Theoretical Notes
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by

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1. INTRODUCTION

The radio flash code developed by Suydam, Longmire, and Longley is based on Maxwell's equations for an axially symmetric problem in polar coordinates r, θ, φ and retarded time $\tau = t - r$ (units are such that the speed of light is unity). The equations in the air and in the ground are slightly different; in the air they are

$$\frac{\partial E_r}{\partial \tau} + 4\pi\sigma E_r = -4\pi J_r + \frac{1}{r\mu}\frac{\partial}{\partial \theta}(\mu B_\varphi)$$

$$(\mu = \sin\theta) , \qquad (1.1)$$

$$\frac{\partial(rE_\theta)}{\partial \tau} + 4\pi\sigma(rE_\theta) = -4\pi rJ_\theta - \frac{\partial(rB_\varphi)}{\partial r} + \frac{\partial(rB_\varphi)}{\partial \tau} , \qquad (1.2)$$

$$\frac{\partial(rB_\varphi)}{\partial \tau} = \frac{\partial E_r}{\partial \theta} - \frac{\partial(rE_\theta)}{\partial r} + \frac{\partial(rE_\theta)}{\partial \tau} . \qquad (1.3)$$

The quantities J_r and J_θ represent the Compton current density and σ the conductivity. The calculation makes partial use of the method of characteristics by introducing two new dependent variables

$$F \overset{\text{def}}{=} r(E_\theta + B_\varphi) ,$$

$$G \overset{\text{def}}{=} r(E_\theta - B_\varphi) ; \qquad (1.4)$$

whereupon equations (1.2) and (1.3) become replaced by

$$\frac{\partial F}{\partial r} + 2\pi\sigma F = \frac{\partial E}{\partial \theta} - 4\pi rJ_\theta - 2\pi\sigma G , \qquad (1.5)$$

$$2\frac{\partial G}{\partial \tau} - \frac{\partial G}{\partial r} + 2\pi\sigma G = -\frac{\partial E}{\partial \theta} - 4\pi rJ_\theta - 2\pi\sigma F . \qquad (1.6)$$

Two systems of finite-difference equations for the system [Eqs. (1.1), (1.5), and (1.6)] are given in Appendices A and B, namely, the original system (per Suydam's manuscript [1]) and the system now in use (per private communication). These systems have exhibited numerical instabilities, and this memo discusses attempts to analyze these instabilities and to remedy them by a modification of the difference equations.

On the tentative assumption that the conductivity of the ground stabilizes the calculation there, only the air equations (given above) will be considered, except in Section 6.

For the stability considerations, it probably suffices to regard σ, J_r, and J_θ as functions of r and t that are known in advance. In the (normal-mode) stability analysis given below, J_r and J_θ are dropped, and σ and r are treated as constants (on the usual supposition that stability is a local phenomenon). All terms are retained, however, in writing down the difference equations in the Appendices A and B, and J_θ (which results from the bending of the paths of the Compton electrons by the field B_φ) is written in the form used in the Flash code, namely,

$$J_\theta = -\frac{\sigma}{2r}\Phi(F - G) , \qquad (1.7)$$

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where

$$\delta \stackrel{\text{def}}{\equiv} \frac{k}{2}\left(-\frac{J_r}{\sigma}\right)\frac{2(1 - \cos k\, B_\varphi)}{(kB_\varphi)^2} , \qquad (1.8)$$

where k is a constant.

The stability analysis presupposes a well-posed mathematical problem before differencing is introduced. Since the use of the retarded time $\tau = t - r$ is slightly unusual, an attempt is made, in Section 2, to study the well-posedness of the problem by investigating a greatly simplified model of the equations. It is concluded that this model presents a well-posed mixed initial-boundary-value problem, in spite of the infinite frequencies to which Longmire has called attention.

In Section 3, some simple difference equations for the simplified model are analyzed; they suggest that the currently used difference equations for the Flash code (Appendices A and B) ought indeed to be unstable, but that a certain modification of them ought to be conditionally stable. A proposed modification of the difference equations for the air part of the Flash code is then given in Appendix C.

In Section 4, a normal-mode stability analysis of the various difference-equation systems is described. Special computer codes for Maniac 2 were written for this purpose.

The proposed new difference equations for the Flash code are implicit; the algorithm for solving them is described in Section 5. Further last-minute comments are made in Section 6.

2. A SIMPLIFIED MODEL

To obtain a workable model (very greatly simplified, to be sure) of the mathematical problem, we replace Maxwell's equations by the scalar wave equation in two cartesian space variables x, y, and we drop the source terms and the conduction terms. We transform the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad (2.1)$$

to new independent variables given by

$$\tau = t - x , \quad x' = x , \quad y' = y , \qquad (2.2)$$

and then we drop the primes:

$$\cancel{\frac{\partial^2 u}{\partial \tau^2}} = \frac{\partial^2 u}{\partial x^2} - 2\,\frac{\partial^2 u}{\partial x \partial \tau} + \cancel{\frac{\partial^2 u}{\partial \tau^2}} + \frac{\partial^2 u}{\partial y^2} . \qquad (2.3)$$

With u and $\partial u/\partial\tau$ given as initial data for $\tau = 0$, this is an improperly posed Cauchy problem (i.e., pure initial-value problem); there is no solution at all unless the given functions u and $\partial u/\partial\tau$ satisfy equation (2.3), and if they do satisfy it, there is no way of obtaining $\partial^2 u/\partial\tau^2$ etc. from Eq. (2.3) without the aid of a boundary condition. This is because the initial surface ($t - x = 0$) is a characteristic surface for the hyperbolic equation (2.1).

We consider instead the mixed problem:

$$\text{I.C.} \quad u \text{ given for } \tau = 0 , \qquad (2.4)$$

$$\text{B.C.} \quad u \to 0 \text{ as } x,y \to \infty . \qquad (2.5)$$

If $u \to 0$ sufficiently rapidly, we can integrate Eq. (2.3) with respect to x to give

$$\frac{\partial u}{\partial \tau} = Lu , \qquad (2.6)$$

where L is the linear integro-differential operator

$$(Lu)(x,y) = \frac{1}{2}\frac{\partial u}{\partial x} + \int_x^\infty \frac{\partial^2 u}{\partial y^2}(x',y)\,dx' ; \qquad (2.7)$$

repeated use of Eq. (2.6) then gives $\partial^2 u/\partial\tau^2$, $\partial^3 u/\partial\tau^3$, etc., so that a formal Taylor's series in τ can be found, as in the Cauchy-Kowalewski theory.

The solution of the mixed problem can be obtained by a Fourier transformation, on the assumption that $u \to 0$ sufficiently rapidly as $x,y \to \pm\infty$: set

$$u(x,y,\tau) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \hat{u}(\alpha,\beta,\tau)e^{i(\alpha x+\beta y)}d\alpha d\beta \ . \qquad (2.8)$$

Substitution into the differential equation (2.3) gives

$$0 = \iint_{-\infty}^{\infty} \left[-(\alpha^2 + \beta^2)\hat{u}(\alpha,\beta,\tau) - 2i\alpha \frac{\partial \hat{u}}{\partial \tau} (\alpha,\beta,\tau) \right]$$

$$\times e^{i(\alpha x+\beta y)}d\alpha d\beta \ , \qquad (2.9)$$

and the vanishing of the square bracket gives a differential equation for $\hat{u}(\alpha,\beta,\tau)$. Since $\hat{u}(\alpha,\beta,0)$ is known as the Fourier transform of the initial function $u(x,y,0)$, the solution of the problem is

$$u(x,y,\tau) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{i\frac{\alpha^2+\beta^2}{2\alpha}\tau} \hat{u}(\alpha,\beta,0)$$

$$\times e^{i(\alpha x+\beta y)}d\alpha d\beta \ . \qquad (2.10)$$

It is now clear that the problem is well posed in the L_2 norm (see [2], Chapter 3): the integral [Eq. (2.10)] exists for any reasonable (say continuous) function $\hat{u}(\alpha,\beta,0)$ in L_2, and the Parseval relation, applied to Eq. (2.10) shows that

$$\iint_{-\infty}^{\infty} |u(x,y,\tau)|^2 dxdy =$$

$$\iint_{-\infty}^{\infty} |\hat{u}(\alpha,\beta,0)|^2 d\alpha d\beta \ , \qquad (2.11)$$

which shows that the solution operator is bounded (and in fact by unity) for all τ.

Although the infinite frequencies that appear in the first exponential in Eq. (2.10) as $\alpha \to 0$ $(\beta \neq 0)$ do not prevent the problem from being well posed, it seems likely that special care may be needed in the construction of difference equations to avoid instabilities.

3. DIFFERENCE EQUATIONS FOR THE MODEL

To put the equations in a form more like Eqs. (1.1), (1.5), (1.6), call

$$E = E(x,y,\tau) = \frac{\partial u}{\partial x} \ ,$$

$$F = F(x,y,\tau) = \frac{\partial u}{\partial y} \ ;$$

then, Eq. (2.3) gives

$$2\frac{\partial E}{\partial \tau} = \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} \ , \qquad (3.1)$$

$$\frac{\partial F}{\partial x} = \frac{\partial E}{\partial y} \ . \qquad (3.2)$$

A simple difference system is

$$\frac{E^{n+1}_{k+\frac{1}{2}\ \ell+\frac{1}{2}} - E^{n-1}_{k+\frac{1}{2}\ \ell+\frac{1}{2}}}{\Delta\tau} = \frac{E^{n}_{k+\frac{3}{2}\ \ell+\frac{1}{2}} - E^{n}_{k-\frac{1}{2}\ \ell+\frac{1}{2}}}{2\Delta x}$$

$$+ \frac{F^{n}_{k+1\ \ell+1} - F^{n}_{k+1\ \ell} + F^{n}_{k\ \ell+1} - F^{n}_{k\ \ell}}{2\Delta y} \ , \qquad (3.3)$$

$$\frac{F^{n}_{k+1\ \ell} - F^{n}_{k\ \ell}}{\Delta x} = \frac{E^{n}_{k+\frac{1}{2}\ \ell+\frac{1}{2}} - E^{n}_{k+\frac{1}{2}\ \ell-\frac{1}{2}}}{\Delta y} \ , \qquad (3.4)$$

where, for any n,k,ℓ (integer or not) $F^{n}_{k\ \ell}$ denotes the computed approximation to $F(k\Delta x,\ell\Delta y,n\Delta\tau)$. By a normal-mode stability analysis it will now be shown that these equations are unstable but that they can be made (conditionally) stable by a modification. Functions of the form

$$\begin{pmatrix} E(x,y,\tau) \\ F(x,y,\tau) \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix} e^{\alpha\tau+i(ax+by)} \qquad (3.5)$$

are substituted into Eqs. (3.3) and (3.4), where e and f are constants; after cancelling out certain common factors from the terms of these equations, we have two homogeneous linear equations for e and f. The condition for the existence of a nontrivial solution is

$$\begin{vmatrix} \gamma^2 - 1 - i\,\frac{\Delta\tau}{\Delta x}(\sin\psi)\gamma & -2i\,\frac{\Delta\tau}{\Delta y}\left(\cos\frac{\psi}{2}\right)\left(\sin\frac{\varphi}{2}\right)\gamma \\ -\frac{\Delta\tau}{\Delta y}\left(\sin\frac{\varphi}{2}\right) & \frac{\Delta\tau}{\Delta x}\left(\sin\frac{\psi}{2}\right) \end{vmatrix} = 0\,, \qquad (3.6)$$

where

$$\gamma \overset{\text{def}}{=} e^{\alpha\Delta\tau} = \underline{\text{amplification}}\ \underline{\text{factor}}\,,$$

$$\psi \overset{\text{def}}{=} a\Delta x\,, \quad \varphi \overset{\text{def}}{=} b\Delta y\,.$$

If $|\gamma| \le 1$ for all real φ and ψ the difference equations are stable, otherwise not (for more precise statements, see [2]). Equation (3.6) is a quadratic equation for γ. As $\psi \to 0$ ($\varphi \neq 0$), the coefficient of γ^2 in this equation $\to 0$, while the other coefficients do not $\to 0$; hence (at least) one root $\gamma \to \infty$, and the equations are unconditionally unstable.

To improve stability, each F^n on the right of Eq. (3.3) can be replaced by $\frac{1}{2}(F^{n+1} + F^{n-1})$ (in each case with the proper subscripts); then the factor γ in the upper right element of the determinant [Eq. (3.6)] is replaced by $\frac{1}{2}(\gamma^2 + 1)$, with the result that the coefficient of γ^2 in the quadratic equation no longer $\to 0$ as $\psi \to 0$. A detailed calculation, based on the quadratic equation, shows that with this modification, $|\gamma| = 1$ for all φ, ψ provided that

$$\frac{\Delta\tau}{\Delta x} \le 2 \quad \left(\text{regardless of the value of } \frac{\Delta\tau}{\Delta y}\right), \qquad (3.7)$$

Eq. (3.7) is evidently the stability condition.

With this modification, the difference equations are of course implicit in both the x and y directions.

A similar modification can be made in the difference equations for the flash code (in the air). In order to avoid having the equations implicit in two directions (here, the r and θ directions), the electric field $E(=E_r)$ is now centered at the points $k\Delta r$ rather than $(k + \frac{1}{2})\Delta r$ - the centering at $(\ell + \frac{1}{2})\Delta\theta$ is retained. This requires some rewriting of the equations; then, each F^n in the E-equation is replaced by $\frac{1}{2}(F^{n+1} + F^{n-1})$ (with the proper subscripts). The resulting difference system is given in Appendix C. The equations are implicit in the θ-direction; the algorithm for solving them is given in Section 5.

4. THE NORMAL-MODE STABILITY ANALYSIS OF THE FLASH-CODE EQUATIONS

Functions of the form

$$\begin{pmatrix} E(x,y,\tau) \\ F(x,y,\tau) \\ G(x,y,\tau) \end{pmatrix} = \begin{pmatrix} e \\ f \\ g \end{pmatrix} e^{\alpha\tau + i(ax+by)} \qquad (4.1)$$

are substituted into the difference equations (Appendix A, B, or C). The resulting secular equation [like Eq. (3.6) for the model] for the amplification factor γ contains a 3×3 determinant and is written out in Appendices D, E, and F for the original equation (Appendix A), for the equations now in use (Appendix B), and for the proposed modified equations (Appendix C). These secular equations are of the third and fourth order in γ, with complex coefficients. Computer codes were written for the Maniac 2 computer, which compute for each φ and each ψ of the set

$$\left\{ \begin{array}{ll} \varphi = \frac{q}{P}\pi & q = 0, 1, 2, \cdots, P \\ \psi = \frac{p}{P}\pi & p = 0, 1, 2, \cdots, P \end{array} \right\}, \qquad (4.2)$$

the coefficients of the algebraic equation for γ, then find the roots of these equations by a subroutine of M. Fraser. The machine then prints out these roots ($3P^2$ or $4P^2$ complex numbers) and the maximum value of $|\gamma|$ and the values of p and q at which the maximum occurs. (In most cases, P was =5, but some larger values were used also.) The input to the program is the values of σ, $\Delta\tau$, Δr, $\Delta\theta$, r (these could have been combined, since only the combinations $\sigma\Delta\tau$, $\sigma\Delta r$, and $\sigma r\Delta\theta$ appear in the calculation).

The results were quite different for the explicit and implicit difference equations. For the explicit equations of Appendices A and B (modifications 0 and 3 of the Maniac code) this (normal-mode) analysis indicates unconditional instability for certain sets of values of σ, Δr, r, and $\Delta\theta$, i.e., for such a set of values of these parameters, Max $|\gamma|$ was >1 for all $\Delta\tau > 0$. Stated differently, for any given values of the ratios $\Delta\tau/\Delta r$ and $\Delta\tau/r\Delta\theta$, the equations are unstable for all σ less than some limiting value $\sigma_0 = \sigma_0(\Delta\tau/\Delta r, \Delta\tau/r\Delta\theta)$, even when the quantity

$$f = \text{Max} \quad \frac{\Delta\tau}{\Delta r}, \frac{\Delta\tau}{r\Delta\theta} \qquad (4.3)$$

(which is often taken as determining for stability) is quite small.

This unconditional instability for small σ can be seen also directly from the secular equations (Appendices D and E). If σ is set equal to zero, the coefficient of γ^3 (or γ^4, respectively) in the equation is $2i \sin \psi/2$, which $\to 0$ as $\psi \to 0$, while other coefficients of the equation are nonzero for $\psi = 0$; therefore one root γ of secular equation becomes infinite as $\psi \to 0$; hence, by the continuous dependence of the roots on the coefficients, Max $|\gamma|$ can be made arbitrarily large by taking σ small enough.

For the equations currently in use (Appendix B - modification 3 of the Maniac code), Figure 1 shows the amplification factor Max $|\gamma|$ as a function of σ in one case, the other parameters being held constant. As $\sigma \to 0$, the amplification factor grows without bound, while the equations are stable for σ greater than or equal to a certain limiting value σ_0. Figure 2 summarizes the values of σ_0 for various values of the other parameters. The Maniac calculations show that for $\sigma \approx \sigma_0$ and $\Delta\tau/r\Delta\theta$ not too large the dominant mode of growth is the one for which $\psi = 0, \varphi = \pi$ (this corresponds to +-+-···· variation in θ and no variation in r); the corresponding γ is near -1. Knowing this, we assume $\psi = 0$, $\varphi = \pi$, $\gamma = -1 + i\epsilon$ ($\epsilon \ll 1$), $\sigma = \sigma_0$ ($\sigma_0\Delta\tau \ll 1$) in the secular equation (Appendix E), and we find

$$\pi\sigma_0 r\Delta\theta \approx \frac{1}{\sqrt{6}} , \quad \epsilon \approx \sqrt{20}\, \pi\sigma_0\Delta\tau , \qquad (4.4)$$

which are in good agreement with the Maniac results for the cases $\Delta\tau/r\Delta\theta = 0.02, 0.1$.

Similar results were obtained for the original difference equations (Appendix A - modification 0 of the Maniac code). In this case, γ is actually equal to -1 at the stability limit, and we get from the secular equation an exact value of σ_0, given by

$$\pi\sigma_0 r\Delta\theta = \frac{1}{2} , \qquad (4.5)$$

which agrees with the Maniac result in all cases that were run.

The (proposed modified) equations of Appendix C, on the other hand, are indicated to be always conditionally stable (modification 2 of the Maniac code): for each set of values of the parameters σ, Δr, $\Delta\theta$, r, there is a positive $\Delta\tau_0$ such that Max $|\gamma|$ is ≤ 1 (in fact always =1) provided $\Delta\tau \leq \Delta\tau_0$. The results are presented in terms of the dimensionless variable $\sigma\Delta\tau$, $\Delta\tau/\Delta r$, and $\Delta\tau/r\Delta\theta$ in Figure 3. For each $\sigma\Delta\tau$, there is a curve in the $\Delta\tau/\Delta r$, $\Delta\tau/r\Delta\theta$ plane; the region of stability is in each case the region to the left of the curve. From these curves, we can take as a rough semi-empirical stability criterion

$$[0.3 - 20.0(\sigma\Delta r)^2]\left(\frac{\Delta\tau}{\Delta r}\right)^2 + 2.25\left(\frac{\Delta\tau}{r\Delta\theta}\right)^2 < 1 . \qquad (4.6)$$

This criterion is very rough, but is on the conservative side; that is, the criterion is violated

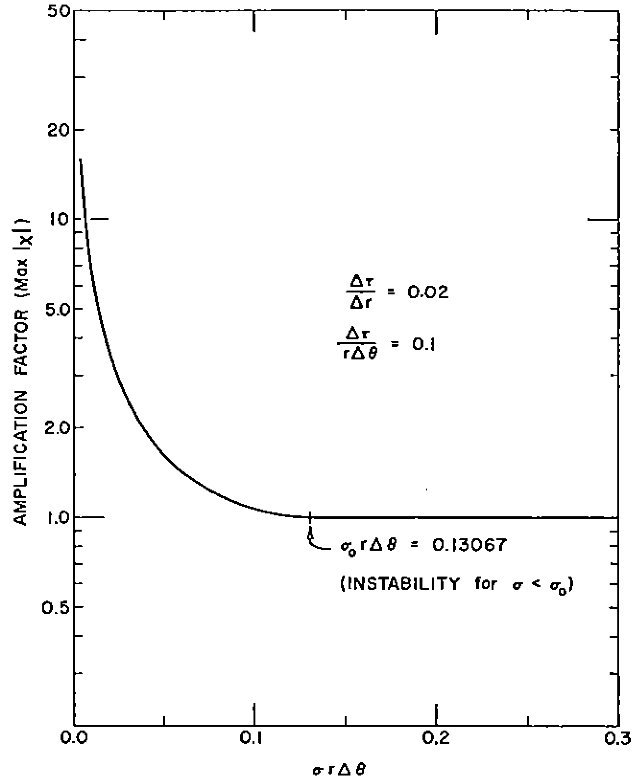

Fig. 1. Stability analysis of the equation of Appendix B.

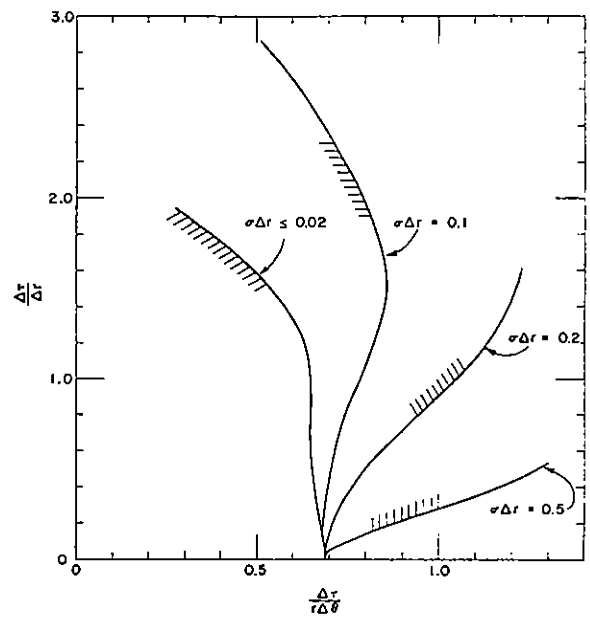

Fig. 3. Stability analysis of proposed new equations (Appendix C). Stability corresponds to parameters in the region to the left of the curve.

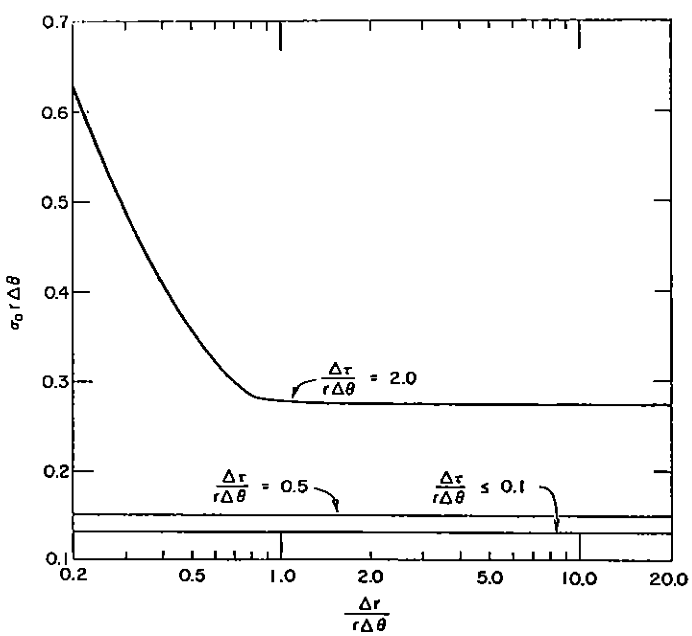

Fig. 2. Limiting values of σ (stability for $\sigma \ge \sigma_0$) (Equations of Appendix B).

in every case in which an amplification factor $|\chi| > 1$ was bound by the normal-mode analysis.

To use this criterion in practice in the Flash (EMP) code, a sweep is made through the r,θ net for each τ; at each point r,θ one calculates the quantity

$$\alpha = 0.3 - 20.0(\sigma\Delta r)^2 + 2.25\left(\frac{\Delta r}{r\Delta\theta}\right)^2 \ ; \qquad (4.7)$$

the procedure is then:

$$\begin{aligned} &\text{if } \alpha \le 0 \ , \quad \text{do nothing} \\ &\text{if } \alpha > 0 \ , \quad \left\{\begin{array}{l} \text{call } y = f_s \Delta r \alpha^{-\frac{1}{2}} \\ \text{if } y < \Delta\tau \ , \quad \text{replace } \Delta\tau \text{ by } y \end{array}\right\} ; \quad (4.8) \end{aligned}$$

here, $\Delta\tau$ is a provisional estimate of the increment $\Delta\tau$ to be used at the next cycle; at the beginning of the sweep mentioned above, $\Delta\tau$ is set to some very large value and is then gradually revised downward during the sweep. The quantity f_s is a number (called "stability fraction") in the interval (0,1) and is one of the input data to the EMP code.

[Modification 1 of the Maniac code was for the original equations of Appendix A but altered by replacing F^n by $\frac{1}{2}(F^{n+1} + F^{n-1})$ in the E equation. Conditional stability was found there, also. For modification 4 of the Maniac code, see Section 6.]

The magnetic tapes for the Maniac codes have been saved - Roger Lazarus knows where they are stored. I can send instructions for running further cases, if they should be desired.

5. METHOD OF SOLVING THE NEW DIFFERENCE EQUATIONS

In this Section, the method of solving the (implicit) equations of Appendix C is discussed, on the assumption that the electric field $E = E_r$ is known on the ground ($\theta = \pi/2$), for all r, all τ. The idea was that the existing EMP (explicit) code could be used to calculate all quantities in the ground and at the ground-air interface. It was believed that the overall system would then be stable, owing to the large conductivity in the ground. [Experiments on the CDC 6600 computer showed that this belief was unjustified, and further modifications of the overall procedure will be mentioned briefly in Section 6.]

Equation (C.5 of the Appendix) is explicit; hence, $G^{n+1}_{k\,\ell}$ can be calculated for all k, all ℓ, befor the equations (C.1) and (C.3) are tackled; then, assuming all quantities known for time $n\Delta\tau$ and assuming furthermore that the circles $r = r_k$ are taken in order of increasing k, the only unknowns in (C.1) and (C.3) for the $k^{\underline{th}}$ circle are the E's and F's with superscript n + 1 and first subscript k (all values of ℓ). The equations can therefore be written in the form

$$E^{n+1}_{k\,\ell+\frac{1}{2}} = \alpha_{\ell+\frac{1}{2}} F^{n+1}_{k\,\ell+1} - \beta_{\ell+\frac{1}{2}} F^{n+1}_{k\,\ell} + \epsilon_{\ell+\frac{1}{2}}$$

$$(\ell = 1, 2, \cdots, L) \ ; \qquad (5.1)$$

$$F^{n+1}_{k\,\ell} = \gamma_\ell E^{n+1}_{k\,\ell+\frac{1}{2}} - \delta_\ell E^{n+1}_{k\,\ell-\frac{1}{2}} + \zeta_\ell$$

$$(\ell = 2, 3, \cdots, L) \ , \qquad (5.2)$$

where the greek letters denote known quantities. The θ mesh has been assumed such that $\theta_1 = 0$, $\theta_{L+\frac{1}{2}} = \pi/2$. The system (5.1), (5.2) is to be solved with the boundary conditions

$$F^{n+1}_{k\,1} = 0 \ ; \qquad (5.3)$$

$$E^{n+1}_{k\,L+\frac{1}{2}} \text{ known} \ . \qquad (5.4)$$

If the unknowns F_1, $E_{\frac{3}{2}}$, F_2, $E_{\frac{5}{2}}$, $\cdots$, F_L, $E_{L+\frac{1}{2}}$ (here, and below, the fixed indices n + 1 and k are omitted) are denoted by u_1, u_2, $\cdots$, u_{2L}, then the system (5.1)-(5.4) is an ordinary 3-term recurrence system and can be solved in the usual way. However, since storage arrays for the E's and F's are already present in the EMP code, it is probably more convenient to proceed as described below.

From each equation (5.2) for $\ell = 2, 3, \cdots, L - 1$ (but not L), $E_{\ell+\frac{1}{2}}$ and $E_{\ell-\frac{1}{2}}$ are eliminated by use of Eq. (5.1) to give an equation of the form

$$-A_\ell F_{\ell+1} + B_\ell F_\ell - C_\ell F_{\ell-1} = D_\ell$$

$$(\ell = 2, 3, \cdots, L - 1) \ , \qquad (5.5)$$

where A_ℓ, B_ℓ, C_ℓ, and D_ℓ are known coefficients given by

$$A_\ell = \gamma_\ell \alpha_{\ell+\frac{1}{2}} ;$$

$$B_\ell = 1 + \gamma_\ell \beta_{\ell+\frac{1}{2}} + \delta_\ell \alpha_{\ell-\frac{1}{2}} ,$$

$$C_\ell = \delta_\ell \beta_{\ell-\frac{1}{2}} ,$$

$$D_\ell = \zeta_\ell + \gamma_\ell \epsilon_{\ell+\frac{1}{2}} - \delta_\ell \epsilon_{\ell-\frac{1}{2}} . \qquad (5.6)$$

From the last equation (5.2) ($\ell = L$), only $E_{L-\frac{1}{2}}$ is eliminated, and the result is written as

$$\underset{\substack{\uparrow \\ A_L}}{-0} \cdot F_{L+1} + \underbrace{(1 + \delta_L \alpha_{L-\frac{1}{2}})}_{B_L} F_L - \underbrace{\delta_L \beta_{L-\frac{1}{2}}}_{C_L} F_{L-1} =$$

$$\underbrace{\zeta_L - \delta_L \epsilon_{L-\frac{1}{2}} + \gamma_L E^{n+1}_{k\ L+\frac{1}{2}}}_{D_L} . \qquad (5.7)$$

With the coefficients identified as indicated (N.B.: $E^{n+1}_{k\ L+\frac{1}{2}}$ is known), this equation is of the same form as Eq. (5.5), so now ℓ runs from 2 to L. The boundary conditions are: $F_1 = 0$, F_{L+1} irrelevant [because $A_L = 0$ in Eq. (5.7)]. To solve the system, one defines inductively

$$e_1 = 0 \qquad\qquad f_1 = 0$$

$$e_\ell = \frac{A_\ell}{B_\ell - C_\ell e_{\ell-1}} \qquad f_\ell = \frac{D_\ell - C_\ell f_{\ell-1}}{B_\ell - C_\ell e_{\ell-1}}$$

$$(\ell = 2, 3, \cdots, L) ; \qquad (5.8)$$

then the solution is given inductively (on decreasing ℓ) by

$$F_{L+1} \quad \text{arbitrary}$$

$$F_\ell = e_\ell F_{\ell+1} + f_\ell$$

$$(\ell = L, L-1, \cdots, 2) . \qquad (5.9)$$

The $E^{n+1}_{k\ \ell+\frac{1}{2}}$ are then given by Eq. (5.1).

6. LAST-MINUTE FURTHER COMMENTS

Numerical tests made by suitably altering the EMP code for the CDC 6600 computer indicated that the procedure outlined in the preceding section is still subject to instabilities, presumably arising from the explicit calculation of E on and in the ground (the instabilities disappeared when the field $E^n_{k\ L+\frac{1}{2}}$ at the ground was taken to be identically zero, as the boundary condition for the air calculation). Jerry Longley has suggested that this remaining instability can probably be overcome by making the entire calculation of E and F implicit all the way from $\theta = 0$ in the air down to some depth in the ground where E can be taken as zero.

The equations to be solved are then of the form

$$\begin{cases} u_{\ell_1} = 0 & (6.1) \\ -A_\ell u_{\ell+1} + B_\ell u_\ell - C_\ell u_{\ell-1} = D_\ell & \\ (\ell = \ell_1 + 1, \ell_1 + 2, \cdots, \ell_2 - 1) & (6.2) \\ \tilde{A} u_{\ell_2+2} - A_{\ell_2} u_{\ell_2+1} + B_{\ell_2} u_{\ell_2} - C_{\ell_2} u_{\ell_2-1} & \\ \qquad + \tilde{C} u_{\ell_2-2} = D_{\ell_2} & (6.3) \\ -A_\ell u_{\ell+1} + B_\ell u_\ell - C_\ell u_{\ell-1} = D_\ell & \\ (\ell = \ell_2 + 1, \ell_2 + 2, \cdots, \ell_3 - 1) & (6.4) \\ u_{\ell_3} = 0 , & (6.5) \end{cases}$$

where the capital letters represent known quantities. The single 5-term equation (6.3) comes from the boundary condition at the air-ground interface. To solve the system, one assumes relations of the form

$$u_\ell = e_\ell u_{\ell+1} + f_\ell$$

$$(\ell_1 \le \ell \le \ell_2 - 1) , \qquad (6.6)$$

$$u_\ell = e_\ell u_{\ell-1} + f_\ell$$

$$(\ell_2 + 1 < \ell \le \ell_3) \; ; \qquad (6.7)$$

the e's and f's are then given inductively by

$$\begin{cases} e_{\ell_1} = 0 & f_{\ell_1} = 0 \\ e_\ell = \dfrac{A_\ell}{B_\ell - C_\ell e_{\ell-1}} & f_\ell = \dfrac{D_\ell - C_\ell f_{\ell-1}}{B_\ell - C_\ell e_{\ell-1}} \\ (\ell = \ell_1 + 1, \ell_1 + 2, \cdots, \ell_2 - 1) & \end{cases} \qquad (6.8)$$

$$\begin{cases} e_{\ell_3} = 0 & f_{\ell_3} = 0 \\ e_\ell = \dfrac{C_\ell}{B_\ell - A_\ell e_{\ell+1}} & f_\ell = \dfrac{D_\ell - A_\ell f_{\ell+1}}{B_\ell - A_\ell e_{\ell+1}} \\ (\ell = \ell_3 - 1, \ell_3 - 2, \cdots, \ell_2 + 1) \; . & \end{cases} \qquad (6.9)$$

Then u_{ℓ_2} (the field at the ground) is given directly by Eq. (6.3) after the other unknowns have been e-limited by use of equation (6.6) with $\ell = \ell_2 - 1$ and and $\ell_2 - 2$ and equation (6.7) with $\ell = \ell_2 + 1$ and $\ell_2 + 2$. Finally, the remaining unknowns are given by inductive use of Eq. (6.6) (decreasing ℓ) and Eq. (6.7) (increasing ℓ).

Jerry Longley has pointed out that if, in addition to replacing F^n by $\frac{1}{2}(F^{n+1} + F^{n-1})$ in the E equation, G^n is also replaced by $\frac{1}{2}(G^{n+1} + G^{n-1})$ in that same equation, no extra calculation is required because the $G^{n+1}_{k\,\ell}$ are all known at the time of the E and F calculations. This procedure seems preferable because (1) it treats F and G on an equal footing in the E equation and (2) it may improve stability. To investigate the latter point, Mod 4 of the Maniac normal-mode analysis code was written. It is based on the same secular equation as in Appendix F, except that the factor χ in the upper right element is replaced by $\frac{1}{2}(\chi^2 + 1)$. Much to my surprise, there is now again unconditional instability when σ is less than a value σ_0. When this occurred, the dominant mode was $\psi = \pi$, $\varphi = \pi$ (+-+- ··· variation in both θ and r), with χ = real < -1. Putting $\psi = \pi$, $\varphi = \pi$, and $\chi = -1$ into the secular equation gives for σ_0 the equation

$$\pi\sigma_0 r\Delta\theta = \frac{1}{2} \qquad (6.10)$$

[compare with Eq. (4.4) and Eq. (4.5)].

The conclusion is that it is not a good idea to replace G^n by $\frac{1}{2}(G^{n+1} + G^{n-1})$ in equation (C.1) of Appendix C.

A possible intuitive explanation of this result is that the G equation, being explicit (leap-frog), contains a germ of instability, so that incipient errors (which are eventually kept under control by the stabilizing influence of the other equations) are more strongly represented in G^{n+1} than in G^n.

A final comment is that the theory of the smoothing procedure

$$(1 - \epsilon)u^{n+1} + 2\epsilon u^n - \epsilon u^{n-1} \rightarrow u^{n+1}$$

$$(\epsilon \approx 0.1) \; ,$$

which has been found so effective in suppressing instabilities in the EMP code, should be investigated more fully sometime. In view of the large amplification factors predicted (for small σ) for the equations that have been used until now (see Figure 1), this smoothing procedure must be rather powerful and can perhaps be of use in many problems.

REFERENCES

1. Manuscript, A New Radio Flash Code for a High Speed Computer, by B. R. Suydam.

2. Difference Methods for Initial-Value Problem, R. D. Richtmyer and K. W. Morton, Wiley-Interscience (1967).

APPENDIX A. ORIGINAL DIFFERENCE EQUATIONS (IN THE AIR)

$$E^{n+1}_{k+\frac{1}{2}\ \ell+\frac{1}{2}} = e^{-S}E^{n-1}_{k+\frac{1}{2}\ \ell+\frac{1}{2}} + \frac{1-e^{-S}}{4\pi\sigma}\left\{-4\pi J + \frac{\mu_{\ell+1}(F^n_{k+1\ \ell+1} + F^n_{k\ \ell+1} - G^n_{k+1\ \ell+1} - G^n_{k\ \ell+1}) - \mu_\ell(F^n_{k+1\ \ell} + F^n_{k\ \ell} - G^n_{k+1\ \ell} - G^n_{k\ \ell})}{4r^2_{k+\frac{1}{2}}\mu_{\ell+\frac{1}{2}}(\theta_{\ell+1} - \theta_\ell)}\right\}, \tag{A.1}$$

where

$$\sigma = \sigma^n_{k+\frac{1}{2}\ \ell+\frac{1}{2}}, \qquad S = 4\pi\sigma(\tau^{n+1} - \tau^{n-1}), \qquad J = (J_r)^n_{k+\frac{1}{2}\ \ell+\frac{1}{2}}; \tag{A.2}$$

$$F^{n+1}_{k+1\ \ell} = e^{-X}F^{n+1}_{k\ \ell} + \frac{1-e^{-X}}{2\pi\sigma(1-\Phi)}\left\{\frac{E^{n+1}_{k+\frac{1}{2}\ \ell+\frac{1}{2}} - E^{n+1}_{k+\frac{1}{2}\ \ell-\frac{1}{2}}}{\theta_{\ell+\frac{1}{2}} - \theta_{\ell-\frac{1}{2}}} - \pi\sigma(1+\Phi)(G^{n+1}_{k+1\ \ell} + G^{n+1}_{k\ \ell})\right\}, \tag{A.3}$$

where

$$\sigma = \sigma^{n+1}_{k+\frac{1}{2}\ \ell}, \qquad X = 2\pi\sigma(r_{k+1} - r_k)(1-\Phi), \qquad \Phi = \Phi^{n+1}_{k+\frac{1}{2}\ \ell}; \tag{A.4}$$

$$G^{n+1}_{k+1\ \ell} = e^{-Y}\left[G^n_{k+1\ \ell} + \frac{\tau^{n+1} - \tau^n}{2(r_{k+2} - r_{k+1})}(G^n_{k+2\ \ell} - G^n_{k+1\ \ell})\right] - \frac{1-e^{-Y}}{2\pi\sigma(1+\Phi)}\left\{\left[\frac{\partial E}{\partial\theta}\right] + \pi\sigma(1-\Phi)\left[F^{n+1}_{k+1\ \ell} + F^n_{k+1\ \ell} + \frac{\tau^{n+1} - \tau^n}{2(r_{k+2} - r_{k+1})}(F^n_{k+2\ \ell} - F^n_{k+1\ \ell})\right]\right\}, \tag{A.5}$$

where

$$\sigma = \frac{1}{2}\left[\sigma^{n+1}_{k+1\ \ell} + \sigma^n_{k+1\ \ell} + \frac{\tau^{n+1} - \tau^n}{2(r_{k+2} - r_{k+1})}(\sigma^n_{k+2\ \ell} - \sigma^n_{k+1\ \ell})\right], \tag{A.6}$$

$$\Phi = \frac{1}{2}\left[\Phi^{n+1}_{k+1\ \ell} + \Phi^n_{k+1\ \ell} + \frac{\tau^{n+1} - \tau^n}{2(r_{k+2} - r_{k+1})}(\Phi^n_{k+2\ \ell} - \Phi^n_{k+1\ \ell})\right], \tag{A.7}$$

$$Y = \pi\sigma(\tau^{n+1} - \tau^n)(1+\Phi), \tag{A.8}$$

and

$$\left[\frac{\partial E}{\partial\theta}\right] = \frac{r_{k+1} - r_k + \frac{\tau^{n+1} - \tau^n}{2}}{2(r_{k+2} - r_k)}\left[\frac{E^{n+1}_{k+\frac{3}{2}\ \ell+\frac{1}{2}} + E^n_{k+\frac{3}{2}\ \ell+\frac{1}{2}} - E^{n+1}_{k+\frac{3}{2}\ \ell-\frac{1}{2}} - E^n_{k+\frac{3}{2}\ \ell-\frac{1}{2}}}{\theta_{\ell+\frac{1}{2}} - \theta_{\ell-\frac{1}{2}}}\right] + \frac{r_{k+2} - r_{k+1} - \frac{\tau^{n+1} - \tau^n}{2}}{2(r_{k+2} - r_k)}\left[\frac{E^{n+1}_{k+\frac{1}{2}\ \ell+\frac{1}{2}} + E^n_{k+\frac{1}{2}\ \ell+\frac{1}{2}} - E^{n+1}_{k+\frac{1}{2}\ \ell-\frac{1}{2}} - E^n_{k+\frac{1}{2}\ \ell-\frac{1}{2}}}{\theta_{\ell+\frac{1}{2}} - \theta_{\ell-\frac{1}{2}}}\right]. \tag{A.9}$$

APPENDIX B. DIFFERENCE EQUATIONS IN USE AUGUST, 1967 (AIR)*

Equations (A.1) and (A.2) are taken over unchanged. The remaining equations are

$$F^{n+1}_{k+1\ \ell} = e^{-X}F^{n+1}_{k\ \ell} + \frac{1 - e^{-X}}{2\pi\sigma(1 - \Phi)} \frac{E^{n+1}_{k+\frac{1}{2}\ \ell+\frac{1}{2}} - E^{n+1}_{k+\frac{1}{2}\ \ell-\frac{1}{2}}}{\theta_{\ell+\frac{1}{2}} - \theta_{\ell-\frac{1}{2}}}$$

$$- \frac{1 + \Phi}{1 - \Phi}\left[\left(1 - \frac{1 - e^{-X}}{X}\right)G^{n+1}_{k+1\ \ell} - \left(e^{-X} - \frac{1 - e^{-X}}{X}\right)G^{n+1}_{k\ \ell}\right] ; \qquad (B.1)$$

Equations (A.4) are unchanged;

$$G^{n+1}_{k+1\ \ell} = e^{-Y'}G^{n-1}_{k+1\ \ell} + \frac{1 - e^{-Y'}}{\pi\sigma}\left\{\frac{G^{n}_{k+2\ \ell} - G^{n}_{k\ \ell}}{2(r_{k+2} - r_{k})}\right.$$

$$\left. - \frac{E^{n}_{k+\frac{3}{2}\ \ell+\frac{1}{2}} + E^{n}_{k+\frac{1}{2}\ \ell+\frac{1}{2}} - E^{n}_{k+\frac{3}{2}\ \ell-\frac{1}{2}} - E^{n}_{k+\frac{1}{2}\ \ell-\frac{1}{2}}}{4(\theta_{\ell+\frac{1}{2}} - \theta_{\ell-\frac{1}{2}})(1 + \Phi)} - \pi\sigma\,\frac{1 - \Phi}{1 + \Phi}\,F^{n}_{k+1\ \ell}\right\}, \qquad (B.2)$$

where

$$\sigma = \sigma^{n}_{k+1\ \ell}\,, \qquad \Phi = \Phi^{n}_{k+1\ \ell}\,, \qquad Y' = \pi\sigma(\tau^{n+1} - \tau^{n-1})(1 - \Phi)\,. \qquad (B.3)$$

*An examination of the FORTRAN listing of the present code reveals that the equations used in the code are not exactly as given here, but the differences are probably of no importance from the point of view of stability.

APPENDIX C. PROPOSED NEW DIFFERENCE EQUATIONS, IN THE AIR

$$E^{n+1}_{k\ \ell+\frac{1}{2}} = e^{-S}E^{n-1}_{k\ \ell+\frac{1}{2}} + \frac{1 - e^{-S}}{4\pi\sigma}\left\{-4\pi J\right.$$

$$\left. + \frac{\mu_{\ell+1}(F^{n+1}_{k\ \ell+1} + F^{n-1}_{k\ \ell+1} - 2G^{n}_{k\ \ell+1}) - \mu_{\ell}(F^{n+1}_{k\ \ell} + F^{n-1}_{k\ \ell} - 2G^{n}_{k\ \ell})}{4r^{2}_{k}\mu_{\ell+\frac{1}{2}}(\theta_{\ell+1} - \theta_{\ell})}\right\}, \qquad (C.1)$$

where

$$\sigma = \sigma^{n}_{k\ \ell+\frac{1}{2}}\,, \qquad S = 4\pi\sigma(\tau^{n+1} - \tau^{n-1})\,, \qquad J = J^{n}_{k\ \ell+\frac{1}{2}}\,, \qquad (C.2)$$

$$F^{n+1}_{k\,\ell} = e^{-X}F^{n+1}_{k-1\,\ell} + \frac{1 - e^{-X}}{2\pi\sigma(1 - \Phi)}\,\frac{E^{n+1}_{k\,\ell+\frac12} + E^{n+1}_{k-1\,\ell+\frac12} - E^{n+1}_{k\,\ell-\frac12} - E^{n+1}_{k-1\,\ell-\frac12}}{2(\theta_{\ell+\frac12} - \theta_{\ell-\frac12})}$$

$$- \frac{1 + \Phi}{1 - \Phi}\left[\left(1 - \frac{1 - e^{-X}}{X}\right)G^{n+1}_{k\,\ell} - \left(e^{-X} - \frac{1 - e^{-X}}{X}\right)G^{n+1}_{k-1\,\ell}\right], \qquad (C.3)$$

where

$$\sigma = \sigma^{n+1}_{k-\frac12\,\ell}, \qquad \Phi = \Phi^{n+1}_{k-\frac12\,\ell}, \qquad X = 2\pi\sigma(r_k - r_{k-1})(1 - \Phi), \qquad (C.4)$$

$$G^{n+1}_{k\,\ell} = e^{-Y'}G^{n-1}_{k\,\ell} + \frac{1 - e^{-Y'}}{\pi\sigma}\left\{\frac{G^n_{k+1\,\ell} - G^n_{k-1\,\ell}}{2(r_{k+1} - r_{k-1})} - \frac{E^n_{k\,\ell+\frac12} - E^n_{k\,\ell-\frac12}}{2(\theta_{\ell+\frac12} - \theta_{\ell-\frac12})(1 + \Phi)} - \pi\sigma\,\frac{1 - \Phi}{1 + \Phi}\,F^n_{k\,\ell}\right\}, \qquad (C.5)$$

where

$$\sigma = \sigma^n_{k\,\ell}, \qquad \Phi = \Phi^n_{k\,\ell}, \qquad Y' = \pi\sigma(\tau^{n+1} - \tau^{n-1})(1 + \Phi). \qquad (C.6)$$

APPENDIX D. NORMAL-MODE STABILITY ANALYSIS OF THE DIFFERENCE EQUATIONS OF APPENDIX A (ORIGINAL SYSTEM) (MOD 0 MANIAC CODE)

The secular equation for the amplification factor χ is

$$\begin{vmatrix} \chi^2 - e^{-S} & \frac{1 - e^{-S}}{4\pi\sigma r\Delta\theta}\left(\sin\frac{\varphi}{2}\right)\left(\cos\frac{\psi}{2}\right)\chi & -\frac{1 - e^{-S}}{4\pi\sigma r\Delta\theta}\left(\sin\frac{\varphi}{2}\right)\left(\cos\frac{\psi}{2}\right)\chi \\ -\frac{1 - e^{-X}}{\pi\sigma r\Delta\theta}\sin\frac{\varphi}{2} & e^{i\frac{\psi}{2}} - e^{-i\frac{\psi}{2} - X} & (1 - e^{-X})\cos\frac{\psi}{2} \\ \frac{1 - e^{-Y}}{2\pi\sigma r\Delta\theta}\left(\sin\frac{\varphi}{2}\right)\left(\cos\frac{\psi}{2} + \frac{\Delta\tau}{2\Delta r}\,i\sin\frac{\psi}{2}\right)(\chi + 1) & \frac{1 - e^{-Y}}{2}\left[\chi + 1 + \frac{\Delta\tau}{2\Delta r}(e^{i\psi} - 1)\right] & \chi - e^{-Y}\left[1 + \frac{\Delta\tau}{2\Delta r}(e^{i\psi} - 1)\right] \end{vmatrix} = 0 .$$

APPENDIX E. NORMAL-MODE STABILITY ANALYSIS OF THE DIFFERENCE EQUATIONS OF APPENDIX B (SYSTEM CURRENTLY IN USE) (MOD 3 MANIAC CODE)

$$\begin{vmatrix} \chi^2 - e^{-S} & \frac{1 - e^{-S}}{4Z}\left(\cos\frac{\psi}{2}\right)\left(\sin\frac{\varphi}{2}\right)\chi & -\frac{1 - e^{-S}}{4Z}\left(\cos\frac{\psi}{2}\right)\left(\sin\frac{\varphi}{2}\right)\chi \\ -\frac{1 - e^{-X}}{Z}\sin\frac{\varphi}{2} & e^{i\frac{\psi}{2}} - e^{-i\frac{\psi}{2} - X} & e^{i\frac{\psi}{2}} - e^{-i\frac{\psi}{2} - X} - 2i\,\frac{1 - e^{-X}}{X}\sin\frac{\psi}{2} \\ \frac{1 - e^{-2Y}}{Z}\left(\cos\frac{\psi}{2}\right)\left(\sin\frac{\varphi}{2}\right)\chi & (1 - e^{-2Y})\chi & \chi^2 - e^{-2Y} - i\,\frac{1 - e^{-2Y}}{X}(\sin\psi)\chi \end{vmatrix} = 0 ,$$

where $Z = \pi\sigma r\Delta\theta$.

APPENDIX F. NORMAL-MODE STABILITY ANALYSIS OF THE DIFFERENCE EQUATIONS OF APPENDIX C (PROPOSED NEW SYSTEM) (MOD 2 MANIAC CODE)

The secular equation for the amplification factor X is

$$
\begin{vmatrix}
X^2 - e^{-S} & \frac{1 - e^{-S}}{8Z}\left(\sin\frac{\varphi}{2}\right)(X^2 + 1) & -\frac{1 - e^{-S}}{4Z}\left(\sin\frac{\varphi}{2}\right)X \\
-\frac{1 - e^{-X}}{Z}\left(\cos\frac{\psi}{2}\right)\left(\sin\frac{\varphi}{2}\right) & e^{i\frac{\psi}{2}} - e^{-i\frac{\psi}{2} - X} & e^{i\frac{\psi}{2}} - e^{-i\frac{\psi}{2} - X} - 2i\,\frac{1 - e^{-X}}{X}\sin\frac{\psi}{2} \\
\frac{1 - e^{-2Y}}{Z}\left(\sin\frac{\varphi}{2}\right)X & (1 - e^{-2Y})X & X^2 - e^{-2Y} - i\,\frac{1 - e^{-2Y}}{X}(\sin\psi)X
\end{vmatrix} = 0 ,
$$

where $Z = \pi\sigma r\Delta\theta$.