x_1}(0,\tau) \simeq -E_o \left[1+\sqrt{\frac{\mu_o\sigma_1}{\mu_1\sigma_o}}\right]^{-1} \tag{116}$$

$$E_{z_o}(0,\tau) \simeq -E_o \left[1+\sqrt{\frac{\mu_1\sigma_o}{\mu_o\sigma_1}}\right]^{-1} \sqrt{\frac{t_{r_o}}{\pi\tau}} \tag{117}$$

$$E_{z_1}(0,\tau) \simeq -E_o \frac{\sigma_o}{\sigma_1} \left[1+\sqrt{\frac{\mu_1\sigma_o}{\mu_o\sigma_1}}\right]^{-1} \sqrt{\frac{t_{r_o}}{\pi\tau}} \tag{118}$$

and

$$H_{y_o}(0,\tau) = H_{y_1}(0,\tau) \simeq J_o c t_{r_o} \left[1+\sqrt{\frac{\mu_1\sigma_o}{\mu_o\sigma_1}}\right]^{-1} 2\sqrt{\frac{\tau}{\pi t_{r_o}}} \tag{119}$$

Since C_o and C_1 are constants these solutions are of the same form as those in Section IV A for $\tau \gg t_{r_o}$. For the retarded time much greater than the relaxation times we can regard C_o as a reduction factor (relative to the infinite σ_1 case) for the vertical electric field above the surface and for the tangential magnetic field at and above the surface. The radial electric field at the surface is reduced, from the value for large positive z, by the factor C_1.

C. Retarded Times Much Less Than Relaxation Times

Consider retarded times much less than the relaxation times (defined in equations (48) through (50)). However, only consider the case of $z = 0$. Assume that $\epsilon_1 > \epsilon_o$ and/or $\mu_1 > \mu_o$ so that $t'_{r_1} > 0$. We then take the limiting forms for large s of the Laplace-transformed quantities. The coefficients from equations (70) and (71) become

$$C_o \simeq \sqrt{\frac{\mu_o \sigma_1}{\mu_1 \sigma_o}} \; \frac{t_{r_1}}{t_{r_o}} \; \frac{1}{\sqrt{st'_{r_1}}} \tag{120}$$

and

$$C_1 \simeq 1 \tag{121}$$

The Laplace-transformed field quantities (equations (63), (64), (66), (67), (72), and (74)) at z = 0 become

$$\tilde{E}_{x_o}(0,s) = \tilde{E}_{x_1}(0,s) \simeq -E_o \frac{C_1}{s^2 t_{r_o}} \tag{122}$$

$$\tilde{E}_{z_o}(0,s) \simeq -E_o \frac{C_o}{s^{3/2}\sqrt{t_{r_o}}} \tag{123}$$

$$\tilde{E}_{z_1}(0,s) \simeq -E_o \sqrt{\frac{\mu_o \sigma_o}{\mu_1 \sigma_1}} \; \frac{C_1}{s^2 \sqrt{t_{r_o} t'_{r_1}}} \tag{124}$$

and

$$\tilde{H}_{y_o}(0,s) = \tilde{H}_{y_1}(0,s) \simeq J_o c t_{r_o} \frac{C_o}{s^{3/2}\sqrt{t_{r_o}}} \tag{125}$$

Substituting for C_o and C_1

$$\tilde{E}_{x_o}(0,s) = \tilde{E}_{x_1}(0,s) \simeq -E_o \frac{1}{s^2 t_{r_o}} \tag{126}$$

$$\tilde{E}_{z_o}(0,s) \simeq -E_o\sqrt{\frac{\mu_o\sigma_1}{\mu_1\sigma_o}}\sqrt{\frac{t_{r_1}}{t_{r_o}}}\sqrt{\frac{t_{r_1}}{t'_{r_1}}}\;\frac{1}{s^2 t_{r_o}} = -E_o\sqrt{\frac{\mu_o\epsilon_1}{\mu_1\epsilon_o}}\left[1-\frac{\mu_o\epsilon_o}{\mu_1\epsilon_1}\right]^{-1/2}\frac{1}{s^2 t_{r_o}} \qquad (127)$$

$$\tilde{E}_{z_1}(0,s) \simeq -E_o\sqrt{\frac{\mu_o\sigma_o}{\mu_1\sigma_1}}\sqrt{\frac{t_{r_o}}{t_{r_1}}}\sqrt{\frac{t_{r_1}}{t'_{r_1}}}\;\frac{1}{s^2 t_{r_o}} = -E_o\sqrt{\frac{\mu_o\epsilon_o}{\mu_1\epsilon_1}}\left[1-\frac{\mu_o\epsilon_o}{\mu_1\epsilon_1}\right]^{-1/2}\frac{1}{s^2 t_{r_o}} \qquad (128)$$

and

$$\tilde{H}_{y_o}(0,s) = \tilde{H}_{y_1}(0,s) \simeq J_o c t_{r_o}\sqrt{\frac{\mu_o\sigma_1}{\mu_1\sigma_o}}\sqrt{\frac{t_{r_1}}{t_{r_o}}}\sqrt{\frac{t_{r_1}}{t'_{r_1}}}\;\frac{1}{s^2 t_{r_o}} = J_o c t_{r_o}\sqrt{\frac{\mu_o\epsilon_1}{\mu_1\epsilon_o}}\left[1-\frac{\mu_o\epsilon_o}{\mu_1\epsilon_1}\right]^{-1/2}\frac{1}{s^2 t_{r_o}} \qquad (129)$$

All the Laplace-transformed field components at $z = 0$ are then of the same form, proportional to s^{-2}, in the limit of large s.

In the retarded-time domain we then have for $\tau \ll t_{r_o}$, t_{r_1}, and t'_{r_1}

$$E_{x_o}(0,\tau) = E_{x_1}(0,\tau) \simeq -E_o\frac{\tau}{t_{r_o}} \qquad (130)$$

$$E_{z_o}(0,\tau) \simeq -E_o\frac{\epsilon_1}{\epsilon_o}\left[\frac{\mu_1\epsilon_1}{\mu_o\epsilon_o}-1\right]^{-1/2}\frac{\tau}{t_{r_o}} \qquad (131)$$

$$E_{z_1}(0,\tau) \simeq -E_o\left[\frac{\mu_1\epsilon_1}{\mu_o\epsilon_o}-1\right]^{-1/2}\frac{\tau}{t_{r_o}} \qquad (132)$$

and

$$H_{y_o}(0,\tau) = H_{y_1}(0,\tau) \simeq J_o c t_{r_o}\frac{\epsilon_1}{\epsilon_o}\left[\frac{\mu_1\epsilon_1}{\mu_o\epsilon_o}-1\right]^{-1/2}\frac{\tau}{t_{r_o}} \qquad (133)$$

If we combine t_{r_o} with E_o (giving J_o/ϵ_o) in equations (130) through (132) we note that no conductivities are present in the solutions. If we compare equation (130) to equation (84) we observe that the radial electric field at the surface is initially unaffected by the presence of the ground or water surface.

D. Approximate Numbers for Time Constants and Coefficients

Finally, let us consider some approximations for various time constants and coefficients in terms of the γ-radiation rate and the conductivity and permittivity of the soil or water. We can use these with the appropriate previous equations for rough estimates of the fields.

Relating the parameters for the air to the radiation rate yields for the air conductivity[4]

$$\sigma_o = e\, n_e\, \mu_e(E) = e\mu_e(E) \times \frac{2.1 \times 10^{15}}{\nu_a} \gamma \tag{134}$$

where e is the electron charge, $\mu_e(E)$ is the electron mobility (a function of E, the magnitude of the electric field), ν_a is the attachment frequency of electrons to O_2 molecules (about 10^8 sec^{-1} at STP), and γ is the γ-ray dose rate expressed in roentgens/sec. Here a steady-state solution is assumed by assuming times long compared to ν_a^{-1}. Also other effects, such as ionic contributions and electron-ion recombination, have been ignored. A typical value of μ_e for small electric fields (less than about 10^3 volts/meter) is about 1 meter^2/(volt-sec) at STP, giving a conductivity

$$\sigma_o \approx 3.4 \times 10^{-12}\, \gamma \tag{135}$$

This is somewhat lowered for typical electric fields (which are larger than 10^3 volts/meter).

The Compton current density can be related to the radiation rate as

$$J_o = -e \frac{r_e}{r_\gamma} n_\gamma \tag{136}$$

where n_γ is the photon current density in photons/meter^2-sec (taken as in the + x direction). The ratio of the mean forward electron range, r_e, to the total scattering γ-ray mean free path, r_γ, is about 1.1×10^{-2} for arbitrarily assumed 2 MeV γ rays.[5] Thus,

$$J_o \approx -1.8 \times 10^{-21}\; n_\gamma \tag{137}$$

Changing n_γ to γ we have for 2 MeV γ-rays that[6]

$$1.2 \times 10^{13} \frac{\text{photons}}{\text{meter}^2} \approx 1 \text{ roentgen} \tag{138}$$

giving

$$J_o \approx -2.1 \times 10^{-8}\, \gamma \tag{139}$$

With J_o and σ_o we can calculate E_o, a characteristic electric field in the solutions. However, at such an electric field μ_e is decreased. For μ_e equal to 0.3

$$\sigma_o \approx 10^{-12}\, \gamma \tag{140}$$

and

$$E_o \approx -2 \times 10^4 \text{ volts/meter} \tag{141}$$

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4. Units are rationalized m.k.s unless otherwise indicated.
5. John S. Malik, EMP Theoretical Note XVI, The Compton Current, Nov. 1965.
6. T. Rockwell, Reactor Shielding Design Manual, 1956.

This mobility and field are consistent with each other.[7] Thus, we use equation (140) for the air conductivity for subsequent calculations. Actually, since σ_o does not instantaneously follow the γ-ray dose rate, the electric field can be significantly higher than E_o.

The time constants for the upper medium (air) are then

$$t_{r_o} = \frac{\varepsilon_o}{\sigma_o} \approx 8.9\ \gamma^{-1} \tag{142}$$

and

$$t_{z_o} = \frac{\mu_o \sigma_o z^2}{4} \approx 3.1 \times 10^{-19}\ z^2\ \gamma \tag{143}$$

We can combine these time constants with other terms for convenience of calculation. From the late-time vertical electric field (equations (96), (117), and (118))

$$\sqrt{\frac{t_{r_o}}{\pi\tau}} \approx \sqrt{\frac{2.8}{\tau\gamma}} = \sqrt{2.8\ \Gamma^{-1}} \tag{144}$$

where we have defined

$$\Gamma \equiv \gamma\tau \tag{145}$$

which is just the γ-ray dose expressed in roentgens. Likewise for the magnetic field (equations (101) and (119))

$$J_o c t_{r_o}\ 2\sqrt{\frac{\tau}{\pi t_{r_o}}} = -\left(J_o^2 c^2 t_{r_o} \frac{4}{\pi}\tau\right)^{1/2} \approx -\sqrt{450\ \Gamma} \tag{146}$$

Excluding the effects of the lower medium we then have rough approximations for the electric and magnetic field components.

To include the effects of the lower medium, consider the two typical cases of NTS soil and sea water as in the following table.

Parameter	NTS Soil (Frenchman Flats)	Sea Water
σ_1 (mhos/meter)	.02	4
$\frac{\varepsilon_1}{\varepsilon_o}$	16	80

Table II. Parameters for Lower Medium

7. See reference 2.

The permeability, μ_1, is taken equal to μ_o in each case.

For NTS the time constants are

$$t_{r_1} = \frac{\epsilon_1}{\sigma_1} \simeq 7.1 \text{ ns} \tag{147}$$

and

$$t_{z_1} = \frac{\mu_1 \sigma_1 z^2}{4} \simeq .63 \times 10^{-8} z^2 \tag{148}$$

We also have the factor

$$\frac{\sigma_o}{\sigma_1} \simeq .5 \times 10^{-10} \gamma \tag{149}$$

which appears in the late time field expressions (equations (110) through (119)).

For sea water the time constants are

$$t_{r_1} = \frac{\epsilon_1}{\sigma_1} \simeq .18 \text{ ns} \tag{150}$$

and

$$t_{z_1} = \frac{\mu_1 \sigma_1 z^2}{4} \simeq 1.3 \times 10^{-6} z^2 \tag{151}$$

The conductivity ratio is

$$\frac{\sigma_o}{\sigma_1} \simeq 2.5 \times 10^{-13} \gamma \tag{152}$$

For these two cases we can see from equations (149) and (152) the γ-ray dose rate at which the air conductivity is about the same as the conductivity of the lower medium. For retarded times much longer than the relaxation times we can see from equations (110) through (119) the effect of the conductivity of the lower medium on the fields. The magnetic field for $z \geq 0$ is reduced by the finite ground conductivity (instead of an infinite ground conductivity). Likewise, the vertical electric field for positive z is decreased, while conversely the radial electric field is increased.

V. Summary

We then have a technique for calculating the early time fields, based on a simplification of Maxwell's equations in a local Cartesian coordinate system. This simplification reduces the time and space variables to two: the vertical coordinate, z, and retarded time, τ (which includes both time and the x coordinate). However, there are possible restrictions on the minimum air conductivity and maximum retarded time for validity of the approximations.

Making some simplifying assumptions regarding the air conductivity and the Compton current density we can obtain analytic solutions for the fields. Such solutions show the influence of the various parameters such as the finite ground conductivity which reduces the magnetic field and increases the horizontal electric field above the ground or water surface, relative to the values for an infinite ground or water conductivity.

However, these simplifying assumptions concerning the various parameters for an analytic solution are not necessary. We can take the simpler form of Maxwell's equations, depending on z and τ, and make a computer calculation. This could include things such as time variation of the air conductivity and Compton current density, dependence of the air conductivity and Compton current density on the field components, and time variation of the ground or water conductivity. Of course, we can only include these dependences to the extent that they are known.