Theoretical Notes
Note 53

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AN ANALYTICALLY SOLVABLE MODEL FOR THE ELECTROMAGNETIC FIELDS PRODUCED BY NUCLEAR EXPLOSIONS

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PREFACE

This study is one result of RAND's continuing interest in the electromagnetic effects of nuclear explosions. It is a contribution to theory, and may be of interest to students of systems intended to function in the environment of such explosions.

SUMMARY

This Memorandum presents a theoretical expanding electromagnetic field structure analogous to that produced by a nuclear explosion. The model displays time-and-space varying conductivity and current, but can be solved analytically. There are certain long-time non-physical effects, but the fields are plausible for an interesting portion of the time scale.

LIST OF FIGURES

LIST OF SYMBOLS

A	function related to vector potential
a	$1/2\ \mu c\sigma_o$ = half the ratio of the conductivity and the free-space characteristic admittance
B	earth's magnetic induction (webers/meter^2)
B_r	radial component of total magnetic induction
B_θ	latitudinal component of total magnetic induction
c	velocity of light (meters/sec)
E_r	radial component of electric field (volts/meter)
E_φ	azimuthal component of electric field
e	electron charge (coulombs)
F	function related to vector potential = r^2A
F_n	coefficient in Legendre polynomial expansion of F
f	gamma ray production rate
J	azimuthal current density (amperes/meter^2)
J_o	constant related to peak value of J
J_n	coefficient in Legendre polynomial expansion of J
m	electron mass (kg)
$n(r,t)$	number of electrons per m^3 at distance r and time t
P_n	Legendre polynomial
p	secondary electron multiplication factor
R	range of Compton electrons (meters)
r	distance from origin (meters)
t	time (sec)
Y	explosion yield (KT)

z	r/ct--similarity variable
β	attachment rate of electrons to oxygen (sec^{-1})
η	fraction of total energy emitted as gamma rays
θ	colatitude
λ	mean free path of gamma rays (meters)
μ	permeability of free space
σ	conductivity (mho/meter)
σ_o	constant related to peak value of σ
φ	azimuthal coordinate
ω_e	electron mobility [(meters/sec)/(volt/meter)]

There have been numerous investigations of the electromagnetic fields produced by nuclear explosions. The normal procedure is to develop a model which is reasonably close to physical fact, then find an approximate mathematical solution to the model. This approximate solution may be purely numerical, which may lead to considerable difficulty in ascertaining how the results change as the parameters of the problem are varied.

The major physical process occurring in atmospheric nuclear explosions which gives rise to electromagnetic fields is non-uniform production of charged particles. The gamma rays from the explosion will be absorbed in the atmosphere, and will produce Compton electrons. Motion of these electrons corresponds to currents, and the atmosphere is also rendered conducting. The time and space variation of these currents then produces electromagnetic fields. Also, the process is complicated by the earth's magnetic field, which deflects the electrons to produce additional currents.

This combination of processes is sufficiently complex that it appears worthwhile to study a model which retains certain features of the physical situation but is capable of being solved analytically. Although the approximation to the actual situation is very rough, and there are certain non-physical aspects, still the feature of time and space variation of conductivity and current is retained. Such a theory is of considerable mathematical interest, over and above the direct physical applications.

In a low-level nuclear explosion, the gamma rays will be emitted with approximate spherical symmetry. The scale height of the atmosphere is about 8 km, so the variation of density with altitude will be neglected. Consideration of the proximity effects caused by the presence of a conducting earth is exceedingly complicated. The effects should not produce more than a factor of 2 difference in the results, so they also will be neglected. As the gamma rays are scattered by the atmosphere, they produce Compton electrons, which retain a large portion of the momentum of the gamma rays and therefore principally move radially outward. These fast primary Compton electrons produce slow-moving secondaries by ionizing the atmosphere. The secondaries will perform three-body attachment reactions with oxygen molecules, where the attachment time constant is about 10^{-8} sec. The major part of the atmospheric conductivity comes from these secondaries.

The earth's magnetic field will deflect the electrons. Since the deflection force is proportional to the velocity, the primaries suffer greater deflection. However, the electron gyration frequency in the earth's field at sea level (0.5 gauss) is 1.4 Mc. The distance the electrons move in an attachment time is thus a small fraction of a gyroradius, and the deflection may be regarded as a straight line. Taking the field to be vertical, the deflection will be azimuthal. A secondary magnetic field will be produced by this azimuthal current, and an azimuthal electric field will develop to oppose the motion.

Take a system of spherical coordinates, with the origin at the center of the burst, and the pole in the direction of the earth's field. There will be radial and azimuthal components of current. The radial current, arising from the radial motion of the primary and secondary Comptons, will be a function of distance and time only.

The azimuthal current is proportional to the component of the earth's magnetic field perpendicular to the radial direction, and therefore to the sine of the colatitude, as well as depending on distance and time. The secondary magnetic field will also produce a deflection, but this deflection is a higher order term.

Under these conditions the Maxwell equations split into two groups, one containing the radial electric field, the other the radial magnetic. This is akin to the separation into transverse electric and transverse magnetic waves. No latitudinal electric or azimuthal magnetic field will be generated. The radial electric field equation states that the time rate of change of the radial component of electric field is proportional to the total radial current. This current is made up of a σE_r term and a driving current term. The equation is a first-order linear ordinary differential equation, which may be solved immediately, and will not be considered further. The second group of equations involves the azimuthal electric field, the radial and latitudinal magnetic fields, and the azimuthal current, and appears as follows:

$$\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \, E_\varphi = - \frac{\partial B_r}{\partial t} \qquad (1)$$

$$\frac{1}{r} \frac{\partial}{\partial r} r \, E'_\varphi = \frac{\partial B_\theta}{\partial t} \qquad (2)$$

$$\frac{1}{r} \frac{\partial}{\partial r} r B_\theta - \frac{1}{r} \frac{\partial B_r}{\partial \theta} = \frac{1}{c^2} \frac{\partial E_\varphi}{\partial t} + \mu[\sigma(r,t) E_\varphi + J(r,t) \sin \theta] \qquad (3)$$

where MKS units have been used throughout.

The problem is to solve these equations with the appropriate initial conditions. Since the situation is a spherically expanding field, when $t < \frac{r}{c}$ the electric field, conductivity, and current vanish, while the magnetic field is uniform and equals the earth's magnetic field B. The angular dependence is established for all time by the factor sin θ in the current. A representation which integrates Eqs. (1) and (2) is:

$$E_\varphi = \frac{\sin \theta}{r} \frac{\partial}{\partial t} F(t,r) \tag{4}$$

$$B_\theta = -B \sin \theta + \frac{\sin \theta}{r} \frac{\partial}{\partial r} F(t,r) \tag{5}$$

$$B_r = B \cos \theta - \frac{2 \cos \theta}{r^2} f(t,r) \tag{6}$$

where F satisfies the wave equation

$$\frac{\partial^2 F}{\partial r^2} - \frac{2F}{r^2} = \frac{1}{c^2} \frac{\partial^2 F}{\partial t^2} + \mu\left[\sigma(r,t) \frac{\partial F}{\partial t} + r\, J(r,t)\right] \tag{7}$$

The analysis to this point has remained reasonably close to the physical situation. However, an attempt to solve Eq. (7), with the appropriate initial conditions, for the actual functional forms of conductivity and current leads to very great complications. Accordingly, the model to be further developed here will use functional forms for conductivity and current which permit analytic solution of Eq. (7). One frequently used form regards the conductivity as infinite inside an expanding sphere and zero outside, with which form Eq. (7) becomes quite simple. However, the physical structure is such that the conductivity and current at a given point must start from zero when the electrons arrive, increase rapidly to a maximum, and then decay in some manner with time. A functional form of this type which still permits analytic solution has been found by ascertaining the conditions under which Eq. (7) possesses a "similarity" solution. The initial conditions are that F and $\frac{\partial F}{\partial t}$ vanish for $r \geq ct$. Let us attempt to find a "similarity" solution of this equation of the form:

$$F = r^2 A(\frac{r}{ct}) = r^2 A(z) \tag{8}$$

The initial conditions are satisfied by setting $A = 0$ for $z \geq 1$. When this form is substituted in the wave equation, with primes representing z derivatives, there results:

$$(1 - z^2) A'' + (\frac{4}{z} - 2z + \mu c \sigma r) A' = \frac{\mu r J}{z^2} \tag{9}$$

For this equation to depend only on z, the conditions must be imposed that

$$\sigma = \frac{1}{r} \sigma(\frac{r}{ct}) = \frac{1}{r} \sigma(z) \tag{10}$$

$$J = \frac{1}{r} J(\frac{r}{ct}) = \frac{1}{r} J(z) \tag{11}$$

The physics requires that at $z = 1$, σ has a simple zero, and J a double zero. Since the conductivity and current must vanish at large times, σ and J must vanish at $z = 0$. Equation (9) can be solved analytically; the result being:

$$A(z) = -\mu \int_z^1 \frac{(1-z_1^2)\, dz_1}{z_1^4} \int_0^{z_1} \frac{z_2^2 J(z_2)\, dz_2}{(1-z_2^2)^2} e^{-\mu c \int_{z_2}^{z_1} \frac{\sigma(z_3) dz_3}{(1-z_3^2)}} \tag{12}$$

The conditions on σ and J make all the integrals converge. The field components are given by:

$$E_\varphi = -c \sin\theta \, z^2 A' \tag{13}$$

$$B_\theta = \sin\theta \, (-B + 2A + zA') \tag{14}$$

$$B_r = \cos\theta \, (B - 2A) \tag{15}$$

The conditions on σ and J make A' bounded at $z = 0$. Therefore, E_φ tends to zero, and B_θ and B_r combine to produce a uniform field, whose value is changed from B to $B - 2A(0)$. For z close to unity, E_φ and $-c\Delta B_\theta$ are approximately equal.

The simplest form for σ and J which satisfies the several conditions and permits simple integrations is:

$$\sigma(z) = \sigma_o \, z(1 - z^2) \tag{16}$$

$$J(z) = J_o \, z(1 - z^2)^2 \tag{17}$$

These tend to a constant ratio as z approaches zero, which reasonably corresponds to the physical situation. The parameters σ_o and J_o are constants.

The peak value of the conductivity, obtained by differentiating Eq. (16), is $.386 \, \sigma_o$, which will be taken to match the peak conductivity at a distance of 1 km from a 1 MT burst. The conductivity equals the product of the number of electrons per cubic meter times the electron charge times the mobility. Experimental data indicate that the mobility is about 0.2 meters^2/volt sec. The number of electrons per cubic meter may be calculated from the expression:[1]

$$n(r,t) = 4 \times 10^{19} \quad \frac{e^{-\frac{r}{\lambda}}}{\lambda\, r^2}\, \eta\, Y \int_0^t dt' f(t^1 - \frac{r}{c})\, e^{-\beta(t-t')} \qquad (18)$$

where λ is the mean free path of the gamma rays in air at sea level in km, r the radial distance in km, Y the yield in KT, η the fraction of energy emitted as gamma rays, β the attachment rate of electrons to oxygen, and f a function describing the temporal variation of the gamma ray production. Assuming an exponential rise and fall, with reasonable constants, the peak value of n at r = 1 km is about 2 x 10^{17} for a 1 MT burst. The conductivity is therefore about 6.6 x 10^{-3}, so the constant σ_o will be taken to be 17 mho. This number should be regarded only as representative.

At long times, the ratio of current to conductivity tends to a constant. Again, using Ref. 1, the constant is determined by:

$$\frac{J_o}{c\sigma_o B} = \frac{e}{m\, \omega_e} \frac{R}{cp} \qquad (19)$$

where e is the electron charge, m the electron mass, ω_e the mobility, R the range of the Compton electrons, and p the secondary electron multiplication factor. All these are known constants, and the combination is equal to 0.1. The constant J_o accordingly is 2.6 x 10^4 amp/meter.

The conductivity σ versus distance r at various times and versus retarded time at various distances is plotted in Figs. 1 and 2, and the current in Figs. 3 and 4. While the qualitative behavior is correct, the time variation is slow compared to the actual time variation. The characteristic time is on the order of microseconds rather than shakes. However the results should still be worthwhile in showing the transition of the fields from the early to late time structure.

When the indicated functional forms are substituted for σ and J, Eq. (12) becomes:

$$A(z) = -\mu J_o \int_z^1 \frac{(1 - z_1^2)\, dz_1}{z_1^4} \int_0^{z_1} z_2^3\, dz_2\, e^{-\frac{1}{2}\mu c \sigma_o (z_1^2 - z_2^2)} \tag{20}$$

For the listed values, $a = \frac{1}{2}\mu c \sigma_o = 3 \times 10^3$. The integral over z_2 may be performed easily, and yields:

$$A(z) = \frac{-\mu J_o}{2a^2} \int_z^1 \frac{(1 - z_1^2)\, dz_1}{z_1^4} \left(a z_1^2 - 1 + e^{-a z_1^2} \right) \tag{21}$$

The explicit evaluation of this integral involves error functions.

The result in full detail is:

$$A(z) = -\frac{J_o}{c\sigma_o} \left[\frac{(1 - z)^2 (3az^2 - 2z - 1)}{3az^3} - \frac{(2az^2 + 3z - 1)}{3az^3} e^{-az^2} + \frac{2}{3a}(1 + a)e^{-a} + \sqrt{\frac{\pi}{a}} \left(1 + \frac{2}{3}a\right) \left(\operatorname{erf}\sqrt{a} - \operatorname{erf}\sqrt{a}\, z\right) \right] \tag{22}$$

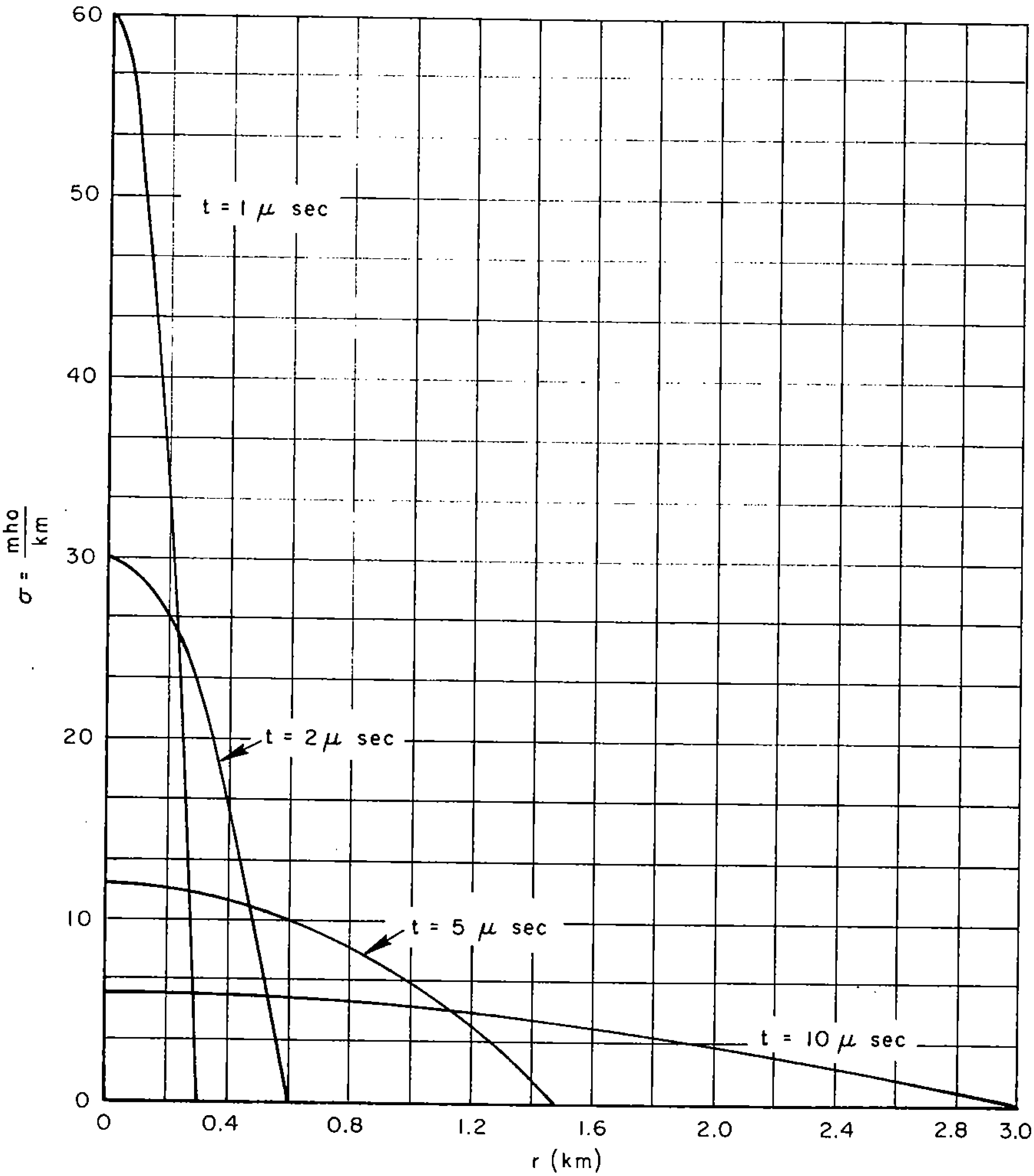

Fig. 1 — Conductivity vs distance
(bomb parameters equivalent to 1 MT)

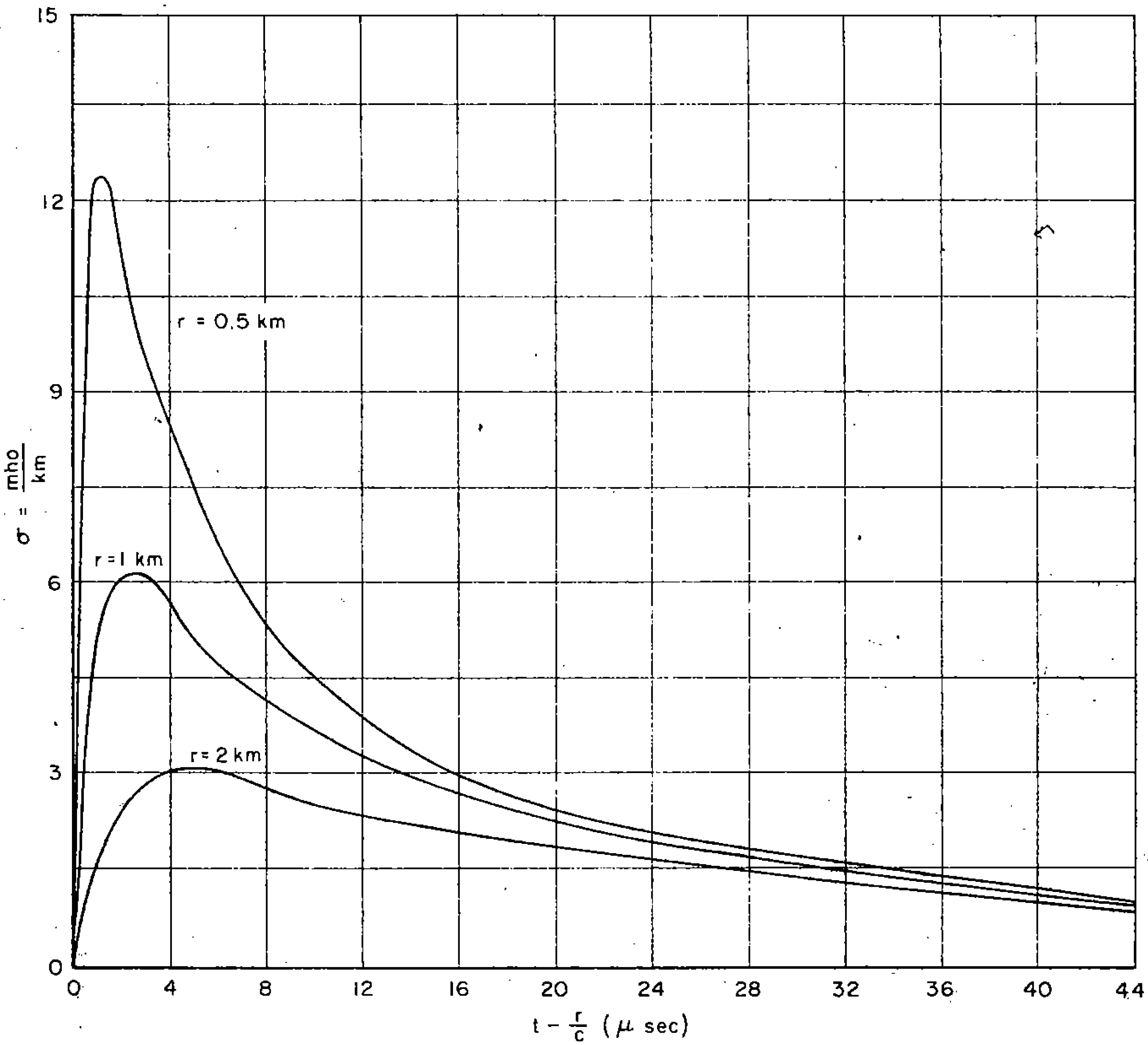

Fig. 2 — Conductivity vs retarded time
(bomb parameters equivalent to 1 MT)

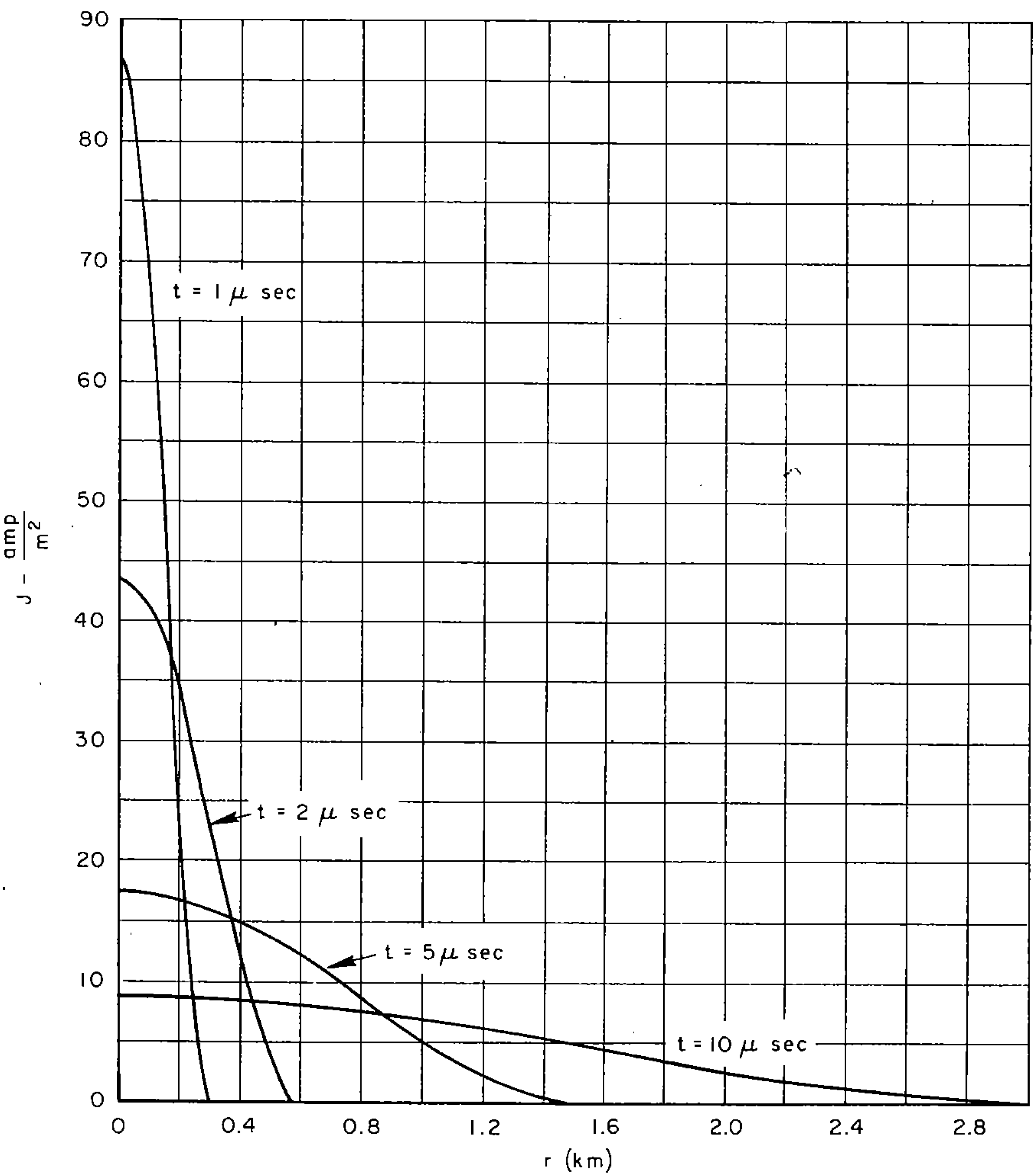

Fig. 3 — Current density vs distance
(bomb parameters equivalent to 1 MT)

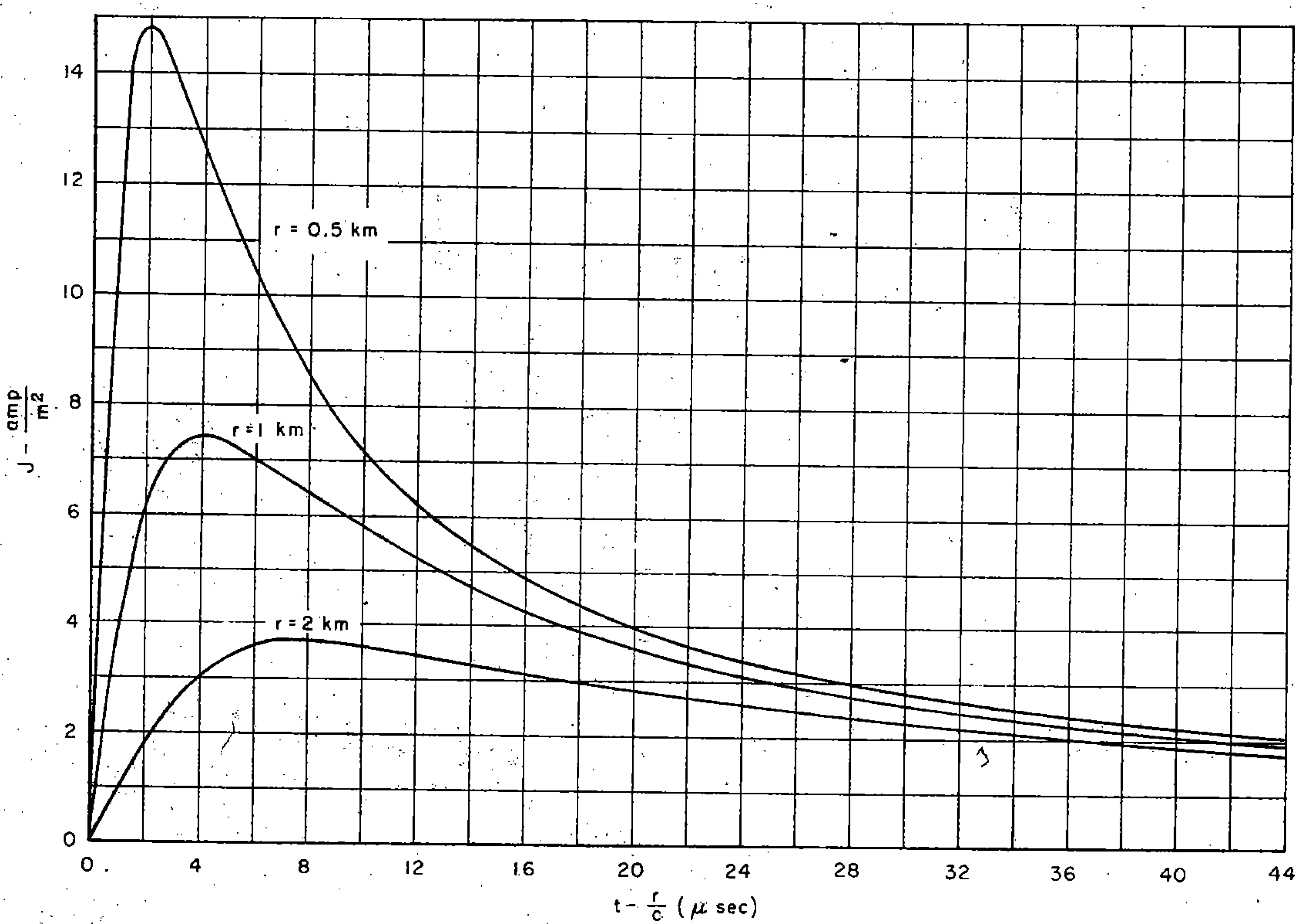

Fig. 4 —Current density vs retarded time
(bomb parameters equivalent to 1 MT)

For $z > .07$, the terms involving exponentials and erfs may be neglected, since all arguments are greater than 5. Thus

$$A(z) = -\frac{J_o}{c\sigma_o}\frac{(1-z)^2}{z}\left(1-\frac{2}{3az}-\frac{1}{3az^2}\right) \qquad z > .07 \qquad (23)$$

$$= -.1B\frac{(1-z)^2}{z}\left(1-\frac{.02}{z}-\frac{.01}{z^2}\right) \qquad z > .07 \qquad (24)$$

To within 10 per cent, the field components are given by:

$$\left.\begin{aligned} E_\varphi &= -.1\, cB(1-z^2)\sin\theta && (25)\\ B_\theta &= -B\left(1-\frac{.1(1-z)(3z-1)}{z}\right)\sin\theta && (26)\\ B_r &= B\left(1+\frac{.1(1-z)^2}{z}\right)\cos\theta && (27)\end{aligned}\right\} \quad z > .07$$

For z very small, the limiting forms of A, A' and the field components are:

$$A(o) = -\frac{2\sqrt{\pi a}}{3}\,.1\,B = -7B \qquad (28)$$

$$A'(o) = .05\,a\,B = 150\,B \qquad (29)$$

$$E_\varphi = -150\,c\,B\,z^2\sin\theta \qquad (30)$$

$$B_\theta = -15\,B\sin\theta \qquad (31)$$

$$B_r = 15\,B\cos\theta \qquad (32)$$

The magnetic field tends to a uniform field 15 times the strength of the unperturbed field, while the electric field tends to zero with a large multiplying constant. Such conditions cannot be a close approach to reality. This long-time behavior represents a non-physical aspect of the model.

The electric field rises to a maximum value of .096 cB, which for the earth's field is about 1.44 kilovolts/meter. The peak is near $az^4 = 1$, which for $a = 3 \times 10^3$, $r = 1$ km occurs at $t = 25$ microseconds (22 microseconds after the leading edge of the pulse arrives). The rise is quite rapid, the field reaching half the peak in one microsecond and nine-tenths of the peak in 6 microseconds. The decay is much slower, being down to nine-tenths after 55 microseconds, and taking until 130 microseconds to the half-field value. The magnetic field exhibits a long slow rise to the large asymptotic value. The electric field in kilovolts per meter is plotted in Fig. 5.

The reason for the large magnetic fields at long times is the large magnetic moment of the configuration. A current distribution of the indicated type has a magnetic moment proportional to t^3, and the results can be expected to be singular.

The analysis of this section can be generalized to more complicated angular dependence. A generalization of the wave Eq. (7) to wider angular dependence is:

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r^2} \frac{\partial}{\partial \theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta F = \frac{1}{c^2} \frac{\partial^2 F}{\partial t^2} + \mu \left[\sigma(r,t) \frac{\partial F}{\partial t} + r J(r,t,\theta) \right] \qquad (33)$$

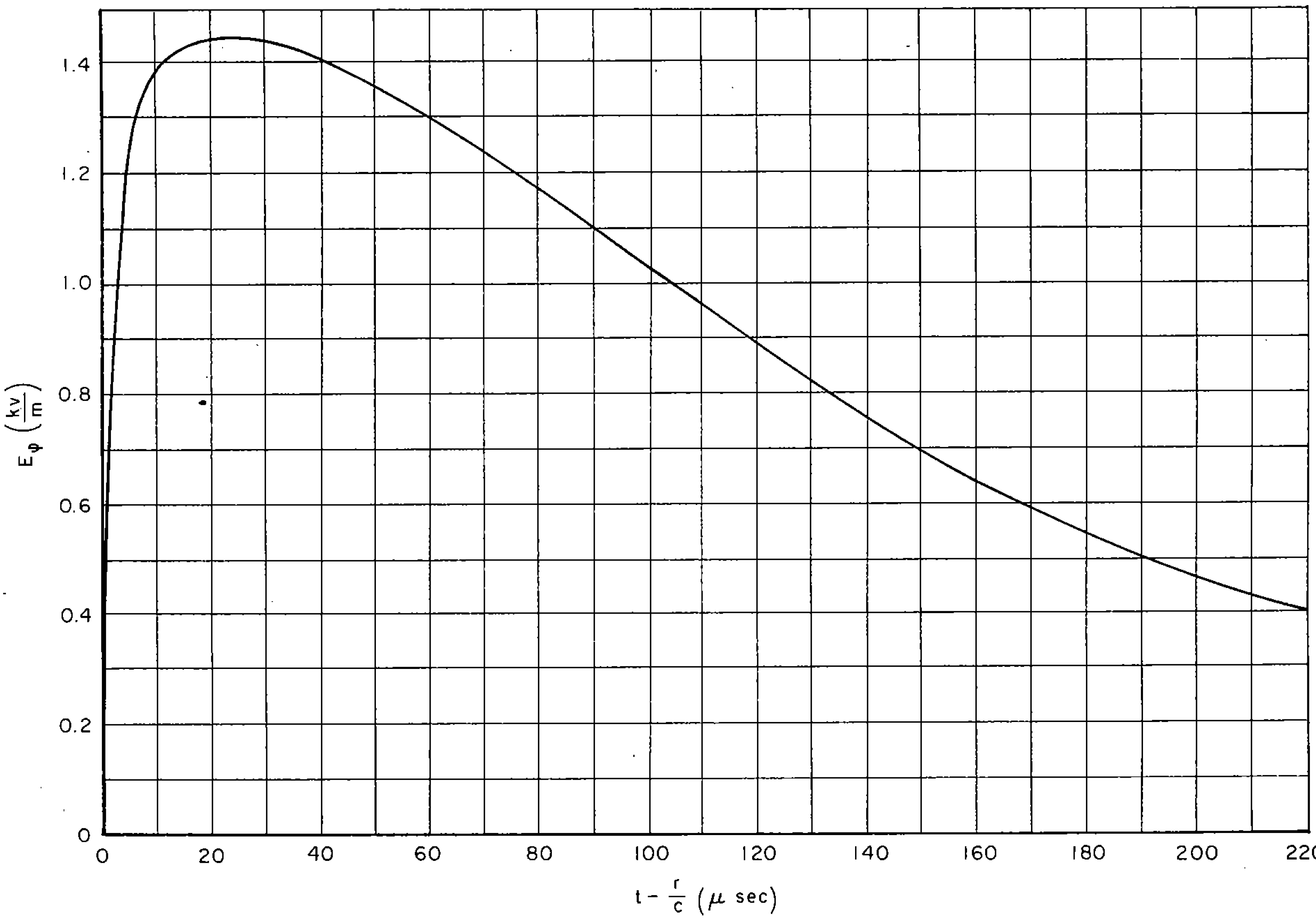

Fig. 5 — Azimuthal electric field at 1 km
(bomb parameters equivalent to 1 MT)

This may be expanded in Legendre polynomials as follows:

$$F = \sum_{1}^{\infty} F_n(r,t) \frac{\partial}{\partial\theta} P_n(\cos\theta) \tag{34}$$

$$J = \sum_{1}^{\infty} J_n(r,t) \frac{\partial}{\partial\theta} P_n(\cos\theta) \tag{35}$$

$$\frac{\partial^2 F_n}{\partial r^2} - \frac{n(n+1)}{r^2} F_n = \frac{1}{c^2}\frac{\partial^2 F_n}{\partial t^2} + \mu\left[\sigma \frac{\partial F_n}{\partial t} + rJ_n(r,t)\right] \tag{36}$$

It has been assumed that σ is independent of θ. Now try a similarity solution as follows:

$$\sigma = \frac{1}{r}\ \sigma\left(\frac{r}{ct}\right) = \frac{1}{r}\sigma(z) \tag{37}$$

$$J_n = r^{n-2} J_n(z) \tag{38}$$

$$F_n = r^{n+1} F_n(z) \tag{39}$$

When this form is substituted into Eq. (36), there results

$$(1 - z^2) F''_n + \left[\frac{2(n+1)}{z} - 2z + \mu c\sigma(z)\right] F'_n = \frac{\mu J_n}{z^2} \tag{40}$$

The solution of this equation and the usual initial conditions is:

$$F_n = -\mu \int_z^1 \frac{(1 - z_1^2)^n \, dz_1}{z_1^{2n+2}} \int_o^{z_1} \frac{z_2^{2n} \, J_n(z_2) dz_2}{(1 - z_2^2)^{n+1}} \, e^{-\mu c \int_{z_2}^{z_1} \frac{\sigma(z_3) dz_3}{1 - z_3^2}} \tag{41}$$

This expression may be inserted back into the expression (34) and the sum over n performed. The result is:

$F(r,t,\theta) =$

$$-\mu r^3 \int_{\frac{r}{ct}}^{1} \frac{dz_1 (1-z_1^2)^2}{z_1^6} \int_o^{z_1} dz_2 z_2^4 (1-z_2^2) \, J\left(\frac{rz_2^2(1-z_1^2)}{z_1^2(1-z_2^2)}, \frac{rz_2(1-z_1^2)}{z_1^2(1-z_2^2)}, \theta\right) e^{-\mu c \int_{z_2}^{z_1} \frac{\sigma(z_3) dz_3}{1-z_3^2}} \tag{42}$$

The complications of this result are sufficient that no further use has been made of it. The condition of validity is that J possess an expression of the type indicated

$$J = \sum_1^{\infty} r^{n-2} \, J_n\left(\frac{r}{ct}\right) \frac{\partial}{\partial \theta} \, P_n(\cos \theta) \tag{43}$$

This can be regrouped into a power series in $\frac{r}{ct}$, yielding

$$J = \frac{1}{r^2} \sum_o^{\infty} \left(\frac{r}{ct}\right)^m \frac{\partial}{\partial \theta} \, \mu_m(r,\theta) \tag{44}$$

where μ_m is a solution of Laplace's equation. This indicates the degree of the generalization.

The model presented here is of both physical and mathematical interest. The physical situation is governed by a number of time constants, which in increasing order are: (1) the time constant ϵ/σ associated with the magnitude of the conductivity, (2) the attachment time constant, (3) the rise and fall time constants associated with the electron production, (4) the reciprocal of the electron gyrofrequency. The relative values of these have been preserved in the model, so a rough correspondence with the physical situation should also be preserved, even though the mathematical functional forms have been changed.

There are certain nonphysical aspects to the model. The functional form for the conductivity is equivalent to an electron charge density which increases logarithmically at large times, which condition of course cannot occur in practice. However, the conductivity model is less singular at the space origin than the usual models, so there is a partial compensation of singularities in the modeling structure.

REFERENCES

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