LA-3571-MS

Theoretical Notes
Note 103

LOS ALAMOS SCIENTIFIC LABORATORY
of the
University of California
LOS ALAMOS • NEW MEXICO

Report written: June 23, 1966

Report distributed: September 27, 1966

Theory of the Radio Flash

Part VII. Finite Ground Conductivity

by

Bergen R. Suydam

Abstract

In Part VI of this series a method was presented for calculating the early part of the electromagnetic field produced by a burst on or near to a perfectly conducting earth. The resulting fields show a sharp spike at around gamma ray maximum. In the present report a method is presented for calculating the modification of this spike produced by losses in a finitely conducting earth.

Contents

1. Introduction

In a recent report[1] an approximation scheme has been developed which enables one to calculate the early portion of the electromagnetic field produced by a burst on or near the ground. In reference (1) the theory was developed for a perfectly conducting flat earth and was seen to agree well with the results of stepwise numerical integration of Maxwell's equations. The present report extends the work of reference (1) by applying the method to the problem of a burst near a ground of finite conductivity.

In reference (1) we saw that the problem reduced essentially to the solution of a diffusion equation

$$\frac{4\pi\sigma}{c}\frac{\partial\Psi_+}{\partial\tau} = \frac{\partial^2\Psi_+}{\partial z^2} + S , \qquad z \geqq 0 , \tag{1.1}$$

where σ denotes air conductivity, c light velocity, τ the retarded time $t - r/c$ and S is a source function involving the Compton current distribution. The exact nature of the diffusing quantity Ψ_+ (the + index indicates that the quantity is evaluated in the air, $z \geqq 0$) and the source S depends on the regime, whether the wave phase when displacement current dominates conduction current, or diffusion phase when conduction current dominates. Finite ground conductivity does not affect the form of Eq. (1.1) but only its boundary conditions.

We solve the problem of finite ground conductivity in three steps. First we reduce Maxwell's equations in the ground to a diffusion equation

$$\frac{4\pi s}{c}\frac{\partial \Psi_-}{\partial \tau} = \frac{\partial^2 \Psi_-}{\partial z^2}, \qquad z \leq 0 , \tag{1.2}$$

where s represents ground conductivity. Using the method of image sources we construct formal solutions to Eqs. (1.1) and (1.2). The second step is to express the conditions that E_r and B_φ be continuous across the ground surface as a pair of conditions on Ψ_+ and Ψ_-. These reduce to an integral equation for the image source. The third step is to solve (approximately) the integral equation. The only complication in this scheme is that the boundary conditions expressed in terms of Ψ_+ change form as a region goes into or out of saturation.

This report is intended to be a continuation of reference (1). Thus when pulse shapes are assumed they are as in reference (1); we have also kept the same notation here so that undefined symbols are as in reference (1).

2. The Fields in the Ground

In the ground we shall denote conductivity and dielectric constant respectively by s and ϵ. Introducing the following notation for the field components

(2.1) $$E \overset{\text{def}}{=} E_r , \qquad F \overset{\text{def}}{=} rE_\theta , \qquad G \overset{\text{def}}{=} rB_\varphi ,$$

Maxwell's equations in polar coordinates are

(2.2) $$\begin{cases} \dfrac{\epsilon}{c}\dfrac{\partial E}{\partial t} + 4\pi s E = \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta G) , \\[2ex] \dfrac{\epsilon}{c}\dfrac{\partial F}{\partial t} + 4\pi s F = -\dfrac{\partial G}{\partial r} , \\[2ex] \dfrac{1}{c}\dfrac{\partial G}{\partial t} = \dfrac{\partial E}{\partial\theta} - \dfrac{\partial F}{\partial r} . \end{cases}$$

We now make the high frequency approximation. First Eqs. (2.2) are re-written in terms of the retarded time T given by

(2.3) $$T = t - \sqrt{\epsilon}(r/c),$$

and then we consistently ignore $\partial/\partial r$ of a quantity as compared with $\sqrt{\epsilon}\partial/c\partial T$ of the same quantity. The last two of Eqs. (2.2) yield

(2.4) $$\begin{cases} G = \sqrt{\epsilon}F + \dfrac{\partial\Phi}{\partial\theta} , \\[2ex] \dfrac{\partial F}{\partial r} + \dfrac{2\pi s}{\sqrt{\epsilon}} F = \dfrac{1}{2}\dfrac{\partial E}{\partial\theta} , \\[2ex] \Phi \overset{\text{def}}{=} \displaystyle\int_{-\infty}^{T} E(T')\,c\,dT' . \end{cases}$$

At this point we limit our attention to a ground sufficiently conducting that

(2.5) $$F = \frac{\sqrt{\epsilon}}{4\pi s}\frac{\partial E}{\partial \theta}$$

is a good approximation to the solution of the second of Eqs. (2.4). As $\partial/\partial r$ may be expected to be of order $1/\lambda$, this means that

(2.6) $$s \gg \sqrt{\epsilon}/2\pi\lambda \ .$$

Typical values might be ϵ about 9 and λ about 3×10^4 cm and the right hand side of Eq. (2.6) becomes $1.6 \times 10^{-5}(\text{cm})^{-1}$ which is equivalent to 5.3×10^{-5} mho meters. Thus condition (2.6) is not unduly restringent.

If we set Eq. (2.5) into the first of Eqs. (2.4) we obtain

(2.7) $$G = \frac{1}{4\pi s}\frac{\partial}{\partial \theta}\left[\frac{\epsilon}{c}\frac{\partial \Phi}{\partial T} + 4\pi s\Phi\right] \ .$$

Setting this into the first of Eqs. (2.2) gives an equation for the radial E-field which can be written in a variety of different ways. For example Eq. (2.7) and the first of Eqs. (2.2) are equivalent to the following pair:*

*Note that $\frac{\partial}{\partial z} = -\frac{\partial}{r\partial\theta}$, $\sin\theta = 1$ within the limits of our approximation.

$$(2.8) \qquad \begin{cases} U = \frac{\epsilon}{c}\frac{\partial\Phi}{\partial T} + 4\pi s\Phi , \\ \frac{4\pi s}{c}\frac{\partial U}{\partial T} = \frac{\partial^2 U}{\partial z^2} . \end{cases}$$

An equivalent form can be obtained by solving the first of Eqs. (2.8), obtaining

$$(2.9) \qquad \Phi = e^{-4\pi scT/\epsilon}\int_{-\infty}^{T} U(T')e^{4\pi scT'/\epsilon} \cdot \frac{cdT'}{\epsilon} \equiv \mathfrak{L}(U) ,$$

where $\mathfrak{L}$ represents the integral operator written out in the center. Operating on the second of Eqs. (2.8) by $\mathfrak{L}$ yields the result that

$$(2.10) \qquad \frac{4\pi s}{c}\frac{\partial\Phi}{\partial T} = \frac{\partial^2\Phi}{\partial z^2} .$$

Finally note that, as r enters only as a parameter, $\partial/\partial T$ equals $\partial/\partial\tau$, where

$$(2.11) \qquad \tau = t - r/c .$$

Thus we have reduced the problem to solving Eq. (1.2). As we have seen we may identify Ψ_- with U or with Φ. Differentiation of Eq. (2.10) by θ

and of Eq. (2.7) by T shows that Ψ_- could also be identified with E or with G. The reason that all this variety is possible and also the reason that we need not distinguish whether conduction or displacement current dominates is that the conductivity is independent of time.

Introducing the new time variable

$$Z = \frac{c\tau}{4\pi s} , \tag{2.12}$$

we can write the solution to Eq. (1.2) in the form

$$\Psi_- = \frac{1}{2} \int_{-\infty}^{Z} g(Z') \left\{ 1 + \text{erf} \left(\frac{z}{2\sqrt{Z - Z'}} \right) \right\} dZ' \tag{2.13}$$

from which we readily obtain

$$\frac{\partial \Psi_-}{\partial \theta} = - \frac{r}{2\sqrt{\pi}} \int_{-\infty}^{Z} \frac{g(Z')dZ'}{\sqrt{Z - Z'}} \, e^{-z^2/4(Z-Z')} , \tag{2.14}$$

where g is at present an arbitrary function which will be determined by boundary conditions. Physically Eqs. (2.13) and (2.14) represent the Green's function solutions to Eq. (1.2) with the arbitrary source term g added in the region $z > 0$, outside the domain of the equation.

3. The Wave Phase

In discussing the fields in the air it is necessary, as in reference (1), to consider separately the wave phase and the diffusion phase. During the wave phase the displacement current dominates conduction current and the diffusing quantity Ψ_+ is the radial field E. Thus the Green's function solution to Eq. (1.1) is

$$(3.1) \qquad E = \frac{1}{2}\int_{-\infty}^{\zeta} S(\zeta')\left[1 + \operatorname{erf}\left(\frac{z}{2\sqrt{\zeta - \zeta'}}\right)\right] d\zeta' + \frac{1}{2}\int_{-\infty}^{\zeta} f(\zeta')\left[1 - \operatorname{erf}\left(\frac{z}{2\sqrt{\zeta - \zeta'}}\right)\right] d\zeta' .$$

For the magnetic field we have

$$(3.2) \quad G = \frac{1}{4\pi\sigma}\frac{\partial E}{\partial \theta} = -\frac{r}{2\sqrt{\pi}} \cdot \frac{1}{4\pi\sigma}\int_{-\infty}^{\zeta} \frac{S(\zeta') - f(\zeta')}{\sqrt{\zeta - \zeta'}}\, d\zeta' \cdot e^{-z^2/4(\zeta-\zeta')} ,$$

where, as before,

$$(3.3) \qquad \zeta \overset{\text{def}}{=} \int \frac{c\,d\tau}{4\pi\sigma}$$

and f is the as yet undetermined image source located in the region $z < 0$.

In discussing the fields in the ground it will be simplest to restrict ourselves, for the time being, to the case of high ground conductivity, such that conduction current dominates displacement current in the ground. This means essentially

$$s \gg \frac{\epsilon\alpha}{4\pi c} , \tag{3.4}$$

which is still well within the range of possible soil conductivities. In Section 5 we shall remove this restriction. Under condition (3.4) we have

$$G = \frac{\partial\Phi}{\partial\theta} \qquad \text{and } E = \frac{\partial\Phi}{c\partial\tau} ; \tag{3.5}$$

therefore it is convenient to identify Ψ_- with Φ. Evaluating Eqs. (2.13) and (2.14) at z = 0 we obtain

$$E_{-0} = \frac{g(\tau)}{4\pi s} \tag{3.6}$$

and

$$G_{-0} = - \frac{r}{2\sqrt{\pi}} \int_{-\infty}^{\tau} \frac{g(\tau')}{\sqrt{Z(\tau) - Z(\tau')}} \cdot \frac{cd\tau'}{4\pi s} \tag{3.6'}$$

for the fields at the ground. An obvious change of dummy variable has been made.

Continuity of E is ensured by equating Eq. (3.6) to the result of evaluating Eq. (3.1) at z = 0 and gives

$$(3.7) \qquad \frac{g(\tau)}{4\pi s} = \int_{-\infty}^{\tau} [S(\tau') + f(\tau')] \frac{c d\tau'}{4\pi\sigma(\tau')} .$$

Setting Eq. (3.7) into Eq. (3.6') and integrating by parts yields

$$(3.8) \quad G_{-0} = - \frac{r}{2\sqrt{\pi}} \cdot 8\pi s \int_{-\infty}^{\tau} \sqrt{Z(\tau) - Z(\tau')}\,[S(\tau') + f(\tau')] \frac{c d\tau'}{4\pi\sigma(\tau')} .$$

The remaining boundary condition is that G be continuous. This is achieved by equating the right hand member of Eq. (3.8) to that of Eq. (3.2), evaluating the latter at z = 0. The result is an integral equation which we may write in the following form:

$$(3.9) \quad \int_{-\infty}^{\zeta} \frac{S(\zeta') - f(\zeta')}{\sqrt{\zeta - \zeta'}} [1 + P(\zeta,\zeta')] d\zeta' = \int_{-\infty}^{\zeta} \frac{2S(\zeta')}{\sqrt{\zeta - \zeta'}} P(\zeta,\zeta') d\zeta' ,$$

where

$$(3.10) \qquad P(\zeta,\zeta') \overset{\text{def}}{=} 32\pi^2 s\sigma(\zeta)\sqrt{[\zeta - \zeta'][Z(\zeta) - Z(\zeta')]} \; .$$

The integral equation for the image current f is a bit complicated; we therefore attempt only an approximate solution. Our method will be based on the fact that the gamma ray pulse, and hence the function S, is very short in time.

We note that the mean value theorem guarantees the existence of numbers ζ_a, ζ_b such that the quantities P can be replaced by their mean values and taken outside the integration, yielding

$$(3.11) \quad [1 + P(\zeta,\zeta_b)] \int_{-\infty}^{\zeta} \frac{S - f}{\sqrt{\zeta - \zeta'}} d\zeta' = P(\zeta,\zeta_a) \int_{-\infty}^{\zeta} \frac{2Sd\zeta'}{\sqrt{\zeta - \zeta'}}$$

in place of Eq. (3.9). Moreover, as S is known, $P(\zeta,\zeta_a)$ can readily be calculated. Calling this quantity $P_0(\zeta)$, we have

$$(3.12) \qquad P_0(\zeta) = \left[\int_{-\infty}^{\zeta} \frac{2SPd\zeta'}{\sqrt{\zeta - \zeta'}} \right] \Bigg/ \left[\int_{-\infty}^{\zeta} \frac{2Sd\zeta'}{\sqrt{\zeta - \zeta'}} \right] .$$

Now S being a sharply spiked function, f the image source is likely to be so also. We might therefore expect that S - f will have a shape similar to that of S, in which case we would have $P(\zeta,\zeta_b) \approx P(\zeta,\zeta_a)$.

Thus as a first order approximation to f, say f_0, we have

$$(3.13) \qquad \int_{-\infty}^{\zeta} \frac{S - f_0}{\sqrt{\zeta - \zeta'}} d\zeta' = \frac{P_0}{1 + P_0} \int_{-\infty}^{\zeta} \frac{2S d\zeta'}{\sqrt{\zeta - \zeta'}} \equiv H_0(\zeta) ,$$

say. Equation (3.13) is readily solved by the method of Laplace transforms and gives (see Appendix A)

$$(3.14) \qquad S - f_0 = \frac{1}{\pi} \int_{-\infty}^{\zeta} \frac{H_0'(\zeta') d\zeta'}{\sqrt{\zeta - \zeta'}} ,$$

the prime on H standing for the derivative. Now, if we wish, we may use $S - f_0$ to make a better estimate of $P(\zeta, \zeta_b)$ and using this solve for $S - f_1$ and so on. We shall stop with Eqs. (3.13) and (3.14).

Direct examination of Eq. (3.9) shows that in the limit $P \gg 1$, as will happen for example when s is very large, the solution is $S - f = 2S$ very nearly. Thus Eq. (3.13) is asymptotically correct in the limit of large P. Similarly for $P \ll 1$ Eq. (3.13) is seen to be asymptotic to the correct solution. Thus if Eq. (3.13) is reasonably accurate for $P_0 \sim 1$ we can have confidence in our result.

As an example we shall consider a moderately close distance where saturation occurs at some time τ_s, either in the α-phase ($\tau_s < 0$) or at the end of the α-phase ($\tau_s = 0$). In this case

$$(3.15) \qquad \zeta = -c/4\pi\alpha\sigma \, , \qquad S = E_\alpha(4\pi\sigma)^2 \, ,$$

which yields immediately

$$(3.16) \qquad P_0 = \frac{4c}{\alpha}\sqrt{\pi s\sigma} \, .$$

Thus Eq. (3.14) becomes

$$(3.17) \qquad S - f_0 = \frac{2kA}{\pi}\int_{-\infty}^{\zeta} \frac{d\zeta'}{(-\zeta')^{5/2}\sqrt{\zeta - \zeta'}} \left\{\frac{\sqrt{-\zeta'} + 3A/4}{[\sqrt{-\zeta'} + A]^2}\right\} \, ,$$

where we have set for short

$$(3.18) \qquad k \overset{\text{def}}{=} \frac{\pi c^2 E_\alpha}{2\alpha^2} \, , \qquad A \overset{\text{def}}{=} \frac{2c}{\alpha}\sqrt{\frac{sc}{\alpha}} \, .$$

The integration is elementary but quite complicated. Therefore we revert to the approximation method given in Appendix B and obtain

$$(3.19) \qquad S(\zeta) - f_0(\zeta) \doteq \frac{8kA}{3\pi(-\zeta)^2} \cdot \frac{\sqrt{-\zeta} + 3A/4}{[\sqrt{-\zeta} + A]^2} \, .$$

For $\sqrt{-\zeta} \leqq A$ the approximation is very good. As $\sqrt{-\zeta}$ increases it becomes progressively worse until in the limit $\sqrt{-\zeta} \gg A$ it is off by 13 percent. Thus Eq. (3.19) is entirely adequate for estimating the next order correction to $P(\zeta,\zeta_b)$. When this is carried out as in Appendix B we find $P_1 \doteq 1.127$ and the error of Eq. (3.13) is about six percent when $P_0 = 1$. Accordingly it seems conservative to claim 10 percent accuracy, especially as P_0 will normally be considerably larger than unity in the most interesting time regime.

4. The Diffusion Phase

We suppose that our observation point is sufficiently near the explosion that at some time $\tau_s \leqq 0$ saturation occurs. Thus for $\tau > \tau_s$ we have $4\pi\sigma \gg \partial/c\partial\tau$. This does not change our formulas for fields in the ground, but our method for calculating fields in the air changes. During this diffusion phase the diffusing quantity in the air is Φ, the same as in the ground. It is convenient to break it into two parts,

$$\Phi = \Phi_0 + \Phi_1 \; ; \; G = G_0 + G_1 \; ; \; \Phi_1(\tau_s) = 0 = G_1(\tau_s) \; . \tag{4.1}$$

The first term, Φ_0, is the result of further diffusion of the field left at $\tau = \tau_s$ by the wave phase. The second term Φ_1 starts zero at $\tau = \tau_s$ and thereafter grows owing to the source S.

At $\tau = \tau_s$ the fields left by the wave phase are

$$(4.2) \qquad G_{s+} = -\frac{r}{2\sqrt{\pi}} \cdot \frac{1}{4\pi\sigma_s} \int_{-\infty}^{\tau_s} \frac{S(\tau') - f(\tau')}{\sqrt{\zeta(\tau_s) - \zeta(\tau')}} \frac{c d\tau'}{4\pi\sigma(\tau')} e^{-z^2/4[\zeta(\tau_s)-\zeta(\tau')]} ,$$

$$z \geq 0 ,$$

in the air and

$$(4.3) \qquad G_{s-} = -\frac{r}{2\sqrt{\pi}} \int_{-\infty}^{\tau_s} \frac{g(\tau')}{\sqrt{Z(\tau_s) - Z(\tau')}} \frac{c d\tau'}{4\pi s} e^{-z^2/4[Z(\tau_s)-Z(\tau')]} ,$$

$$z \leq 0 ,$$

in the ground. The functions f and g are determined as in the preceding section so that G_{s+} and G_{s-} are known functions of z. The standard method(2) of determining the future behavior of G_0 is to sum the contributions of instantaneous point sources distributed according to Eqs. (4.2) and (4.3). In general this summation is awkward because of the jump of the diffusion constant at z = 0. If however we restrict our attention to points on the ground plane no contribution to $G_0(\tau)$ crosses the discontinuity and the result can be written simply as

$$(4.4) \qquad G_0(\tau) = \int_0^\infty \frac{G_{s+}(z')}{\sqrt{4\pi[\zeta(\tau) - \zeta(\tau_s)]}} e^{-(z')^2/4[\zeta(\tau)-\zeta(\tau_s)]} dz' +$$

$$+ \int_{-\infty}^0 \frac{G_{s-}(z')}{\sqrt{4\pi[Z(\tau) - Z(\tau_s)]}} e^{-(z')^2/4[Z(\tau)-Z(\tau_s)]} dz' \; ; \; z = 0 \; ,$$

where G_{s+}, G_{s-} are given by Eqs. (4.2) and (4.3). We can easily perform the z' integration, set in the value of g given by Eq. (3.7), and perform a partial integration. The result is

$$(4.5) \qquad G_0(\tau) = - \frac{r}{4\sqrt{\pi}} \left[\frac{1}{4\pi\sigma_s} \int_{-\infty}^{\tau_s} \frac{S - f}{\sqrt{\zeta(\tau) - \zeta(\tau')}} \frac{cd\tau'}{4\pi\sigma} + \right.$$

$$\left. + 8\pi s \int_{-\infty}^{\tau_s} \sqrt{Z(\tau) - Z(\tau')} \; [S + f] \frac{cd\tau'}{4\pi\sigma} \right] .$$

This, of course, gives G_0 only on the ground plane.

The function G_1 is determined in the same manner as was the wave-phase field, now however using τ_s rather than $-\infty$ as the initial value of τ. The diffusing quantities both in the air and on the ground are

Φ and $\partial\Phi/\partial\theta = G$. Thus in the air we have

$$(4.6)\quad \left\{ \begin{aligned} \Phi_{1+} &= \frac{1}{2}\int_{\tau_s}^{\tau} [S(\tau') + f(\tau')]\,\frac{c d\tau'}{4\pi\sigma(\tau')} \\ G_{1+} &= -\frac{r}{2\sqrt{\pi}}\int_{\tau_s}^{\tau} \frac{S'(\tau') - f(\tau')}{\sqrt{\zeta(\tau) - \zeta(\tau')}}\,\frac{c d\tau'}{4\pi\sigma(\tau')} \end{aligned} \right\}, \quad \text{for } z = 0 ,$$

and in the ground

$$(4.7)\quad \left\{ \begin{aligned} \Phi_{1-} &= \frac{1}{2}\int_{\tau_s}^{\tau} g(\tau') \cdot \frac{c d\tau'}{4\pi s} \\ G_{1-} &= -\frac{r}{2\sqrt{\pi}}\int_{\tau_s}^{\tau} \frac{g(\tau')}{\sqrt{Z(\tau) - Z(\tau')}}\,\frac{c d\tau'}{4\pi s} \end{aligned} \right\}, \quad \text{for } z = 0 .$$

In using the same symbols f, g as in the preceding section, we do not wish to imply any kind of continuity at $\tau = \tau_s$.

The boundary conditions that E and G be continuous are equivalent to

(4.8) $$\Phi_{1+} = \Phi_{1-} \quad \text{and } G_{1+} = G_{1-} .$$

The first of these determines g as

(4.9) $$\frac{g(\tau)}{4\pi s} = \frac{1}{4\pi\sigma} [S(\tau) - f(\tau)] .$$

Setting this into the second of Eqs. (4.7) and the result into the second boundary condition yields the following integral equation:

(4.10) $$\int_{\tau_s}^{\tau} \frac{S(\tau') - f(\tau')}{\sqrt{\zeta(\tau) - \zeta(\tau')}} [1 + P(\tau,\tau')] \frac{c d\tau'}{4\pi\sigma(\tau')} = \int_{\tau_s}^{\tau} \frac{2S(\tau')P(\tau,\tau')}{\sqrt{\zeta(\tau) - \zeta(\tau')}} \frac{c d\tau'}{4\pi\sigma(\tau')} ,$$

similar to Eq. (3.9), but now

(4.11) $$P(\tau,\tau') \overset{\text{def}}{=} \sqrt{\frac{\zeta(\tau) - \zeta(\tau')}{Z(\tau) - Z(\tau')}} .$$

Again, use of the same symbol P as in Section 3 does not imply any sort of continuity.

Equation (4.10) we solve in the same fashion as we treated Eq. (3.9). Thus in lowest order we have

$$(4.12) \qquad \int_{\zeta_s}^{\zeta} \frac{S - f_0}{\sqrt{\zeta - \zeta'}} d\zeta' = \frac{P_0}{1 + P_0} \int_{\zeta_s}^{\zeta} \frac{2Sd\zeta'}{\sqrt{\zeta - \zeta'}} ,$$

where

$$(4.13) \qquad P_0(\zeta) \overset{\text{def}}{=} \left[\int_{\zeta_s}^{\zeta} \frac{2SPd\zeta'}{\sqrt{\zeta - \zeta'}} \right] \Bigg/ \left[\int_{\zeta_s}^{\zeta} \frac{2Sd\zeta'}{\sqrt{\zeta - \zeta'}} \right] .$$

As an example we shall consider a very close observation point where saturation occurs long before the peak of the gamma ray curve. Thus we may ignore what happens before saturation. We have

$$(4.14) \qquad S = E_s \cdot (4\pi\sigma) ,$$

where E_s is the saturation voltage (E_α for $\tau < 0$; E_κ for $\tau > 0$), so that

$$(4.15) \qquad \int_{\zeta_s}^{\zeta} \frac{2SP}{\sqrt{\zeta - \zeta'}} d\zeta' = \int_{\tau_s}^{\tau} \frac{2E_s \cdot cd\tau'}{\sqrt{\frac{c}{4\pi s} (\tau - \tau')}}$$

or

$$
(4.16) \quad \int_{\zeta_s}^{\zeta} \frac{2SP}{\sqrt{\zeta - \zeta'}}\, d\zeta' = \begin{cases} 4E_\alpha \sqrt{\dfrac{4\pi c_s}{\alpha}} \cdot \sqrt{\alpha(\tau - \tau_s)} \;, & \text{for } \tau < 0 \\[2ex] 4E_\alpha \sqrt{\dfrac{4\pi c_s}{\alpha}} \left\{ \sqrt{-\alpha\tau_s} + \dfrac{E_\kappa}{E_\alpha} \sqrt{\alpha\tau} \right\}, & \text{for } \tau > 0 \;. \end{cases}
$$

The denominator of Eq. (4.13) has been calculated in reference (1). For example

$$
(4.17) \quad \int_{\zeta_s}^{\zeta} \frac{2S d\zeta'}{\sqrt{\zeta - \zeta'}} = \frac{4cE_\alpha}{\alpha\sqrt{-\zeta}} \tan^{-1} \sqrt{\frac{\zeta_s}{\zeta} - 1} \;, \quad \text{for } \tau < 0
$$

whence, if τ_s is much less than zero

$$
(4.18) \quad P_0 = \frac{2}{\pi}\sqrt{\frac{s}{\sigma}} \cdot \sqrt{\alpha(\tau - \tau_s)} \;, \quad \text{for } \tau < 0 \;.
$$

For $\tau > 0$ the expressions become longer.

At very close distances σ becomes very large and P_0 consequently becomes small. At such a close distance we have

$$G \approx \frac{r}{2\sqrt{\pi}} \int_{\zeta_s}^{\zeta} \frac{2SPd\zeta'}{\sqrt{\zeta - \zeta'}}, \qquad (P_0 \ll 1) \; ; \tag{4.19}$$

and, as we see from Eq. (4.16), the spike is completely lost. At larger distances it begins to reappear, but smaller than would be the case were the ground conductivity infinite.

At $\tau = \tau_1$ the conductivity and the Compton current level off and are essentially constant for $\tau > \tau_1$. When τ is sufficiently greater than τ_1, Eq. (3.27) of reference 1 gives

$$\int_{\zeta_s}^{\zeta} \frac{2Sd\zeta'}{\sqrt{\zeta - \zeta'}} \sim 4E_\kappa \sqrt{\frac{4\pi c\sigma_1}{\alpha}} \cdot \sqrt{\alpha\tau} \; , \qquad \text{for } \tau \to \infty \; . \tag{4.20}$$

Combining with Eq. (4.16) we see that

$$P_0(\tau) \sim \sqrt{\frac{s}{\sigma_1}} \; , \qquad \text{for } \tau \to \infty \; ; \tag{4.21}$$

that is to say, as the semistatic phase is approached, our formula approaches the result derived by Longmire[3] for the semistatic phase.

At this point we have calculated G only on the ground for the period $\tau > \tau_s$. Should desaturation occur at some time τ_u we would need to know G at τ_u for all z in order to calculate G_0 for $\tau > \tau_u$. This can be readily calculated by the standard methods of heat flow theory (see e.g. reference 2, p. 356). In each medium we calculate separately from the known initial value $G(z,\tau_s)$ at all z and from the known boundary value $G(0,\tau)$ for $\tau_s \leqq \tau \leqq \tau_u$. We do not give the results here, but they go through exactly as above except that the formulas get longer. In particular S is a rather complicated function after desaturation [see reference 1, Eqs. (3.42) to (3.52)].

5. The General Theory

In Section 3 we imposed the restriction of Eq. (3.4), or more generally $4\pi s \gg \epsilon\partial/c\partial\tau$, which we now remove. Naturally the theory will be more complicated. We start with Eqs. (2.8). Evaluation of the solution of these equations on the ground plane yields

$$
(5.1) \quad \begin{cases} U_{-0} = \frac{1}{2} \displaystyle\int_{-\infty}^{\tau} g(\tau') \frac{c d\tau'}{4\pi s} , \\[2ex] \Phi_{-0} = e^{-\Sigma\tau} \displaystyle\int_{-\infty}^{\tau} U(\tau') e^{\Sigma\tau'} \frac{c d\tau'}{\epsilon} , \quad \Sigma \overset{\text{def}}{=} 4\pi s c/\epsilon . \end{cases}
$$

Evaluation of Eq. (2.7) on the ground plane gives

$$(5.2) \qquad G_{-0} = \left(\frac{\partial U}{\partial \theta}\right)_{-0} = - \frac{r}{2\sqrt{\pi}} \int_{-\infty}^{\tau} \frac{g(\tau')}{\sqrt{Z(\tau) - Z(\tau')}} \frac{c d\tau'}{4\pi s} .$$

Setting the first of Eqs. (5.1) into the second and partially integrating yields

$$(5.3) \qquad \Phi_{-0} = \frac{1}{4\pi s} \int_{-\infty}^{\tau} [1 - e^{-\Sigma(\tau-\tau')}] \frac{1}{2} g(\tau') \frac{c d\tau'}{4\pi s} ,$$

or, equivalently

$$(5.4) \qquad E_{-0} = \frac{e^{-\Sigma\tau}}{2\epsilon} \int_{-\infty}^{\tau} e^{\Sigma\tau'} g(\tau') \frac{c d\tau'}{4\pi s} .$$

In the wave phase we found the following expressions for the fields on the ground:

$$(5.5) \qquad \begin{cases} E_{+0} = \frac{1}{2} \int_{-\infty}^{\tau} [S(\tau') + f(\tau')] \frac{c d\tau'}{4\pi\sigma} , \\ G_{+0} = - \frac{r}{2\sqrt{\pi}} \int_{-\infty}^{\tau} \frac{S(\tau') - f(\tau')}{\sqrt{\zeta(\tau) - \zeta(\tau')}} \frac{c d\tau'}{4\pi\sigma} . \end{cases}$$

Continuity of the E component yields

$$(5.6) \qquad \frac{g(\tau)}{4\pi s} = \frac{\epsilon(S + f)}{4\pi\sigma} + 4\pi s \int_{-\infty}^{\tau} (S + f) \frac{cd\tau'}{4\pi\sigma}$$

(wave phase) .

Setting this into the expression for G_{-0} and partially integrating yields

$$(5.7) \quad G_{-0} = -\frac{r}{2\sqrt{\pi}} \int_{-\infty}^{\tau} [S + f] \frac{cd\tau'}{4\pi\sigma} \left\{ 8\pi s\sqrt{Z(\tau) - Z(\tau')} + \frac{\epsilon}{4\pi s\sqrt{Z(\tau) - Z(\tau')}} \right\}$$

(wave phase) .

Thus, continuity of G recovers for us Eq. (3.9) but now P represents the slightly more complicated function

$$(5.8) \quad P(\tau,\tau') = 32\pi^2 s\sigma(\tau)\sqrt{[Z(\tau) - Z(\tau')][\zeta(\tau) - \zeta(\tau')]} + $$

$$+ \frac{\epsilon\sigma(\tau)}{s}\sqrt{\frac{\zeta(\tau) - \zeta(\tau')}{Z(\tau) - Z(\tau')}}$$

(wave phase) .

An approximate solution to Eq. (3.9) is now found in the same way as previously.

For the diffusion phase calculation, we again break the field components into two parts as in Section 4 [see Eq. (4.1)]. The quantity $G_0(\tau)$ is calculated precisely as before and is given by Eq. (4.5). For the source driven terms we have Eqs. (4.6) for the fields in the air. In the ground Φ_{1-} and G_{1-} are given by Eqs. (5.3) and (5.2) respectively upon replacing the lower integration limit ∞ with τ_s. Continuity of E is equivalent to continuity of Φ and yields

$$\frac{g(\tau)}{4\pi s} = 4\pi s\left[\frac{S(\tau) + f(\tau)}{4\pi\sigma}\right] + \frac{\epsilon}{c}\frac{\partial}{\partial\tau}\left[\frac{S(\tau) + f(\tau)}{4\pi\sigma}\right] \tag{5.9}$$

(diffusion phase) .

Setting this into the expression for G_{1-}, we find the following condition for the continuity of G_1:

$$\int_{\tau_s}^{\tau} \frac{S(\tau') - f(\tau')}{\sqrt{\zeta(\tau) - \zeta(\tau')}} [1 + P(\tau,\tau')] \frac{cd\tau'}{4\pi\sigma(\tau')} + \tag{5.10}$$

$$+ \frac{\epsilon}{c}\int_{\tau_s}^{\tau} \frac{1}{\sqrt{Z(\tau) - Z(\tau')}} \frac{cd\tau'}{4\pi s} \frac{\partial}{\partial\tau'}\left[\frac{S(\tau') + f(\tau')}{4\pi\sigma(\tau')}\right] =$$

$$= \int_{\tau_s}^{\tau} \frac{2S(\tau')P(\tau,\tau')}{\sqrt{\zeta(\tau) - \zeta(\tau')}} \frac{cd\tau'}{4\pi\sigma(\tau')} + \frac{\epsilon}{c}\int_{\tau_s}^{\tau} \frac{1}{\sqrt{Z(\tau) - Z(\tau')}} \frac{cd\tau'}{4\pi s} \frac{\partial}{\partial\tau'}\left[\frac{2S(\tau')}{4\pi\sigma(\tau')}\right] ,$$

where now

(5.11) $$P(\tau,\tau') \overset{def}{=} \sqrt{\frac{\zeta(\tau) - \zeta(\tau')}{Z(\tau) - Z(\tau')}}$$

(diffusion phase) .

We can write Eq. (5.10) in another form which will be more convenient for obtaining an approximate solution. In Appendix A is given a formula for differentiating a commonly occurring integral. Applying the result to the two terms containing derivatives Eq. (5.10) is seen to be equivalent to

(5.12) $$\int_{\tau_s}^{\tau} \frac{S - f}{\sqrt{\zeta(\tau) - \zeta(\tau')}} [1 + P] \frac{cd\tau'}{4\pi\sigma} + \frac{\epsilon}{4\pi s} \frac{\partial}{c\partial\tau} \int_{\tau_s}^{\tau} \frac{(S - f)P}{\sqrt{\zeta(\tau) - \zeta(\tau')}} \frac{cd\tau'}{4\pi\sigma} =$$

$$= \left(1 + \frac{\epsilon}{4\pi s} \frac{\partial}{c\partial\tau}\right) \int_{\tau_s}^{\tau} \frac{2SP}{\sqrt{\zeta(\tau) - \zeta(\tau')}} \frac{cd\tau'}{4\pi\sigma} - \frac{\epsilon}{4\pi s\sqrt{Z - Z_s}} \left[\frac{S + f}{4\pi\sigma}\right]_s ,$$

the subscript s standing for evaluation at $\tau = \tau_s$. This equation is handled in the same manner as our previous integral equations. First we write

(5.13) $$\int_{\zeta_s}^{\zeta} \frac{S - f}{\sqrt{\zeta - \zeta'}} d\zeta' = H(\zeta)$$

for short. Once $H(\zeta)$ is obtained we can invert Eq. (5.13) obtaining

(5.14) $$S(\zeta) - f(\zeta) = \frac{1}{\pi} \int_{\zeta_s}^{\zeta} \frac{H'(\zeta')d\zeta'}{\sqrt{\zeta - \zeta'}} ,$$

whence it follows that

(5.15) $$f(\zeta_s) = S(\zeta_s) ,$$

and the term on the extreme right of Eq. (5.12) is evaluated. We now apply the mean value theorem to Eq. (5.12) obtaining

(5.16) $$[1 + P(\zeta,\zeta_b)]H + \frac{\epsilon}{4\pi s} P(\zeta,\zeta_d) \frac{\partial H}{c\partial\tau} = P(\zeta,\zeta_a)I +$$

$$+ \frac{\epsilon}{4\pi s} P(\zeta,\zeta_c) \frac{\partial I}{c\partial\tau} - \frac{\epsilon}{4\pi s} \frac{2S_s}{4\pi\sigma_s\sqrt{Z - Z_s}} ,$$

where I is short for

(5.17) $$I \overset{\text{def}}{=} \int_{\zeta_s}^{\zeta} \frac{2S(\zeta')d\zeta'}{\sqrt{\zeta - \zeta'}}$$

and is a known function and ζ_a, ζ_b, ζ_c, and ζ_d are certain values of ζ' all lying in the interval (ζ_s, ζ). The functions $P_a \equiv P(\zeta, \zeta_a)$ and $P_c \equiv P(\zeta, \zeta_c)$ can be calculated. As before, insofar as S - f is similar in shape to 2S the approximation $P(\zeta, \zeta_b) = P_a$ and $P(\zeta, \zeta_d) = P_c$ is a reasonably good one and Eq. (5.16) is replaced by

(5.18) $$\frac{\epsilon P_c(\tau)}{4\pi s} \frac{\partial H_0}{c\partial\tau} + [1 + P_a(\tau)]H_0 = P_a I + \frac{\epsilon P_c}{4\pi s} \frac{\partial I}{c\partial\tau} - \frac{2\epsilon S(\tau_s)}{4\pi s \cdot 4\pi\sigma(\tau_s)\sqrt{Z(\tau) - Z(\tau_s)}} .$$

This is an ordinary linear equation and is soluble by quadrature. Once the lowest order approximation to H_0 is obtained we can calculate $S - f_0$ by Eq. (5.14), make better estimates of $P(\zeta, \zeta_b)$, $P(\zeta, \zeta_d)$ and so on. Alternatively stopping with the lowest order gives

(5.19) $$G_1(\tau) = - \frac{r}{2\sqrt{\pi}} H_0 .$$

In the limit

$$\frac{\epsilon}{4\pi s} \frac{\partial}{c\partial\tau} \ll 1 , \tag{5.20}$$

we recover exactly the theory of Sections 3 and 4. In the opposite limit

$$\frac{\epsilon}{4\pi s} \frac{\partial}{c\partial\tau} \gg 1 , \tag{5.21}$$

we obtain simply

$$H_0(\tau) = I(\tau) - \frac{2S(\tau_s)}{4\pi\sigma(\tau_s)} \int_{\tau_s}^{\tau} \frac{c d\tau'}{P_a(\tau')\sqrt{Z(\tau') - Z(\tau_s)}} . \tag{5.22}$$

Although this section removes any restriction regarding relative sizes of displacement and conduction currents in the ground, ground conductivity is still restricted by Eq. (2.6). Thus any conclusions one might draw by setting s = 0 in any of our formulas must be viewed with caution.

References

1. B. R. Suydam, "Theory of the Radio Flash - Part VI" Los Alamos Scientific Laboratory Report LA-3532-MS, April 1966.

2. Carslow and Jaeger, "The Conduction of Heat in Solids, Chapter XIV," Oxford at the University Press, 1959.

3. C. L. Longmire, "Close-In E.M. Effects," Lectures I to IX, Los Alamos Scientific Laboratory Report LA-3072-MS, April 1964.

Appendix A

On a Certain Integral

Consider the integral

(A.1)
$$I(\zeta) = \int_a^{\zeta} \frac{2S(\zeta')d\zeta'}{\sqrt{\zeta - \zeta'}} .$$

If we attempt to calculate the derivative by the usual rule we find

(A.2)
$$I' \equiv \frac{\partial I}{\partial \zeta} = \lim_{\zeta' \to \zeta} \left[\frac{2S(\zeta')}{\sqrt{\zeta - \zeta'}} \right] - \int_a^{\zeta} \frac{S(\zeta')d\zeta'}{(\zeta - \zeta')^{3/2}} ,$$

both terms of which are singular. We get rid of the singularity upon integration by parts; thus

(A.3)
$$I = \frac{2S(a)}{\sqrt{\zeta - a}} + \int_a^{\zeta} \frac{2S'(\zeta')}{\sqrt{\zeta - \zeta'}} d\zeta' .$$

This method works, in fact, for any kernel function of the type $K(\zeta - \zeta')$ and depends simply on the relation

(A.4) $$\frac{\partial K}{\partial \zeta} = - \frac{\partial K}{\partial \zeta'} .$$

We next consider Eq. (A.1) as an integral equation; I is known and we wish to determine S. The result is well known but the solution is very simple and so we give it here. In terms of shifted variables

(A.5) $$x = \zeta - a , \qquad y = \zeta' - a ,$$

we may write Eq. (A.1) as

(A.6) $$I(x) = \int_0^x \frac{2S(y)\,dy}{\sqrt{x - y}} .$$

Now applying the Laplace transform, and denoting the transforms of I and S respectively by $\mathcal{I}$ and $\mathcal{S}$, we obtain

(A.7) $$\mathcal{I}(s) = \sqrt{\frac{\pi}{s}}\, 2\mathcal{S}(s)$$

or

(A.8) $$2\mathcal{S}(s) = \sqrt{\frac{s}{\pi}}\, \mathcal{I}(s) = \frac{1}{\pi}\sqrt{\frac{\pi}{s}} \cdot s\mathcal{I}(s) .$$

Now as

$$s\mathcal{J}(s) = \mathcal{L}\left[\frac{\partial I}{\partial x}\right] \tag{A.9}$$

we can apply the inverse transform. Changing variables back to ζ and ζ' we have the result

$$2S(\zeta) = \frac{1}{\pi}\int_a^{\zeta} \frac{I'(\zeta')d\zeta'}{\sqrt{\zeta - \zeta'}} . \tag{A.10}$$

When $a = -\infty$ the result still holds as one sees readily by proceeding to the limit. Alternatively the two sided Laplace transform could be employed.

Appendix B

Method of Estimating Error

In Section 3 we saw that the error may be estimated by calculating H_0 from Eq. (3.13), then calculating $S - f_0$ from Eq. (3.14). When this has been done we can calculate the next order value of $P(\zeta,\zeta_b)$, say $P_1(\zeta)$, by the equation

$$\text{(B.1)} \quad P_1(\zeta) = \left[\int_{-\infty}^{\zeta} \frac{P(S - f_0)}{\sqrt{\zeta - \zeta'}} \, d\zeta' \right] \Bigg/ \left[\int_{-\infty}^{\zeta} \frac{S - f_0}{\sqrt{\zeta - \zeta'}} \, d\zeta' \right] .$$

Finally comparing $P_0/(1 + P_1)$ with $P_0/(1 + P_0)$ gives an estimate of the error in our computation of G using f_0 instead of the true f.

The exact computation of these integrals is often quite laborious. As we are only trying to estimate an error which, it is hoped, is not large, accuracy is not needed and we can use an approximation scheme. Let us write

$$\text{(B.2)} \qquad I(\zeta) \overset{\text{def}}{=} \int_{-\infty}^{\zeta} \frac{2S(\zeta')d\zeta'}{\sqrt{\zeta - \zeta'}}$$

for short. Then Eq. (3.13) gives

$$\text{(B.3)} \qquad H_0 = \frac{IP_0}{1 + P_0}$$

whence

$$\text{(B.4)} \qquad H_0' = I'\Phi \; ; \; \Phi \overset{\text{def}}{=} \left[\frac{P_0(P_0 + 1) + IP_0'/I'}{(P_0 + 1)^2} \right] ,$$

and we suppose that the function Φ is slowly varying as compared with I'. Thus the integral

(B.5) $$S - f_0 = \frac{1}{\pi} \int_{-\infty}^{\zeta} \frac{\Phi(\zeta')I'(\zeta')}{\sqrt{\zeta - \zeta'}} d\zeta'$$

can be approximated by

(B.6) $$S - f_0 = \frac{\Phi(\zeta)}{\pi} \int_{-\infty}^{\zeta} \frac{I'(\zeta)d\zeta'}{\sqrt{\zeta - \zeta'}} = 2S\Phi ,$$

the last step being the inversion of Eq. (B.2). Now, in order to evaluate the integrals of Eq. (B.1), we expand $\Phi(\zeta')$ as follows

(B.7) $$\Phi(\zeta') = \Phi(\zeta) - \Phi'(\zeta) \cdot (\zeta - \zeta') .$$

Setting Eq. (B.6) into Eq. (B.1) and Eq. (B.7) into the result one obtains

$$
(B.8) \quad P_1 = P_0 + \frac{\Phi'(\zeta)}{\Phi(\zeta)} \left\{ \frac{\int_{-\infty}^{\zeta} 2S\sqrt{\zeta - \zeta'}\, d\zeta'}{\int_{-\infty}^{\zeta} \frac{2S}{\sqrt{\zeta - \zeta'}}\, d\zeta'} - \frac{\int_{-\infty}^{\zeta} 2SP\sqrt{\zeta - \zeta'}\, d\zeta'}{\int_{-\infty}^{\zeta} \frac{2SP}{\sqrt{\zeta - \zeta'}}\, d\zeta'} \right\} .
$$

The integrals in the denominators were evaluated in calculating P_0; those in the numerators are normally relatively easily evaluated. The reliability of this method seems to depend on whether Φ is in fact sufficiently slowly varying. But if it is not, Eq. (B.8) will say that the error is large, which in fact it is.

We have written everything out as though we were considering the initial wave phase. Needless to say, we have only to substitute an appropriate lower limit to apply the result also to other stages of the calculation. In some cases a modification of Eq. (B.7) may be advisable. That is, we may find it better to expand $\Phi(\zeta')$ about some point other than $\zeta' = \zeta$; for instance we might expand about the point where $S(\zeta')/\sqrt{\zeta - \zeta'}$ is maximum, or probably even better about $\zeta' = \zeta_a$. The changes thus induced in Eq. (B.8) are obvious.