EMP Theoretical Notes
Note 62

NUMERICAL SOLUTIONS OF MAXWELL'S EQUATIONS WITH AZIMUTHAL SYMMETRY IN PROLATE SPHEROIDAL COORDINATES

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ABSTRACT

Two digital computer codes are described that calculate the electromagnetic pulse generated by a low altitude nuclear explosion in the presence of a finitely conducting earth. The codes assume azimuthal symmetry and employ a finite difference technique for solving Maxwell's equations in prolate spheroidal coordinates. A test problem with some typical results is included.

I. INTRODUCTION

Two digital computer codes, which calculate the electromagnetic pulse generated by a low altitude nuclear burst in the presence of a finitely conducting earth, are discussed. It is assumed in the codes that the currents and conductivities are known functions of space and time, which have azimuthal symmetry. With this input, Maxwell's equations are solved in prolate spheroidal coordinates by a finite difference technique. This coordinate system was chosen because it more nearly fits the geometry of the problem than other orthogonal coordinate systems. For example, the earth and the z-axis are described by constant coordinates and the wave front of the wave propagating from the weapon as well as the wave reflected from the ground are determined by linear functions of the coordinates.

The codes differ in that one is written in real-time whereas the other employs the retarded-time of the burst point as an independent variable. A primary advantage of the real-time code is that a definite physical boundary condition exists at the wave front in space (i.e., the fields are zero for $r > ct$). Advantages of the retarded-time code which at present are considered decisive are: (1) regridding at the wave front is much simpler than for the real-time code, and (2) the spatial derivatives at the wave front, $r = ct$, are somewhat smaller than for the real-time case. For these reasons a production version of the retarded-time code is presently being produced.

In Section III a version of the retarded-time code which is specialized to take advantage of the simplification that arises by neglecting the density gradient of the atmosphere will be discussed in addition to the more generalized code. This simplified version runs much more rapidly on the computer. The approximation made is accurate provided that the height of burst of the weapon is not too great.

II. REAL-TIME FINITE DIFFERENCE CODE

Maxwell's Equations

In M. K. S. units Maxwell's equations are

$$\nabla \cdot \vec{D} = \rho \tag{1}$$

$$\nabla \cdot \vec{B} = 0 \tag{2}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3}$$

$$\nabla \times \vec{H} = j + \frac{\partial \vec{D}}{\partial t} \tag{4}$$

The divergence equations may be treated as initial conditions since Maxwell's equations predict that if they are initially satisfied they will remain so for all time. Our attention will thus be focused on the curl equations. In prolate spheroidal coordinates they are: (The geometry is described in Section IV.)

$$\frac{\hat{\xi}}{\sqrt{(\zeta^2 - \xi^2 a^2)(1 - \xi^2)}} \left[\frac{\partial}{\partial \zeta} \left(\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\, E_\phi \right) - \frac{\partial}{\partial \phi} \left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{\zeta^2 - a^2}}\, E_\zeta \right) \right]$$

$$+ \frac{\hat{\zeta}}{\sqrt{(\zeta^2 - \xi^2 a^2)(\zeta^2 - a^2)}} \left[\frac{\partial}{\partial \phi} \left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{1 - \xi^2}}\, E_\xi \right) - \frac{\partial}{\partial \xi} \left(\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\, E_\phi \right) \right]$$

$$+ \frac{\hat{\phi} \sqrt{(\zeta^2 - a^2)(1 - \xi^2)}}{\zeta^2 - \xi^2 a^2} \left[\frac{\partial}{\partial \xi} \left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{\zeta^2 - a^2}}\, E_\zeta \right) - \frac{\partial}{\partial \zeta} \left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{1 - \xi^2}}\, E_\xi \right) \right]$$

$$= -\frac{\partial}{\partial t} \left(B_\xi \hat{\xi} + B_\zeta \hat{\zeta} + B_\phi \hat{\phi} \right) \tag{5}$$

and

$$\frac{\hat{\xi}}{\sqrt{(\zeta^2 - \xi^2 a^2)(1 - \xi^2)}} \left[\frac{\partial}{\partial \zeta} \left(\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\, H_\phi \right) - \frac{\partial}{\partial \phi} \left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{\zeta^2 - a^2}}\, H_\zeta \right) \right]$$

$$+ \frac{\hat{\zeta}}{\sqrt{(\zeta^2 - \xi^2 a^2)(\zeta^2 - a^2)}} \left[\frac{\partial}{\partial \phi} \left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{1 - \xi^2}}\, H_\xi \right) - \frac{\partial}{\partial \xi} \left(\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\, H_\phi \right) \right]$$

$$+ \hat{\phi}\, \frac{\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}}{\zeta^2 - \xi^2 a^2} \left[\frac{\partial}{\partial \xi} \left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{\zeta^2 - a^2}}\, H_\zeta \right) - \frac{\partial}{\partial \zeta} \left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{1 - \xi^2}}\, H_\xi \right) \right]$$

$$= (j_\xi \hat{\xi} + j_\zeta \hat{\zeta} + j_\phi \hat{\phi}) + \frac{\partial}{\partial t} (D_\xi \hat{\xi} + D_\zeta \hat{\zeta} + D_\phi \hat{\phi}) \qquad (6)$$

We assume the constitutive equations

$$\vec{D} = \epsilon \vec{E}, \qquad \vec{B} = \mu_o \vec{H} \qquad (7)$$

The problem considered has azimuthal symmetry. Thus no physical quantity is a function of ϕ. By using Eq. (7) and this symmetry condition, the curl equations are reduced to

$$\frac{\hat{\xi}}{\sqrt{(\zeta^2 - \xi^2 a^2)(1 - \xi^2)}} \frac{\partial}{\partial \zeta} \left(\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\, E_\phi \right) - \frac{\hat{\zeta}}{\sqrt{(\zeta^2 - \xi^2 a^2)(\zeta^2 - a^2)}}$$

$$\frac{\partial}{\partial \xi} \left(\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\, E_\phi \right) + \hat{\phi}\, \frac{\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}}{\zeta^2 - \xi^2 a^2} \left[\frac{\partial}{\partial \xi} \left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{\zeta^2 - a^2}}\, E_\zeta \right) \right.$$

$$\left. - \frac{\partial}{\partial \zeta} \left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{1 - \xi^2}}\, E_\xi \right) \right] = - \frac{\partial}{\partial t} (B_\xi \hat{\xi} + B_\zeta \hat{\zeta} + B_\phi \hat{\phi}) \qquad (8)$$

$$\frac{c^2\hat{\xi}}{\sqrt{(\zeta^2 - \xi^2 a^2)(1 - \xi^2)}} \frac{\partial}{\partial \zeta} \left(\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\, B_\phi\right) - \frac{c^2\hat{\zeta}}{\sqrt{(\zeta^2 - \xi^2 a^2)(\zeta^2 - a^2)}}$$

$$\frac{\partial}{\partial \xi} \left(\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\, B_\phi\right) + c^2\hat{\phi} \frac{\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}}{\zeta^2 - \xi^2 a^2} \left[\frac{\partial}{\partial \xi}\left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{\zeta^2 - a^2}}\, B_\zeta\right)\right.$$

$$\left. - \frac{\partial}{\partial \zeta}\left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{1 - \xi^2}}\, B_\xi\right)\right] = \frac{1}{\epsilon}\left(j_\xi \hat{\xi} + j_\zeta \hat{\zeta} + j_\phi \hat{\phi}\right) + \frac{\partial}{\partial t}\left(E_\xi \hat{\xi} + E_\zeta \hat{\zeta} + E_\phi \hat{\phi}\right) \quad (9)$$

where

$$c = \frac{1}{\sqrt{\mu_o \epsilon}} \quad (10)$$

In scalar form, Eqs. (8) and (9) separate into two independent sets of coupled equations. One set involves the variables E_ξ, E_ζ, B_ϕ, j_ξ, and j_ζ. The other set relates E_ϕ, B_ξ, B_ζ, and j_ϕ. Maxwell's divergence equations behave similarly. We assume that for our problem $j_\phi = 0$. It then follows that if E_ϕ, B_ξ, and B_ζ are initially zero, they will remain so. Thus we set

$$E_\phi = B_\xi = B_\zeta = j_\phi = 0 \quad (11)$$

The curl equations become

$$\hat{\phi} \frac{\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}}{\zeta^2 - \xi^2 a^2} \left[\frac{\partial}{\partial \xi}\left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{\zeta^2 - a^2}}\, E_\zeta\right) - \frac{\partial}{\partial \zeta}\left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{1 - \xi^2}}\, E_\xi\right)\right]$$

$$= - \frac{\partial B_\phi}{\partial t} \hat{\phi} \quad (12)$$

$$\frac{c^2\hat{\xi}}{\sqrt{(\zeta^2 - \xi^2 a^2)(1 - \xi^2)}} \frac{\partial}{\partial \zeta}\left(\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\, B_\phi\right) \frac{c^2\hat{\zeta}}{\sqrt{(\zeta^2 - \xi^2 a^2)(\zeta^2 - a^2)}}$$

$$\frac{\partial}{\partial \xi}\left(\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\, B_\phi\right) = \frac{1}{\epsilon}\left(j_\xi \hat{\xi} + j_\zeta \hat{\zeta}\right) + \frac{\partial}{\partial t}\left(E_\xi \hat{\xi} + E_\zeta \hat{\zeta}\right) \tag{13}$$

In order to simplify these, the fields will be transformed according to

$$\begin{pmatrix} E_\xi \\ j_\xi \end{pmatrix} = \sqrt{(\zeta^2 - \xi^2 a^2)(1 - \xi^2)} \begin{pmatrix} E'_\xi \\ j'_\xi \end{pmatrix} \tag{14}$$

$$\begin{pmatrix} E_\zeta \\ j_\zeta \end{pmatrix} = \sqrt{(\zeta^2 - \xi^2 a^2)(\zeta^2 - a^2)} \begin{pmatrix} E'_\zeta \\ j'_\zeta \end{pmatrix} \tag{15}$$

$$B_\phi = \sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\; B'_\phi \tag{16}$$

in which the primed quantities are those that have been used previously. In component form, the field equations then become

$$\frac{\partial E_\xi}{\partial t} = -\frac{j_\xi}{\epsilon} + c^2 \frac{\partial B_\phi}{\partial \zeta} \tag{17}$$

$$\frac{\partial E_\zeta}{\partial t} = -\frac{j_\zeta}{\epsilon} - c^2 \frac{\partial B_\phi}{\partial \xi} \tag{18}$$

$$\frac{\partial B_\phi}{\partial t} = \left(\frac{\zeta^2 - a^2}{\zeta^2 - \xi^2 a^2}\right) \frac{\partial E_\xi}{\partial \zeta} - \left(\frac{1 - \xi^2}{\zeta^2 - \xi^2 a^2}\right) \frac{\partial E_\zeta}{\partial \xi} \tag{19}$$

Make the substitution

$$j \longrightarrow j + \sigma E \tag{20}$$

Henceforth j will represent that portion of the current not resulting from conductivity of the medium.

Finally, we have

$$\frac{\partial E_\xi}{\partial t} = -\frac{\sigma}{\epsilon} E_\xi - \frac{j_\xi}{\epsilon} + c^2 \frac{\partial B_\phi}{\partial \zeta} \tag{21}$$

$$\frac{\partial E_\zeta}{\partial t} = -\frac{\sigma}{\epsilon} E_\zeta - \frac{j_\zeta}{\epsilon} - c^2 \frac{\partial B_\phi}{\partial \xi} \tag{22}$$

$$\frac{\partial B_\phi}{\partial t} = \left(\frac{\zeta^2 - a^2}{\zeta^2 - \xi^2 a^2}\right)\frac{\partial E_\xi}{\partial \zeta} - \left(\frac{1 - \xi^2}{\zeta^2 - \xi^2 a^2}\right)\frac{\partial E_\zeta}{\partial \xi} \tag{23}$$

This set of equations will be solved numerically below by a finite difference technique.

Difference Equations

Let

$$\xi = 1 + (1-i)\Delta\xi, \qquad \zeta = a + (j-1)\Delta\zeta, \qquad t = k\Delta t \tag{24}$$

so that at a mesh point a function $f(\xi, \zeta, t)$ may be labeled $f(i, j, k)$. Maxwell's curl equations (21) through (23) will be replaced by difference equations centered at $(i, j, k-1/2)$. For mesh points in the air, we find

$$\frac{1}{\Delta t}\Big[E_\xi(i, j, k) - E_\xi(i, j, k-1)\Big] = \frac{c^2}{2\Delta\zeta}\Big[B_\phi(i, j+1, k-1) - B_\phi(i, j, k-1)$$

$$+ B_\phi(i, j, k) - B_\phi(i, j-1, k)\Big] - \frac{\sigma(i, j, k-1/2)}{2\epsilon_0}\Big[E_\xi(i, j, k) + E_\xi(i, j, k-1)\Big]$$

$$- \frac{1}{\epsilon_0} j_\xi(i, j, k-1/2) \qquad (25)$$

$$\frac{1}{\Delta t}\Big[E_\zeta(i, j, k) - E_\zeta(i, j, k-1)\Big] = -\frac{c^2}{2\Delta\xi}\Big[B_\phi(i-1, j, k) - B_\phi(i, j, k)$$

$$+ B_\phi(i, j, k-1) - B_\phi(i+1, j, k-1)\Big] - \frac{\sigma(i, j, k-1/2)}{2\epsilon_0}\Big[E_\zeta(i, j, k) + E_\zeta(i, j, k-1)\Big]$$

$$- \frac{1}{\epsilon_0} j_\zeta(i, j, k-1/2) \qquad (26)$$

$$\frac{1}{\Delta t}\Big[B_\phi(i, j, k) - B_\phi(i, j, k-1)\Big] = \frac{1}{\zeta^2 - \xi^2 a^2}\Bigg\{\frac{\zeta^2 - a^2}{2\Delta\zeta}\Big[E_\xi(i, j+1, k-1)$$

$$- E_\xi(i, j, k-1) + E_\xi(i, j, k) - E_\xi(i, j-1, k)\Big] - \frac{1 - \xi^2}{2\Delta\xi}\Big[E_\zeta(i-1, j, k)$$

$$- E_\zeta(i, j, k) + E_\zeta(i, j, k-1) - E_\zeta(i+1, j, k-1)\Big]\Bigg\} \qquad (27)$$

Provided the values of the fields are known on the z axis, on the line j-1, and for values of ξ greater than $1 + (1-i)\Delta\xi$, the above are three equations that can be solved simultaneously for the fields $E_\xi(i, j, k)$, $E_\zeta(i, j, k)$, and $B_\phi(i, j, k)$. In order for the calculation to proceed in this manner it is necessary to first determine the fields on the z axis. From the symmetry of the problem it is easy to establish the following limits

$$\lim_{\substack{\zeta \to a \\ -1 < \xi < 1}} E_\zeta(\xi, \zeta, t) = 0, \qquad \lim_{\substack{\zeta \to a \\ -1 < \xi < 1}} B_\phi(\xi, \zeta, t) = 0 \qquad (28)$$

The field component $E_\xi(i, 1, k)$ is nonzero at $\zeta \to a$ and in the air is determined simultaneously with the three field components at $(i, 2, k)$ by using

$$\frac{1}{\Delta t}\left[E_\xi(i, 1, k) - E_\xi(i, 1, k-1)\right] = \frac{c^2}{\Delta\zeta} B_\phi(i, 2, k) - \frac{\sigma(i, 1, k-1/2)}{2\epsilon}\left[E_\xi(i, 1, k)\right.$$

$$\left. + E_\xi(i, 1, k-1)\right] - \frac{1}{\epsilon} j_\xi(i, 1, k-1/2) \qquad (29)$$

together with Eqs. (25), (26), and (27) evaluated at $(i, 2, k-1/2)$.

On the portion of the z axis above the weapon we have

$$\lim_{\substack{\xi \to 1 \\ \zeta > a}} E_\xi(\xi, \zeta, t) = 0, \qquad \lim_{\substack{\xi \to 1 \\ \zeta > a}} B_\phi(\xi, \zeta, t) = 0 \qquad (30)$$

The value of E_ζ at $\xi \to 1$ is obtained by solving

$$\frac{1}{\Delta t}\left[E_\zeta(1, j, k) - E_\zeta(1, j, k-1)\right] = \frac{c^2}{\Delta\xi} B_\phi(2, j, k) - \frac{\sigma}{2\epsilon}\left[E_\zeta(1, j, k)\right.$$

$$\left. + E_\zeta(1, j, k-1)\right] - \frac{1}{\epsilon} j_\zeta(1, j, k-1/2) \qquad (31)$$

Simultaneously with Eqs. (25), (26), and (27) evaluated at $(2, j, k-1/2)$ for the four field components involved. The boundary conditions at the z axis in the ground are handled in an analogous manner. The difference equations used in the ground are

$$\frac{1}{\Delta t}\left[E_\xi(i, j, k) - E_\xi(i, j, k-1)\right] = \frac{u^2}{2\Delta\zeta}\Big[B_\phi(i, j, k) - B_\phi(i, j-1, k)$$

$$+ B_\phi(i, j+1, k-1) - B_\phi(i, j, k-1)\Big] - \frac{\sigma(i, j, k-1/2)}{2\epsilon}\Big[E_\xi(i, j, k)$$

$$+ E_\xi(i, j, k-1)\Big] - \frac{1}{\epsilon} j_\xi(i, j, k-1/2) \qquad (32)$$

where u is the velocity of propagation in the ground.

$$\frac{1}{\Delta t}\left[E_\zeta(i, j, k) - E_\zeta(i, j, k-1)\right] = -\frac{u^2}{2\Delta\xi}\Big[B_\phi(i, j, k) - B_\phi(i+1, j, k)$$

$$+ B_\phi(i-1, j, k-1) - B_\phi(i, j, k-1)\Big] - \frac{\sigma(i, j, k-1/2)}{2\epsilon}\Big[E_\zeta(i, j, k)$$

$$+ E_\zeta(i, j, k-1)\Big] - \frac{1}{\epsilon} j_\zeta(i, j, k-1/2) \qquad (33)$$

$$\frac{1}{\Delta t}\left[B_\phi(i, j, k) - B_\phi(i, j, k-1)\right] = \frac{1}{\zeta^2 - \xi^2 a^2}\Bigg\{\frac{\zeta^2 - a^2}{2\Delta\zeta}\Big[E_\xi(i, j, k)$$

$$- E_\xi(i, j-1, k) + E_\xi(i, j+1, k-1) - E_\xi(i, j, k-1)\Big] - \frac{1-\xi^2}{2\Delta\xi}\Big[E_\zeta(i, j, k)$$

$$- E_\zeta(i+1, j, k) + E_\zeta(i-1, j, k-1) - E_\zeta(i, j, k-1)\Big]\Bigg\} \qquad (34)$$

The calculation proceeds in the following manner. $E_\xi(i, 1, k)$ and the three fields at $(i, 2, k)$ are determined simultaneously for all i. Then fields are obtained along successive lines of constant j in the order: E_ζ at $\xi = 1$ together with the three field components at $i = 2$, fields in the air mesh in the direction of increasing i, E_ζ at $\xi = -1$ together with E_ξ, E_ζ, and B_ϕ at the adjoining mesh point, fields in the ground in the direction of decreasing i, and finally the fields at the ground boundary.

To treat the air-ground boundary, we define two lines of points labeled i_a and i_g for which $\xi = 0$. Even though these points have the same ξ coordinates, the first set is assumed to be in air and the other in the ground. The mesh points near the ground are labeled as shown in Figure 1.

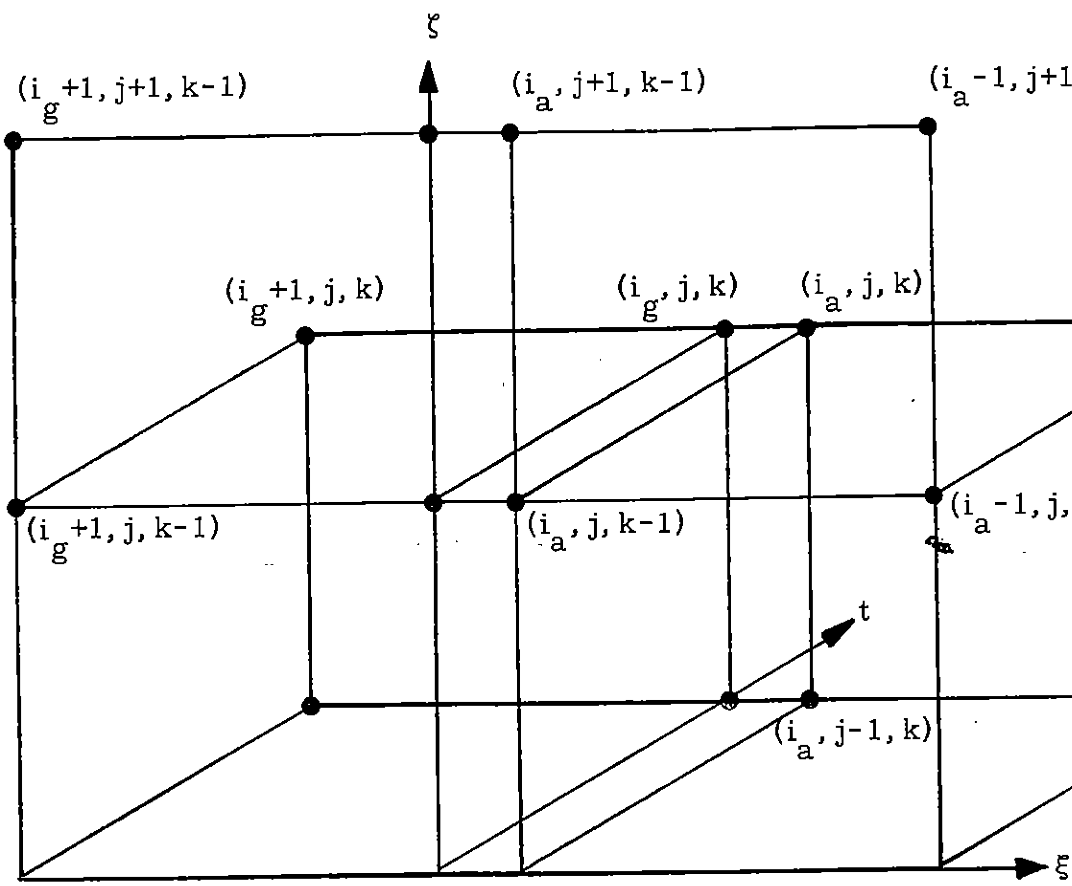

Figure 1

A Portion of the ξ - ζ Mesh Near the Ground

The magnetic permeability of the ground is sufficiently close to μ_o that we may assume they are identical. Then the appropriate boundary conditions are that E_ζ and B_ϕ are continuous. Since this must hold for all time, we infer that $\partial E_\zeta / \partial t$ and $\partial B_\phi / \partial t$ must be continuous. Whence

$$\frac{\partial E_\zeta(i_a, j, k-1/2)}{\partial t} = \frac{\partial E_\zeta(i_g, j, k-1/2)}{\partial t} \approx \frac{1}{2}\left[\frac{\partial E_\zeta(i_a-1/2, j, k-1/2)}{\partial t} + \frac{\partial E_\zeta(i_g+1/2, j, k-1/2)}{\partial t}\right] \tag{35}$$

and

$$\frac{\partial B_\phi(i_a, j, k-1/2)}{\partial t} = \frac{\partial B_\phi(i_g, j, k-1/2)}{\partial t} \approx \frac{1}{2}\left[\frac{\partial B_\phi(i_a-1/2, j, k-1/2)}{\partial t} + \frac{\partial B_\phi(i_g+1/2, j, k-1/2)}{\partial t}\right] \tag{36}$$

Or

$$\frac{1}{\Delta\tau}\left[E_\zeta(i_a, j, k) - E_\zeta(i_a, j, k-1)\right] = -\frac{u^2}{8\Delta\xi}\Big[B_\phi(i_g, j+1, k-1)$$

$$- B_\phi(i_g+1, j+1, k-1) + B_\phi(i_g, j, k-1) - B_\phi(i_g+1, j, k-1)$$

$$+ B_\phi(i_g, j, k) - B_\phi(i_g+1, j, k) + B_\phi(i_g, j-1, k) - B_\phi(i_g+1, j-1, k)\Big]$$

$$- \frac{c^2}{8\Delta\xi}\Big[B_\phi(i_a-1, j+1, k-1) - B_\phi(i_a, j+1, k-1) + B_\phi(i_a-1, j, k-1)$$

$$- B_\phi(i_a, j, k-1) + B_\phi(i_a-1, j, k) - B_\phi(i_a, j, k) + B_\phi(i_a+1, j-1, k)$$

$$- B_\phi(i_a, j-1, k)\Big] - \frac{\sigma(i_g+1/2, j, k-1/2)}{8\epsilon}\Big[E_\zeta(i_g, j, k-1) + E_\zeta(i_g, j, k)$$

$$+ E_\zeta(i_g+1, j, k) + E_\zeta(i_g+1, j, k-1)\Big] - \frac{\sigma(i_a-1/2, j, k-1/2)}{8\epsilon_0}\Big[E_\zeta(i_a-1, j, k)$$

$$+ E_\zeta(i_a, j, k) + E_\zeta(i_a, j, k-1) + E_\zeta(i_a-1, j, k-1)\Big] - \frac{1}{2\epsilon} j_\zeta(i_g+1/2, j, k-1/2)$$

$$- \frac{1}{2\epsilon_0} j_\zeta(i_a-1/2, j, k-1/2) \qquad (37)$$

$$\frac{1}{\Delta t}\Big[B_\phi(i_a, j, k) - B_\phi(i_a, j, k-1)\Big] = \frac{1}{8\Delta\zeta}\,\frac{\zeta^2 - a^2}{\zeta^2 - a^2\Delta\xi^2/4}\Big[E_\xi(i_g+1, j+1, k-1)$$

$$- E_\xi(i_g+1, j, k-1) + E_\xi(i_g+1, j, k) - E_\xi(i_g+1, j-1, k) + E_\xi(i_g, j+1, k-1)$$

$$- E_\xi(i_g, j, k-1) + E_\xi(i_g, j, k) - E_\xi(i_g, j-1, k) + E_\xi(i_a, j+1, k-1)$$

$$- E_\xi(i_a, j, k-1) + E_\xi(i_a, j, k) - E_\xi(i_a, j-1, k) + E_\xi(i_a-1, j+1, k-1)$$

$$- E_\xi(i_a-1, j, k-1) + E_\xi(i_a-1, j, k) - E_\xi(i_a-1, j-1, k)\Big]$$

$$- \frac{1}{8\Delta\xi}\,\frac{1 - \Delta\xi^2/4}{\zeta^2 - a^2\Delta\xi^2/4}\Big[E_\zeta(i_a-1, j, k) + E_\zeta(i_a-1, j, k-1) - E_\zeta(i_g+1, j, k-1)$$

$$- E_\zeta(i_g+1,\ j,\ k) + E_\zeta(i_a-1,\ j+1,\ k-1) + E_\zeta(i_a-1,\ j-1,\ k) - E_\zeta(i_g+1,\ j+1,\ k-1)$$

$$- E_\zeta(i_g+1,\ j-1,\ k)\Big] \qquad (38)$$

Evaluate $\partial E_\xi / \partial t$ at $(i_a,\ j,\ k-1/2)$

$$\frac{1}{\Delta t}\Big[E_\xi(i_a,\ j,\ k) - E_\xi(i_a,\ j,\ k-1)\Big] = \frac{c^2}{2\Delta\zeta}\Big[B_\phi(i_a,\ j+1,\ k-1) - B_\phi(i_a,\ j,\ k-1)$$

$$+ B_\phi(i_a,\ j,\ k) - B_\phi(i_a,\ j-1,\ k)\Big] - \frac{\sigma(i_a,\ j,\ k-1/2)}{2\epsilon_0}\Big[E_\xi(i_a,\ j,\ k)$$

$$+ E_\xi(i_a,\ j,\ k-1)\Big] - \frac{1}{\epsilon_0} j_\xi(i_a,\ j,\ k-1/2) \qquad (39)$$

Evaluate $\partial E_\phi / \partial t$ at $(i_g,\ j,\ k-1/2)$

$$\frac{1}{\Delta t}\Big[E_\xi(i_g,\ j,\ k) - E_\xi(i_g,\ j,\ k-1)\Big] = \frac{u^2}{2\Delta\zeta}\Big[B_\phi(i_g,\ j+1,\ k-1) - B_\phi(i_g,\ j,\ k-1)$$

$$+ B_\phi(i_g,\ j,\ k) - B_\phi(i_g,\ j-1,\ k)\Big] - \frac{\sigma(i_g,\ j,\ k-1/2)}{2\epsilon}\Big[E_\xi(i_g,\ j,\ k)$$

$$+ E_\xi(i_g,\ j,\ k-1)\Big] - \frac{1}{\epsilon} j_\xi(i_g,\ j,\ k-1/2) \qquad (40)$$

After computing fields in the air and ground meshes for given i and k, Eqs. (37) through (40) can be solved simultaneously to give the four independent field components at the boundary.

III. RETARDED-TIME EMP CODE FOR LOW ALTITUDE NUCLEAR BURSTS

Maxwell's Equations

The time variable in this code will be taken as the retarded time of the burst point for waves that propagate in air. Thus put

$$\tau = t - \frac{r}{c} = t - \frac{\zeta - a\xi}{c} \tag{41}$$

where τ is the retarded time, t is the real time, c is the velocity of light in free space, and ξ and ζ are described in Section IV. The corresponding transformations for the derivatives are

$$\left.\frac{\partial}{\partial \tau}\right|_{\xi,\zeta} = \left.\frac{\partial}{\partial t}\right|_{\xi,\zeta} \tag{42}$$

$$\nabla = \nabla_1 - \frac{\hat{r}}{c}\frac{\partial}{\partial \tau} \tag{43}$$

Here ∇_1 is the gradient operator at constant τ and $\hat{r}$ is the unit radial vector in a spherical polar coordinate system centered at the burst point of the weapon. After applying Eqs. (41) through (43), Maxwell's equations in M.K.S. units become

$$\nabla_1 \cdot \vec{D} - \frac{1}{c}\hat{r} \cdot \frac{\partial \vec{D}}{\partial \tau} = \rho \tag{44}$$

$$\nabla_1 \cdot \vec{B} - \frac{1}{c}\hat{r} \cdot \frac{\partial \vec{B}}{\partial \tau} = 0 \tag{45}$$

$$\nabla_1 \times \vec{E} - \frac{1}{c}\hat{r} \times \frac{\partial \vec{E}}{\partial \tau} = -\frac{\partial \vec{B}}{\partial \tau} \tag{46}$$

$$\nabla_1 \times \vec{H} - \frac{1}{c}\hat{r} \times \frac{\partial \vec{H}}{\partial \tau} = \vec{j} + \frac{\partial \vec{D}}{\partial \tau} \tag{47}$$

In the section on geometry transformation coefficients, α_{ji} are determined such that

$$\hat{e}_i = \sum_j \alpha_{ji} \hat{u}_j \tag{48}$$

Where $\hat{e}_i$ are spherical polar unit vectors and $\hat{u}_j$ are prolate spheroidal unit vectors. In particular

$$\hat{r} = \alpha_{11}\hat{\xi} + \alpha_{21}\hat{\zeta} \tag{49}$$

Substitute this into Eqs. (46) and (47) to obtain

$$\frac{\partial \vec{D}}{\partial \tau} = -\vec{j} + \nabla_1 \times \vec{H} - \frac{1}{c}(\alpha_{11}\hat{\xi} + \alpha_{21}\hat{\zeta}) \times \frac{\partial}{\partial \tau}(H_\xi \hat{\xi} + H_\zeta \hat{\zeta} + H_\phi \hat{\phi}) \tag{50}$$

$$\frac{\partial \vec{B}}{\partial \tau} = -\nabla_1 \times \vec{E} + \frac{1}{c}(\alpha_{11}\hat{\xi} + \alpha_{21}\hat{\zeta}) \times \frac{\partial}{\partial \tau}(E_\xi \hat{\xi} + E_\zeta \hat{\zeta} + E_\phi \hat{\phi}) \tag{51}$$

Recall that the prolate spheroidal unit vectors are cyclic in order ξ-ζ-ϕ, then these reduce to

$$\frac{\partial \vec{D}}{\partial \tau} = -\vec{j} + \nabla_1 \times \vec{H} - \frac{1}{c}\left[\alpha_{21}\frac{\partial H_\phi}{\partial \tau}\hat{\xi} - \alpha_{11}\frac{\partial H_\phi}{\partial \tau}\hat{\zeta} + \left(\alpha_{11}\frac{\partial H_\zeta}{\partial \tau} - \alpha_{21}\frac{\partial H_\xi}{\partial \tau}\right)\hat{\phi}\right] \tag{52}$$

$$\frac{\partial \vec{B}}{\partial \tau} = -\nabla_1 \times \vec{E} + \frac{1}{c}\left[\alpha_{21}\frac{\partial E_\phi}{\partial \tau}\hat{\xi} - \alpha_{11}\frac{\partial E_\phi}{\partial \tau}\hat{\zeta} + \left(\alpha_{11}\frac{\partial E_\zeta}{\partial \tau} - \alpha_{21}\frac{\partial E_\xi}{\partial \tau}\right)\hat{\phi}\right] \tag{53}$$

Assume azimuthal symmetry and write the remaining terms of the above equations as functions of ξ, ζ, and ϕ. Also, set

$$\vec{D} = \epsilon\vec{E}, \qquad \vec{B} = \mu_o\vec{H} \tag{54}$$

Then

$$\frac{\partial}{\partial\tau}\left[E_\xi\hat{\xi} + E_\zeta\hat{\zeta} + E_\phi\hat{\phi}\right] = -\frac{1}{\epsilon}\left[j_\xi\hat{\xi} + j_\zeta\hat{\zeta} + j_\phi\hat{\phi}\right]$$

$$+ \frac{u^2\hat{\xi}}{\sqrt{(\zeta^2 - \xi^2 a^2)(1 - \xi^2)}}\frac{\partial}{\partial\zeta}\left(\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\; B_\phi\right)$$

$$- \frac{u^2\hat{\zeta}}{\sqrt{(\zeta^2 - \xi^2 a^2)(\zeta^2 - a^2)}}\frac{\partial}{\partial\xi}\left(\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\; B_\phi\right)$$

$$+ u^2\hat{\phi}\,\frac{\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}}{\zeta^2 - \xi^2 a^2}\left[\frac{\partial}{\partial\xi}\left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{\zeta^2 - a^2}}\; B_\zeta\right)\right.$$

$$\left. - \frac{\partial}{\partial\zeta}\left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{1 - \xi^2}}\; B_\xi\right)\right] - \frac{u^2}{c}\left[\alpha_{21}\frac{\partial B_\phi}{\partial\tau}\hat{\xi} - \alpha_{11}\frac{\partial B_\phi}{\partial\tau}\hat{\zeta}\right.$$

$$\left. + \left(\alpha_{11}\frac{\partial B_\zeta}{\partial\tau} - \alpha_{21}\frac{\partial B_\xi}{\partial\tau}\right)\hat{\phi}\right] \tag{55}$$

$$\frac{\partial}{\partial \tau}(B_\xi \hat{\xi} + B_\zeta \hat{\zeta} + B_\phi \hat{\phi}) = - \frac{\hat{\xi}}{\sqrt{(\zeta^2 - \xi^2 a^2)(1 - \xi^2)}} \frac{\partial}{\partial \zeta}\left(\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\, E_\phi\right)$$

$$+ \frac{\hat{\zeta}}{\sqrt{(\zeta^2 - \xi^2 a^2)(\zeta^2 - a^2)}} \frac{\partial}{\partial \xi} \sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\, E_\phi - \hat{\phi} \frac{\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}}{\zeta^2 - \xi^2 a^2}$$

$$\left[\frac{\partial}{\partial \xi}\left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{\zeta^2 - a^2}}\, E_\zeta\right) - \frac{\partial}{\partial \zeta}\left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{1 - \xi^2}}\, E_\xi\right)\right] + \frac{1}{c}\left[\alpha_{21} \frac{\partial E_\phi}{\partial \tau} \hat{\xi}\right.$$

$$\left. - \alpha_{11} \frac{\partial E_\phi}{\partial \tau} \hat{\zeta} + \left(\alpha_{11} \frac{\partial E_\zeta}{\partial \tau} - \alpha_{21} \frac{\partial E_\xi}{\partial \tau}\right) \hat{\phi}\right] \qquad (56)$$

The velocity of light, u, in the medium is determined by

$$u = \frac{1}{\sqrt{\mu_o \epsilon}} \qquad (57)$$

Note that in component form, Eq. (57) and (58) separate into two uncoupled sets of equations involving E_ξ, E_ζ, j_ξ, j_ζ, B_ϕ, and E_ϕ, j_ϕ, B_ξ, and B_ζ. We assume that $j_\phi = 0$. Provided that the second set of fields is initially zero, Maxwell's equations predict they will remain so. Hence, we put

$$j_\phi = E_\phi = B_\xi = B_\zeta = 0 \qquad (58)$$

The field equations then reduce to

$$\frac{\partial E_\xi}{\partial \tau} = - \frac{j_\xi}{\epsilon} + \frac{u^2}{\sqrt{(\zeta^2 - \xi^2 a^2)(1 - \xi^2)}} \frac{\partial}{\partial \zeta}\left(\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\, B_\phi\right) - \frac{u^2 \alpha_{21}}{c} \frac{\partial B_\phi}{\partial \tau} \qquad (59)$$

$$\frac{\partial E_\zeta}{\partial \tau} = -\frac{j_\zeta}{\epsilon} - \frac{u^2}{\sqrt{(\zeta^2 - \xi^2 a^2)(\zeta^2 - a^2)}} \frac{\partial}{\partial \xi} \sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\, B_\phi + \frac{u^2 \alpha_{11}}{c} \frac{\partial B_\phi}{\partial \tau} \tag{60}$$

$$\frac{\partial B_\phi}{\partial \tau} = \frac{\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}}{\zeta^2 - \xi^2 a^2} \left[\frac{\partial}{\partial \zeta} \left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{1 - \xi^2}}\, E_\xi \right) - \frac{\partial}{\partial \xi} \left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{\zeta^2 - a^2}}\, E_\zeta \right) \right]$$

$$+ \frac{\alpha_{11}}{c} \frac{\partial E_\zeta}{\partial \tau} - \frac{\alpha_{21}}{c} \frac{\partial E_\xi}{\partial \tau} \tag{61}$$

Make the substitution

$$\vec{j} \rightarrow \vec{j} + \sigma \vec{E} \tag{62}$$

Henceforth $\sigma \vec{E}$ will give the conduction portion of the current and $\vec{j}$ will denote only that part caused by Compton electrons. Also at this point, substitute the proper values of α_{ij} from Section IV. Obtain

$$\frac{\partial E_\xi}{\partial \tau} = -\frac{\sigma}{\epsilon} E_\xi - \frac{j_\xi}{\epsilon} + \frac{u^2}{\sqrt{(\zeta^2 - \xi^2 a^2)(1 - \xi^2)}} \frac{\partial}{\partial \zeta} \left(\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\, B_\phi \right)$$

$$- \frac{u^2}{c} \sqrt{\frac{\zeta^2 - a^2}{\zeta^2 - \xi^2 a^2}} \frac{\partial B_\phi}{\partial \tau} \tag{63}$$

$$\frac{\partial E_\zeta}{\partial \tau} = -\frac{\sigma}{\epsilon} E_\zeta - \frac{j_\zeta}{\epsilon} - \frac{u^2}{\sqrt{(\zeta^2 - \xi^2 a^2)(\zeta^2 - a^2)}} \frac{\partial}{\partial \xi}\left(\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\, B_\phi\right) - \frac{au^2}{c}\sqrt{\frac{1 - \xi^2}{\zeta^2 - \xi^2 a^2}}\frac{\partial B_\phi}{\partial \tau} \tag{64}$$

$$\frac{\partial B_\phi}{\partial \tau} = \frac{\sqrt{(\zeta^2 - a^2)(1 - \xi^2)}}{\zeta^2 - \xi^2 a^2}\; \frac{\partial}{\partial \zeta}\left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{1 - \xi^2}}\, E_\xi\right) - \frac{\partial}{\partial \xi}\left(\sqrt{\frac{\zeta^2 - \xi^2 a^2}{\zeta^2 - a^2}}\, E_\zeta\right) - \frac{a}{c}\sqrt{\frac{1 - \xi^2}{\zeta^2 - \xi^2 a^2}}\frac{\partial E_\zeta}{\partial \tau} - \frac{1}{c}\sqrt{\frac{\zeta^2 - a^2}{\zeta^2 - \xi^2 a^2}}\frac{\partial E_\xi}{\partial \tau} \tag{65}$$

Transform the fields according to

$$\begin{pmatrix} E_\xi \\ j_\xi \end{pmatrix} = \sqrt{(\zeta^2 - \xi^2 a^2)(1 - \xi^2)} \begin{pmatrix} E'_\xi \\ j'_\xi \end{pmatrix} \tag{66}$$

$$\begin{pmatrix} E_\zeta \\ j_\zeta \end{pmatrix} = \sqrt{(\zeta^2 - \xi^2 a^2)(\zeta^2 - a^2)} \begin{pmatrix} E'_\zeta \\ j'_\zeta \end{pmatrix} \tag{67}$$

$$B_\phi = \sqrt{(\zeta^2 - a^2)(1 - \xi^2)}\; B'_\phi \tag{68}$$

in which the primed fields are those used previously. Maxwell's curl equations then become

$$\frac{\partial E_\xi}{\partial \tau} = -\frac{\sigma}{\epsilon} E_\xi - \frac{j_\xi}{\epsilon} + u^2 \frac{\partial B_\phi}{\partial \zeta} - \frac{u^2}{c}\frac{\partial B_\phi}{\partial \tau} \tag{69}$$

$$\frac{\partial E_\zeta}{\partial \tau} = -\frac{\sigma}{\epsilon} E_\zeta - \frac{j_\zeta}{\epsilon} - u^2 \frac{\partial B_\phi}{\partial \xi} - \frac{au^2}{c} \frac{\partial B_\phi}{\partial \tau} \tag{70}$$

$$\frac{\partial B_\phi}{\partial \tau} = \frac{\zeta^2 - a^2}{\zeta^2 - \xi^2 a^2} \frac{\partial E_\xi}{\partial \zeta} - \frac{1 - \xi^2}{\zeta^2 - \xi^2 a^2} \frac{\partial E_\zeta}{\partial \xi} - \frac{a}{c} \frac{1 - \xi^2}{\zeta^2 - \xi^2 a^2} \frac{\partial E_\zeta}{\partial \tau} - \frac{1}{c} \frac{\zeta^2 - a^2}{\zeta^2 - \xi^2 a^2} \frac{\partial E_\xi}{\partial \tau} \tag{71}$$

Boundary Conditions in Retarded Time

In the limit as the distance between two points approaches zero, their associated retarded times become equal. Thus intuitively we expect that boundary conditions will remain the same in retarded time as they are in real time. This is demonstrated below in a more rigorous manner.

From Maxwell's differential equations (44) through (47), equivalent integral equations for retarded time may be derived. By integrating Eq. (44) over a three-dimensional space volume v, we find

$$\int_v \nabla_1 \cdot \vec{D}\, dv = \int_v \rho dv + \frac{1}{c} \int_v \hat{r} \cdot \frac{\partial \vec{D}}{\partial \tau} dv \tag{72}$$

Apply Gauss's divergence theorem to the left side, obtain

$$\int_s \vec{D} \cdot d\vec{s} = q + \frac{1}{c} \frac{\partial}{\partial \tau} \int_v \hat{r} \cdot \vec{D}\, dv \tag{73}$$

where q is the charge in v. This is the equivalent of Gauss's law in retarded time. From Eq. (45), a similar law may be found for $\vec{B}$

$$\int_s \vec{B} \cdot \vec{ds} = \frac{1}{c} \frac{\partial}{\partial \tau} \int_V \hat{r} \cdot \vec{B} \, dv \tag{74}$$

Thus in retarded time $\vec{B}$ is not necessarily solenoidal.

From Eq. (46)

$$\int_s \nabla_1 \times \vec{E} \cdot \vec{ds} = -\int_s \frac{\partial \vec{B}}{\partial \tau} \cdot \vec{ds} + \frac{1}{c} \frac{\partial}{\partial \tau} \int_s \hat{r} \times \vec{E} \cdot \vec{ds} \tag{75}$$

Applying Stoke's theorem yields

$$\int_c \vec{E} \cdot \vec{d\ell} = -\frac{\partial}{\partial \tau} \int_s \vec{B} \cdot \vec{ds} + \frac{1}{c} \frac{\partial}{\partial \tau} \int_s \hat{r} \times \vec{E} \cdot \vec{ds} \tag{76}$$

in which c is the curve that bounds the surface s. This, of course, is the retarded time analog of Faraday's law of induction. Similarly from Eq. (47) one finds,

$$\int_c \vec{H} \cdot \vec{d\ell} = \int_s \vec{j} \cdot \vec{ds} + \frac{\partial}{\partial \tau} \int_s \vec{D} \cdot \vec{ds} + \frac{1}{c} \frac{\partial}{\partial \tau} \int_s \hat{r} \times \vec{H} \cdot \vec{ds} \tag{77}$$

which is a generalization of Ampere's law in retarded time.

Now consider the boundary conditions in retarded time. From Gauss's law, Eq. (73), and the pillbox shown in Figure 2, one finds

$$\frac{1}{\Delta s} \int_{\Delta s_\ell} \vec{D} \cdot \vec{ds} + (D_1 - D_2) \cdot \hat{n} = \frac{q}{\Delta s} + \frac{\delta}{2c} \frac{\partial}{\partial \tau} (\vec{D}_1 + \vec{D}_2) \cdot \hat{r} \tag{78}$$

Here it is assumed that $\vec{D}$ may be discontinuous on the boundary, but that it is continuous in regions 1 and 2 above and below the boundary. The lateral surface of the pillbox is labeled Δs_ℓ and Δs is the area of each end. Take the limit $\delta \longrightarrow 0$.

$$(\vec{D}_1 - \vec{D}_2) \cdot \hat{n} = \lambda \qquad (79)$$

where λ is the surface charge density.

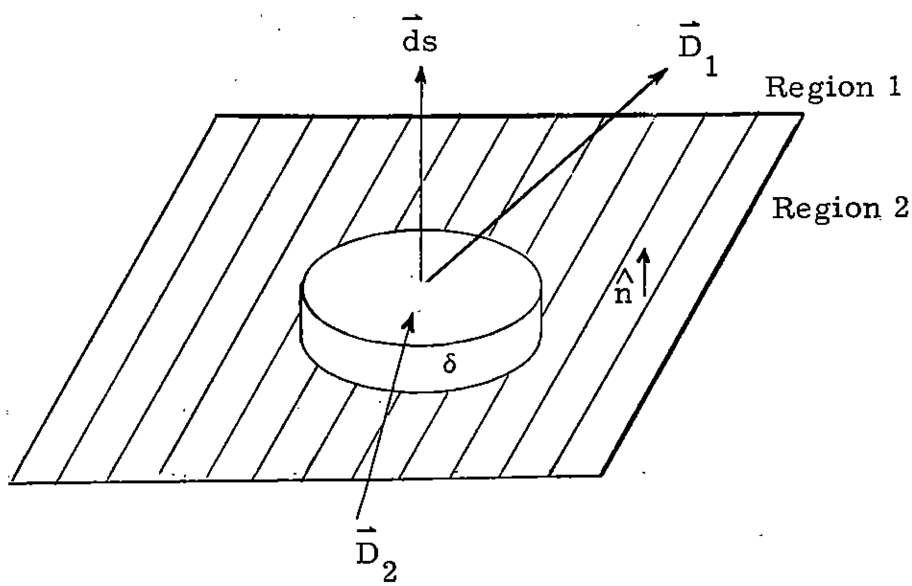

Figure 2

Gaussian Pillbox for the Determination of Boundary Conditions for Normal Field Components

Similarly, from Eq. (74), one finds

$$(\vec{B}_1 - \vec{B}_2) \cdot \hat{n} = 0 \qquad (80)$$

From Eq. (76) and the contour shown in Figure 3, it follows that

$$(\vec{E}_1 - \vec{E}_2) \cdot \hat{t}\Delta\ell = -\frac{1}{2}\frac{\partial}{\partial t}(\vec{B}_1 + \vec{B}_2) \cdot \hat{n} \times \hat{t}\Delta\ell\delta + \frac{1}{2c}\frac{\partial}{\partial \tau}\hat{r}$$

$$\times (\vec{E}_1 + \vec{E}_2) \cdot \hat{n} \times \hat{t}\Delta\ell\delta \qquad (81)$$

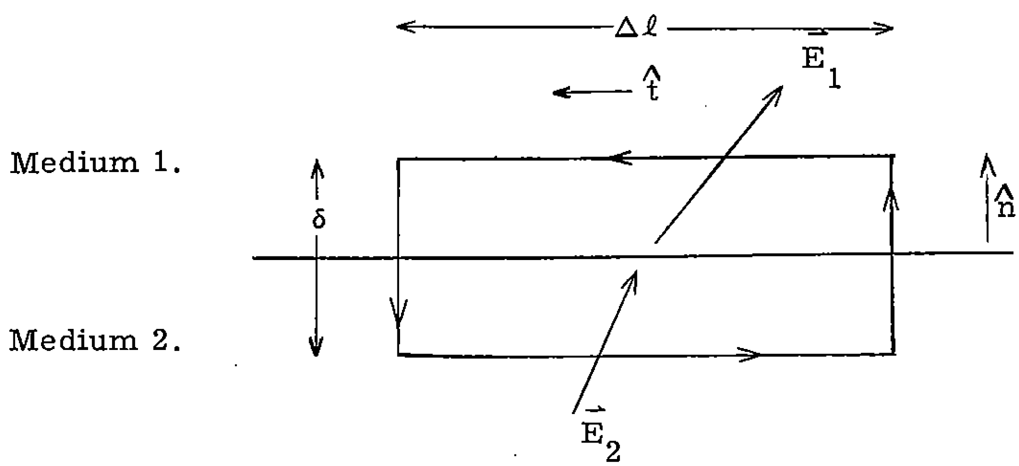

Figure 3

Contour Used in Determination of Boundary Conditions for Tangential Field Components

Cancel $\Delta\ell$ and take the limit $\delta \longrightarrow 0$, then

$$(\vec{E}_1 - \vec{E}_2) \cdot \hat{t} = 0 \qquad (82)$$

From Eq. (77), a similar argument gives

$$(\vec{H}_1 - \vec{H}_2) \cdot \hat{t} = \vec{k} \cdot \hat{n} \times \hat{t} \qquad (83)$$

where $\vec{k}$ is the surface current, $\hat{n}$ is the unit normal vector, and $\hat{t}$ is the unit tangent vector. Equations (79), (80), (82), and (83) will be recognized as the boundary conditions in real time.

For problems with azimuthal symmetry, it is a simple matter to show in real time that B_ϕ vanishes on the z axis. Let us look at the derivation in retarded time. Integrate Eq. (77) over the contour shown in Figure 4.

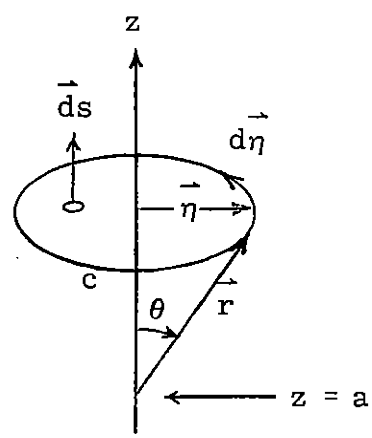

Figure 4
Contour for Evaluating $\vec{B}$ on the z Axis

For our problem, $\vec{B} = B_\phi \hat{\phi}$. The azimuthal symmetry ensures that the magnitude of $\vec{B}$ is constant on c. Therefore,

$$\int_c \vec{B} \cdot \vec{d\eta} = 2\pi\eta \, B_\phi \Big|_{\eta, z} \tag{84}$$

Here η is a coordinate measured perpendicular to the z axis. Provided η is small enough that $\vec{B}$ does not change appreciably over the surface s, we can, for points on the z axis other than the burst point, set

$$\frac{\partial}{\partial \tau} \int_s \hat{r} \times \vec{B} \cdot \vec{ds} \simeq \frac{\partial B_\phi}{\partial \tau} \Big|_{\eta=0, z} \int_s \sin\theta \, ds \tag{85}$$

Use the approximation

$$\sin\theta \simeq \frac{\eta}{z - a} \tag{86}$$

which is valid for $\eta \ll z - a$. Then Eq. (85) becomes

$$\frac{\partial}{\partial\tau} \int_S \hat{r} \times \vec{B} \cdot d\vec{s} \simeq \frac{2\pi\eta^3}{3(z - a)} \left.\frac{\partial B_\phi}{\partial\tau}\right|_{\eta = 0, z} \tag{87}$$

Assume that j_z, $\partial E_z/\partial\tau$, and $\partial B_\phi/\partial\tau$ are finite on the z axis and that the constitutive equations (54) hold. Then in the limit $\eta \longrightarrow 0$, Eqs. (77), (84), and (87) yield

$$\lim_{\eta \to 0} (2\pi\eta)\, B_\phi(\eta, z, \tau) = \lim_{\eta \to 0} \left[\mu\pi\eta^2 j_z(\eta, z, \tau) + \frac{\pi\eta^2}{c^2} \frac{\partial E_z(\eta, z, \tau)}{\partial\tau} + \frac{2\pi\eta^3}{3c(z - a)} \frac{\partial B_\phi(\eta, z, \tau)}{\partial\tau} \right] \tag{88}$$

or

$$\lim_{\eta \to 0} B_\phi(\eta, z, \tau) = 0 \tag{89}$$

Another obvious symmetry condition is that

$$\lim_{\eta \to 0} E_\eta(\eta, z, \tau) = 0 \tag{90}$$

Difference Equations

At a mesh point the independent variables may be written

$$\xi = 1 + (1 - i)\Delta\xi\,, \qquad \zeta = a + (j - 1)\Delta\zeta\,, \qquad t = k\Delta t \tag{91}$$

With this notation, Maxwell's curl equations (69), (70), and (71) may be replaced by difference equations centered at (i, j, k-1/2). We find

$$\frac{1}{\Delta\tau}\left[E_\xi(i, j, k) - E_\xi(i, j, k-1)\right] = -\frac{\sigma(i, j, k-1/2)}{2\epsilon}\Big[E_\xi(i, j, k)$$

$$+ E_\xi(i, j, k-1)\Big] - \frac{j_\xi(i, j, k-1/2)}{\epsilon} + \frac{u^2}{2\Delta\zeta}\Big[B_\phi(i, j+1, k-1) - B_\phi(i, j, k-1)$$

$$+ B_\phi(i, j, k) - B_\phi(i, j-1, k)\Big] - \frac{u^2}{c\Delta t}\left[B_\phi(i, j, k) - B_\phi(i, j, k-1)\right] \tag{92}$$

$$\frac{1}{\Delta\tau}\left[E_\zeta(i, j, k) - E_\zeta(i, j, k-1)\right] = -\frac{\sigma(i, j, k-1/2)}{2\epsilon}\Big[E_\zeta(i, j, k)$$

$$+ E_\zeta(i, j, k-1)\Big] - \frac{j_\zeta(i, j, k-1/2)}{\epsilon} - \frac{u^2}{2\Delta\xi}\Big[B_\phi(i-1, j, k) - B_\phi(i, j, k)$$

$$+ B_\phi(i, j, k-1) - B_\phi(i+1, j, k-1)\Big] - \frac{au^2}{c\Delta\tau}\left[B_\phi(i, j, k) - B_\phi(i, j, k-1)\right] \tag{93}$$

$$\frac{1}{\Delta\tau}\Big[B_\phi(i, j, k) - B_\phi(i, j, k-1)\Big] = \frac{1}{2\Delta\zeta}\left(\frac{\zeta^2 - a^2}{\zeta^2 - \xi^2 a^2}\right)\Bigg|_{i,j} \Big[E_\xi(i, j+1, k-1)$$

$$- E_\xi(i, j, k-1) + E_\xi(i, j, k) - E_\xi(i, j-1, k)\Big]$$

$$+ \frac{1}{2\Delta\xi}\left(\frac{1 - \xi^2}{\zeta^2 - \xi^2 a^2}\right)\Bigg|_{i,j} \Big[E_\zeta(i+1, j, k-1) - E_\zeta(i, j, k-1) + E_\zeta(i, j, k)$$

$$- E_\zeta(i-1, j, k)\Big] - \frac{a}{c\Delta\tau}\left(\frac{1 - \xi^2}{\zeta^2 - \xi^2 a^2}\right)\Bigg|_{i,j} \Big[E_\zeta(i, j, k) - E_\zeta(i, j, k-1)\Big]$$

$$- \frac{1}{c\Delta\tau}\left(\frac{\zeta^2 - a^2}{\zeta^2 - \xi^2 a^2}\right)\Bigg|_{i,j} \Big[E_\xi(i, j, k) - E_\xi(i, j, k-1)\Big] \qquad (94)$$

These equations may be used both in the air and the ground meshes provided appropriate values of σ, and ϵ are inserted. Equations (92), (93), and (94) are solved for the fields in a manner analogous to that used in the real-time code. The one difference in the methods is that in the retarded time code the fields are determined in the air mesh along a line of constant j, the boundary conditions at the ground are applied, then the fields are obtained in the ground mesh along this same line in the direction of decreasing ξ ; whereas in the real-time code, the boundary conditions are applied last.

To facilitate handling the boundary conditions at the ground, two lines of points are defined which have $\xi = 0$. One set of these is assumed to be in air and the other to be in the ground. The mesh near the boundary is labeled as shown in Figure 1.

The boundary conditions at this interface are that E_ζ and B_ϕ are continuous. This implies that $\partial E_\zeta / \partial\tau$ and $\partial B_\phi / \partial\tau$ are continuous. Thus we set

$$\frac{\partial E_\zeta(i_a, j, k-1/2)}{\partial \tau} = \frac{\partial E_\zeta(i_g, j, k-1/2)}{\partial \tau} \approx \frac{1}{2}\left[\frac{\partial E_\zeta(i_a-1/2, j, k-1/2)}{\partial \tau}\right.$$

$$\left. + \frac{\partial E_\zeta(i_g+1/2, j, k-1/2)}{\partial \tau}\right] \tag{95}$$

$$\frac{\partial B_\phi(i_a, j, k-1/2)}{\partial \tau} = \frac{\partial B_\phi(i_g, j, k-1/2)}{\partial \tau} \approx \frac{1}{2}\left[\frac{\partial B_\phi(i_a-1/2, j, k-1/2)}{\partial \tau}\right.$$

$$\left. + \frac{\partial B_\phi(i_g+1/2, j, k-1/2)}{\partial \tau}\right] \tag{96}$$

By writing Eqs. (95) and (96) in difference form, we obtain

$$\frac{1}{\Delta\tau}\left[E_\zeta(i_a, j, k) - E_\zeta(i_a, j, k-1)\right] = -\frac{\sigma(i_g+1/2, j, k-1/2)}{8\epsilon}\left[E_\zeta(i_g, j, k)\right.$$

$$\left. + E_\zeta(i_g+1, j, k) + E_\zeta(i_g+1, j, k-1) + E_\zeta(i_g, j, k-1)\right] - \frac{j_\zeta(i_g+1/2, j, k-1/2)}{2\epsilon}$$

$$-\frac{u^2}{4\Delta\xi}\left[B_\phi(i_a, j, k) - B_\phi(i_g+1, j, k) + B_\phi(i_a, j, k-1) - B_\phi(i_g+1, j, k-1)\right]$$

$$-\frac{au^2}{4c\Delta\tau}\left[B_\phi(i_g+1, j, k) - B_\phi(i_g+1, j, k-1) + B_\phi(i_a, j, k) - B_\phi(i_a, j, k-1)\right]$$

$$-\frac{\sigma(i_a-1/2, j, k-1/2)}{8\epsilon_0}\left[E_\zeta(i_a-1, j, k) + E_\zeta(i_a, j, k) + E_\zeta(i_a, j, k-1)\right.$$

$$+ E_\zeta(i_a-1, j, k-1)\Big] - \frac{j_\zeta(i_a-1/2, j, k-1/2)}{2\epsilon_0} - \frac{c^2}{4\Delta\xi}\Big[B_\phi(i_a-1, j, k)$$

$$- B_\phi(i_a, j, k) + B_\phi(i_a-1, j, k-1) - B_\phi(i_a, j, k-1)\Big] - \frac{ac}{4\Delta\tau}\Big[B_\phi(i_a-1, j, k)$$

$$- B_\phi(i_a-1, j, k-1) + B_\phi(i_a, j, k) - B_\phi(i_a, j, k-1)\Big] \tag{97}$$

and

$$\frac{1}{\Delta\tau}\Big[B_\phi(i_a, j, k) - B_\phi(i_a, j, k-1)\Big] = \frac{1}{8\Delta\zeta}\,\frac{\zeta^2 - a^2}{\zeta^2 - a^2\Delta\xi^2/4}\Big[E_\xi(i_a, j+1, k-1)$$

$$- E_\xi(i_a, j, k-1) + E_\xi(i_a, j, k) - E_\xi(i_a, j-1, k) + E_\xi(i_a-1, j+1, k-1)$$

$$- E_\xi(i_a-1, j, k-1) + E_\xi(i_a-1, j, k) - E_\xi(i_a-1, j-1, k) + E_\xi(i_g+1, j+1, k-1)$$

$$- E_\xi(i_g+1, j, k-1) + E_\xi(i_g+1, j, k) - E_\xi(i_g+1, j-1, k) + E_\xi(i_g, j+1, k-1)$$

$$- E_\xi(i_g, j, k-1) + E_\xi(i_g, j, k) - E_\xi(i_g, j-1, k)\Big] - \frac{1}{4\Delta\xi}$$

$$\frac{1 - \Delta\xi^2/4}{\zeta^2 - a^2\Delta\xi^2/4}\Big[E_\zeta(i_a-1, j, k) + E_\zeta(i_a-1, j, k-1) - E_\zeta(i_g+1, j, k)$$

$$- E_\zeta(i_g+1, j, k-1)\Big] - \frac{1}{4c\Delta\tau}\,\frac{\zeta^2 - a^2}{\zeta^2 - a^2\Delta\xi^2/4}\Big[E_\xi(i_a, j, k) - E_\xi(i_a, j, k-1)$$

$$+ E_\xi(i_a-1, j, k) - E_\xi(i_a-1, j, k-1) + E_\xi(i_g, j, k) - E_\xi(i_g, j, k-1)$$

$$+ E_\xi(i_g+1, j, k) - E_\xi(i_g+1, j, k-1)\Big] - \frac{a}{4c\Delta\tau}\,\frac{1 - \Delta\xi^2/4}{\zeta^2 - a^2\Delta\xi^2/4}\Big[2E_\zeta(i_a, j, k)$$

$$-2E_\zeta(i_a, j, k-1) + E_\zeta(i_a-1, j, k) - E_\zeta(i_a-1, j, k-1) + E_\zeta(i_g+1, j, k)$$

$$- E_\zeta(i_g+1, j, k-1)\Big] \qquad (98)$$

Evaluate $\partial E_\xi/\partial\tau$ at $(i_a, j, k-1/2)$

$$\frac{1}{\Delta\tau}\Big[E_\xi(i_a, j, k) - E_\xi(i_a, j, k-1)\Big] = -\frac{\sigma(i_a, j, k-1/2)}{2\epsilon_0}\Big[E_\xi(i_a, j, k)$$

$$+ E_\xi(i_a, j, k-1)\Big] - \frac{j_\xi(i_g, j, k-1/2)}{\epsilon_0} + \frac{c^2}{2\Delta\zeta}\Big[B_\phi(i_a, j+1, k-1) - B_\phi(i_a, j, k-1)$$

$$+ B_\phi(i_a, j, k) - B_\phi(i_a, j-1, k)\Big] - \frac{c}{\Delta\tau}\Big[B_\phi(i_a, j, k) - B_\phi(i_a, j, k-1)\Big] \qquad (99)$$

Evaluate $\partial E_\xi/\partial\tau$ at $(i_g, j, k-1/2)$

$$\frac{1}{\Delta\tau}\Big[E_\xi(i_g, j, k) - E_\xi(i_g, j, k-1)\Big] = -\frac{\sigma(i_g, j, k-1/2)}{2\epsilon}\Big[E_\xi(i_g, j, k)$$

$$+ E_\xi(i_g, j, k-1)\Big] - \frac{j_\xi(i_g, j, k-1/2)}{\epsilon} + \frac{u^2}{2\Delta\zeta}\Big[B_\phi(i_g, j+1, k-1) - B_\phi(i_g, j, k-1)$$

$$+ B_\phi(i_g, j, k) - B_\phi(i_g, j-1, k)\Big] - \frac{u^2}{c\Delta\tau}\Big[B_\phi(i_g, j, k) - B_\phi(i_g, j, k-1)\Big] \qquad (100)$$

Evaluate Maxwell's curl equations at $(i_g+1, j, k-1/2)$

$$\frac{1}{\Delta\tau}\left[E_\xi(i_g+1, j, k) - E_\xi(i_g+1, j, k-1)\right] = -\frac{\sigma(i_g+1, j, k-1/2)}{2\epsilon}\left[E_\xi(i_g+1, j, k)\right.$$

$$\left.+ E_\xi(i_g+1, j, k-1)\right] - \frac{j_\xi(i_g+1, j, k-1/2)}{\epsilon} + \frac{u^2}{2\Delta\zeta}\left[B_\phi(i_g+1, j+1, k-1)\right.$$

$$\left.- B_\phi(i_g+1, j, k-1) + B_\phi(i_g+1, j, k) - B_\phi(i_g+1, j-1, k)\right] - \frac{u^2}{c\Delta\tau}\left[B_\phi(i_g+1, j, k)\right.$$

$$\left.- B_\phi(i_g+1, j, k-1)\right] \qquad (101)$$

$$\frac{1}{\Delta\tau}\left[E_\zeta(i_g+1, j, k) - E_\zeta(i_g+1, j, k-1)\right] = -\frac{\sigma(i_g+1, j, k-1/2)}{2\epsilon}\left[E_\zeta(i_g+1, j, k)\right.$$

$$\left.+ E_\zeta(i_g+1, j, k-1)\right] - \frac{j_\zeta(i_g+1, j, k-1/2)}{\epsilon} - \frac{u^2}{2\Delta\xi}\left[B_\phi(i_g, j, k) - B_\phi(i_g+1, j, k)\right.$$

$$\left.+ B_\phi(i_g+1, j, k-1) - B_\phi(i_g+2, j, k-1)\right] - \frac{au^2}{c\Delta\tau}\left[B_\phi(i_g+1, j, k)\right.$$

$$\left.- B_\phi(i_g+1, j, k-1)\right] \qquad (102)$$

$$\frac{1}{\Delta\tau}\left[B_\phi(i_g+1, j, k) - B_\phi(i_g+1, j, k-1)\right] = \frac{1}{2\Delta\zeta}\,\frac{\zeta^2 - a^2}{\zeta^2 - a^2\Delta\xi^2}\left[E_\xi(i_g+1, j+1, k-1)\right.$$

$$\left.- E_\xi(i_g+1, j, k-1) + E_\xi(i_g+1, j, k) - E_\xi(i_g+1, j-1, k)\right] - \frac{1}{2\Delta\xi}$$

$$\frac{1 - \Delta\xi^2}{\zeta^2 - a^2\Delta\xi^2}\left[E_\zeta(i_g, j, k) - E_\zeta(i_g+1, j, k) + E_\zeta(i_g+1, j, k-1)\right.$$

$$\left. - E_\zeta(i_g+2, j, k-1)\right] - \frac{1}{c\Delta\tau}\frac{\zeta^2 - a^2}{\zeta^2 - a^2\Delta\xi^2}\left[E_\xi(i_g+1, j, k) - E_\xi(i_g+1, j, k-1)\right]$$

$$- \frac{a}{c\Delta\tau}\frac{1 - \Delta\xi^2}{\zeta^2 - a^2\Delta\xi^2}\left[E_\zeta(i_g+1, j, k) - E_\zeta(i_g+1, j, k-1)\right] \qquad (103)$$

For given j and k, Eqs. (97) through (103) are solved simultaneously for the four independent field quantities at the air-ground boundary and for the three field quantities at the point (i_g+1, j, k).

A Difference Code for Problems With Certain Additional Symmetry

Provided the electromagnetic fields in the region above the wave reflected from the ground has spherical symmetry, considerable savings in storage and running time of the retarded time code may be achieved. One may realistically assume this symmetry if (1) the magnetic field of the earth is neglected, (2) the burst is low enough that the density gradient of the atmosphere is negligible, and (3) the weapon involved would produce a spherically symmetric explosion in a homogeneous medium.

The code takes advantage of the symmetry mentioned by using the simple differential equation,

$$\frac{\partial E_r}{\partial \tau} = - \frac{j_r}{\epsilon} - \frac{\sigma E_r}{\epsilon} \qquad (104)$$

which is valid for spherically symmetric source distributions, to calculate fields on a curve just above the wave front reflected from the ground. These are then used as boundary conditions for the two-dimensional retarded time code which calculates fields only in the region occupied by the reflected wave and in the ground. Thus, storage and calculation of fields and sources above this reflected wave are ignored. Another advantage of handling the problem in this way is that one avoids the usual artificial device of placing a perfectly conducting sphere around the burst.

The geometry of the retarded time code is particularly well suited to this problem. The wave front of the wave reflected from the ground is a portion of the circumference of a sphere whose center is the image point of the burst and is therefore described by

$$\zeta - a\xi = ct \tag{105}$$

combining this with Eq. (41) yields

$$\tau = \frac{2a\xi}{c} \tag{106}$$

Thus for a given retarded time of the burst point, the wave front reflected from the ground lies on the hyperbola ξ = constant. This greatly facilitates applying the boundary conditions obtained from the solution of Eq. (104). That solution is obtained by writing the differential equation in difference form*.

* See Section V for a finite difference solution of Eq. (104).

IV. GEOMETRY FOR THE LOW ALTITUDE EMP CODES

By referring to Figure 5, define prolate spheroidal coordinates as

$$\xi = \frac{1}{2a}(r_1 - r), \qquad \zeta = \frac{1}{2}(r_1 + r), \qquad \phi = \tan^{-1}\frac{y}{x} \tag{107}$$

Note that surfaces of constant ξ are hyperboloids of revolution and those of constant ζ are prolate spheroids (i.e., prolate ellipsoids of revolution). Also observe that all real space lies within the range of coordinates

$$\zeta \geq a, \qquad -1 \leq \xi \leq 1 \tag{108}$$

The z axis is separated into three segments determined by

$$x = y = 0 \quad \text{or} \quad \begin{cases} \xi = 1 & \text{for } z < a \quad (109) \\ \zeta = a & \text{for } -a < z < a \quad (110) \\ \xi = -1 & \text{for } z < -a. \quad (111) \end{cases}$$

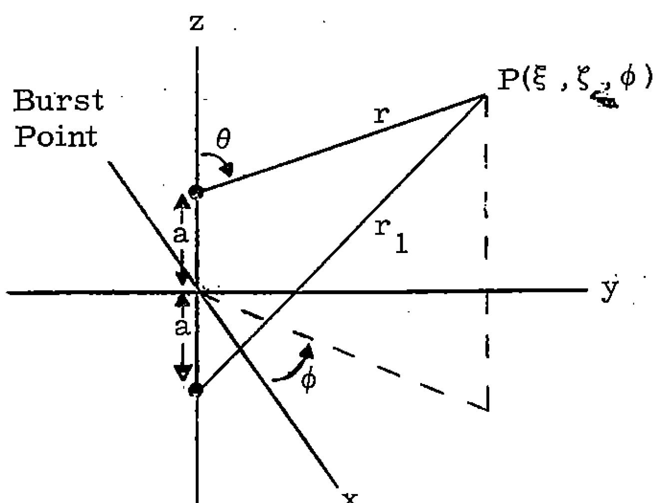

Figure 5

Geometry for the Prolate Spheroidal Coordinate System

The x-y plane is taken as the ground-air interface and according to Eq. (107) is the degenerate hyperboloid determined by $\xi = 0$. The burst point of the weapon is on the z axis at the point z = a which is also described by $\zeta = a$, $\xi = 1$. The wave front of the electromagnetic pulse generated by the weapon spreads out in air from the burst point with velocity c and is determined by r = ct, or by

$$\zeta - a\xi = ct \tag{112}$$

Thus for fixed t the wave front describes a straight line in the prolate spheroidal coordinate system.

From Eq. (107) we find

$$r_1 = \zeta + a\xi \tag{113}$$

$$r = \zeta - a\xi \tag{114}$$

Now

$$r_1^2 = (\zeta + a\xi)^2 = x^2 + y^2 + (z + a)^2 \tag{115}$$

$$r^2 = (\zeta - a\xi)^2 = x^2 + y^2 + (z - a)^2 \tag{116}$$

From these it follows that the transformation equations from cartesian coordinates to prolate spheroidal coordinates are

$$\xi = \frac{1}{2a}\sqrt{x^2 + y^2 + (z + a)^2} - \frac{1}{2a}\sqrt{x^2 + y^2 + (z - a)^2} \tag{117}$$

$$\zeta = \frac{1}{2}\sqrt{x^2 + y^2 + (z + a)^2} + \frac{1}{2}\sqrt{x^2 + y^2 + (z - a)^2} \tag{118}$$

$$\phi = \tan^{-1} \frac{y}{x} \tag{119}$$

where principal square roots are intended. The inverse transformations are

$$x = \sqrt{(\zeta^2 - a^2)(1 - \xi^2)} \cos\phi \tag{120}$$

$$y = \sqrt{(\zeta^2 - a^2)(1 - \xi^2)} \sin\phi \tag{121}$$

$$z = \xi \zeta \tag{122}$$

Transformation equations from a spherical polar coordinate system centered at the burst point to prolate spheroidal coordinates may be obtained in a similar manner, we find

$$\xi = \frac{2(a/r + \cos\theta)}{1 + \sqrt{1 + (4a/r)(a/r + \cos\theta)}} \tag{123}$$

$$\zeta = \frac{r}{2}\left[1 + \sqrt{1 + (4a/r)(a/r + \cos\theta)}\right] \tag{124}$$

The azimuthal angle is the same in both coordinate systems. The inverse transformations are

$$r = \zeta - a\xi \tag{125}$$

$$\theta = \cos^{-1}\left(\frac{\xi\zeta - a}{\zeta - a\xi}\right) \tag{126}$$

We now seek the scale factors h_ξ, h_ζ, and h_ϕ such that

$$ds^2 = h_\xi^2\, d\xi^2 + h_\zeta^2\, d\zeta^2 + h_\phi^2\, d\phi^2 \tag{127}$$

where ds is an element of length. To find these, differentiate Eqs. (120) through (122) and substitute into

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{128}$$

Obtain

$$ds^2 = \frac{\zeta^2 - \xi^2 a^2}{1 - \xi^2}\, d\xi^2 + \frac{\zeta^2 - \xi^2 a^2}{\zeta^2 - a^2}\, d\zeta^2 + (\zeta^2 - a^2)(1 - \xi^2)\, d\phi^2 \tag{129}$$

comparing Eqs. (127) and (129) gives

$$h_\xi = \sqrt{\frac{\zeta^2 - \xi^2 a^2}{1 - \xi^2}} \tag{130}$$

$$h_\zeta = \sqrt{\frac{\zeta^2 - \xi^2 a^2}{\zeta^2 - a^2}} \tag{131}$$

$$h_\phi = \sqrt{(\zeta^2 - a^2)(1 - \xi^2)} \tag{132}$$

Because Eq. (129) does not contain cross terms in $d\xi$, $d\zeta$, and $d\phi$ it follows that we are dealing with an orthogonal coordinate system. Incidentally, the coordinates are cyclic in the order $\xi - \zeta - \phi$.

By using Eqs. (130) through (132) one easily finds expressions for the various vector operations in prolate spheroidal coordinates.

$$\nabla f = \sqrt{\frac{1-\xi^2}{\zeta^2-\xi^2 a^2}}\frac{\partial f}{\partial \xi}\hat{\xi} + \sqrt{\frac{\zeta^2-a^2}{\zeta^2-\xi^2 a^2}}\frac{\partial f}{\partial \zeta}\hat{\zeta} + \sqrt{(\zeta^2-a^2)(1-\xi^2)}\frac{\partial f}{\partial \phi}\hat{\phi} \quad (133)$$

$$\nabla \cdot \vec{f} = \frac{1}{\zeta^2-\xi^2 a^2}\left\{\frac{\partial}{\partial \xi}\left(\sqrt{(\zeta^2-\xi^2 a^2)(1-\xi^2)}\, f_\xi\right) + \frac{\partial}{\partial \zeta}\left(\sqrt{(\zeta^2-\xi^2 a^2)(\zeta^2-a^2)}\, f_\zeta\right)\right.$$

$$\left. + \frac{\partial}{\partial \phi}\left(\frac{\zeta^2-\xi^2 a^2}{\sqrt{(\zeta^2-a^2)(1-\xi^2)}}\, f_\phi\right)\right\} \quad (134)$$

$$\nabla \times \vec{f} = \frac{\hat{\xi}}{\sqrt{(\zeta^2-\xi^2 a^2)(1-\xi^2)}}\left[\frac{\partial}{\partial \zeta}\left(\sqrt{(\zeta^2-a^2)(1-\xi^2)}\, f_\phi\right)\right.$$

$$\left. - \frac{\partial}{\partial \phi}\left(\sqrt{\frac{\zeta^2-\xi^2 a^2}{\zeta^2-a^2}}\right) f_\zeta\right] + \frac{\hat{\zeta}}{\sqrt{(\zeta^2-\xi^2 a^2)(\zeta^2-a^2)}}\left[\frac{\partial}{\partial \phi}\left(\sqrt{\frac{\zeta^2-\xi^2 a^2}{1-\xi^2}}\, f_\xi\right)\right.$$

$$\left. - \frac{\partial}{\partial \xi}\left(\sqrt{(\zeta^2-a^2)(1-\xi^2)}\, f_\phi\right)\right] + \hat{\phi}\,\frac{\sqrt{(\zeta^2-a^2)(1-\xi^2)}}{\zeta^2-\xi^2 a^2}\left[\frac{\partial}{\partial \xi}\left(\sqrt{\frac{\zeta^2-\xi^2 a^2}{\zeta^2-a^2}}\, f_\zeta\right)\right.$$

$$\left. - \frac{\partial}{\partial \zeta}\left(\sqrt{\frac{\zeta^2-\xi^2 a^2}{1-\xi^2}}\, f_\xi\right)\right] \quad (135)$$

$$\nabla^2 f = \frac{1}{\zeta^2 - \xi^2 a^2}\left\{\frac{\partial}{\partial \xi}\left[(1 - \xi^2)\frac{\partial f}{\partial \xi}\right] + \frac{\partial}{\partial \zeta}\left[(\zeta^2 - a^2)\frac{\partial f}{\partial \zeta}\right] + \frac{\partial}{\partial \phi}\left[\frac{\zeta^2 - \xi^2 a^2}{(\zeta^2 - a^2)(1 - \xi^2)}\frac{\partial f}{\partial \phi}\right]\right\} \tag{136}$$

Let us now consider the vector transformation equations between prolate spheroidal coordinates and other orthogonal systems. For convenience denote the base vectors in the prolate spheroidal system by $\hat{u}_i$ and those of the other system by $\hat{e}_i$. We may expand $\hat{e}_i$ in terms of the $\hat{u}_i$.

$$\hat{e}_i = \sum_j \alpha_{ji}\, \hat{u}_j \tag{137}$$

But

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = \sum_{k,\ell} (\alpha_{ki}\hat{u}_k) \cdot (\alpha_{\ell j}\hat{u}_\ell) = \sum_{k,\ell} \alpha_{ki}\alpha_{\ell j}(\hat{u}_k \cdot \hat{u}_\ell)$$

$$= \sum_{k,\ell} \alpha_{ki}\alpha_{\ell j}\delta_{k\ell} = \sum_k \alpha_{ki}\alpha_{kj} \tag{138}$$

i.e.,

$$\sum_k \alpha_{ki}\alpha_{kj} = \delta_{ij} \tag{139}$$

similarly it can be shown that

$$\sum_i \alpha_{ki}\alpha_{\ell i} = \delta_{k\ell} \tag{140}$$

Equations (139) and (140) are the orthogonality relations for the transformation coefficients α_{ij}. Multiply Eq. (137) by α_{ki} and sum over i.

$$\sum_i \alpha_{ki}\hat{e}_i = \sum_{i,j} \alpha_{ki}\alpha_{ji}\hat{u}_j = \sum_j \delta_{kj}\hat{u}_j = \hat{u}_k \tag{141}$$

Equation (140) has been used. Thus

$$\hat{u}_i = \sum_j \alpha_{ij}\hat{e}_j \tag{142}$$

Any vector may be expanded in terms of either set of base vectors.

$$\vec{f} = \sum_i f_i \hat{e}_i = \sum_{i,j} f_i \alpha_{ji}\hat{u}_j = \sum_j f'_j \hat{u}_j \tag{143}$$

Here f'_j are the components of f in prolate spheroidal coordinates and f_i are those in the system with base vectors $\hat{e}_i$. Observe that

$$f'_i = \sum_j \alpha_{ij} f_j \tag{144}$$

Multiply this by α_{ik}, sum over i, and use the orthogonality relation. Obtain

$$f_i = \sum_j \alpha_{ji} f'_j \tag{145}$$

An easy way to determine the transformation matrix is to find an expression for the prolate spheroidal unit vectors in terms of those of the other coordinate system and compare to Eq. (142). Now,

$$\hat{\xi} = h_\xi \nabla\xi, \qquad \hat{\zeta} = h_\zeta \nabla\zeta, \qquad \hat{\phi} = h_\phi \nabla\phi \tag{146}$$

we wish to find the transformation to spherical polar coordinates centered at the burst point. By taking the gradients of Eqs. (123) and (124) and employing Eqs. (130), (131), and (146) we find

$$\hat{\xi} = -\frac{a}{h_\xi}\hat{r} - \frac{1}{h_\zeta}\hat{\theta} \tag{147}$$

$$\hat{\zeta} = \frac{1}{h_\zeta}\hat{r} - \frac{a}{h_\xi}\hat{\theta} \tag{148}$$

Of course $\hat{\phi}$ is identical to the corresponding spherical polar unit vector. By comparing Eq. (142) with Eqs. (147) and (148) the desired transformation maxtrix is obtained

$$(\alpha_{ij}) = \begin{pmatrix} -\frac{a}{h_\xi} & -\frac{1}{h_\zeta} \\ \frac{1}{h_\zeta} & -\frac{a}{h_\xi} \end{pmatrix} \tag{149}$$

Because of the orthogonality of the transformation, the inverse matrix is

$$(\alpha_{ij})^{-1} = (\alpha_{ji}) = \begin{pmatrix} -\frac{a}{h_\xi} & \frac{1}{h_\zeta} \\ -\frac{1}{h_\zeta} & -\frac{a}{h_\xi} \end{pmatrix} \tag{150}$$

V. A TEST PROBLEM

The following test problem was devised to provide a check on the codes. Current and conductivities produced by the weapon are approximated by

$$\vec{J}(r,\tau) = \frac{J_o}{r^2} \frac{\exp\left[\alpha(\tau - \tau_p) - r/R_\gamma\right]}{1 + \exp\left[(\alpha + \beta)(\tau - \tau_p)\right]} \hat{r} \tag{151}$$

$$\sigma(r,\tau) = \frac{\sigma_o}{r^2} \frac{\exp\left[\alpha(\tau - \tau_p) - r/R_e\right]}{1 + \exp\left[(\alpha + \beta)(\tau - \tau_p)\right]} \tag{152}$$

in which r is the distance from the burst, τ is the retarded time of the burst point, and the other quantities are suitable parameters. With these inputs, causality implies that in the region above the wave reflected from the ground the solution of the problem is spherically symmetric.

For spherically symmetric problems, the only time dependent field is determined by

$$\frac{\partial E_r}{\partial t} = -\frac{j_r}{\epsilon} - \frac{\sigma E_r}{\epsilon} \tag{153}$$

A numerical solution of this equation may be obtained by writing it in the finite difference form

$$E_r(j,k) = \frac{-\dfrac{j_r(j,\ k-1/2)\Delta t}{\epsilon} + \left(1 - \dfrac{\sigma(j,\ k-1/2)\Delta t}{2\epsilon}\right) E_r(j,\ k-1)}{1 + \dfrac{\sigma(j,\ k-1/2)\Delta t}{2\epsilon}}$$

An exact solution of Eq. (153) is*

$$E_r(r,t) = -\int^{t} j_r(t') \exp\left\{-\int_{t'}^{t} \sigma(t'')dt''\right\} dt' \tag{155}$$

with inputs given by Eqs. (151) and (152), numerical results obtained from Eq. (154) and Eq. (155) compare favorably with the output of the two dimensional codes in regions where the latter codes have the proper symmetry.

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* Longmire, C. L., Close In E.M. Effects, LAMS-3072, E-3, May 1964.