EMP Theoretical Notes

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The Amplitude Distribution of an Electromagnetic Pulse Propagated Through the Ionosphere

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ABSTRACT

With the goal of calculating the peak of an ionospherically dispersed electromagnetic pulse, the author performs an analysis which suggests a method for deriving a probability distribution function of signal amplitudes. This technique avoids the necessity for extensive calculations and is not sensitive to details of specific ionospheric models. From the distribution function, the probability that the resultant amplitude exceeds any given percentage of the total energy of the received pulse can be calculated.

THE AMPLITUDE DISTRIBUTION OF AN
ELECTROMAGNETIC PULSE PROPAGATED
THROUGH THE ATMOSPHERE

Consider the EMP generated by a nuclear burst in the atmosphere which at some reference position $\bar{x} = 0$ is described by a known Fourier transform F (ω) through the relation [1]

$$f(0,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(0,\omega)e^{i\omega t}\, d\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} G^2(\omega)\cos(\omega t + \phi(\omega))d\omega, \quad (1)$$

where $F(0,\omega) = G(\omega)e^{i\phi(\omega)}$. The total energy in the pulse can be found using Parseval's relation

$$E_0 = \int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} G^2(\omega)d = \frac{1}{\pi} \int_0^{\infty} G^2(\omega)d\omega. \quad (2)$$

The effect of atmospheric propagation on pulse shape can, in principle, be evaluated by trating the atmosphere as a linear system with system function

$$G(\omega) = A(\omega)e^{i\theta(\omega)}. \quad (3)$$

Then at $\bar{x} = \bar{x}_1$, the time waveform is given by

$$g(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(0,\omega)\; A(s,\omega)e^{i(\theta+\omega t)}d\omega, \quad (4)$$

where $s(\omega)$ is the distance along the ray path for each frequency. Variation of path length with frequency is a result of Fermat's Principle which states that the path between 0 and $\bar{x}_1$, is given by

$$\delta p = \delta \int_0^{\bar{x}_1} n ds = 0 \quad , \quad (5)$$

where $n(\omega)$ is the index of refraction along the path, which in general varies with frequency. It is clearly a formidable task to approach this problem directly considering the number of variables which need to be taken into account in any propagation mode. Any detailed waveshape $g(t,\bar{x})$ clearly depends on assumptions regarding specific atmospheric properties. Extensive numerical calculations would need to be performed to adquately assess the effect of each variable parameter on the waveshape of $g(t)$.

One of the most important properties of the distorted pulse is its peak amplitude as a function of position relative to the burst point. Peak amplitude is often the critical parameter in the burst point. Peak amplitude is often the critical parameter in the vulnerability analysis of electronic circuits in satellites or ground systems. The following analysis suggests a method for deriving a probability distribution function of signal amplitudes based on use of the central limit theorem [2]. This technique avoids the necessity for extensive calculations and is not sensitive to details of specific atmospheric models.

The shape of the pulse $g(t\vec{x})$ is determined by two atmospheric processes:

1. Attenuation (including reflection) as described by

   $A(s(\omega),\omega)$ and

2. Phase delay resulting from the variation of

   $V_{Phase} = \frac{C}{n(\omega)}$ with frequency.

However, the effects of attenuation are not difficult to calculate, generally being adequately modeled by

$$e^{-\alpha(\omega)s}$$

which is a slowly varying function of path length. $\alpha(\omega)$ is known over the frequency band of interest (less than a few hundred MHz) [3] in both the troposphere and ionosphere.

The real difficulty lies in generating relative phase delays over the bandwidth of the pulse. A one-way transmission through the ionosphere can generate time delays in excess of 1μ sec for a pulse containing frequency components near 100 MC (.1 - .01μsec pulse width). This is a relative phase shift of hundreds of cycles [4].

To avoid this difficulty, assume that the propagation of each frequency component $\cos(\omega t + \phi(\omega))$,

$$f(t) = \frac{1}{\pi} \int_0^\infty G(\omega) \cos(\omega t + \phi(\omega)) d\omega , \tag{6}$$

results in a random phase distribution. This is, at the position $\bar{x}$ let each frequency component be modeled as a random variable

$$y(\omega) = \bar{G}(\omega) \cos(\omega t + \beta) \tag{7}$$

where β is a random phase uniformly distributed $(-\pi, \pi)$,

$$f_\beta(\beta) = \frac{1}{2\pi} \quad \text{on } (-\pi, \pi) , \tag{8}$$

and $\bar{G}(\omega)$ is the attenuated value of $G(\omega)$, i.e.

$$\bar{G}(\omega) = e^{-\alpha(\omega)s} G(\omega) \tag{9}$$

The probability density function of $y(\omega)$ can be shown to be [2]

$$f_y(y) = \frac{1}{\pi\sqrt{\bar{G}^2 - y^2}} , -\bar{G}(\omega) < y < \bar{G}(\omega) . \tag{10}$$

The mean and variance of $Y(\omega)$ are

$$\bar{y}(\omega) = 0 \tag{11}$$

and

$$\sigma^2(\omega) = \frac{\bar{G}^2(\omega)}{2} \quad \text{, respectively.} \tag{12}$$

The central limit theorem [2] asserts that the probability density function of the random variable $S = y_1 + y_2 + \cdots + y_n$ tends in the limit as $n \longrightarrow \infty$ to

$$\sigma \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-(S-M)^2}{2\sigma^2}\right] \tag{13}$$

where

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$$

and

$$M = \bar{y}_1 + \bar{y}_2 + \cdots + \bar{y}_n$$

Applying this to the integral

$$g(t) = \frac{1}{\pi} \int_0^\infty y(\omega)\, d\omega \tag{14}$$

we obtain

$$f_g(g) = \sqrt{\frac{1}{2E}}\, e^{-g^2/2E} = \frac{1}{\sqrt{2rE_0}}\, e^{-g^2/2rE_0}, \tag{15}$$

where

$$\sigma^2 = \int_0^\infty \frac{1}{2} \bar{G}(\omega) d\omega = \pi E(x) = \pi r E_o ,$$

where $rE_0 = E(x)$ is the energy in the distorted pulse at x and $0 \le r \le 1$ due to the attenuation.

From this distribution function the probability that the resultant amplitude y exceeds any given percentage of the total energy of the received pulse can be calculated. For example:

$$P(y > \sqrt{E_0}) = .158$$

$$P(y > 2\sqrt{E_0}) = .022$$

$$P(y > 3\sqrt{E_0}) = .0013$$

It should be realized that the value of this approach lies in the fact that, although we don't know what the specific waveform will be in a given case <u>all</u> possible amplitudes are contained in $f_g(g)$ since this ensemble includes every possible relative phase distribution for the component frequencies.

It is important to realize that energy conservation is not being violated with these large amplitudes ($>\sqrt{rE_0}$) since it is the total energy which must satisfy

$$rE_o = E(\bar{x}) = \int_{-\infty}^{\infty} g^2(t)dt = \frac{1}{\pi}\int_0^{\infty} \bar{G}^2(\omega)d\omega \quad . \tag{16}$$

However for improved numerical calculations, the infinite tails of the Gaussian distribution may need to be eliminated. We note that the maximum amplitude of any received signal must obey.

$$|g(t)| \leq \frac{1}{\pi} \int_0^{\infty} PG(\omega)d\omega = V/\pi \tag{17}$$

Therefore the central limit theorem must be formulated to generate a probability density function which vanishes outside this interval.

To achieve this, make a linear change of variable to

$$z(t) = \frac{g(t) - \frac{V}{\pi}}{\frac{2V}{\pi}} = \frac{\pi g(t)}{2V} - \frac{1}{2} \quad , \qquad (18)$$

so that $0<z(t)<1$. If $g(t)$ is normal $(0,\sigma^2)$ then $z(t)$ is normal $(\frac{1}{2}, \frac{\sigma 2}{V/\pi})$.

It can be shown [2] that the beta density

$$\begin{aligned} P(z) &= Mz^{\alpha} (1 - z)^{\beta} \\ &= 0 \end{aligned} \qquad (19)$$

$0<z<1$ elsewhere tends to a Gaussian function as $n \to \infty$ and can therefore be used to approximate the correct probability density function.

The constants α and β are determined in this case from [2]

$$\alpha + 1 = \beta + 1 = \frac{\frac{1}{2}\left(\frac{1}{2} - \frac{1}{4} - \frac{\pi^2 E(x)}{4V^2}\right)}{\frac{\pi^2 E^2(x)}{V^2}} \qquad (20)$$

with the result that $\alpha=\beta$. M is determined from the requirement that $\int_0^1 P(z)dz = 1$ to be

$$M = \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\ \Gamma(\beta + 1)} = \frac{\Gamma(2\alpha + 2)}{[\Gamma(\alpha + 1)]^2} \qquad (21)$$

Finally, the distribution of amplitudes $g(t)$ is obtained as

$$P_g(g) = M\left[\frac{\pi g(t)}{2V} - \frac{1}{2}\right]^2 \left[\frac{1}{2} + \frac{\pi g(t)}{2V}\right]^2$$

$$= M\left[\frac{\pi^2 g^2}{4V^2} - \frac{1}{4}\right]^2 \qquad (22)$$

on the interval

$$|g| < \frac{V}{\pi}$$

REFERENCES

[1] Papoulis, A., Probability Random Variables and Stochastic Process, McGraw Hill, 1965.

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[3] Glasstone, S., The Effects of Nuclear Weapons (rev) Atomic Energy Commission, 1964

[4] Modern Radar, Edited by Berkowitz, John Wiley and Sons, 1965.