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Theoretical Notes

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THE ELECTROMAGNETIC FIELDS PRODUCED BY A GENERAL CURRENT DISTRIBUTION IN A CONDUCTIVE ENVIRONMENT UNDER CERTAIN SYMMETRY CONDITIONS*

by

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THE ELECTROMAGNETIC FIELDS PRODUCED BY A GENERAL CURRENT DISTRIBUTION IN A CONDUCTIVE ENVIRONMENT UNDER CERTAIN SYMMETRY CONDITIONS

ABSTRACT

The detonation of a nuclear weapon near the surface of the ground generates large electromagnetic fields. The distributed current and time and space varying conductivity caused by the detonation make analytic solutions to Maxwell's equations intractable. A numerical technique is presented in this paper which allows one to find the nuclear-weapon-generated electromagnetic fields with the aid of a high-speed digital computer. The technique is also applicable to problems of antennas in conductive environments subject to certain symmetry conditions. Stability and accuracy of the numerical technique are discussed, an internal consistency check is derived , and a test problem for the technique is developed. It is concluded that this numerical technique has proved successful in calculating the electromagnetic fields generated by a nuclear weapon detonation.

SECTION I

INTRODUCTION

It has been observed that the detonation of a nuclear weapon creates large electric and magnetic fields. The most significant part of these electromagnetic fields is produced by a neutron- and gamma-ray-induced current. This current comes from the outward motion of electrons ejected from air molecules by hard radiation through the process of Compton scattering. Since the mean free path for this hard radiation is much greater in the atmosphere than it is in the ground, a detonation near the surface of the ground will create a hemisphere of outward-moving electron current about the burst point (see figure 1). If the ground is of sufficiently high conductivity, then, in solutions for the fields in the upper hemisphere, the ground may be replaced by a perfectly conducting ground plane. This ground plane in turn may be replaced by a lower hemisphere wherein currents are the image of the currents in the upper hemisphere again as long as only the fields in the upper hemisphere are to be found (see figure 2).

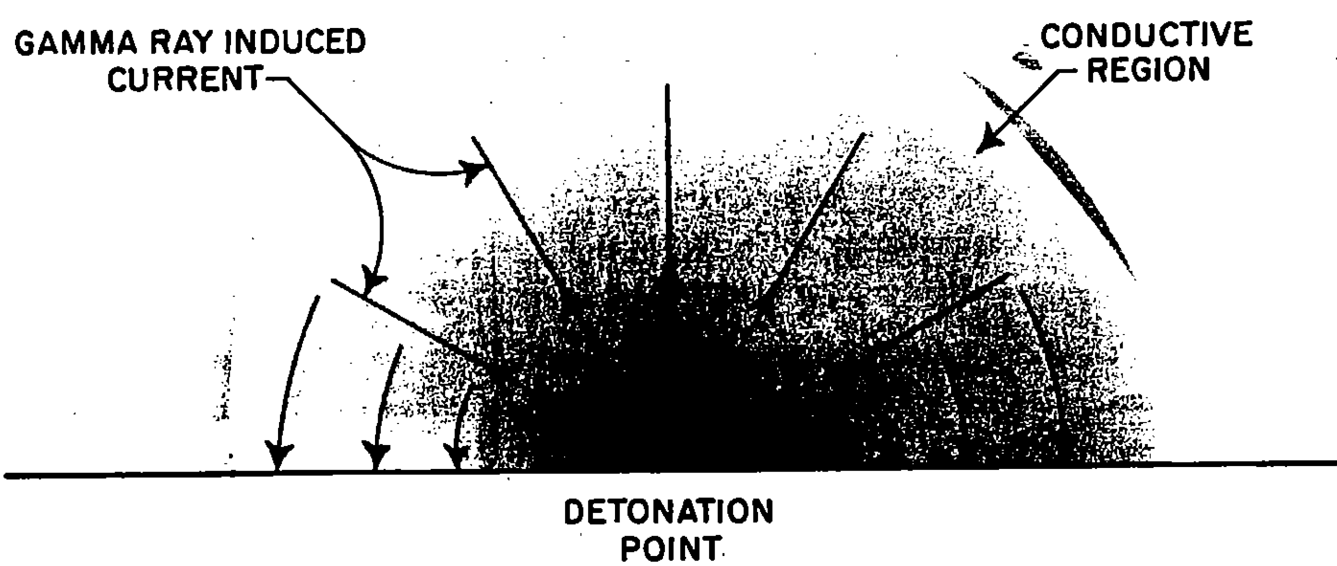

Figure 1. The Electromagnetic Environment of a Nuclear Detonation

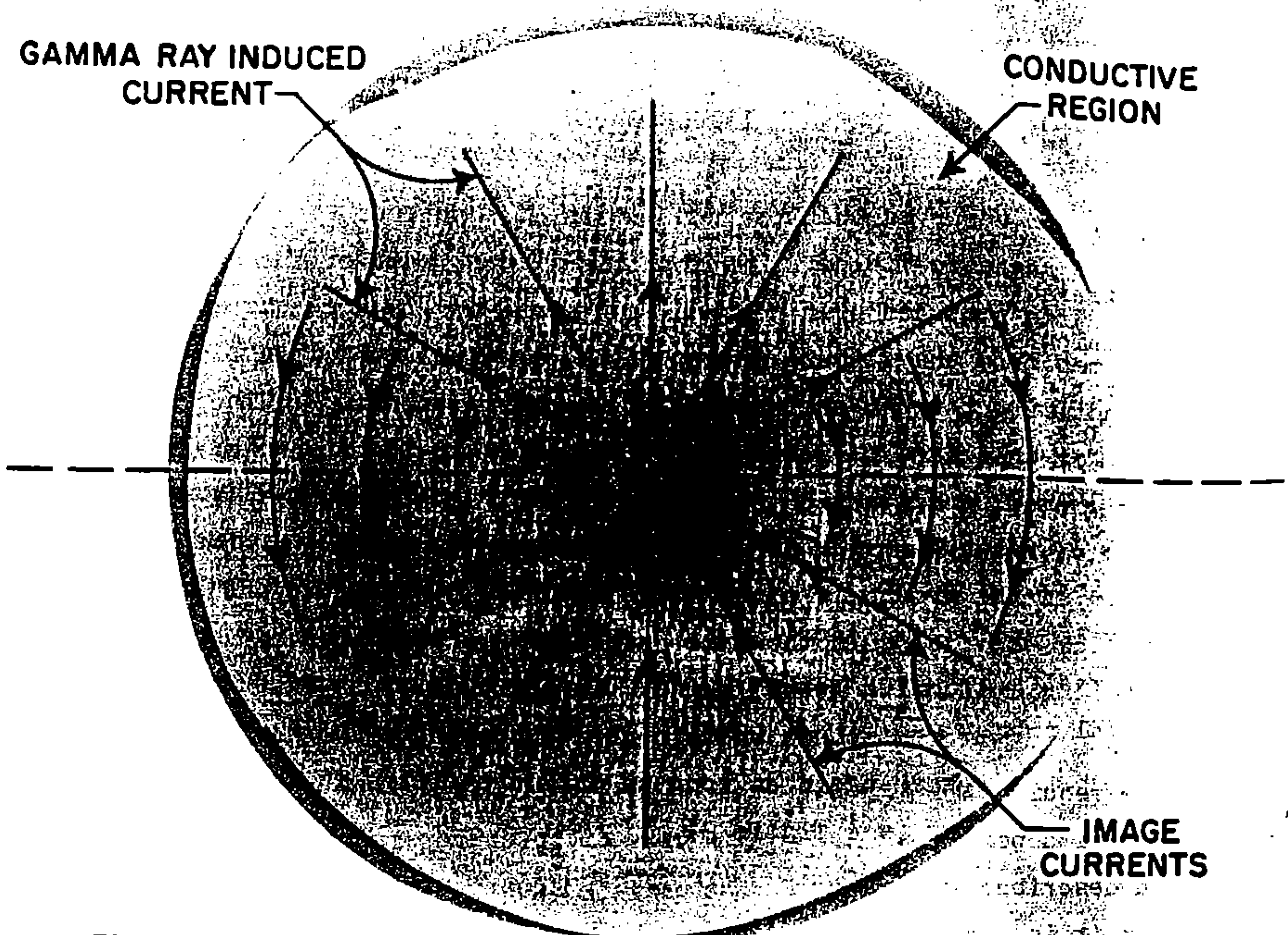

Figure 2. Simplified Schematic Representation of the EM Environment Produced by a Surface Nuclear Detonation

Maxwell's equations, which describe the electromagnetic field, must then be solved. However, one severe complication frustrates attempts to solve these equations analytically: the ejected electrons lose energy by ionizing many air molecules in their paths, and this ionization makes the atmosphere conductive. The conductivity is a function of both space and time, and such a conductivity greatly complicates finding the solution to Maxwell's equations.

Even though the equations are for the most part intractable analytically, their solution can be found numerically with the assistance of a high-speed digital computer. This report presents a numerical technique for solving Maxwell's equations for the area adjacent to a nuclear detonation. The technique described is applicable as well to a wide variety of problems concerned with the operation of antennas in conducting environments.

SECTION II

DEFINITION OF THE PROBLEM

A straightforward application of the theory of the vector potential allows one to calculate the electromagnetic fields from an arbitrary current distribution, provided that the conductivity, dielectric permeability, and magnetic permeability of the propagating medium are uniform and time independent.

The problem is much more difficult when the conductivity is an arbitrary function of space and time. In such a case, no general analytical formalism for the solution exists. The numerical technique discussed in this report has been developed to solve Maxwell's equations in a conductive environment under certain restraints upon the symmetries of the current distribution and the electrical properties of the surrounding media.

The required symmetry conditions are most easily expressed in terms of functional dependencies. The one symmetry property common to the entire method described is azimuthal symmetry. Thus, for each parameter P in the problem it is required that

(in spherical coordinates)

$$P(r,\theta,\phi,t) = P(r,\theta,t) \quad (1)$$

For the radial current J_r, a further condition is

$$J_r(r,\theta,t) = -J_r(r,\pi-\theta,t) \quad (2)$$

and for the polar current J_θ

$$J_\theta(r,\theta,t) = J_\theta(r,\pi-\theta,t) \quad (3)$$

These symmetry conditions are met by an arbitrary, azimuthally independent current about a point on the surface of a perfectly conducting plane, if one uses the method of images to find the fields above the plane.

The remaining conditions are $J_\phi = 0$,

$$\sigma(r,\theta,t) = \sigma(r,t); \ \varepsilon = \text{a constant}, \ \mu = \text{a constant}. \tag{4}$$

That is, all electrical properties of the medium are spherically symmetric about the origin. These symmetry conditions are also met by the image model which may be used to calculate the fields above an infinitely conducting plane given that the conductivity in that half space is hemispherically symmetric about the origin on the plane. Under these symmetry conditions the nonzero field components are[1]

$$H_\phi(r,\theta,t) = +H_\phi(r,\pi-\theta,t) \tag{5}$$

$$E_r(r,\theta,t) = -E_r(r,\pi-\theta,t) \tag{6}$$

$$E_\theta(r,\theta,t) = +E_\theta(r,\pi-\theta,t) \tag{7}$$

The determination of J_r, J_θ, and σ for the case of a nuclear detonation is beyond the scope of this work, and will be taken up in a later publication.

[1] These statements follow directly from Maxwell's equations and the assumed symmetries, as will be demonstrated in the next section.

SECTION III

MAXWELL'S EQUATIONS

The basic equations which govern the behavior of the electromagnetic fields are Maxwell's equations, commonly written in the vector equation form, in the rationalized MKS system,

$$\nabla \times \overline{H} = \overline{J}_d + \sigma\overline{E} + \frac{\partial \overline{D}}{\partial t} \tag{8}$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \tag{9}$$

where $\overline{H}$ = the magnetic field intensity

$\overline{J}_d$ = the driving current. In the nuclear detonation, $\overline{J}_d$ is the "Compton" current

$\overline{E}$ = electric field intensity

$\overline{D}$ = electric flux density

$\overline{B}$ = magnetic flux density

σ = electrical conductivity

Making use of the constitutive equations

$$\overline{B} = \mu\overline{H} \tag{10}$$

$$\overline{D} = \epsilon\overline{E}, \tag{11}$$

where μ = the magnetic permeability of the medium

ϵ = the electric permeability of the medium[2]

then Maxwell's equations may be written

$$\nabla \times \left(\frac{1}{\mu}\overline{B}\right) = \overline{J}_d + \sigma\overline{E} + \frac{\partial(\epsilon\overline{E})}{\partial t} \tag{12}$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \tag{13}$$

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[2] ϵ is sometimes misleadingly referred to as the dielectric constant of the medium.

If one now assumes that μ and ε are constants, the equations reduce to the final form:

$$\frac{1}{\mu}\nabla \times \bar{B} = \bar{J}_d + \sigma\bar{E} + \varepsilon\frac{\partial \bar{E}}{\partial t} \tag{14}$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \tag{15}$$

From the assumed symmetry conditions, the first equation has only r and θ components and the second only a ϕ component. All terms of the ϕ component of the first of Maxwell's equations and of the r and θ components of the second equation contain either E_ϕ, H_r, or H_θ. If the initial conditions specify that all these fields are zero, then Maxwell's equations predict that these field components will remain zero. Thus, the scalar forms of the above equations are, with $\bar{J}_d = \bar{r}J_r + \bar{\theta}J_\theta$,

r component:

$$\frac{1}{\mu r \sin\theta}\frac{\partial}{\partial\theta}\left(B_\phi \sin\theta\right) = J_r + \sigma E_r + \varepsilon\frac{\partial E_r}{\partial t} \tag{16}$$

θ component:

$$-\frac{1}{\mu r}\frac{\partial}{\partial r}\left(rB_\phi\right) = J_\theta + \sigma E_\theta + \varepsilon\frac{\partial E_\theta}{\partial t} \tag{17}$$

ϕ component:

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rE_\theta\right) - \frac{\partial E_r}{\partial\theta}\right] = -\frac{\partial B_\phi}{\partial t} \tag{18}$$

The symmetries of the problem posed are just sufficient to allow one to remove the explicit θ dependence from Maxwell's equations. This is done by first representing each field as a series expansion in Legendre polynomials.[3] Under the assumed symmetries, only the odd n coefficients will be nonzero.

$$J_r = \sum_{n=1}^{\infty} J_{r_n} P_n^0(\cos\theta) \tag{19}$$

$$J_\theta = \sum_{n=1}^{\infty} J_{\theta_n} P_n^1(\cos\theta) \tag{20}$$

[3] These expansions were first proposed by Dr. B. R. Suydam.

$$E_r = \sum_{n=1}^{\infty} E_{r_n} P_n^0 (\cos\theta) \tag{21}$$

$$E_\theta = \sum_{n=1}^{\infty} E_{\theta_n} P_n^1 (\cos\theta) \tag{22}$$

$$B_\phi = \sum_{n=1}^{\infty} B_{\phi_n} P_n^1 (\cos\theta) \tag{23}$$

Using the above Legendre expansions of the fields, the three coupled partial differential equations become:

r component:[4]

$$\frac{1}{\mu r \sin\theta} \frac{\partial}{\partial\theta} \left(\sum_{n=1}^{\infty} B_{\phi_n} P_n^1 (\cos\theta) \sin\theta \right) = \sum_{n=1}^{\infty} J_{r_n} P_n^0 (\cos\theta) +$$

$$\sigma \sum_{n=1}^{\infty} E_{r_n} P_n^0 (\cos\theta) + \epsilon \frac{\partial}{\partial t} \sum_{n=1}^{\infty} E_{r_n} P_n^0 (\cos\theta) \tag{24}$$

Since

$$P_n^1 (\cos\theta) = -\frac{\partial}{\partial\theta} P_n^0 (\cos\theta) \tag{25}$$

This may be put into the left hand side of the r component equation giving

$$-\frac{1}{\mu r \sin\theta} \frac{\partial}{\partial\theta} \sum_{n=1}^{\infty} B_{\phi_n} \sin\theta \frac{\partial}{\partial\theta} P_n^0 (\cos\theta) =$$

$$-\frac{1}{\mu r} \sum_{n=1}^{\infty} B_{\phi_n} \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} P_n^0 (\cos\theta) \right) \right] \tag{26}$$

Differentiating this with respect to θ gives

[4] See Appendix

$$-\frac{1}{\mu r \sin\theta}\frac{\partial}{\partial\theta}\sum_{n=1}^{\infty} B_{\phi_n} \sin\theta \frac{\partial}{\partial\theta} P_n^0 (\cos\theta) =$$

$$-\frac{1}{\mu r}\sum_{n=1}^{\infty} B_{\phi_n}\left[\frac{\partial^2 P_n^0}{\partial\theta^2} + \cot\theta \frac{\partial P_n^0}{\partial\theta}\right] \tag{27}$$

Since the defining equation for the Legendre polynomial is

$$-\frac{\partial^2 P_n^0}{\partial\theta^2} - \cot\theta \frac{\partial P_n^0}{\partial\theta} = n(n+1) P_n^0 \quad ; \tag{28}$$

thus,

$$\frac{1}{\mu r}\sum_{n=1}^{\infty} B_{\phi_n} n(n+1) P_n^0 = \sum_{n=1}^{\infty} J_{r_n} P_n^0 (\cos\theta) +$$

$$\sigma \sum_{n=1}^{\infty} E_{r_n} P_n^0 (\cos\theta) + \varepsilon \frac{\partial}{\partial t}\sum_{n=1}^{\infty} E_{r_n} P_n^0 (\cos\theta) \tag{29}$$

Integrating the above equation multiplied by P_n^0 (cos θ) sin θ from 0 to π, the final result is

$$\frac{n(n+1)}{\mu r} B_{\phi_n} = J_{r_n} + \sigma E_{r_n} + \varepsilon \frac{\partial}{\partial t} E_{r_n} \tag{30}$$

Similarly, the θ equation becomes

$$-\frac{1}{\mu r}\frac{\partial}{\partial r}\left(r B_{\phi_n}\right) = J_{\theta_n} + \sigma E_{\theta_n} + \varepsilon \frac{\partial E_{\theta_n}}{\partial t} \; ; \tag{31}$$

and the ϕ equation becomes

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r E_{\theta_n}\right) + E_{r_n}\right] = -\frac{\partial B_{\phi_n}}{\partial t} \tag{32}$$

The equations may be written more simply with the substitutions

$$B_{\phi_n} = r\tilde{B}_{\phi_n} \tag{33}$$

(Where the ~ designates the variable as the one used previous to this point in the discussion)

$$E_{\theta_n} = r\tilde{E}_{\theta_n} \tag{34}$$

$$E_{r_n} = E_{r_n} \tag{35}$$

giving for the equations to be differenced the following set of three coupled, first-order, partial differential equations in two independent variables, r and t, and three dependent variables:

$$\frac{J_{r_n}}{\varepsilon} + \frac{\sigma}{\varepsilon} E_{r_n} + \frac{\partial E_{r_n}}{\partial t} - \frac{c^2 n(n+1)}{r^2} B_{\phi_n} = 0 \tag{36}$$

$$\frac{r}{\varepsilon} J_{\theta_n} + \frac{\sigma}{\varepsilon} E_{\theta_n} + \frac{\partial E_{\theta_n}}{\partial t} + c^2 \frac{\partial}{\partial r} B_{\phi_n} = 0 \tag{37}$$

$$\frac{\partial}{\partial r} E_{\theta_n} + E_{r_n} + \frac{\partial B_{\phi_n}}{\partial t} = 0 \tag{38}$$

where, of course,

$$c = \frac{1}{\sqrt{\mu\varepsilon}}$$

SECTION IV

THE FINITE DIFFERENCING FORMALISM

To solve a set of differential equations with the aid of a digital computer, the differential equations must first be approximated by finite-difference equations. The resulting finite-difference equations must be numerically tractable and stable. In addition they should give solutions which converge to the correct solution within a reasonable computation time. This section will discuss the finite-difference equations to be used and their accuracy, while the next section will consider the closely related topic of stability.

To understand the constraints on the differencing scheme, an inspection of the domain over which the differential equations are to be solved is in order. Figure 3 shows the domain for the problem (c is the speed of light in the medium):

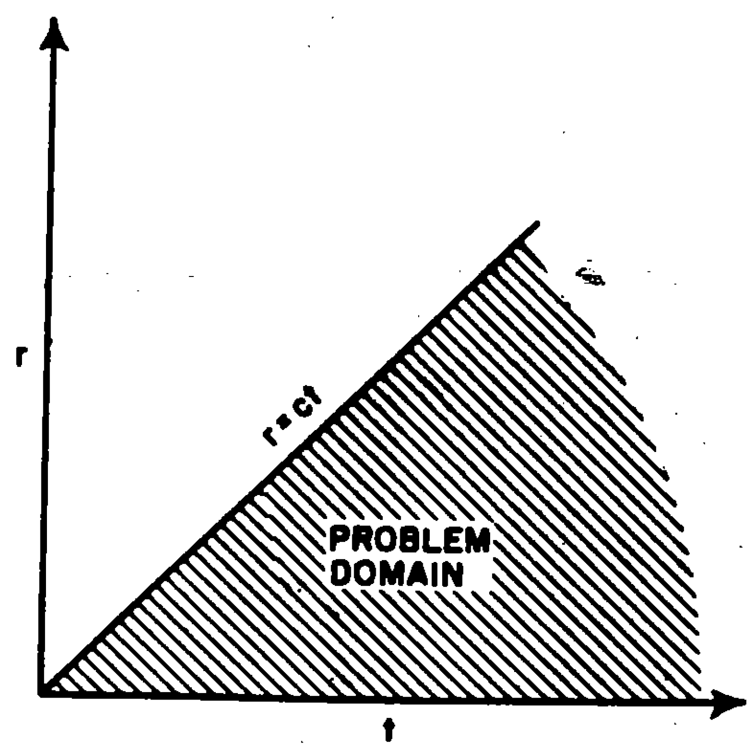

Figure 3. The Domain of the Problem

Without loss of generality, it may be specified that the source for the fields, J, is turned on at the origin at time $t = \delta$ and elsewhere (r) at a retarded time of δ (i.e., $t-r/c = \delta$). Along the curve $t-r/c = 0$, all fields are zero, since the fields will propagate no faster than c, the speed of light. Therefore, we have the condition that for $t-r/c = 0$, all fields are zero.

To avoid numerical difficulty along the r = 0 boundary, the variables used in the computation,

$$E_{r_n}$$

$$\left(r\tilde{E}_{\theta_n}\right)$$

$$\left(r\tilde{B}_{\phi_n}\right)$$

must approach the r origin in the following menner:

$$\lim_{r\to 0} E_{r_n} = a, \ \lim_{r\to 0}\left(r\tilde{E}_{\theta_n}\right) = b, \ \lim_{r\to 0}\left(r\tilde{B}_{\phi_n}\right) = c \tag{39}$$

(a, b, c finite)

and a, b, and c must be known. These conditions are not very restrictive; they apply, for example, to the linear dipole antenna of finite length (i.e., any physical linear dipole).

In general, the electromagnetic fields will propagate to infinity in both r and t, so we cannot close the third side of our boundary in the r, t plane.

The differencing scheme is built upon the grid shown in figure 4:

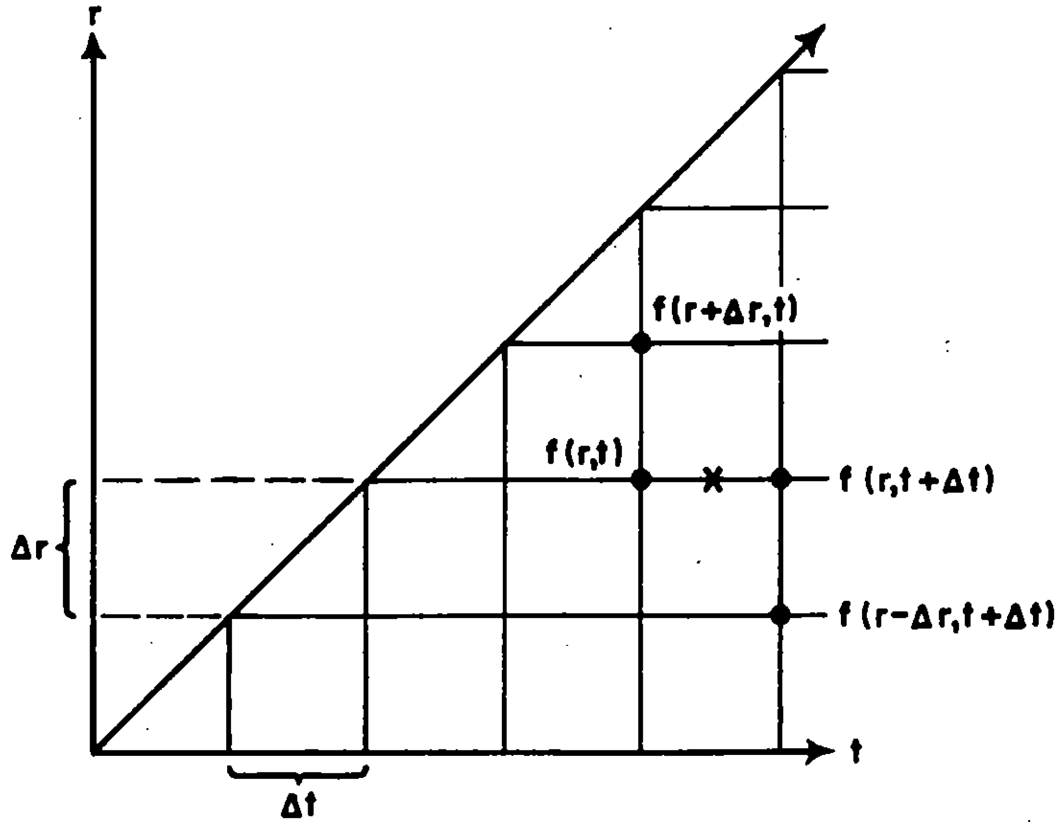

Figure 4. The Grid for the Differencing Scheme

An inductive process will be used to show how the electromagnetic fields are found. Assume that by the process to be described, the fields (denoted by f(r,t) in the figure) at all grid points (intersections of two grid lines) to the left of the point f(r,t+Δt) and directly below the point f(r,t+Δt) have been found. Then the next point at which the fields must be found is the point f(r,t+Δt). To do this a set of difference equations are written which are centered about the point x on figure 4. Explicitly, the equations are

$$\frac{1}{\varepsilon}J_{r_n}\left(r,t+\frac{\Delta t}{2}\right)+\frac{\sigma}{2\varepsilon}\left(r,t+\frac{\Delta t}{2}\right)\cdot\left[E_{r_n}(r,t)+E_{r_n}(r,t+\Delta t)\right]+\frac{1}{\Delta t}$$

$$\cdot\left[E_{r_n}(r,t+\Delta t)-E_{r_n}(r,t)\right]-\frac{c^2 n(n+1)}{2r^2}\left[B_{\phi_n}(r,t)+B_{\phi_n}(r,t+\Delta t)\right]=0 \qquad (40)$$

$$\frac{r}{\epsilon}J_{\theta_n}\left(r,t+\frac{\Delta t}{2}\right)+\frac{\sigma}{2\epsilon}\left(r,t+\frac{\Delta t}{2}\right)\cdot\left[E_{\theta_n}(r,t)+E_{\theta_n}(r,t+\Delta t)\right]+\frac{1}{\Delta t}$$

$$\cdot\left[E_{\theta_n}(r,t+\Delta t)-E_{\theta_n}(r,t)\right]+\frac{c^2}{2\Delta r}\left[B_{\phi_n}(r+\Delta r,t)-B_{\phi_n}(r,t)\right.$$

$$\left.+B_{\phi_n}(r,t+\Delta t)-B_{\phi_n}(r-\Delta r,t+\Delta t)\right] = 0 \qquad (41)$$

$$\frac{1}{\Delta t}\left[B_{\phi_n}(r,t+\Delta t)-B_{\phi_n}(r,t)\right]+\frac{1}{2}\cdot\left[E_{r_n}(r,t+\Delta t)+E_{r_n}(r,t)\right]+\frac{1}{2\Delta r}$$

$$\cdot\left[E_{\theta_n}(r+\Delta r,t)-E_{\theta_n}(r,t)+E_{\theta_n}(r,t+\Delta t)-E_{\theta_n}(r-\Delta r,t+\Delta t)\right] = 0 \qquad (42)$$

Multiplying by various constants and rearranging terms, these equations are conveniently written

$$E_{r_n}(r,t+\Delta t)\left[\frac{\sigma\Delta t}{2\epsilon}+1\right]-B_{\phi_n}(r,t+\Delta t)\left[\frac{c^2\Delta t n(n+1)}{2r^2}\right]$$

$$= -\frac{\Delta t}{\epsilon}J_{r_n}-E_{r_n}(r,t)\left[\frac{\sigma\Delta t}{2\epsilon}-1\right]+B_{\phi_n}(r,t)\left[\frac{c^2\Delta t n(n+1)}{2r^2}\right] \qquad (43)$$

$$B_{\phi_n}(r,t+\Delta t)\left[\frac{c^2\Delta t}{2\Delta r}\right]+E_{\theta_n}(r,t+\Delta t)\left[\frac{\sigma\Delta t}{2\epsilon}+1\right] = -\frac{r\Delta t}{\epsilon}J_{\phi_n}-\frac{c^2\Delta t}{2\Delta r}\left[B_{\phi_n}(r+\Delta r,t)-B_{\phi_n}(r,t)\right.$$

$$\left.-B_{\phi_n}(r-\Delta r,t+\Delta t)\right]-E_{\theta_n}(r,t)\left[\frac{\sigma\Delta t}{2\epsilon}-1\right] \qquad (44)$$

$$E_{r_n}(r,t+\Delta t)\left[\frac{\Delta t}{2}\right]+B_{\phi_n}(r,t+\Delta t) + E_{\theta_n}(r,t+\Delta t)\ \frac{\Delta t}{2\Delta r}$$

$$= -\frac{\Delta t}{2}E_{r_n}(r,t)+B_{\phi_n}(r,t)-\frac{\Delta t}{2\Delta r}\left[E_{\theta_n}(r+\Delta r,t)-E_{\theta_n}(r,t)-E_{\theta_n}(r-\Delta r,t+\Delta t)\right] \quad (45)$$

All of the fields on the right-hand sides of the above equations are known, and all of the fields on the left-hand sides are unknown. Thus, there are three independent linear algebraic equations and three unknowns, which may immediately be solved for

$E_{r_n}(r,t+\Delta t)$

$E_{\theta_n}(r,t+\Delta t)$

$B_{\phi_n}(r,t+\Delta t)$

The solutions are

$$E_{r_n}(r,t+\Delta t) = \frac{1}{DET}\left\{D_1\left(\frac{c\Delta t}{2\Delta r}\right)^2-D_3\frac{c^2\Delta t n(n+1)}{2r^2}\left[\frac{\sigma\Delta t}{2\varepsilon}+1\right]-D_1\left[\frac{\sigma\Delta t}{2\varepsilon}+1\right]\right\} \quad (46)$$

$$B_{\phi_n}(r,t+\Delta t) = \frac{1}{DET}\left\{D_2\frac{\Delta t}{2\Delta r}\left[\frac{\sigma\Delta t}{2\varepsilon}+1\right]+D_1\frac{\Delta t}{2}\left[\frac{\sigma\Delta t}{2\varepsilon}+1\right]-D_3\left[\frac{\sigma\Delta t}{2\varepsilon}+1\right]^2\right\} \quad (47)$$

$$E_{\theta_n}(r,t+\Delta t) = \frac{1}{DET}\left\{D_3\frac{c^2\Delta t}{2\Delta r}\left[\frac{\sigma\Delta t}{2\varepsilon}+1\right]-D_2\left(\frac{c\Delta t}{2r}\right)^2 n(n+1)-D_1\frac{c^2(\Delta t)^2}{4\Delta r}\right.$$

$$\left.-D_2\left[\frac{\sigma\Delta t}{2\varepsilon}+1\right]\right\} \quad (48)$$

where

$$DET = \left(\frac{c\Delta t}{2\Delta r}\right)^2\left[\frac{\sigma\Delta t}{2\varepsilon}+1\right]-\left(\frac{c\Delta t}{2r}\right)^2 n(n+1)\left[\frac{\sigma\Delta t}{2\varepsilon}+1\right]-\left[\frac{\sigma\Delta t}{2\varepsilon}+1\right]^2 \quad (49)$$

$$D_1 = -\frac{\Delta t}{\varepsilon} J_{r_n} - E_{r_n}(r,t)\left[\frac{\sigma \Delta t}{2\varepsilon} - 1\right] + B_{\phi_n}(r,t)\frac{\varepsilon^2 \Delta t n(n+1)}{2r^2} \quad (50)$$

$$D_2 = -\frac{r\Delta t}{\varepsilon} J_{\theta_n} - \frac{c^2 \Delta t}{2\Delta r}\left[B_{\phi_n}(r+\Delta r,t) - B_{\phi_n}(r,t) - B_{\phi_n}(r-\Delta r,t+\Delta t)\right]$$

$$-E_{\theta_n}(r,t)\left[\frac{\sigma \Delta t}{2\varepsilon} - 1\right] \quad (51)$$

$$D_3 = -\frac{\Delta t}{2} E_{r_n}(r,t) + B_{\phi_n}(r,t) - \frac{\Delta t}{2\Delta r}\left[E_{\theta_n}(r+\Delta r,t) - E_{\theta_n}(r,t)\right.$$

$$\left. - E_{\theta_n}(r-\Delta r,t+\Delta t)\right] \quad (52)$$

In this manner the calculation advances the field solutions, progressing outward through the range r at a fixed origin time t until the light cone is reached, then advances the time by Δt, and progresses outward from the origin once more. The principal machine memory requirement is that each of the three components of the field be stored for each n considered at the latest time for which it is available.

The Truncation Error of the Differencing Scheme

It is of interest to find the order in Δt and Δr to which the finite-difference equations are accurate. If the equations were accurate to all orders, one would obtain the exact solution independent of the grid size, so long as the differencing scheme was stable. Practical differencing schemes will be accurate only to some finite order; the higher the order, the larger the Δt and Δr steps that may be taken to obtain answers within a specified accuracy. To demonstrate the method for determining the order of accuracy, detailed calculations will be performed with the equation

$$\frac{r}{\varepsilon} J_{\theta_n} + \frac{\sigma}{\varepsilon} E_{\theta_n} + \frac{\partial E_{\theta_n}}{\partial t} + c^2 \frac{\partial B_{\phi_n}}{\partial r} = 0 \quad (53)$$

Let J_{θ_n} and be the current and conductivity at the point about which the equations are differenced. Call this point (r,t). Let B_{ϕ_n}, E_{θ_n}, E_{r_n} be the exact solutions to Maxwell's equations at this point. Then, using a Taylor expansion for the variables appearing in our finite differencing scheme in two dimensions,

$$B_{\phi_n}\left(r+\Delta r, t-\frac{\Delta t}{2}\right) = B_{\phi_n} + \Delta r \frac{\partial B_{\phi_n}}{\partial r} - \frac{\Delta t}{2}\frac{\partial B_{\phi_n}}{\partial t} - \frac{\Delta r \Delta t}{2}\frac{\partial^2 B_{\phi_n}}{\partial r \partial t} + \frac{1}{2}\left(\frac{\Delta t}{2}\right)^2 \frac{\partial^2 B_{\phi_n}}{\partial t^2} + \frac{(\Delta r)^2}{2}\frac{\partial^2 B_{\phi_n}}{\partial r^2} + o\left((\Delta r, \Delta t)^3\right) \tag{54}$$

Expanding the finite difference form of the equation under consideration in this manner, the truncation is found:

$$\frac{r}{\varepsilon}J_{\theta_n} + \frac{\sigma}{\varepsilon}\left[E_{\theta_n} + \frac{\Delta t}{2}\frac{\partial E_{\theta_n}}{\partial t} + \frac{1}{2}\left(\frac{\Delta t}{2}\right)^2 \frac{\partial^2 E_{\theta_n}}{\partial t^2} - \frac{\Delta t}{2}\frac{\partial E_{\theta_n}}{\partial t} + \frac{1}{2}\left(\frac{\Delta t}{2}\right)^2\frac{\partial^2 E_{\theta_n}}{\partial t^2}\right.$$

$$\left. + o\left((\Delta t)^3\right)\right] + \frac{1}{\Delta t}\left[E_{\theta_n} + \frac{\Delta t}{2}\frac{\partial E_{\theta_n}}{\partial t} + \frac{1}{2}\left(\frac{\Delta t}{2}\right)^2\frac{\partial^2 E_{\theta_n}}{\partial t^2} - E_{\theta_n} + \frac{\Delta t}{2}\frac{\partial E_{\theta_n}}{\partial t}\right.$$

$$\left. -\frac{1}{2}\left(\frac{\Delta t}{2}\right)^2\frac{\partial^2 E_{\theta_n}}{\partial t^2} + o\left((\Delta t)^3\right)\right] + \frac{c^2}{2\Delta r}\left[B_{\phi_n} + \Delta r\frac{\partial B_{\phi_n}}{\partial r} - \frac{\Delta t}{2}\frac{\partial B_{\phi_n}}{\partial t}\right.$$

$$-\Delta r\left(\frac{\Delta t}{2}\right)\frac{\partial^2 B_{\phi_n}}{\partial r \partial t} + \frac{1}{2}\left(\frac{\Delta t}{2}\right)^2\frac{\partial^2 B_{\phi_n}}{\partial t^2} + \frac{(\Delta r)^2}{2}\frac{\partial^2 B_{\phi_n}}{\partial r^2} - B_{\phi_n} + \frac{\Delta t}{2}\frac{\partial B_{\phi_n}}{\partial t}$$

$$-\frac{1}{2}\left(\frac{\Delta t}{2}\right)^2 \frac{\partial^2 B_{\phi_n}}{\partial t^2} + B_{\phi_n} + \frac{\Delta t}{2}\frac{\partial B_{\phi_n}}{\partial t} + \frac{1}{2}\left(\frac{\Delta t}{2}\right)^2 \frac{\partial^2 B_{\phi_n}}{\partial t^2} - B_{\phi_n} + \Delta r \frac{\partial B_{\phi_n}}{\partial r}$$

$$\left. -\frac{\Delta t}{2}\frac{\partial B_{\phi_n}}{\partial t} + \frac{\Delta r \Delta t}{2}\frac{\partial^2 B_{\phi_n}}{\partial r \partial t} - \frac{1}{2}\left(\frac{\Delta t}{2}\right)^2 \frac{\partial^2 B_{\phi_n}}{\partial t^2} - \frac{(\Delta r)^2}{2}\frac{\partial^2 B_{\phi_n}}{\partial r^2} + 0\left((\Delta r, \Delta t)^3\right)\right]$$

$$= 0 + 0 \left\{(\Delta t)^2, (\Delta r)^3\right\} \tag{55}$$

Therefore, the equation is accurate to order $(\Delta t)^2$ in time and to order $(\Delta r)^3$ in r.

A similar reduction of the other two equations shows that they are accurate to the same order in time and to at least an order of $(\Delta r)^3$ in r.

SECTION V

STABILITY

Stability of a set of finite-difference equations is the property that perturbations of the solutions do not grow as the computation proceeds.

Convergence is the somewhat more nearly ideal property in that as the grid size (Δt and Δr in this case) approaches zero in some specified manner, then the solutions to the finite difference equations approach arbitrarily close to the solutions of the differential equations which they approximate. A theorem, known as Lax's equivalence theorem, ties together these two properties: given a properly posed initial value problem and a finite-difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence.

Two conditions must be met before this theorem is applicable: first, the initial value problem must be properly posed, and second, a consistency condition on the finite-difference equations must be satisfied. A properly posed initial value problem is one in which the fields on the boundaries satisfy the equations to be solved (i.e., are physically possible). Stated more completely, a properly posed problem is one for which a unique solution exists, this solution being a continuous function of the initial data. The consistency condition requires that as Δr and Δt approach zero, the finite-difference equations approach the original differential equations. Since these conditions are met in the scheme proposed, then by Lax's theorem they should converge to the correct solution as the grid size approaches zero. It should be kept in mind that in actual numerical computation, the grid size must not become too small, or differences taken between fields at two adjacent grid points will be overwhelmed by the roundoff error of the machine.

Therefore, the problem remaining is to understand the stability of the scheme. The subject of stability is best approached through a consideration of small perturbations, δf, upon the exact solution to the differential equations, wherein

B_{ϕ_n} goes to $B_{\phi_n} + \delta B_{\phi_n}$

E_{θ_n} goes to $E_{\theta_n} + \delta E_{\theta_n}$

E_{r_n} goes to $E_{r_n} + \delta E_{r_n}$

These variational fields satisfy the homogeneous form of the original equations. Furthermore, in the finite-difference scheme, the variational fields will satisfy the homogeneous form of the difference equations. Consider a perturbation of the form[5]

$$\delta f(r,t) = \delta f_o e^{ikr+\alpha t} \tag{56}$$

where k may have any value from zero to infinity. This expression is one term of a Fourier series in r (k is real) and to achieve stability, the real part of α must be less than or equal to zero for any k. The relation between α and k can be found through the homogeneous finite-difference equations. First, expand the variational fields about the point (r,t), so that, for example,

$$\delta E(r+\Delta r, t+\Delta t) = \delta E(r,t) e^{ik\Delta r+\alpha\Delta t} \tag{57}$$

Using this expansion, the variational equations become

$$\delta E_{r_n}(r,t) \left\{ \frac{\sigma\Delta t}{2\varepsilon}\left[1+e^{\alpha\Delta t}\right] + \left(e^{\alpha\Delta t}-1\right) \right\} - \delta B_{\phi_n}(r,t) \left\{ \frac{c^2\Delta t n(n+1)}{2r^2}\left[1+e^{\alpha\Delta t}\right] \right\} = 0 \tag{58}$$

$$\delta E_{\theta_n}(r,t) \left\{ \frac{\sigma\Delta t}{2\varepsilon}\left[1+e^{\alpha\Delta t}\right] + \left(e^{\alpha\Delta t}-1\right) \right\} + \delta B_{\phi_n}(r,t) \frac{c^2\Delta t}{2\Delta r}\left[e^{ik\Delta r}-1+e^{\alpha\Delta t} - e^{-ik\Delta r+\alpha\Delta t}\right] = 0 \tag{59}$$

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[5]For a detailed discussion of the theory of the stability of finite-difference equations, see Richtmyer, R. D., "Difference Methods for Initial Value Problems," Interscience Publishers, 1957.

$$\delta E_{r_n}(r,t)\left\{\frac{\Delta t}{2}\left[e^{\alpha\Delta t}+1\right]\right\}+\delta B_{\phi_n}\left\{e^{\alpha\Delta t}-1\right\}+\delta E_{\theta_n}\left\{\frac{\Delta t}{2\Delta r}\left[e^{ik\Delta r}-1+e^{\alpha\Delta t}-e^{ik\Delta r+\alpha\Delta t}\right]\right\}=0 \tag{60}$$

For these equations to have a nontrivial solution, it is necessary that the determinant of the coefficient matrix be zero. For ease in manipulation, the following variables are defined:

$$x = e^{\alpha\Delta t}$$

$$a = \frac{\sigma\Delta t}{\varepsilon}$$

$$b = \frac{c^2\Delta t n(n+1)}{r^2}$$

$$g = \frac{\Delta t}{\Delta r}$$

$$m = k\Delta r$$

$$A = 1-e^{im}$$

$$A^* = 1-e^{-im} \tag{61}$$

Substituting these variables, the determinantal equation becomes,

$$\left[a(x+1)+2(x-1)\right]\left\{c^2g^2\left[-A+xA^*\right]^2-b\Delta t(x+1)^2-2(x-1)\left[a(x+1)+2(x-1)\right]\right\}=0 \tag{62}$$

If the first factor is to be zero, then

$$x = \frac{2-a}{2+a}$$

Since $x = e^{\alpha_r\Delta t+i\alpha_i\Delta t} = Re^{i\theta}$, a general representation of a complex number, then the magnitude of x is

$$|x| = \left|e^{\alpha\Delta t}\right| = \left|Re^{i\theta}\right| = \left|e^{\alpha_r\Delta t}\right| \tag{63}$$

Clearly, if there are to be no growing perturbations in the computed solutions, then the real part of α must be less than or equal to zero. This is always the case for the above equation, so that for this factor, the stability condition is met. If the second factor is to be zero, then

$$x = \frac{c^2g^2AA^*+b\Delta t-4\pm\sqrt{c^2g^2b\Delta t(A+A^*)^2+4c^2g^2(A-A^*)^2+2ac^2g^2\left(A^2-A^{*2}\right)+4a^2-16b\Delta t}}{c^2g^2A^{*2}-b\Delta t-2a-4} \tag{64}$$

The stability given by this equation will be considered for very small and very large conductivities. The first step will be to find the stability with the conductivity set equal to zero, in which case

$$x = \frac{c^2g^2AA^*+b\Delta t-4\pm\sqrt{c^2g^2b\Delta t(A+A^*)^2+4c^2g^2(A-A^*)^2-16b\Delta t}}{c^2g^2A^{*2}-b\Delta t-4} \tag{65}$$

Assume that c^2g^2 is ≤ 1. Then the maximum value of the sum under the square root must be less than or equal to zero. Therefore, the condition for stability, found from considering the square of the magnitude of x, is

$$\left(c^2g^2A^{*2}-b\Delta t-4\right)\left(c^2g^2A^2-b\Delta t-4\right)\geq\left(c^2g^2AA^*+b\Delta t-4\right)\left(c^2g^2AA^*+b\Delta t-4\right)-c^2g^2b\Delta t(A+A^*)^2$$
$$-4c^2g^2(A-A^*)^2+16b\Delta t \tag{66}$$

Expanding this one finds that there is an equality between the magnitudes of the numerator and the denominator, and thus the stability is marginal. That is to say, an error introduced into the field values neither grows nor decays with advancing time. Now assume that there is a small nonzero conductivity. Then, keeping only the two lowest order terms in a, the numerator of our stability expression is, schematically

$$N = w\pm\sqrt{u+iv} \tag{67}$$

$$|N|^2 = \left[w\pm\left(u^2+v^2\right)^{1/4}\cos\frac{\phi}{2}\right]^2+\left(u^2+v^2\right)^{1/2}\sin^2\frac{\phi}{2} = w^2\pm2w\left(u^2+v^2\right)^{1/4}\cos\frac{\phi}{2}+\left(u^2+v^2\right)^{1/2} \tag{68}$$

where

$$\varepsilon^2 = 4a^2$$

$$w = c^2g^2AA^*+2b\Delta t-4$$

$$u = c^2g^2b\Delta t(A+A^*)^2+4c^2g^2(A-A^*)^2-16b\Delta t+\varepsilon$$

$$v = \varepsilon c^2g^2\left(A^2-A^{*2}\right)$$

$$\phi = \tan^{-1}\left(\frac{v}{u}\right) \tag{69}$$

Note that $\frac{v}{u}$ is of order ε.

Since the real part of the expression under the square root sign in the numerator is still negative, $\phi \approx \pi$ and $\tan^{-1}\phi$ is π plus terms of $O(\varepsilon)$. Therefore, $\cos\frac{\phi}{2}$ is of $O(\varepsilon)$, and so $2w\left(u^2+v^2\right)^{1/4}\cos\frac{\phi}{2}$ is of $O\left(\varepsilon^3\right)$, as can be seen from a binomial expansion of $\left(u^2+v^2\right)^{1/4}$. But $\left(u^2+v^2\right)^{1/2}$ is of $O\left(\varepsilon^2\right)$; therefore, only the term $\left(u^2+v^2\right)^{1/2}$ will be kept, and higher order terms will be disregarded. Thus

$$|N|^2 = w^2+\left(u^2+v^2\right)^{1/2} \tag{70}$$

$$|N|^2 \simeq w^2+|u|+\frac{1}{2}\ \frac{v^2}{|u|} \tag{71}$$

Under the small conductivity approximation the magnitude of the denominator of equation (64) squared may be written

$$|D|^2 = (s+\varepsilon)^2+t^2 \tag{72}$$

where

$$s = 4-\frac{c^2g^2}{2}\left(A^{*2}+A^2\right) \geq 0,$$

$$t = \frac{c^2g^2}{2}\left(A^{*2}-A^2\right) \tag{73}$$

Therefore, the stability condition is

$$s^2+2s\varepsilon+\varepsilon^2+t^2 \geq v^2+|u|+\frac{1}{2}\frac{v^2}{|u|} \tag{74}$$

From the results of the discussion of the zero conductivity case, this reduces immediately to

$$2s\varepsilon+\varepsilon^2 \geq \frac{1}{2}\frac{v^2}{|u|} \tag{75}$$

This is a strict inequality for nonzero a, and therefore, the finite-difference equations form an unconditionally stable system. Now consider the opposite limiting case, where the conductivity becomes arbitrarily large. Only the two highest order terms in "a" will be kept. As before, the two highest order terms are necessary, since keeping only the highest order term predicts marginal stability, and it is desirable to know whether marginal stability is approached from a condition of stability or instability. In the limit of large a, the numerator of expression (64) is approximated by

$$N = w \pm \left(u^2+v^2\right)^{1/4} e^{i\frac{\phi}{2}} \tag{76}$$

where

$$w = c^2g^2AA^*+b\Delta t-4$$

$$u = 4a^2$$

$$v = 2ac^2g^2\left(A^2-A^{*2}\right)$$

$$\phi = \tan^{-1}\frac{v}{u} \tag{77}$$

then

$$|N|^2 = \left[w \pm \left(u^2+v^2\right)^{1/4}\cos\frac{\phi}{2}\right]^2 + \left(u^2+v^2\right)^{1/2}\sin\frac{2\phi}{2} \tag{78}$$

The terms in this which are of the highest order "a" are

$$|N|^2 \approx \pm 2wu^{1/2}\cos\frac{\phi}{2} + \left(u^2+v^2\right)^{1/2} \tag{79}$$

Squaring this again, and keeping only the two highest order terms in "a"

$$|N|^4 \approx u^2 \pm 4wu^{3/2}\cos\frac{\phi}{2} \tag{80}$$

$$|N|^4 \approx \left(4a^2\right)^2 \pm 32a^3\left[-c^2g^2AA^* - b\Delta t + 4\right]\cos\frac{\phi}{2} \tag{81}$$

By a similar process applied to the denominator,

$$|D|^4 \approx \left(4a^2\right)^2 + 32a^3\left[b\Delta t + 4 - \frac{c^2g^2}{2}\left(A^{*2}+A^2\right)\right] \tag{82}$$

The stability expression becomes

$$|x|^4 = \frac{|N|^4}{|D|^4} = \frac{\left(4a^2\right)^2 \pm 32a^3\left[4 - b\Delta t - c^2g^2AA^*\right]\cos\frac{\phi}{2}}{\left(4a^2\right)^2 + 32a^3\left[4 + b\Delta t - \frac{c^2g^2}{2}\left(A^{*2}+A^2\right)\right]} \tag{83}$$

Since

$$\frac{A^2+A^{*2}}{2} = 2\cos\theta(\cos\theta - 1) \tag{84}$$

and

$$AA^* = 2\,(1-\cos\theta) \tag{85}$$

then

$$|x|^4 = \frac{a^4 \pm 2a^3\left[4 - \left[b\Delta t + 2c^2g^2(1-\cos\theta)\right]\right]\cos\frac{\phi}{2}}{a^4 + 2a^3\left[4 + b\Delta t + 2c^2g^2\cos\theta(1-\cos\theta)\right]} \tag{86}$$

Since $a^4 > 0$, the differencing scheme will be stable so long as

$$4+b\Delta t+2c^2g^2\cos\theta(1-\cos\theta) \geq \pm \left\{ 4-\left[b\Delta t+2c^2g^2(1-\cos\theta)\right] \right\} \cos\frac{\phi}{2} \qquad (87)$$

The right-hand side of this expression will be greatest for either

$\phi = 0$ or $\phi = 2\pi$

Therefore, the two possible forms of equation (87) are

$$4+b\Delta t+2c^2g^2\cos\theta(1-\cos\theta) \geq 4-b\Delta t-2c^2g^2(1-\cos\theta) \qquad (88)$$

$$2b\Delta t \geq -2c^2g^2(1-\cos^2\theta) \qquad (89)$$

(This is a strict inequality and implies stability.)

And

$$4+b\Delta t+2c^2g^2\cos\theta(1-\cos\theta) \geq -4+b\Delta t+2c^2g^2(1-\cos\theta) \qquad (90)$$

or

$$8 \geq 2c^2g^2(1-\cos\theta)^2 \qquad (91)$$

(This is a strict inequality for $c^2g^2 < 1$.)

Thus, the finite-differencing scheme is stable for large values of the conductivity, as before, under the condition that $c^2g^2<1$.

SECTION VI

A QUADRATIC ACCURACY INDICATOR

Lax's equivalence theorem gives an indication that as long as the computation converges, one need only use a fine enough mesh size to obtain any desired accuracy. But the question that still remains is how small that mesh size should be for a particular accuracy requirement on a particular problem. The following discussion will derive a relationship which can be used to check the internal consistency of a solution as that solution is being computed, so that both the solution and a check upon its consistency may be taken as the problem output.

Returning to Maxwell's equations

$$\frac{1}{\mu}\nabla x\overline{B} = \overline{J}_d + \sigma\overline{E} + \varepsilon\frac{\partial\overline{E}}{\partial t}$$

$$\nabla x\overline{E} = -\frac{\partial\overline{B}}{\partial t} ,$$

one forms the scalar product of $\overline{E}$ with the first equation, and $\overline{B}$ with the second equation. The two resulting equations are then subtracted, giving

$$\frac{1}{\mu}\overline{E}\cdot\nabla x\overline{B} - \frac{1}{\mu}\overline{B}\cdot\nabla x\overline{E} = \overline{J}_d\cdot\overline{E} + \sigma E^2 + \frac{1}{2}\varepsilon\frac{\partial E^2}{\partial t} + \frac{1}{2\mu}\frac{\partial B^2}{\partial t}$$

Since

$$\nabla\cdot(\overline{E}x\overline{B}) = \overline{B}\cdot\nabla x\overline{E} - \overline{E}\cdot\nabla x\overline{B},$$

$$\overline{J}_d\cdot\overline{E} + \sigma E^2 + \frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right) + \frac{1}{\mu}\nabla\cdot(\overline{E}x\overline{B}) = 0$$

This may be interpreted as a statement of the conservation of energy. The first term is the power being taken from the fields by the current J_d; the second is the ohmic power loss; the third is the rate of change of energy in the fields; and the fourth is the divergence of the poynting vector.

Remembering that only the E_r, E_θ, and B_ϕ components of the fields exist according to the original symmetry conditions, the expression becomes

$$J_rE_r+J_\theta E_\theta+\sigma E_r^2+\sigma E_\theta^2+\frac{1}{r^2\mu}\frac{\partial}{\partial r}\left(r^2E_\theta B_\phi\right)-\frac{1}{\mu r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta E_rB_\phi\right)$$

$$+\frac{1}{2}\frac{\partial}{\partial t}\left\{\varepsilon\left[E_r^2+E_\theta^2\right]+\frac{1}{\mu}B_\phi^2\right\}=0$$

If the field components are then replaced with their Legendre expansions, and the result is multiplied by sin θ and integrated over θ from zero to π, the integrals to be evaluated are essentially the Legendre orthogonality integrals.[6] Thus, the quadratic expression above results in a separate equation for the coefficients of each order of polynomial, the equation being

$$E_{r_n}J_{r_n}+n(n+1)E_{\theta_n}J_{\theta_n}+\sigma E_{r_n}^2+\sigma n(n+1)E_{\theta_n}^2+\frac{n(n+1)}{\mu r^2}\frac{\partial}{\partial r}\left(r^2E_{\theta_n}B_{\phi_n}\right)$$

$$+\frac{1}{2}\frac{\partial}{\partial t}\left\{\varepsilon E_{r_n}^2+n(n+1)\left[\varepsilon E_{\theta_n}^2+\frac{1}{\mu}B_{\phi_n}^2\right]\right\}=0$$

If now part of this equation is transposed to the right-hand side and the equation divided by that transposed quantity, the comparison of the resultant quotient with unity gives a check on the internal consistency of the solution.

To obtain a sensitive accuracy indicator, care must be taken to subtract dominant terms in the above expression.

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[6] See Appendix

SECTION VII

TEST PROBLEM

In numerical analysis, a test problem is a problem whose solution is known analytically but which still closely approximates the more difficult problems that cannot be solved analytically. The following problem was chosen because it is typical of a large class of electromagnetic problems and because its solution requires the use of all of the parameters in the problem; furthermore it gives a clear indication of the effect of grid size upon accuracy.

It was decided that an exponential time dependence for the test problem fields would be used. Such a time dependence can be used to represent either periodic or pulsed fields, and furthermore it simplifies the analysis of the effects of grid size. An H_ϕ and E_θ distance dependence whose leading term was 1/r is desirable in that this dependence represents the radiation field.

If the form

$$\overline{B} = \overline{\phi}B_\phi = \frac{\mu}{r}\, e^{\alpha(t-r/c)} P_n^1(\cos\theta)\ \overline{\phi}$$

is assumed, and σ restricted to an arbitrary real positive constant, the equation satisfied by $\overline{J}_d$ is

$$\nabla\times\overline{J} = \nabla\times\nabla\times\overline{B}+\sigma\dot{\overline{B}}+\epsilon\ddot{\overline{B}}$$

Substituting the above form for $\overline{B}$, we find that an appropriate J_r is, with $J_\theta = 0$,

$$J_r = -\left\{ \frac{\alpha^2}{c^2} - \frac{n(n+1)}{r^2} - (\sigma+\alpha\epsilon)\alpha\mu \right\} e^{\alpha(t-r/c)} P_n^o(\cos\theta)$$

Then from the r component of the equation

$$\frac{1}{\mu}\nabla\times\overline{B} = \overline{J}+\sigma\overline{E}+\epsilon\dot{\overline{E}}$$

a solution for E_r of the form

$$E_r = \frac{1}{\sigma+\alpha\varepsilon}\left\{\frac{\alpha^2}{c^2} - (\sigma+\alpha\varepsilon)\alpha\mu\right\} e^{\alpha(t-r/c)}P_n^o(\cos\theta)$$

is obtained. Finally, from the θ component of the same equation,

$$E_\theta = \frac{\alpha}{rc(\sigma+\alpha\varepsilon)} e^{\alpha(t-r/c)}P_n^1(\cos\theta)$$

To this point, we have exhibited a particular solution to Maxwell's equations. There are still an infinite number of solutions to the homogeneous equation which may be added to this particular solution, the sum still satisfying Maxwell's equations. These solutions to the homogeneous equations may be set to zero by a particular specification of the boundary condition and initial conditions. Both the field values and their derivatives must be specified at the origin. This is done in finite difference form by specifying the fields at the origin and on the first grid line away from the origin. Since the field variables actually used are rB_ϕ, rE_θ, and E_r, all variables are finite at the origin and elsewhere.

Actual calculations are run forward from some starting time t_o. The example, however, assumes that the current source has been in existence from $t = -\infty$. Therefore, it is necessary to insert the analytic values for the fields at the first grid points such that $t_o - r/c \leq 0$ in order to conform to the analytical problem. The problem now has a properly posed set of boundary and initial conditions, and its solution may be compared directly to the analytic solution.

Example:

The curves shown in figures 5, 6, and 7 are the results of a numerical calculation performed on the test problem with the following parameters:

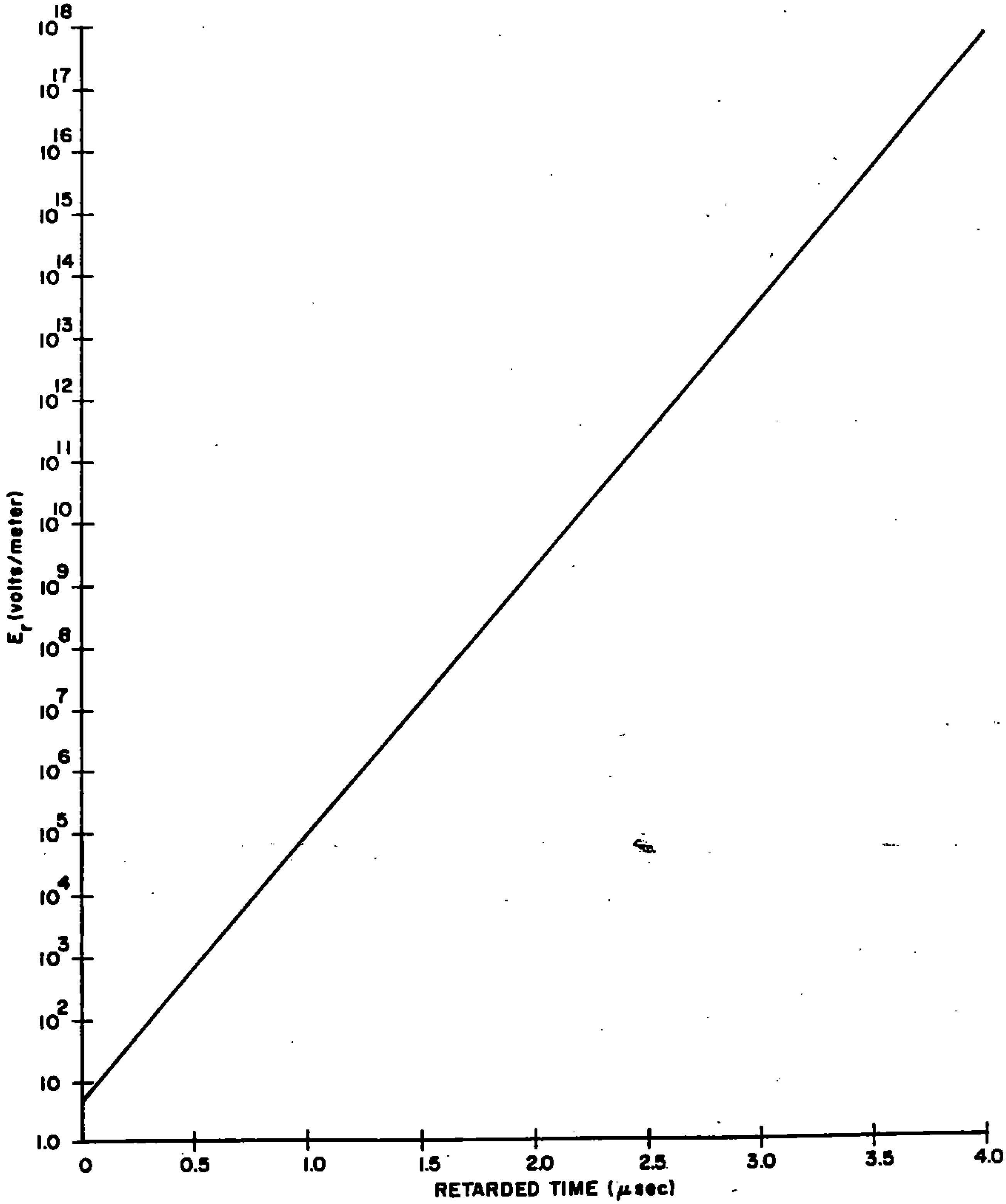

Figure 5. Test Problem, E_r vs. Time

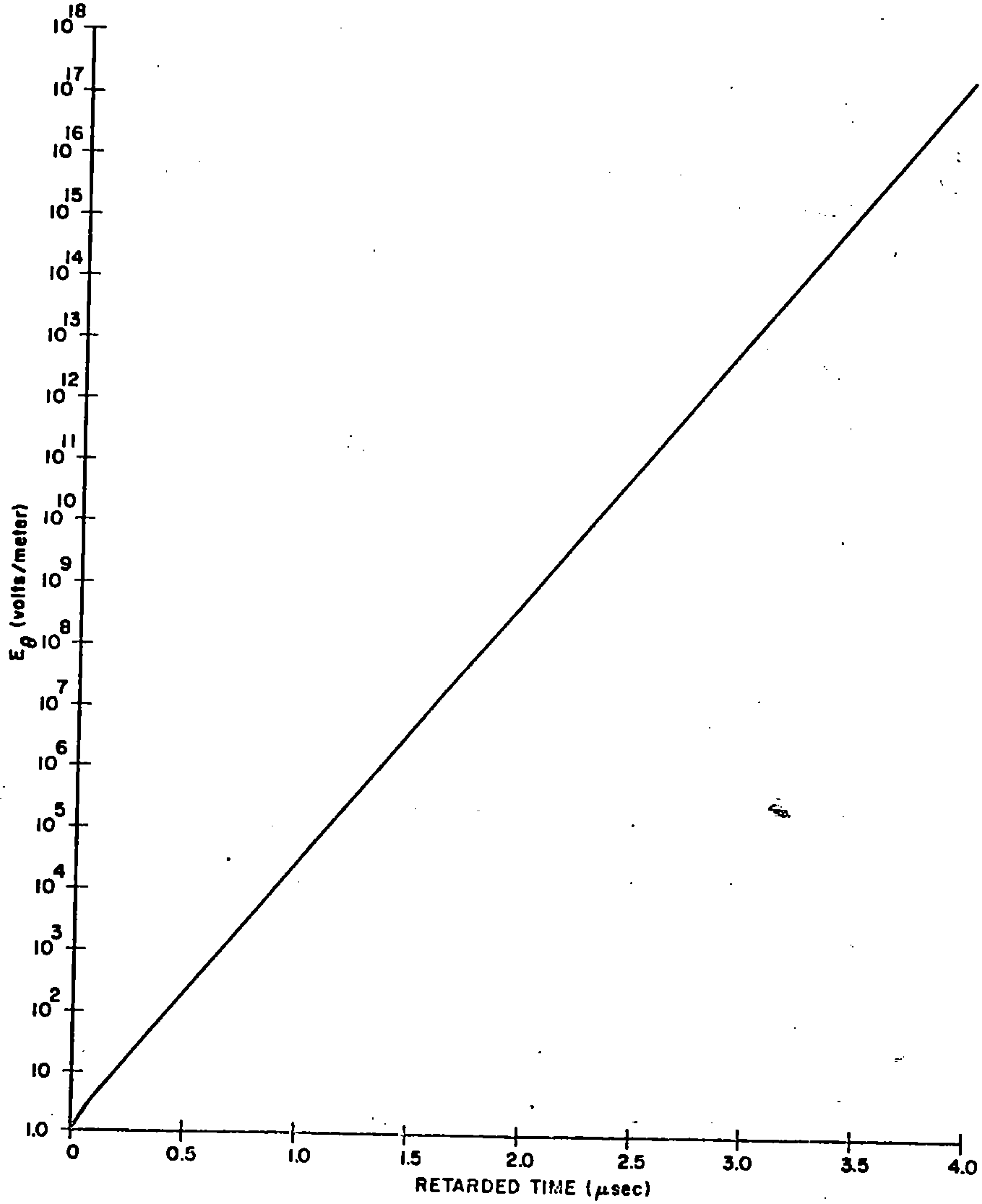

Figure 6. Test Problem, E_θ vs. Time

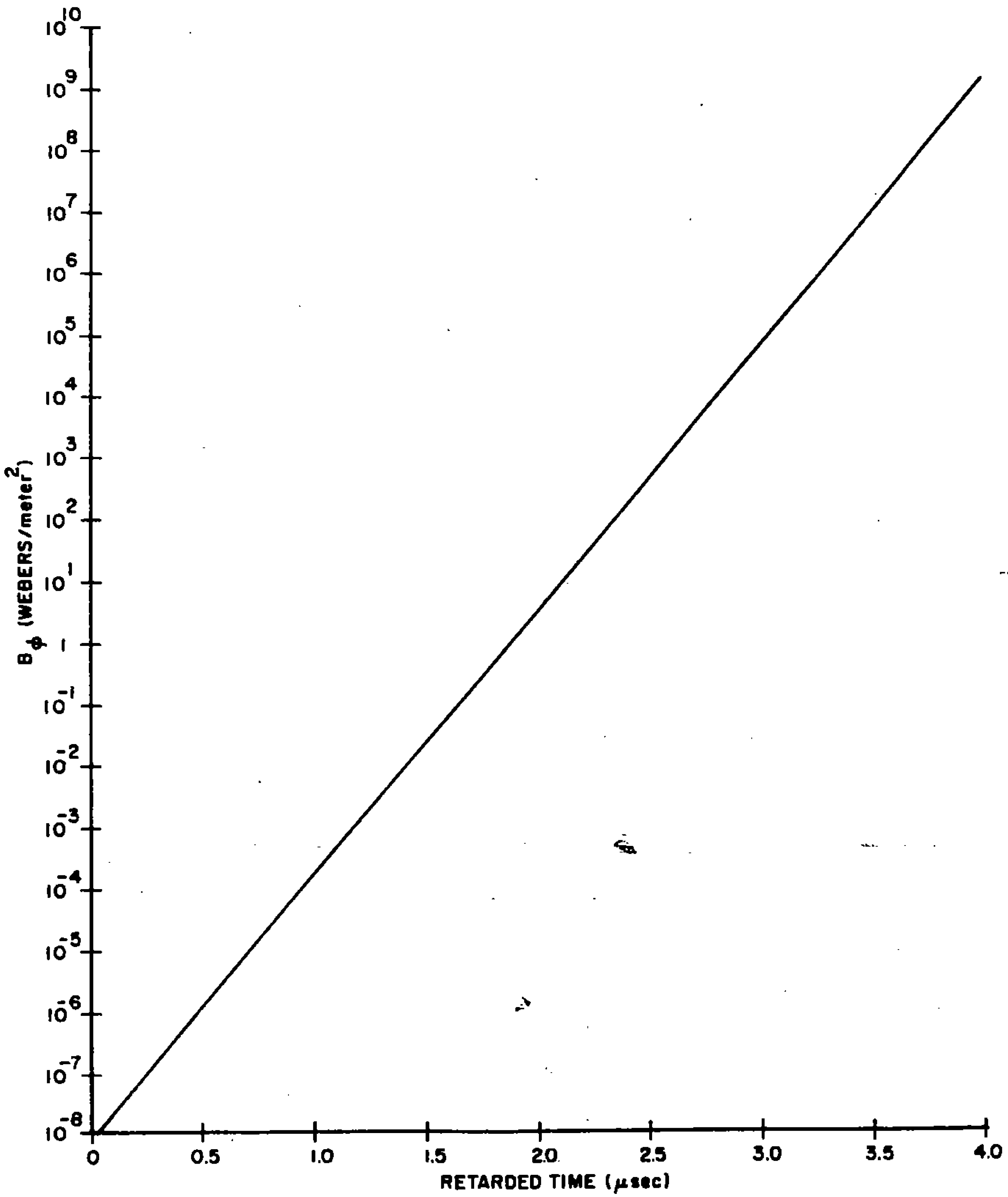

Figure 7. Test Problem, B_ϕ vs. Time

$N = 1$

$\sigma = 8.85 \times 10^{-5}\ [\Omega m]^{-1}$

$\varepsilon = \varepsilon_o$

$\mu = \mu_o$

$\alpha = 10^7\ \text{sec}^{-1}$

$\Delta t = 2.0 \times 10^{-8}$ sec

$\Delta r = 12$ meters

The range for the fields in the curves was 120 meters. The exact solutions for the fields fall directly on the numerically calculated curves over the 18 decades shown. To give a better indication of the convergence of the numerical solutions in this problem, the relative error,

$$\frac{f_N - f_E}{f_E}$$

where

f_N = numerical field solutions

f_E = exact field solutions

is plotted in figures 8, 9, 10, 11, 12, and 13. In figures 8, 9, and 10, a grid size of

$\Delta t = 2.0 \times 10^{-8}$ sec

$\Delta r = 12$ meters

was used, while in figures 11, 12, and 13 a grid size of

$\Delta t = 4.0 \times 10^{-8}$ sec

$\Delta r = 24$ meters

was used. In both cases, the range at which the fields were taken was 120 m. As expected, the relative errors were greater when the larger grid size was used, but even a time step of 40 percent of the current efolding time gave answers correct to within 40 percent.

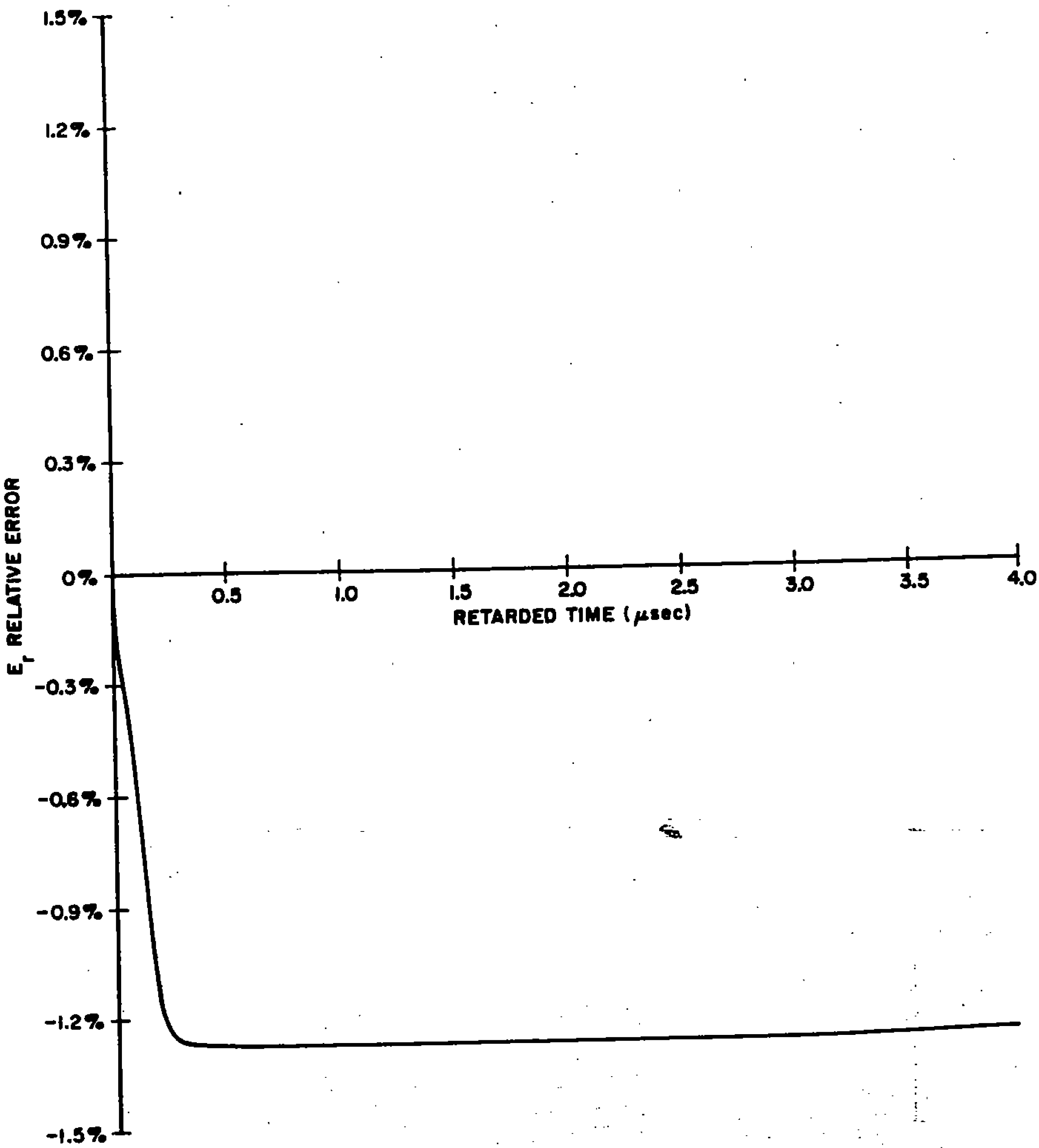

Figure 8. Finer Grid: Relative Error in E_r Computation vs. Time, Range = 120m

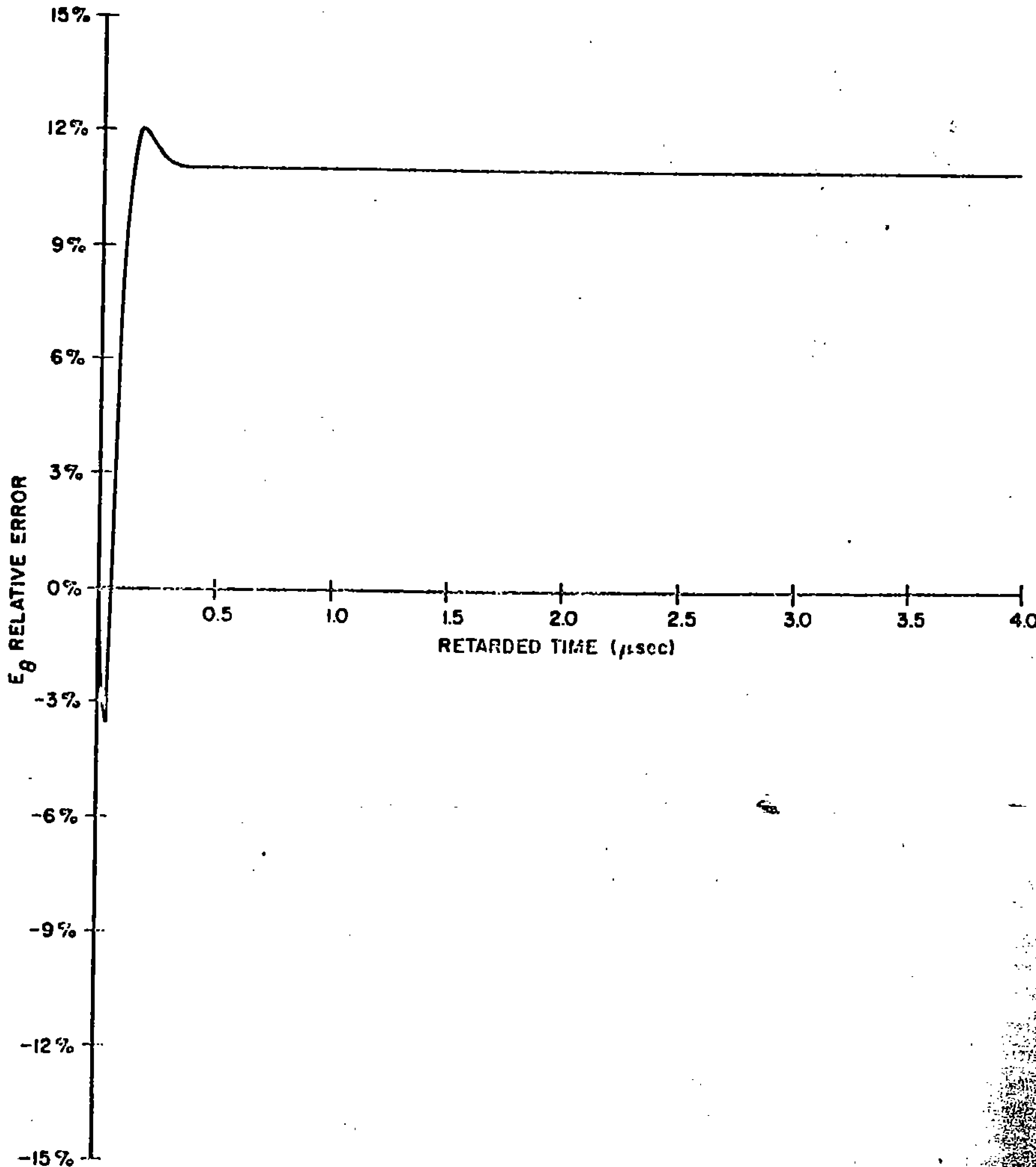

Figure 9. Finer Grid: Relative Error in E_θ Computation vs. Time, Range = 120 m

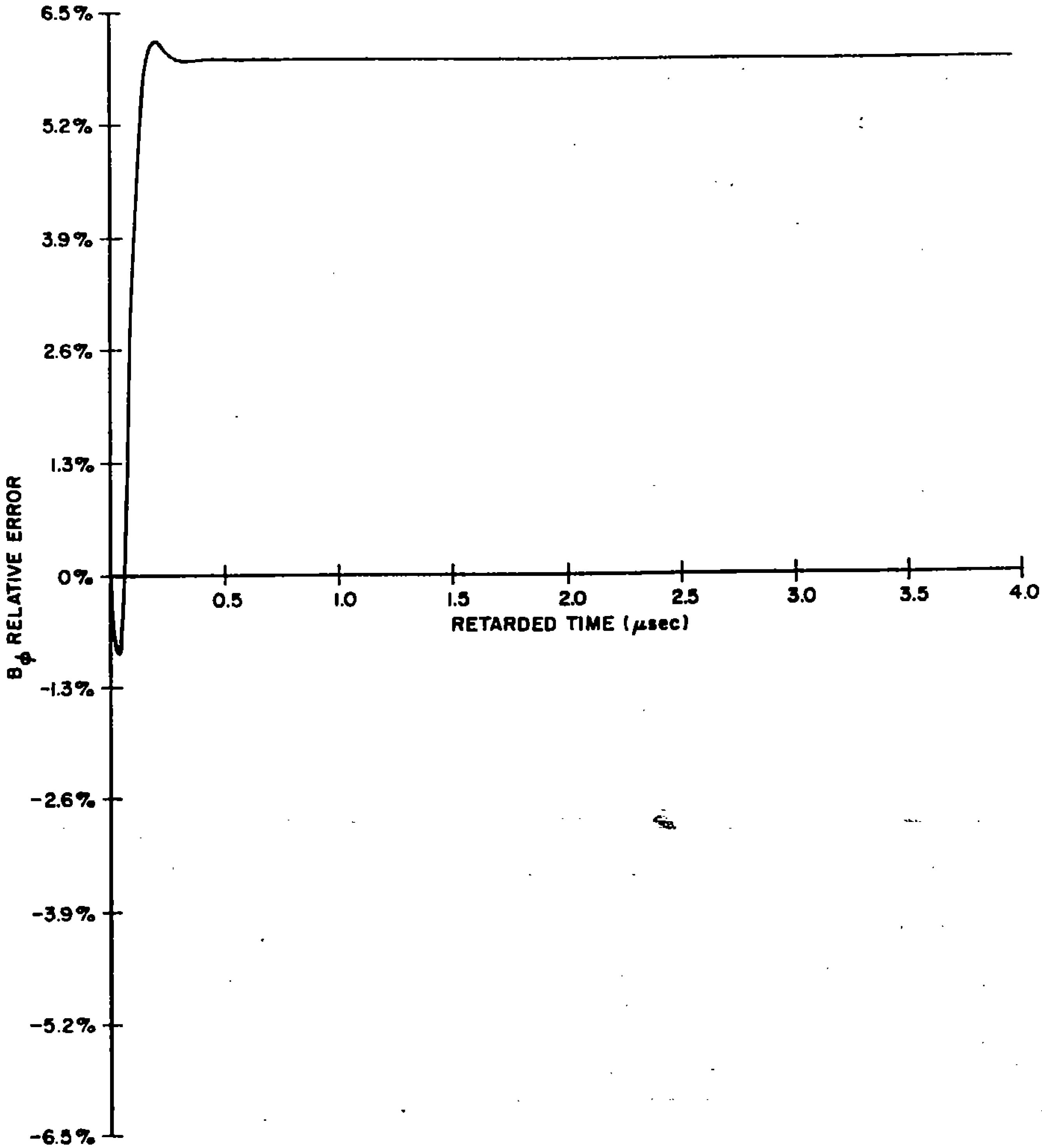

Figure 10. Finer Grid: Relative Error in B_ϕ Computation vs. Time, Range = 120m

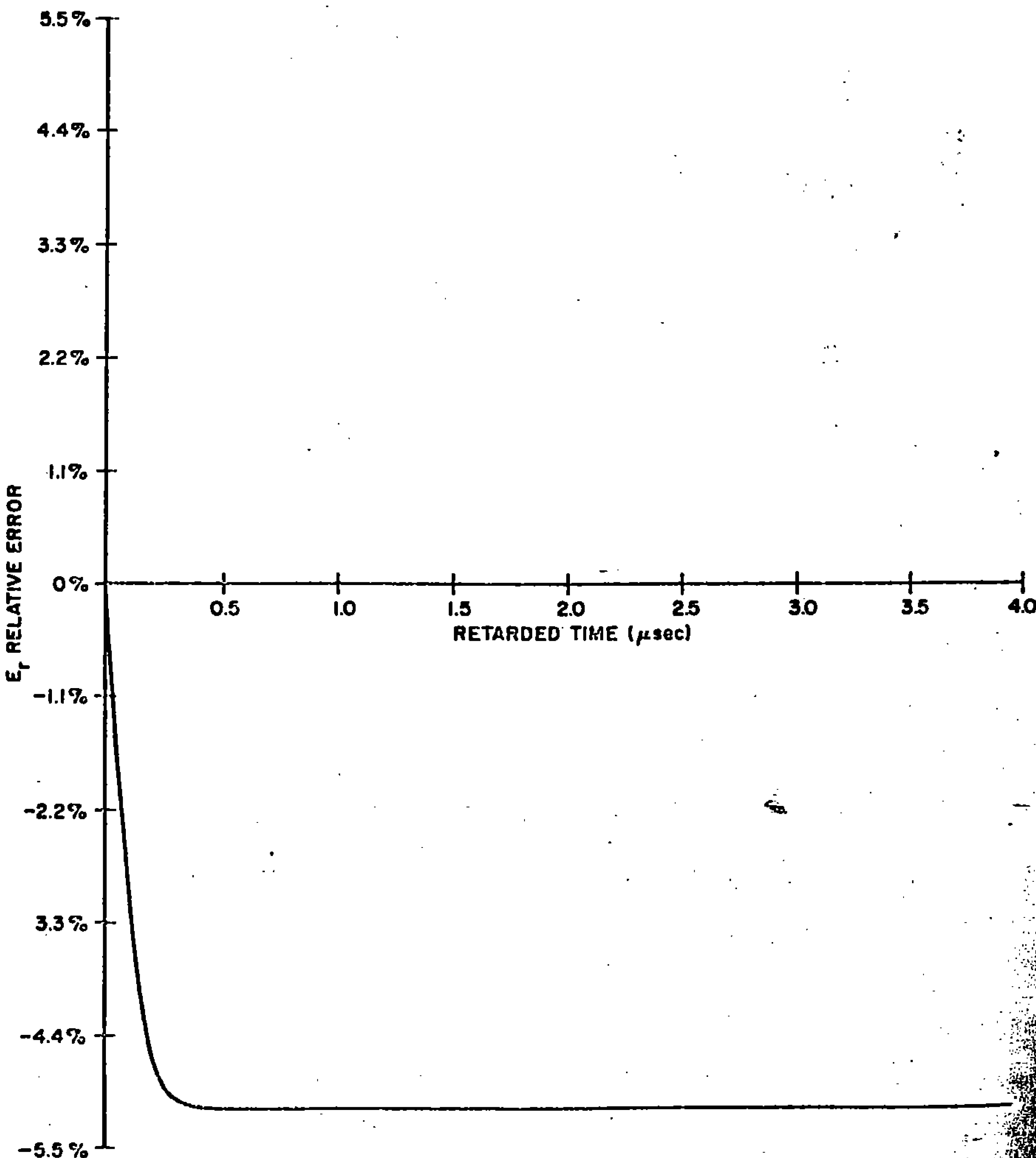

Figure 11. Coarser Grid: Relative Error in E_r Computation vs. Time, Range = 120m

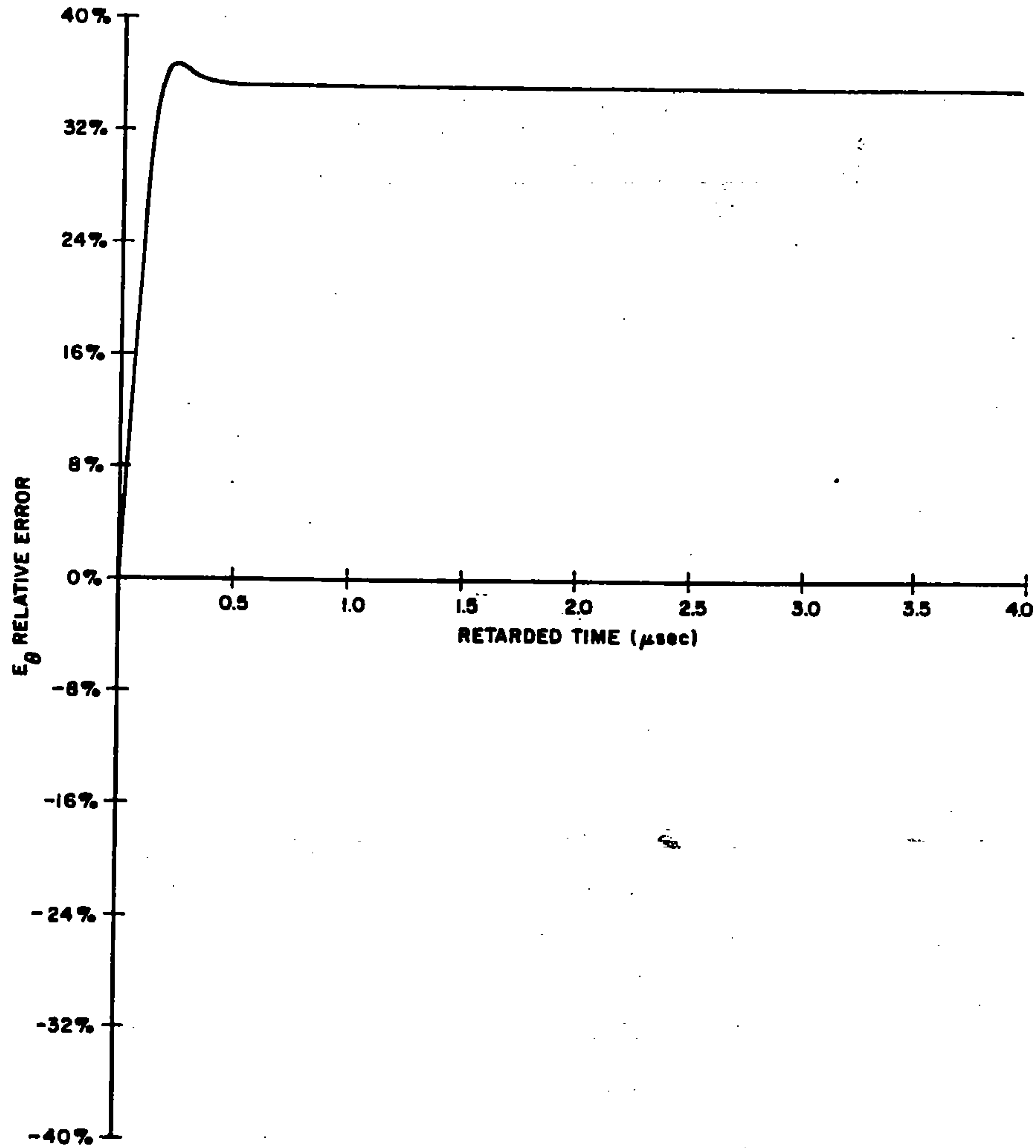

Figure 12. Coarser Grid: Relative Error in E_θ Computation vs. Time, Range = 120m

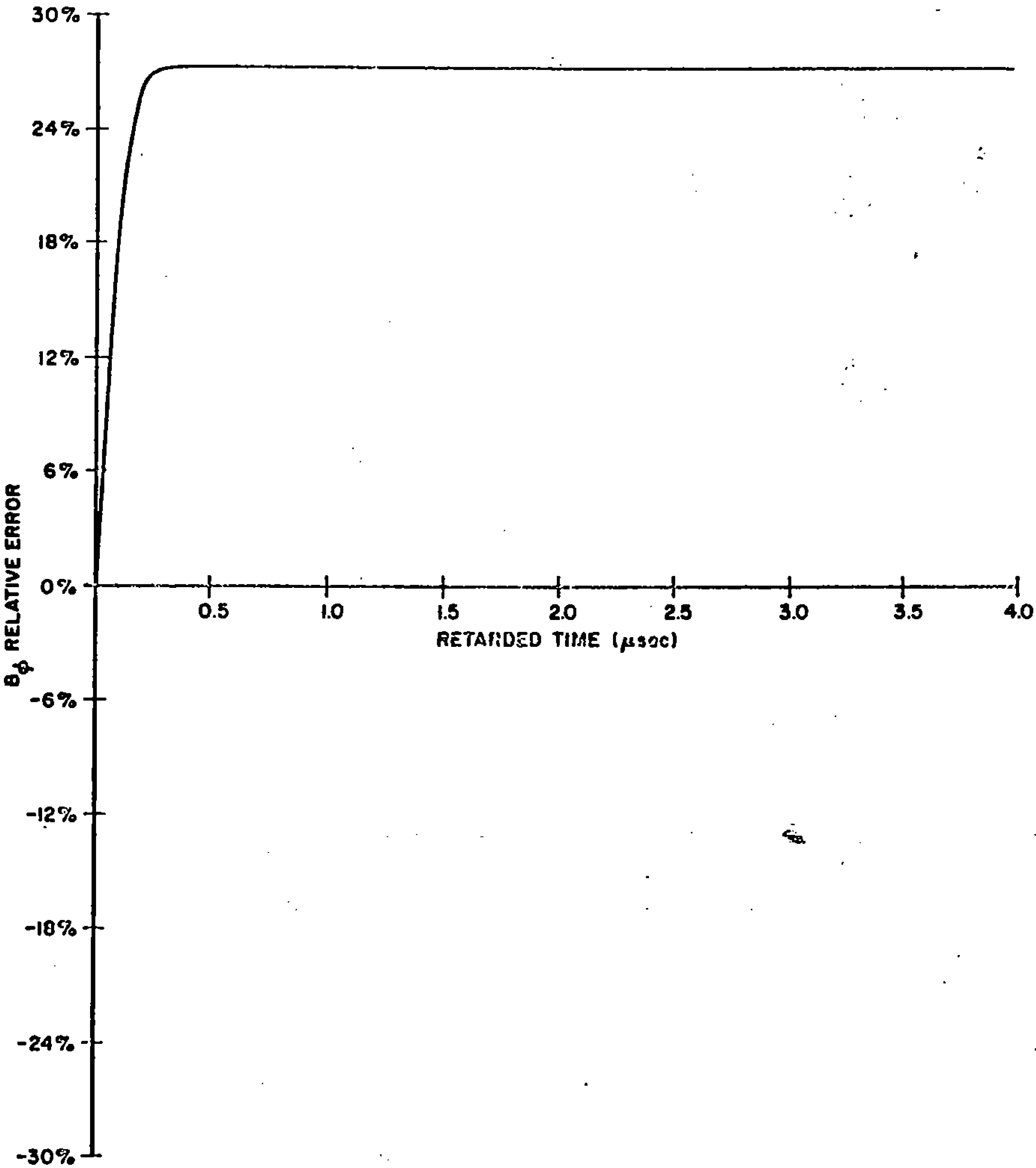

Figure 13. Coarser Grid: Relative Error in B_ϕ Computation vs. Time, Range = 120m

SECTION VIII

CONCLUSION

This numerical technique has proved successful in calculating the electromagnetic fields generated by a nuclear weapon detonation. A report covering the results of these field calculations will be available in the near future.

APPENDIX

LEGENDRE POLYNOMIALS

Ordinary Legendre Polynomials: The ordinary Legendre polynomials $P_n^o(\cos\theta)$ arise as the theta dependent component of the orthogonal functions into which Laplace's equation, $\nabla^2 f = 0$ separates in the spherical coordinate system. The equation satisfied by $P_n^o(\cos\theta)$ is

$$\frac{d^2 P_n^o(\cos\theta)}{d\theta^2} + \frac{1}{\tan\theta}\frac{dP_n^o(\cos\theta)}{d\theta} + n(n+1)P_n^o(\cos\theta) = 0$$

A general relation for finding the ordinary Legendre polynomials is

$$P_n^o(\cos\theta) = \frac{1}{2^n n!}\left[\frac{d}{d(\cos\theta)}\right]^n \left(\cos^2\theta - 1\right)^n$$

The ordinary Legendre polynomials form a complete set for expanding bounded functions in the interval $0 \le \theta \le \pi$. Over this range, the orthogonality condition is

$$\int_0^\pi P_n^o(\cos\theta)P_m^o(\cos\theta)\sin\theta d\theta = 0 \quad (m \neq n)$$

$$\int_0^\pi \left[P_n^o(\cos\theta)\right]^2 \sin\theta d\theta = \frac{2}{2n+1}$$

The first few ordinary Legendre polynomials are

$$P_0^o(\cos\theta) = 1$$

$$P_1^o(\cos\theta) = \cos\theta$$

$$P_2^o(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$

$$P_3^o(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

$$P_4^o(\cos\theta) = \frac{1}{8}(35\cos^4\theta-30\cos^2\theta+3)$$

$$P_5^o(\cos\theta) = \frac{1}{8}(63\cos^5\theta-70\cos^3\theta+15\cos\theta)$$

Associated Legendre Polynomials: The associated Legendre polynomial is a more general type of Legendre polynomial and arises in the separation of variables in the solution of the Helmholtz equation,

$$\nabla^2 f + kf = 0$$

when performed in spherical coordinates. It is obvious that the ordinary Legendre polynomials result from this equation in the special case of $k = 0$.

The equation satisfied by the associated Legendre polynomial $P_n^k(\cos\theta)$ is

$$\frac{d^2\left[P_n^k(\cos\theta)\right]}{d\theta^2} + \frac{1}{\tan\theta}\frac{d\left[P_n^k(\cos\theta)\right]}{d\theta} + \left[n(n+1)-\frac{k^2}{\sin^2\theta}\right]P_n^k(\cos\theta) = 0$$

A general relation for finding the first associated Legendre polynomials is

$$P_n^1(\cos\theta) = -\frac{d}{d\theta}P_n^o(\cos\theta)$$

The first few associated Legendre polynomials of the first kind are

$$P_0^1(\cos\theta) = 0$$

$$P_1^1(\cos\theta) = \sin\theta$$

$$P_2^1(\cos\theta) = 3\sin\theta\cos\theta$$

$$P_3^1(\cos\theta) = \frac{3}{2}\sin\theta\left(5\cos^2\theta-1\right)$$

$$P_4^1(\cos\theta) = \frac{5}{2}\sin\theta\left(7\cos^3\theta-3\cos\theta\right)$$

$$P_5^1(\cos\theta) = \frac{15}{8}\sin\theta\left[21\cos^4\theta-14\cos^2\theta+1\right]$$