Theoretical Notes
Note 171

AN ELECTROMAGNETIC BOUNDARY VALUE PROBLEM IN AN INHOMOGENEOUS MEDIUM

Thomas Franklin Ezell

ACKNOWLEDGMENTS

The author expresses his sincere appreciation for the interest and enthusiasm proffered throughout this study by Professor A. Erteza, the dissertation supervisor, who suggested the theory developed herein, and to Sandia Laboratories who supported this research.

Sincere appreciation is expressed to Professors W. J. Byatt and M. D. Bradshaw at the University of New Mexico, and to Dr. David E. Merewether of Mission Research Corporation for their advice, assistance, and encouragement.

A special note of appreciation is also extended to Dr. C. J. MacCallum, Dr. J. R. NiCastro, Dr. D. L. Mangan, and Dr. G. J. Scrivner of Sandia Laboratories for their continuing interest and assistance in this study. The author is also grateful to Mr. J. L. Rea of Sandia Laboratories, who helped in many ways with the computer programming.

A special note of gratitude is extended by the author to his wife, Colleen, and to their children, Teri and Kevin, and their foster daughter, Rita, for their patience and encouragement.

Finally, the patient and very excellent help of Mrs. Elaine Cockelreas, Mrs. Nena Brannan, and Mr. C. K. Lumpkin in preparation of the manuscript is deeply appreciated.

AN ELECTROMAGNETIC BOUNDARY VALUE PROBLEM IN AN INHOMOGENEOUS MEDIUM

BY
Thomas Franklin Ezell

ABSTRACT OF DISSERTATION

Submitted in Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy in Electrical Engineering
in the Graduate School of
The University of New Mexico
Albuquerque, New Mexico
May, 1972

AN ELECTROMAGNETIC BOUNDARY VALUE PROBLEM
IN AN INHOMOGENEOUS MEDIUM*

Thomas Franklin Ezell, Ph.D.
Department of Electrical Engineering
The University of New Mexico, 1972

ABSTRACT

This dissertation presents a general theory for the solution of electromagnetic boundary value problems for regions which are not homogeneous. The theory begins with the wave equation in Fourier frequency domain for the electric field in the interior of a closed volume; the electromagnetic property parameters, specifically conductivity and dielectric constant, are written as functions of position. The wave equation, which holds throughout the interior of the closed volume, is then converted to an integral equation by use of a Green's function for the same volume containing a homogeneous medium. Boundary conditions between homogeneous regions inside the closed volume appear as sources in the integral equation. The same theory applies to a closed region in which the parameters vary smoothly, rather than discontinuously, as a function of position.

The theoretical development is first presented, and the remainder of the paper illustrates the theory in the solution of a problem arising from the study of internal electromagnetic pulse phenomena. The problem consists of determining the electric field in the interior of a two-dimensional rectangular cavity excited by a source current density specified throughout the cavity. The walls of the cavity are assumed to be perfectly conducting. The cavity contains a single rectangular inhomogeneity, or object. The example problem is worked in rectangular coordinates for clarity of presentation. Although the object treated in the presentation is rectangular, any other object of regular shape could be treated just as well in this coordinate system. The choice of coordinate system is determined by the homogeneous cavity walls.

In rectangular coordinates, the integral equation for each component of the electric field reduces to an algebraic equation. In this paper, the algebraic equations are solved by an iterative process which requires that the parameter changes in the inhomogeneity be small. Results are presented for the cavity containing a conductive inhomogeneity and for the cavity containing a dielectric inhomogeneity with a higher dielectric constant than the rest of the cavity.

Further applications of the theory are suggested.

*This research supported by the Atomic Energy Commission.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

A_{mp}, B_{mp}	Constant values of integrals (43, 48)
a, b	Dimensions of rectangular cavity (Figure 1)
a_1, a_2, b_1, b_2	Dimensions of rectangular inhomogeneity (Figure 2)
$\overleftrightarrow{G}$	Dyadic Green's function (14)
G_x, G_y	Green's functions for x and y components of the electric field (18, 36)
J_{ij}	Fourier coefficient for J_x (62)
K_{im}, L_{im}, M_{im}, N_{im}	Constant values of integrals (71, 72, 77, 78)
K_x, K_y	Kernels of integral equations (42, 47)
u_m	Functional x dependence of E_x (40)
v_m	Functional y dependence of E_y (41)
W	Wronskian (30)
α_m, β_m, γ_m, δ_m	Separation constants (22, 24, 37)
δ	Dirac delta function (14)
ϵ	Permittivity, farads/meter (4)
η_{pm}, ξ_{pm}	Integrals describing spatial dependence of inhomogeneity (45, 49)
μ	Magnetic permeability, henrys/meter (5)
ρ	Charge density (6)
σ	Conductivity, either constant or function of position, mhos/meter (4)
φ_{pm}, ψ_{pm}	Fourier coefficients (73, 74)
ω	Radian frequency or Fourier transform variable (3)

Note: Numbers in parentheses are equation numbers.

CHAPTER I. INTRODUCTION

Solutions to electromagnetic boundary value problems[1-5] provide the values of electromagnetic fields and associated charge and current densities within a given region, finite or infinite. Maxwell's equations, which govern the fields, are usually applied first without reference to boundaries to determine the nature of the solution.[6] The solutions are obtained by separating the partial differential equations to give ordinary second order differential equations which can be solved. When boundaries are introduced, unique solutions can be picked out of the infinite number of solutions to the differential equations by selecting only those solutions which satisfy the boundary conditions. The dependence of the solutions on the electromagnetic sources and on the values of the fields at the boundaries is demonstrated by Helmholtz's theorem[7] in which the scalar potential and rectangular components of the vector potentials and the field intensities satisfy

$$\nabla\psi + k^2\psi = -g \tag{1}$$

where g is the specified source distribution and the equation is the inhomogeneous Helmholtz wave equation which applies throughout a homogeneous volume V bounded by the surface S. Then the solution for ψ can be written in general as

$$\psi(x', y', z') = \frac{1}{4\pi}\int_V \frac{ge^{jkR}}{R}\,dv + \frac{1}{4\pi}\int_S \left[\frac{\partial\psi}{\partial n}\frac{e^{jkR}}{R} - \psi\frac{\partial}{\partial n}\left(\frac{e^{jkR}}{R}\right)\right] dS \quad . \tag{2}$$

The first integral in Equation 2 accounts for the fields arising from the source distribution g, and the second integral accounts for the effects of the boundaries, i.e., reflections, etc. The terms inside the surface integral give the boundary conditions at the surfaces.

Examples of problems which can be treated by this approach are (1) scattering problems, in which a source a great distance away produces waves which are reflected by a conducting or dielectric object, or (2) cavity problems, in which a source produces waves inside a closed homogeneous cavity with conducting boundaries and the presence of the boundary greatly affects the fields. Problems involving inhomogeneous media cannot be adequately treated by use of the Helmholtz theorem, because Equation 1 applies only to a region in which the medium is homogeneous. The theory presented in this study is analogous to that of Helmholtz but includes the capability to treat inhomogeneous media.

This dissertation presents a general theory for the solution of electromagnetic boundary value problems for regions which are not homogeneous. The theory begins with the wave equation in the Fourier frequency domain for the electric field in the interior of a closed region; the electromagnetic property parameters, specifically conductivity and dielectric constant, are written as functions of position. The wave equation, which holds throughout the interior of the closed volume, is then converted to an integral equation by use of a Green's function for the same volume containing a homogeneous medium. Boundary conditions at boundaries between homogeneous regions inside the closed volume appear as sources in the integral equation. The same theory applies to a closed region in which the parameters vary smoothly, rather than discontinuously, as a function of position. The theory is illustrated in the solution of an internal electromagnetic pulse (EMP) problem.

When a guided missile is exposed inflight to a transient radiation pulse produced by the detonation of a nearby nuclear device, spatially distributed electron current densities are generated inside the missile as a result of the interaction of the radiation with materials in the missile. These current densities generate electromagnetic fields which then couple electrical energy into electronic circuits in the missile system. In Reference 8, an estimate of the generated current densities is presented, and estimates of the energy contained in the fields produced by these current densities in various shapes of cavities are presented. The problem posed by this phenomenon led to the analysis presented in this study, which was performed to devise an analytical method for investigating the effects of this electrical energy on the circuitry.

Electronic circuits in a missile system are generally compartmentalized into small regions bounded by conductive surfaces to provide electrical isolation. In the mathematical modeling used in this analysis, these regions are considered to be cavities and the circuit components inhomogeneities in the cavities. The work in Reference 9 modeled these regions as homogeneous cavities. This study investigates an electromagnetic boundary value problem for a cavity filled with an inhomogeneous medium. Electromagnetic fields in the cavity are excited by a source current density which is specified throughout the interior of the cavity.

In Chapter II, the theory which provides the basis for the technique described in this study is developed. In Chapter III, a specific problem is defined and the Green's function for that problem is derived. Chapter IV presents the mathematical solution of the problem. The results of this study are presented in Chapter V and the conclusions in Chapter VI.

Related Work

This type of problem is normally treated by solving a wave equation for each region in which the parameters are constant and the fields are matched at the boundaries where the parameters change values. This technique can be used when the boundaries correspond to constant values of coordinates in coordinate systems in which the wave equation is separable. However, if this is not the case, some other technique must be used. The wave equation (in the frequency domain) for the time-varying electromagnetic fields is the vector Helmholtz wave equation; there are only six coordinate systems in which this equation is separable.[10]

A technique which has been used for problems in which boundaries do not correspond to constant values in one of the six coordinate systems is that of numerical finite difference computations.[11]

Several examples of techniques, used in solving for both static and dynamic fields, are presented in Reference 12; in these techniques, integral equations are solved by numerical methods. These are primarily external problems, i.e., scattering problems; rather than interior problems such as those related to the cavities in this study.

Another technique is used to solve the problem of a waveguide having periodic discontinuities and of a resonant cavity containing a solid dielectric disc in the center of the cavity. This technique[13, 14] is similar to that described in this study in that the parameters are treated as functions of position; however, the solution is obtained by numerical solution of differential equations as compared to the solution of an integral equation used in this study.

CHAPTER II. DERIVATION OF INTEGRAL EQUATIONS

The analysis in this study applies to sinusoidally time-varying fields in that it is done in the Fourier frequency domain. From the results of the analysis, transient problems can be treated by use of the inverse Fourier transform. The form of the Fourier transform used in the analysis is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt \tag{3}$$

The Fourier transforms of Maxwell's equations are

$$\nabla \times \vec{H} = \vec{J} + (\sigma + j\omega\epsilon)\vec{E} \tag{4}$$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \tag{5}$$

$$\nabla \cdot \vec{D} = \rho \tag{6}$$

$$\nabla \cdot \vec{B} = 0 \tag{7}$$

The current density in Equation 4 is written in three parts: the source current density, $\vec{J}$, which is specified throughout the region of interest; the conduction or induced current density, $\sigma\vec{E}$; and the displacement current density, $j\omega\epsilon\vec{E}$.

The wave equation for the electric field is derived by taking the curl of Equation 5 and then substituting Equation 4 for $\nabla \times \vec{H}$. If the magnetic permeability, μ, is a function of position, it must be differentiated when the curl of the right-hand side of Equation 5 is taken. However, in many problems of practical interest, the magnetic permeability is constant. This restriction will apply here: conductivity σ and electric permittivity ϵ can vary with position, but the permeability must remain constant. With this restriction, the wave equation for the electric field is

$$\nabla \times \nabla \times \vec{E} - k_t^2(\vec{r})\vec{E} = -j\omega\mu\vec{J} \tag{8}$$

where

$$k_t^2(\vec{r}) = \omega^2\mu\epsilon(\vec{r}) - j\omega\mu\sigma(\vec{r}) \quad . \tag{9}$$

Application of the vector identity

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A} \tag{10}$$

to Equation 8 results in

$$\nabla^2\vec{E} + k_t^2(\vec{r})\vec{E} = j\omega\mu\vec{J} + \nabla(\nabla \cdot \vec{E}) \quad . \tag{11}$$

In order to obtain a solution to Equation 11, define

$$k_t^2(\vec{r}) = k^2 + k_1^2(\vec{r}) \tag{12}$$

where k^2 is constant and the variation of the material parameters is included in $k_1^2(\vec{r})$. In the example problem which is presented in this report, it is assumed that k_1^2 is of the same order of magnitude as k^2; however, some additional methods of solution will be described in which this assumption could possibly be removed. Equation 12 is next substituted into Equation 11 to give

$$\nabla^2\vec{E} + k^2\vec{E} = j\omega\mu\vec{J} + \nabla(\nabla \cdot \vec{E}) - k_1^2\vec{E} = \vec{P}(\vec{r}) \quad . \tag{13}$$

If the right-hand side of Equation 13 is considered simply as a source (even though it depends on the unknown $\vec{E}$), the left side is an operator for which a Green's function can be found. This Green's function, which must be a dyadic for a vector wave equation,[15] satisfies

$$\nabla^2\overset{\rightrightarrows}{G}(\vec{r},\vec{r}_o) + k^2\overset{\rightrightarrows}{G}(\vec{r},\vec{r}_o) = -\ \delta(\vec{r} - \vec{r}_o)\overset{\rightrightarrows}{I} \tag{14}$$

where δ is the three-dimensional Dirac delta function, and $\overset{\rightrightarrows}{I}$ is the identity tensor or dyadic. Because dot products involving dyadics are not commutative, the order of multiplication, left or right, is important. Subtracting the left dot product of $\vec{E}$ and the terms of Equation 14 from the right dot product of $\overset{\rightrightarrows}{G}$ and the terms of Equation 13, integrating the resulting equation over the volume of the cavity, and making use of the properties of the delta function, results in

$$\vec{E}(\vec{r}) = -\int_V \vec{P}(\vec{r}_o) \cdot \overset{\rightrightarrows}{G}(\vec{r},\vec{r}_o)\, d\vec{v}_o + \int_V \left[\nabla^2\vec{E}(\vec{r}_o) \cdot \overset{\rightrightarrows}{G}(\vec{r},\vec{r}_o) - \vec{E}(\vec{r}_o) \cdot \nabla^2\overset{\rightrightarrows}{G}(\vec{r},\vec{r}_o)\right] dv_o \ . \tag{15}$$

The second volume integral in Equation 15 can be changed to a surface integral by use of a vector Green's theorem:[16]

$$\int_V (\vec{E} \cdot \nabla^2\overset{\rightrightarrows}{G} - \nabla^2\vec{E} \cdot \overset{\rightrightarrows}{G})\, dv = \int_S \Big[(\vec{n} \cdot \vec{E})(\nabla \cdot \overset{\rightrightarrows}{G}) - (\nabla \cdot \vec{E})(n \cdot \overset{\rightrightarrows}{G})$$

$$+ (\vec{n} \times \vec{E}) \cdot (\nabla \times \overset{\rightrightarrows}{G}) - (\nabla \times \vec{E}) \cdot (\vec{n} \times \overset{\rightrightarrows}{G})\Big]\, dS \tag{16}$$

where the surface is that of the volume V, and $\bar{n}$ is the outward-pointing normal to the surface. When this substitution is made in Equation 15, the resulting equation

$$\vec{E}(\vec{r}) = -\int_V \vec{P}(\vec{r}_o) \cdot \vec{\vec{G}}(\vec{r}, \vec{r}_o)\, dv_o - \int_S \Big[(\vec{n} \cdot \vec{E})(\nabla \cdot \vec{\vec{G}}) - (\nabla \cdot \vec{E})(\vec{n} \cdot \vec{\vec{G}})$$

$$+ (\vec{n} \times \vec{E}) \cdot (\nabla \times \vec{\vec{G}}) - (\nabla \times \vec{E}) \cdot (\vec{n} \times \vec{\vec{G}}) \Big]\, dS \qquad (17)$$

expresses $\vec{E}$ as an integral of the source function $\vec{P}$ and the Green's function $\vec{\vec{G}}$ integrated throughout the volume of the sources and various products of the Green's function and the fields evaluated on the surface and integrated over the surface, i.e., boundary values. This is the normal application of the Green's function to the solution of a boundary value problem. However, it is somewhat different in that the source function $\vec{P}$ includes both the electric field $\vec{E}$ and derivatives of $\vec{E}$; therefore, Equation 15 is really an integro-differential equation for $\vec{E}$.

An interesting observation can now be made: the ruse of writing the parameters as a function of position, using the Green's function to derive an integral equation, etc., has converted the boundary conditions at any boundaries between different media (for example, conducting objects) inside the cavity into the integral on the right side which is a source integral. In other words, the boundary conditions can be thought of as sources for the fields.[17]

The Green's function $\vec{\vec{G}}$ is that for a homogeneous cavity; and, if the walls of the cavity correspond to one of the six coordinate systems in which the vector wave equation is separable, the determination of $\vec{\vec{G}}$ is a standard boundary value problem which can be solved by separation of variables.[18] The surface integral in Equation 17 determines the boundary conditions which $\vec{\vec{G}}$ must satisfy; these are chosen so that the evaluation of the surface integral is as simple as possible.

The remainder of this study is concerned with the solution of a specific problem in order to further explain and demonstrate the technique of solution.

CHAPTER III. DERIVATION OF A GREEN'S FUNCTION FOR A SPECIFIC PROBLEM

Definition of Problem

In this chapter, an application of the theory developed in Chapter II is presented. The development there applies to general three-dimensional coordinate systems; however, in order to demonstrate the theory, it is desirable to avoid as much mathematical complexity as possible. Therefore, the problem to be investigated is defined in rectangular coordinates to avoid the complexity of tensor mathematics and is restricted to two dimensions in order to reduce the number of manipulations. In summary then, the problem to be investigated is a rectangular "cavity" in two dimensions infinite in the z direction, and the source current density is defined to be in the x direction (Figure 1). Finally, a very important restriction is that the walls are assumed to be perfectly conducting. This problem simulates a wave guide containing an inhomogeneous dielectric with source current density passing through the guide perpendicular to its axis. The inhomogeneities which can be treated will be discussed in the next chapter.

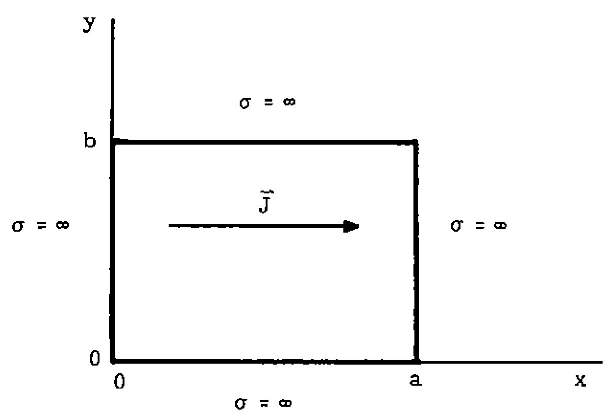

Figure 1. Geometric Definition of Sample Problem

Green's Function

The use of rectangular coordinates means that the wave equation (Equation 13) can be written in component form. Because the Green's function for each component of the electric field is a scalar, the use of tensor mathematics is not required. In that the derivation of the Green's function for the x component will be presented in some detail, the y component of the Green's function can be written by a simple interchange of variables. This derivation follows closely that given in Reference 19.

Because the cavity is infinite in the z direction and the source current density is defined to be independent of z, as well as confined to the x-y plane, all functions are independent of z, derivatives with respect to z are zero, and the z component of the electric field is zero. Thus when the gradient or curl operators are written, they are understood to include only derivatives with respect to x and y.

The wave equation which the x component of the Green's function must satisfy is

$$\nabla^2 G_x(\vec{r}, \vec{r}_o) + k^2 G_x(\vec{r}, \vec{r}_o) = -\delta(x - x_o)\delta(y - y_o) \quad . \tag{18}$$

The boundary conditions which G_x must satisfy are chosen so that the surface integral in Equation 17 in which G_x appears is simplified:

$$\int_S \left[G_x(\vec{r}, \vec{r}_o) \frac{\partial E_x(r_o)}{\partial n_o} - E_x(\vec{r}_o) \frac{\partial G_x(\vec{r}, \vec{r}_o)}{\partial n_o} \right] dS_o \quad . \tag{19}$$

Because the walls are assumed perfectly conducting and because the boundary condition is that the tangential component vanishes at a perfectly conducting surface, E_x is zero on the two walls described by $y = 0$ and $y = b$. When G_x is chosen to be zero on these two walls, the surface integral over these two walls vanishes. On the other two walls, E_x is a normal component, and the boundary condition on the normal component of the electric field at a perfectly conducting surface can be expressed in terms of the normal derivative of the electric field (this will be shown later). Thus, the normal derivative of E_x is known, and if the normal derivative of the Green's function is set equal to zero, the surface integral can be evaluated. In summary,

$$G_x = 0 \begin{cases} y = 0 \\ y = b \end{cases} \qquad \frac{\partial G_x}{\partial x} = 0 \begin{cases} x = 0 \\ x = a \end{cases} \quad . \tag{20}$$

Note that the manner in which these boundary conditions have been applied implies that the boundary condition is the same all over a given surface, e.g., the wall described by $x = 0$. This leads to the restriction that the inhomogeneities (which will be discussed in Chapter IV) cannot be located against the wall; otherwise, the boundary condition would not be constant for the whole surface. Therefore, the inhomogeneities can exist only in the interior of the cavity and not against the walls, i.e., in the region defined by

$$0 < x < a; \qquad 0 < y < b \quad .$$

A function which satisfies the boundary conditions at $y = 0$ and $y = b$ is

$$G_x = \sum_{m=1}^{\infty} h(x) \sin \alpha_m y \tag{21}$$

where

$$\alpha_m = \frac{m\pi}{b} \quad . \tag{22}$$

Substitution of Equation 21 into the wave equation for G_x (Equation 18) gives

$$\sum_{m=1}^{\infty} \left\{ \frac{d^2h}{dx^2} + (-\alpha_m^2 + k^2)\, h \right\} \sin \alpha_m y = -\delta(x - x_o)\,\delta(y - y_o) \quad . \tag{23}$$

The validity of interchange of the order of differentiation and summation will be discussed later.

Let

$$\beta_m^2 = k^2 - \alpha_m^2 \quad ; \tag{24}$$

multiply both sides of Equation 23 by $\sin \alpha_n y$, and integrate from $y = 0$ to $y = b$. In that the sine functions are orthogonal on this interval, the only term in the series which is non-zero is the term for which $n = m$. Again, the validity of the interchange of the order of integration and summation will be examined later. The result is

$$\frac{d^2h(x)}{dx^2} + \beta_m^2\, h(x) = -\frac{2}{b} \sin \alpha_m y_o\, \delta(x - x_o) \quad . \tag{25}$$

Equation 25 is an ordinary second-order differential equation, which can be solved according to the classical theory of ordinary differential equations.[20] The solution to the homogeneous equation (i.e., Equation 25 with the right-hand side equal to zero) is

$$h_1(x) = A_1 \cos \beta_m x + B_1 \sin \beta_m x \qquad x \le x_o \tag{26}$$

$$h_2(x) = A_2 \cos \beta_m x + B_2 \sin \beta_m x \qquad x_o \le x \quad . \tag{27}$$

The boundary conditions which h must satisfy (Equation 20) require that

$$h_1(x) = A_1 \cos \beta_m x \qquad x \le x_o \tag{28}$$

and

$$h_2(x) = \frac{B_2 \cos \beta_m (a - x)}{\sin \beta_m a} \qquad x_o \le x \quad . \tag{29}$$

The Wronskian W is given by

$$W(x) = h_1(x)\frac{dh_2}{dx} - h_2(x)\frac{dh_1}{dx} = A_1B_1\beta_m \quad , \tag{30}$$

and the solution to

$$\frac{d^2h}{dx^2} + \beta_m^2 h = g(x) \tag{31}$$

is given by

$$h(x) = \int \left[\frac{h_2(x)h_1(x') - h_1(x)h_2(x')}{W(x')}\right] g(x')\, dx' \quad . \tag{32}$$

So the solution to Equation 25 is

$$h(x) = \frac{-2 \sin \alpha_m y_o}{b\beta_m \sin \beta_m a} \times \left[\int_{x_1}^{x} \cos \beta_m(a - x) \cos \beta_m x' \delta(x' - x_o)\, dx' - \int_{x_2}^{x} \cos \beta_m x \cos \beta_m(a - x')\delta(x' - x_o)\, dx'\right] \tag{33}$$

If the limits x_1 and x_2 are chosen carefully, the solution can be made to fit the boundary conditions without the addition of solutions of the homogeneous equation. If $x_1 < x_o$ and $x_2 > x_o$ are chosen, the first integral is zero, unless $x > x_o$, and the second integral is zero unless $x < x_o$. In applying this choice, i.e., $x_1 = 0$ and $x_2 = a$, and by making use of the properties of the delta function, the result for h(x) is

$$h(x) = \frac{-2 \sin \alpha_m y_o}{\beta_m b \sin \beta_m b} \begin{cases} \cos \beta_m(a - x) \cos \beta_m x_o & x \ge x_o \\ \cos \beta_m x \cos \beta_m(a - x_o) & x \le x_o \end{cases} \quad . \tag{34}$$

This solution satisfies the boundary conditions and the inhomogeneous Equation 25. Therefore, the Green's function is given by substituting Equation 34 into Equation 21.

$$G_x(\vec{r}, \vec{r}_o) = \frac{-2}{b} \sum_{m=1}^{\infty} \frac{\sin \alpha_m y_o \sin \alpha_m y}{\beta_m \sin \beta_m b} \begin{cases} \cos \beta_m(a - x) \cos \beta_m x_o & x \ge x_o \\ \cos \beta_m x \cos \beta_m(a - x_o) & x \le x_o \end{cases} \quad . \tag{35}$$

The derivation of the y component of the Green's function, which follows the same pattern as that for the x component, can now be written by simply substituting x for y and y for x, with the proper modification of the constants:

$$G_y(\vec{r}, \vec{r}_o) = \frac{-2}{a} \sum_{m=1}^{\infty} \frac{\sin \gamma_m x_o \sin \gamma_m x}{\delta_m \sin \delta_m a} \begin{cases} \cos \delta_m (b - y) \cos \delta_m y_o & y_o \leq y \\ \cos \delta_m y \cos \delta_m (b - y_o) & y \leq y_o \end{cases} \tag{36}$$

with

$$\gamma_m = \frac{m\pi}{a} ; \qquad \delta_m^2 = k^2 - \gamma_m^2 \quad . \tag{37}$$

If $|k^2| < |\gamma_m^2|$, then δ_m is imaginary and the trigonometric functions become hyperbolic functions. The same thing happens in Equation 35.

This completes the derivation of the Green's function for the problem described in the early part of this chapter. The solution of the integral equation (Equation 17) for this problem will be presented in the next chapter.

CHAPTER IV. SOLUTION OF THE INTEGRAL EQUATION FOR A TWO DIMENSIONAL RECTANGULAR CAVITY

It was noted in Chapter II that in rectangular coordinates the vector wave equation could be separated into component scalar equations. Also, the integral equation for $\vec{E}$, Equation 17, can be written as component equations. The equation for E_x is

$$E_x(\vec{r}) = -\int_V \left[j\omega\mu J_x + \frac{\partial(\nabla \cdot \vec{E})}{\partial x_o} - k_1^2 E_x \right] G_x \, dv_o + \int_S \left[G_x \frac{\partial E_x}{\partial n_o} - E_x \frac{\partial G_x}{\partial n_o} \right] dS_o \quad . \tag{38}$$

The equation for E_y (J_y is set equal to zero) is

$$E_y(\vec{r}) = -\int_V \left[\frac{\partial(\nabla \cdot \vec{E})}{\partial y_o} - k_1^2 E_y \right] G_y \, dv_o + \int_S \left[G_y \frac{\partial E_y}{\partial n_o} - E_y \frac{\partial G_y}{\partial n_o} \right] dS_o \quad . \tag{39}$$

It should be observed here that there is no mathematical difficulty in allowing J_y to be non-zero; it is simply set to zero to simplify the presentation.

When the Green's functions derived in Chapter II are substituted into the above integral equations, and the order of summation is interchanged with integration and with differentiation (this operation again will be examined later), it can be seen that, because of the separability of the Green's functions, part of the functional dependence on $\vec{r}$ can be brought outside the integrals. For example, in Equation 38, the only part of the integrand which depends on y is the factor in G_x, $\sin \alpha_m y$. This factor is constant with respect to the integration which is over x_o and y_o and can therefore be brought out of the integral. The remaining integral is of course then constant with respect to y, and the form of E_x can be written

$$E_x(\vec{r}) = \sum_{m=1}^{\infty} u_m(x) \sin \alpha_m y \tag{40}$$

and E_y,

$$E_y(\vec{r}) = \sum_{m=1}^{\infty} v_m(y) \sin \gamma_m x \quad . \tag{41}$$

These expressions are then substituted into the integral equations (38 and 39) in order to determine the unknown functions u_m and v_m. In each equation, part of the integrations can now be performed; the resulting equation depends on one variable only, rather than two, as Equations 38 and 39 do. These equations, taken term by term, are

$$u_m(x) = -\sum_{p=1}^{\infty} \frac{\gamma_p A_{mp} \cos \gamma_p x}{\gamma_p^2 - \beta_m^2} + \left.\frac{du_m}{dx}\right|_{x=0} \cos \beta_m (a - x) - \left.\frac{du_m}{dx}\right|_{x=a} \cos \beta_m x$$

$$+ \int_0^a \left[\hat{J}_m(x_o) + \frac{d^2 u_m}{dx_o^2} - \sum_{p=1}^{\infty} u_p(x_o) \eta_{pm}(x_o) \right] K_x(x, x_o)\, dx_o \tag{42}$$

where

$$A_{mp} = \frac{2}{b} \int_0^b \frac{dv_p}{dy_o} \sin \alpha_m y_o \, dy_o \quad , \tag{43}$$

$$\hat{J}_m(x) = j\omega\mu \left(\frac{2}{b}\right) \int_0^b J_x(x, y_o) \sin \alpha_m y_o \, dy_o \quad , \tag{44}$$

$$\eta_{pm}(x) = \frac{2}{b} \int_0^b k_1^2(x, y_o) \sin \alpha_m y_o \sin \alpha_p y_o \, dy_o \quad , \tag{45}$$

and

$$K_x(x, x_o) = \frac{1}{\beta_m \sin \beta_m b} \begin{cases} \cos \beta_m (a - x) \cos \beta_m x_o & x \geq x_o \\ \cos \beta_m x \cos \beta_m (a - x_o) & x \leq x_o \end{cases} \quad . \tag{46}$$

$$v_m(y) = \sum_{p=1}^{\infty} \frac{\alpha_p B_{mp} \cos \alpha_p y}{\delta_m^2 - \alpha_p^2} + \int_0^b \left[\frac{d^2 v_m}{dy_o^2} - \sum_{p=1}^{\infty} v_p(y_o) \xi_{pm}(y_o) \right] K_y(y, y_o)\, dy_o \tag{47}$$

where

$$B_{mp} = \frac{2}{a} \int_0^a \frac{du_p}{dx_o} \sin \gamma_m x_o \, dx_o \quad , \tag{48}$$

$$\xi_{pm}(y) = \frac{2}{a} \int_0^a k_1^2(x_o, y) \sin \gamma_m x_o \sin \gamma_p x_o \, dx_o \quad , \tag{49}$$

and

$$K_y(y, y_o) = \frac{1}{\delta_m \sin \delta_m a} \begin{cases} \cos \delta_m (b - y) \cos \delta_m y_o & y_o \leq y \\ \cos \delta_m y \cos \delta_m (b - y_o) & y \leq y_o \end{cases} \quad . \tag{50}$$

The surface integral for E_y is zero because there is no y component of the current density. Note that Equations 42 and 47 are dependent because of the constants A_{mp} and B_{mp}.

Next, the two integral equations are differentiated by using the following formula:

if

$$h(x) = \int_o^a \varphi(x_o) K_x(x, x_o)\, dx_o \quad , \tag{51}$$

then

$$\frac{d^2h(x)}{dx^2} = -\beta_m^2 h(x) + \varphi(x) \quad . \tag{52}$$

The resulting equations are

$$u_m(x) = \frac{1}{\beta_m^2}\left[\hat{J}_m(x) + \sum_{p=1}^{\infty} \left(\gamma_p A_{mp} \cos \gamma_p x - u_p(x)\eta_{pm}(x)\right)\right] \tag{53}$$

and

$$v_m(y) = \frac{1}{\delta_m^2} \sum_{p=1}^{\infty} \left(\alpha_p B_{mp} \cos \alpha_p y - v_p(y)\xi_{pm}(y)\right) \quad . \tag{54}$$

Equations 43 and 48 can each be integrated by parts once to give

$$A_{mp} = -\frac{2\alpha_m}{b} \int_o^b v_p(y_o) \cos \alpha_m y_o\, dy_o \quad , \tag{55}$$

$$B_{mp} = -\frac{2\gamma_m}{a} \int_o^a u_p(x_o) \cos \gamma_m x_o\, dx_o \quad , \tag{56}$$

and then Equations 53 and 54 can be substituted into Equations 55 and 56 to give

$$A_{mp} = -\frac{\alpha_m^2 B_{pm}}{\delta_p^2} + \frac{2\alpha_m}{b\delta_p^2} \sum_{q=1}^{\infty} \int_o^b v_q(y_o)\xi_{qp}(y_o) \cos \alpha_m y_o\, dy_o \tag{57}$$

and

$$B_{mp} = -\frac{\gamma_m^2 A_{pm}}{\beta_p^2} - \frac{2\gamma_m}{a\beta_p^2} \int_o^a \left[\hat{J}_p(x_o) - \sum_{q=1}^{\infty} u_q(x_o)\eta_{qp}(x_o)\right] \cos \gamma_m x_o\, dx_o \quad . \tag{58}$$

The subscripts in Equation 57, i.e., m and p, can be interchanged to give A_{pm} which can then be substituted into Equation 58; B_{mp} can then be found in terms of integrals of u_m and v_m. The same thing can be done for A_{mp}. The results are

$$A_{mp} = \left(\frac{\alpha_m \beta_m^2}{\delta_p^2 \beta_m^2 - \alpha_m^2 \gamma_p^2}\right)\left[\frac{2\gamma_p \alpha_m}{a\beta_m^2} \int_o^a \left(\hat{J}_m(x_o) - \sum_{i=1}^{\infty} u_i(x_o)\eta_{im}(x_o)\right) \cos \gamma_p x_o \, dx_o\right.$$

$$\left. + \frac{2}{b} \sum_{i=1}^{\infty} \int_o^b v_i(y_o)\xi_{ip}(y_o) \cos \alpha_m y_o \, dy_o\right] \tag{59}$$

and

$$B_{mp} = \left(\frac{-\gamma_m \delta_m^2}{\beta_p^2 \delta_m^2 - \gamma_m^2 \alpha_p^2}\right)\left[\frac{2\alpha_p \gamma_m}{b\delta_m^2} \sum_{i=1}^{\infty} \int_o^b v_i(y_o)\xi_{im}(y_o) \cos \alpha_p y_o \, dy_o\right.$$

$$\left. + \frac{2}{a} \int_o^a \left(\hat{J}_p(x_o) - \sum_{i=1}^{\infty} u_i(x_o)\eta_{ip}(x_o)\right) \cos \gamma_m x_o \, dx_o\right] \tag{60}$$

Up to this point, the development has been entirely general for the problem being discussed; no restrictions have been placed on the source current density J_x or on the inhomogeneities (other than that they be in the interior and not touching the walls), described by $k_1^2(x, y)$ in η_{pm} and ξ_{pm}. Several computational methods could probably be used to solve Equations 53 and 54. For example, the functions u and v can be expanded in Fourier series with arbitrary coefficients. When these series (truncated) are substituted into the equations, a matrix equation for the Fourier coefficients can be derived. When this matrix is inverted, the solutions are obtained by summing the Fourier series.

The method which will be used in this study is an iterative perturbation scheme which, for the parameters within a certain range, converges to an exact solution. Because it is a perturbation scheme, the change of parameters in the inhomogeneities must be small, i.e.,

$$\frac{|k_1^2|}{|k^2|} < 1 \quad . \tag{61}$$

Next, forms for the source current density J_x and then for the description of the inhomogeneities k_1^2 are specified for the example problem. The current density is assumed to be expanded in a series of eigenfunctions for the cavity:

$$J_x(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} J_{ij} \cos \gamma_i x \sin \alpha_j y \quad . \tag{62}$$

Many functions can be approximated with this expansion by simply specifying values for J_{ij}; here J_{ij} is assumed to be specified.

In Chapter III, it was stated that the boundary condition for the normal component of the electric field could be stated in terms of its normal derivative. To show this, the continuity equation is needed. Because inhomogeneities cannot be against the walls and because the boundary conditions apply only near the walls, derivatives of the parameters of conductivity and dielectric constant can be ignored in deriving the boundary conditions. Therefore, to derive the continuity equation, the divergence of Equation 4 gives

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \left(\vec{J} + (\sigma + j\omega\epsilon)\vec{E}\right) = 0 \quad . \tag{63}$$

This expression is equal to zero because the divergence of the curl of any vector is zero, an identity from vector calculus. Therefore, the continuity equation near the walls is

$$\nabla \cdot \vec{E} = \frac{-\nabla \cdot \vec{J}}{\sigma + j\omega\epsilon} \quad . \tag{64}$$

The tangential component of $\vec{E}$ at a perfectly conducting wall is zero; and, because it is zero at every point on the wall, its derivative is zero too. Therefore, the only remaining non-zero term on the left side of Equation 63 is the derivative of the normal component; therefore, the boundary condition may be stated

$$\frac{\partial\vec{E}}{\partial n} = \frac{-\nabla \cdot \vec{J}}{\sigma + j\omega\epsilon} \quad . \tag{65}$$

The current density specified in Equation 62 leads to the boundary condition that the normal derivative is zero, because the divergence of J_x has a sine dependence in both x and y:

$$\sin \gamma_i x \sin \alpha_j y \Big|_{\substack{x=0 \text{ or } a \text{ or} \\ y=0 \text{ or } b}} = 0 \quad .$$

The form of the description of the parameters is

$$k_1^2(x, y) = \hat{k}_1^2 u_x(x) u_y(y) \tag{66}$$

where $\hat{k}_1^2$ is a constant, and $u_x(x)$ is a combination of unit step functions:

$$u_x(x) = u(x - a_1) - u(x - a_2) \quad . \tag{67}$$

Thus the inhomogeneity described is a rectangular object with its electrical parameters given by $\hat{k}_1^2$ (Figure 2).

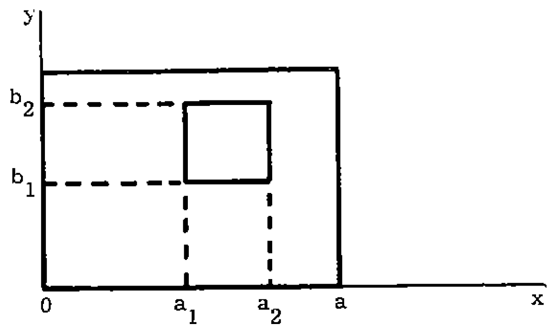

Figure 2. Definition of Inhomogeneity

It is not necessary to choose such a simple form for $k_1^2(x, y)$; the theory holds for much more general functions. To keep the mathematical manipulations as simple as possible, this form was chosen.

With these definitions of J_x and k_1^2, several integrals can be evaluated:

$$\hat{J}_m(x) = j\omega\mu \sum_{i=1}^{\infty} J_{im} \cos \gamma_i x \quad , \tag{68}$$

$$\eta_{pm}(x) = \hat{k}_1^2 K_{pm} u_x(x) \quad , \tag{69}$$

and

$$\xi_{pm}(x) = \hat{k}_1^2 L_{pm} u_y(y) \tag{70}$$

where

$$K_{pm} = \frac{2}{b} \int_{b_1}^{b_2} \sin \alpha_m y_o \sin \alpha_p y_o \, dy_o \tag{71}$$

and

$$L_{pm} = \frac{2}{a} \int_{a_1}^{a_2} \sin \gamma_m x_o \sin \gamma_p x_o \, dx_o \quad . \tag{72}$$

Note that K_{pm} and L_{pm} are symmetric, i.e., $K_{pm} = K_{mp}$.

The equations to be solved are Equations 53 and 54. By expanding u_m and v_m in Fourier series and substituting these expansions into Equations 53 and 54, two dependent equations for the Fourier coefficients can be obtained. Express u and v as

$$u_m(x) = \sum_{i=1}^{\infty} \varphi_{im} \cos \gamma_i x \tag{73}$$

$$v_m(y) = \sum_{i=1}^{\infty} \psi_{im} \cos \alpha_i y \quad . \tag{74}$$

It is to be noted that A_{mp} and B_{mp} are proportional to the Fourier coefficients for u and v because of the way they were defined (Equations 55 and 56). Therefore,

$$B_{mp} = -\gamma_m \varphi_{mp} \tag{75}$$

$$A_{mp} = -\alpha_m \psi_{mp} \quad , \tag{76}$$

and when the expansions for u_m and v_m (Equations 73 and 74) are substituted into Equations 59 and 60), coupled equations which involve only φ and ψ and known constants are obtained. More integrals similar to K_{pm} and L_{pm} (Equations 71 and 72) can be defined:

$$M_{pm} = \frac{2}{a}\int_{a_1}^{a_2} \cos \gamma_p x_o \cos \gamma_m x_o \, dx_o \tag{77}$$

and

$$N_{pm} = \frac{2}{b}\int_{b_1}^{b_2} \cos \alpha_m y_o \cos \alpha_p y_o \, dy_o \quad . \tag{78}$$

The two coupled equations for φ and ψ which result are

$$\varphi_{pm} = \frac{j\omega\mu J_{pm} \delta_p^2}{\delta_p^2 \beta_m^2 - \gamma_p^2 \alpha_m^2} - \hat{k}_1^2 \frac{\delta_p^2}{\delta_p^2 \beta_m^2 - \gamma_p^2 \alpha_m^2} \sum_{q=1}^{\infty} \sum_{i=1}^{\infty} \left(K_{qm} M_{ip} \varphi_{iq} - \frac{\gamma_p \alpha_m}{\delta_p^2} L_{qp} N_{im} \psi_{iq} \right) \tag{79}$$

and

$$\psi_{pm} = \frac{-j\omega\mu J_{mp} \alpha_p \gamma_m}{\beta_p^2 \delta_m^2 - \alpha_p^2 \gamma_m^2} + \hat{k}_1^2 \frac{\beta_p^2}{\beta_p^2 \delta_m^2 - \alpha_p^2 \gamma_m^2} \sum_{q=1}^{\infty} \sum_{i=1}^{\infty} \left(\frac{\gamma_m \alpha_p}{\beta_p^2} K_{qp} M_{im} \varphi_{iq} - L_{qm} N_{ip} \psi_{iq} \right) . \tag{80}$$

To this point, there has been no reason for the restriction given in Equation 61, but it will be given now in order to solve Equations 79 and 80. These make up a matrix equation which could possibly be solved by matrix inversion for all values of $\hat{k}_1^2$; however, now $\hat{k}_1^2$ will be assumed to be small (Equation 61) in order to solve Equations 79 and 80 by an iteration method. Once φ and ψ are determined, the fields can be found by simply evaluating Equations 40 and 41.

Method of Iteration

Because $\hat{k}_1^2$ is assumed small, the predominant terms for φ and ψ in Equations 79 and 80 are the first terms. This suggests substituting the first term into the summations, then adding the correction term inside the summations to get a further correction. This can be continued indefinitely to give more accurate values. The procedure will be carried out in detail for φ_{pm}, and the question as to whether the iteration converges and under what conditions will be answered.

Let

$$\varphi^o_{pm} = \frac{j\omega\mu J_{pm}\delta_p^2}{\delta_p^2\beta_m^2 - \gamma_p^2\alpha_m^2} , \tag{81}$$

$$\psi^o_{pm} = \frac{-j\omega\mu J_{mp}\alpha_p\gamma_m}{\delta_m^2\beta_p^2 - \gamma_m^2\alpha_p^2} , \tag{82}$$

and

$$\Gamma_{pm} = \frac{1}{\delta_p^2\beta_m^2 - \gamma_p^2\alpha_m^2} . \tag{83}$$

Then Equations 79 and 80 become

$$\varphi_{pm} = \varphi^o_{pm} - \hat{k}_1^2\delta_p^2\Gamma_{pm}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(K_{jm}M_{ip}\varphi_{ij} - \frac{\gamma_p\alpha_m}{\delta_p^2} L_{jp}N_{im}\psi_{ij}\right) \tag{84}$$

and

$$\psi_{pm} = \psi^{o}_{pm} + \hat{k}_1^2 \beta_p^2 \Gamma_{mp} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{\gamma_m \alpha_p}{\beta_p^2} K_{jp} M_{im} \varphi_{ij} - L_{jm} N_{ip} \psi_{ij} \right) \quad . \tag{85}$$

Now the iteration process is begun by taking Equations 84 and 85 with φ^{o} and ψ^{o} in the place of φ and ψ on the right side as approximations for φ_{pm} and ψ_{pm} and substituting them into Equations 84 and 85 again and again to obtain an infinite series of φ^{o}_{ij} and ψ^{o}_{ij}:

$$\begin{aligned}
\varphi_{pm} = \varphi^{o}_{pm} - k_1^2 \delta_p^2 \Gamma_{pm} \sum_i \sum_j \Bigg\{ K_{jm} M_{ip} \Bigg[\varphi^{o}_{ij} \\
- \hat{k}_1^2 \delta_i^2 \Gamma_{ij} \sum_q \sum_r \left(K_{rj} M_{qi} \varphi_{qr} - \frac{\gamma_i \alpha_j}{\delta_i^2} L_{ri} N_{qj} \psi_{qr} \right) \Bigg] \\
- \frac{\gamma_p \alpha_m}{\delta_p^2} L_{jp} N_{im} \Bigg[\psi^{o}_{ij} + \hat{k}_1^2 \beta_i^2 \Gamma_{ji} \sum_q \sum_r \left(\frac{\gamma_j \alpha_i}{\beta_i^2} K_{ri} M_{qj} \varphi_{qr} - L_{rj} N_{qi} \psi_{qr} \right) \Bigg] \Bigg\} \\
= \varphi^{o}_{pm} - \hat{k}_1^2 \delta_p^2 \Gamma_{pm} \sum_i \sum_j \left(K_{jm} M_{ip} \varphi^{o}_{ij} - \frac{\gamma_p \alpha_m}{\delta_p^2} L_{jp} N_{im} \psi^{o}_{ij} \right) \\
+ \hat{k}_1^4 \delta_p^2 \Gamma_{pm} \sum_i \sum_j \Bigg[\delta_i^2 \Gamma_{ij} K_{jm} M_{ip} \sum_q \sum_r \left(K_{rj} M_{qi} \varphi_{qr} - \frac{\gamma_i \alpha_j}{\delta_i^2} L_{ri} N_{qj} \psi_{qr} \right) \\
+ \frac{\gamma_p \alpha_m}{\delta_p^2} \delta_i^2 \Gamma_{ji} L_{jp} N_{im} \sum_q \sum_r \left(\frac{\gamma_j \alpha_i}{\beta_i^2} K_{ri} M_{qj} \varphi_{qr} - L_{rj} N_{qi} \psi_{qr} \right) \Bigg] \quad , \text{ etc.}
\end{aligned} \tag{86}$$

$$\varphi_{pm} = \varphi^{o}_{pm} - \hat{k}_1^2 \delta_p^2 \Gamma_{pm} \sum_i \sum_j \left(K_{jm} M_{ip} \varphi^{o}_{ij} - \frac{\gamma_p \alpha_m}{\delta_p^2} L_{jp} N_{jm} \psi^{o}_{ij} \right)$$

$$+ \hat{k}_1^4 \delta_p^2 \Gamma_{pm} \sum_i \sum_j \left[\delta_1^2 \Gamma_{ij} K_{jm} M_{ip} \sum_q \sum_r \left(K_{rj} M_{qi} \varphi_{qr} - \frac{\gamma_i \alpha_j}{\delta_i^2} L_{ri} N_{qj} \psi^{o}_{qr} \right) \right.$$

$$\left. - \frac{\gamma_p \alpha_m}{\delta_p^2} \beta_i^2 \Gamma_{ji} L_{jp} N_{im} \sum_q \sum_r \left(\frac{\gamma_j \alpha_i}{\beta_i^2} K_{ri} M_{qj} \varphi^{o}_{qr} - L_{rj} N_{qi} \psi^{o}_{qr} \right) \right]$$

$$+ \hat{k}_1^6 \delta_p^2 \Gamma_{pm} \sum_i \sum_j \left\{ \delta_i^2 \Gamma_{ij} K_{jm} M_{ip} \sum_q \sum_r \left[\delta_q^2 \Gamma_{qr} K_{rj} M_{qi} \sum_s \sum_t K_{tr} M_{sq} \varphi^{o}_{st} \right. \right.$$

$$- \frac{\gamma_i \alpha_j}{\delta_i^2} \gamma_r \alpha_q \Gamma_{rq} L_{ri} N_{qj} \sum_s \sum_t K_{tq} M_{sr} \varphi^{o}_{st}$$

$$\left. + \gamma_q \alpha_r \Gamma_{qr} K_{rj} M_{qi} \sum_s \sum_t L_{tq} N_{sr} \psi^{o}_{st} - \frac{\gamma_i \alpha_j}{\delta_i^2} \beta_q^2 \Gamma_{rq} L_{ri} N_{qj} \sum_s \sum_t L_{tr} N_{sq} \psi^{o}_{st} \right]$$

$$+ \beta_i^2 \Gamma_{ji} L_{jm} N_{ip} \sum_q \sum_r \left[\frac{\gamma_j \alpha_i}{\beta_i^2} \delta_q^2 \Gamma_{qr} K_{ri} M_{qj} \sum_s \sum_t K_{tr} M_{sq} \varphi^{o}_{st} \right.$$

$$- \gamma_r \alpha_q \Gamma_{rq} L_{rj} N_{qi} \sum_s \sum_t K_{tq} M_{sr} \varphi^{o}_{st}$$

$$- \frac{\gamma_j \alpha_i}{\beta_i^2} \gamma_q \alpha_r \Gamma_{qr} K_{ri} M_{qj} \sum_s \sum_t L_{tq} N_{sr} \psi^{o}_{st}$$

$$\left. \left. + \beta_q^2 \Gamma_{rq} L_{rj} N_{qi} \sum_s \sum_t L_{tr} N_{sq} \psi^{o}_{st} \right] \right\} + \ldots \qquad (87)$$

In the same way, an expression for ψ_{pm} can be derived:

$$\psi_{pm} = \psi^{o}_{pm} + \hat{k}_1^2\beta_p^2\Gamma_{mp}\sum_i\sum_j\left(\frac{\gamma_m\alpha_p}{\beta_p^2}K_{jp}M_{im}\varphi^{o}_{ij} - L_{jm}N_{ip}\psi^{o}_{ij}\right)$$

$$- \hat{k}_1^4\beta_p^2\Gamma_{mp}\sum_i\sum_j\left[\frac{\gamma_m\alpha_p}{\beta_p^2}\delta_i^2\Gamma_{ij}K_{jp}M_{im}\sum_q\sum_r\left(K_{rj}M_{qi}\varphi^{o}_{qr} - \frac{\gamma_i\alpha_j}{\delta_i^2}L_{ri}N_{qi}\psi^{o}_{qr}\right)\right.$$

$$\left. + \beta_i^2\Gamma_{ji}L_{jm}N_{ip}\sum_q\sum_r\left(\frac{\gamma_j\alpha_i}{\beta_i^2}K_{ri}M_{qj}\varphi^{o}_{qr} - L_{rj}N_{qi}\psi^{o}_{qr}\right)\right]$$

$$- \hat{k}_1^6\beta_p^2\Gamma_{mp}\sum_i\sum_j\left\{\frac{\gamma_m\alpha_p}{\beta_p^2}\delta_i^2\Gamma_{ij}K_{jp}M_{im}\sum_q\sum_r\left[\delta_q^2\Gamma_{qr}K_{rj}M_{qi}\right.\right.$$

$$\times\sum_s\sum_t\left(K_{tr}M_{sq}\varphi^{o}_{st} - \frac{\gamma_q\alpha_r}{\delta_q^2}L_{tq}N_{sr}\psi^{o}_{st}\right)$$

$$\left. + \frac{\gamma_m\alpha_p}{\beta_p^2}\frac{\gamma_i\alpha_j}{\delta_i^2}\beta_q^2\Gamma_{rq}L_{ri}N_{qj}\sum_s\sum_t\left(\frac{\gamma_r\alpha_q}{\beta_q^2}K_{tq}M_{sr}\varphi^{o}_{st} - L_{tr}N_{sq}\psi^{o}_{st}\right)\right]$$

$$+ \beta_i^2\Gamma_{ji}L_{jm}N_{ip}\sum_q\sum_r\left[\frac{\gamma_j\alpha_i}{\beta_i^2}\delta_q^2\Gamma_{qr}K_{ri}M_{qj}\sum_s\sum_t\left(K_{tr}M_{sq}\varphi^{o}_{st} - \frac{\gamma_q\alpha_r}{\delta_q^2}L_{tq}N_{sr}\psi^{o}_{st}\right)\right.$$

$$\left.\left. + \delta_q^2\Gamma_{qr}L_{rj}N_{qi}\sum_s\sum_t\left(\frac{\gamma_r\alpha_q}{\beta_q^2}K_{tq}M_{sr}\varphi^{o}_{st} - L_{tr}N_{sq}\psi^{o}_{st}\right)\right]\right\} + \dots \tag{88}$$

In order to show that the series converges, an upper bound on K_{mn} is obtained:

$$K_{mn} = \frac{2}{b}\int_{b_1}^{b_2} \sin \alpha_m y \sin \alpha_n y \, dy$$

$$= \frac{2}{\pi}\int_{\pi b_1/b}^{\pi b_2/b} \sin mx \sin nx \, dx$$

$$\leq \frac{2}{\pi}\int_{\pi b_1/b}^{\pi b_2/b} dx = \frac{2(b_2 - b_1)}{b}$$

$$\left|K_{mn}\right| \leq \frac{2(b_2 - b_1)}{b} \quad . \tag{89}$$

The same thing can be done for L_{mn}, M_{mn}, and N_{mn} to give

$$\left|L_{mn}\right| \leq \frac{2(a_2 - a_1)}{a} \quad ; \quad \left|M_{mn}\right| \leq \frac{2(a_2 - a_1)}{a} \quad ; \quad \left|N_{mn}\right| \leq \frac{2(b_2 - b_1)}{b} \quad . \tag{90}$$

To investigate convergence of the series in Equations 87 and 88, the upper bounds for K, L, M, and N are factored out of each term. The last factor in each term then is either $\sum_p \sum_m \varphi^o_{pm}$ or $\sum_p \sum_m \psi^o_{pm}$. The constants φ^o and ψ^o are defined in Equations 81 and 82. There it can be seen that convergence of the series above depends on the definition of J_{pm} (Equation 62). In order for the series for J_x to converge, J_{mp} must be proportional to $1/pm$ (or higher powers of p and m) for large values of m and p. Therefore, φ^o_{mp} is proportional to $p/(m(p^2 + m^2))$, and ψ^o_{pm} is proportional to $1/(p^2 + m^2)$ for large values of p and m. The series involving ψ^o converges absolutely.[22] However, the situation for φ^o is different and must be examined in more detail. For large values of $p > m$, a better upper bound on K_{pm} (and L, M, and N) is obtained. The integral (Equation 71) is easily evaluated:

$$K_{pm} = \frac{1}{b}\left(\frac{\sin (\alpha_m - \alpha_p) b_2 - \sin (\alpha_m - \alpha_p) b_1}{\alpha_m - \alpha_p} - \frac{\sin (\alpha_m + \alpha_p) b_2 - \sin (\alpha_m + \alpha_p) b_1}{\alpha_m + \alpha_p}\right) \quad . \tag{91}$$

Then, the trigonometric functions are expanded, a common denominator is found, and corresponding terms are added. Because p is much greater than m, the terms which are multiplied by α_m are neglected:

$$K_{pm} \approx \frac{2\alpha_p}{b}\left(\frac{\sin \alpha_m b_2 \cos \alpha_p b_2 - \sin \alpha_m b_1 \cos \alpha_p b_1}{\alpha_m^2 - \alpha_p^2}\right) \quad . \tag{92}$$

Again, by neglecting α_m in the denominator, the result for large values of $p >> m$ is

$$\left|K_{pm}\right| < \frac{4}{b\alpha_p} = \frac{4}{p\pi} \quad . \tag{93}$$

A similar result is obtained for L, M, and N. With this upper bound on K, φ^o_{mp} is then proportional to $1/(m(p^2 + m^2))$ for large values of p, and the series involving φ^o also converges absolutely.

The remaining factors in each term in the series are of the form $\sum_q \sum_r \Gamma_{qr}$, and, because Γ_{qr} is inversely proportional to q^2 and to r^2, this series converges absolutely. Thus, the general nth term of the series in Equations 87 and 88 is of the form

$$\left[k_1^2\left(\frac{a_2 - a_1}{a}\right) \sum_q \sum_r \Gamma_{qr}\right]^n \sum_i \sum_j \varphi^o_{ij} \quad . \tag{94}$$

A series whose terms are of this form, i.e., $\sum_n d^n$, is a geometric series which converges absolutely if $|d| < 1$.

The preceding argument is not a mathematically rigorous proof that the series converge; however, it indicates that the series should converge more rapidly for smaller values of $\hat{k}_1^2$ and also for smaller dimensions of the inhomogeneity.

In earlier chapters, the operations of interchanging the order of summation, integration, and differentiation were performed on series which are now seen to be Fourier series. A Fourier series for f(x) converges uniformly where f(x) is continuous,[23] and a series which converges uniformly can be integrated term by term; and, if the series for the derivative converges, it converges uniformly and can be differentiated term by term.[24]

Now that φ and ψ are determined, the solution is complete. The final form of the electric field components is

$$E_x(x, y) = \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \varphi_{pm} \cos \gamma_p x \sin \alpha_m y \quad , \tag{95}$$

and

$$E_y(x, y) = \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \psi_{pm} \sin \gamma_m x \cos \alpha_p y \quad . \tag{96}$$

Chapter V presents the results of a computer calculation for this problem.

CHAPTER V. CALCULATIONAL RESULTS

Calculations were performed with a computer program called FIELD to evaluate the equations for a particular problem described in Chapters III and IV. The results obtained by these calculations are discussed in this chapter.

The program, written in Fortran, has been adapted to run on both the Sandia Laboratories CDC 6600 and DEC PDP-10 computers. The program listing for the PDP-10 is presented in the Appendix. The method of solution was to substitute the values for φ_{pm}^{o} and ψ_{pm}^{o}, given in Equations 81 and 82, into Equations 84 and 85 to give new values for φ_{pm} and ψ_{pm}, which were again substituted into Equations 84 and 85 to again give new values for φ_{pm} and ψ_{pm}. This iterative process was continued until the new values for φ_{pm} and ψ_{pm} differed from the last values by less than a specified percentage.

In the example given in this chapter, the size and shape of the cavity and inhomogeneity are shown in Figure 3. The dielectric was assumed to be vacuum and the inhomogeneity to have a conductivity of 5×10^{-3} mhos/meter and dielectric constant the same as the rest of the cavity. The frequency at which the calculations were made was 1×10^{8} Hz, well below the lowest cutoff frequency for the cavity. The value of $\left|\hat{k}_1^2\right| / \left|k^2\right|$ for the problem presented here was 0.9.

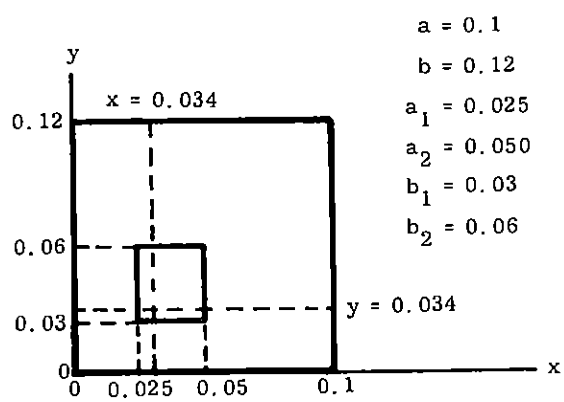

Figure 3. Cavity and Inhomogeneity Dimensions (in meters)

The source for the problem was a constant current source, described as a function of position by $J_x(x, y)$ (Equation 62). The program was set up to include all terms of the series as given in Equation 62; however, only one term of the series was used in the calculations: the term for $i = 1$, $j = 2$. The physical situation can be described as analogous to an electrical circuit in which the energy source is a constant current source applied across a long resistor which represents the impedance of the cavity and the conductivity of the inhomogeneity in the cavity is a shunt resistor across part of the long resistor (Figure 4). The voltage per length (electric field) along the resistor is unchanged except along the length where the shunt resistor is added; here it will decrease according to the ratio of resistances.

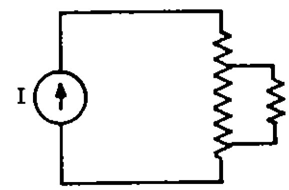

Figure 4. Circuit Analog

Behavior analogous to that of the circuit analog can be observed in Figures 5, 6, 7, and 8, where the magnitudes of the x and y components of the electric field were plotted as functions of x and y for homogeneous and inhomogeneous cavities. The value of y was chosen so that the plots were through the inhomogeneity (Figure 3). The magnitudes of the field components, which were about the same as those for the homogeneous cavity outside of the inhomogeneity, differed considerably inside the object. Thus, the fields were essentially unchanged away from the inhomogeneity and decreased inside the inhomogeneity, as expected.

The number of terms which were taken in the Fourier series was 400, 20 in the sum over x and 20 in the sum over y. The approximate computer time for 20 iterations on φ and ψ was one hour on the PDP-10. The 6600 run time was approximately 1/15th of that for the PDP-10, i.e., about four minutes for evaluation of φ and ψ. The summing of the series to evaluate the fields as a function of position required about two minutes for 50 field points on the PDP-10.

In Figures 9 through 12, plots similar to those in Figure 5 through 8 are presented. In Figures 9 through 12, the inhomogeneity is a dielectric rather than a conductor; the cavity dimensions, frequency, etc. are the same. The value of $|\hat{k}_1^2|/|k^2|$ for this problem was 0.7; this corresponds to a relative dielectric constant for the inhomogeneity of 1.7. Although the effects of the dielectric object were not so pronounced as those for the conductive object, they were similar.

The curves in Figures 5 through 12 present the magnitudes of the components of the electric fields. The phase information is of course included in the complex values of the field components and is necessary for the inverse Fourier transform required to give functions of time. However, the magnitude plots show the effects of the inhomogeneity simply and clearly.

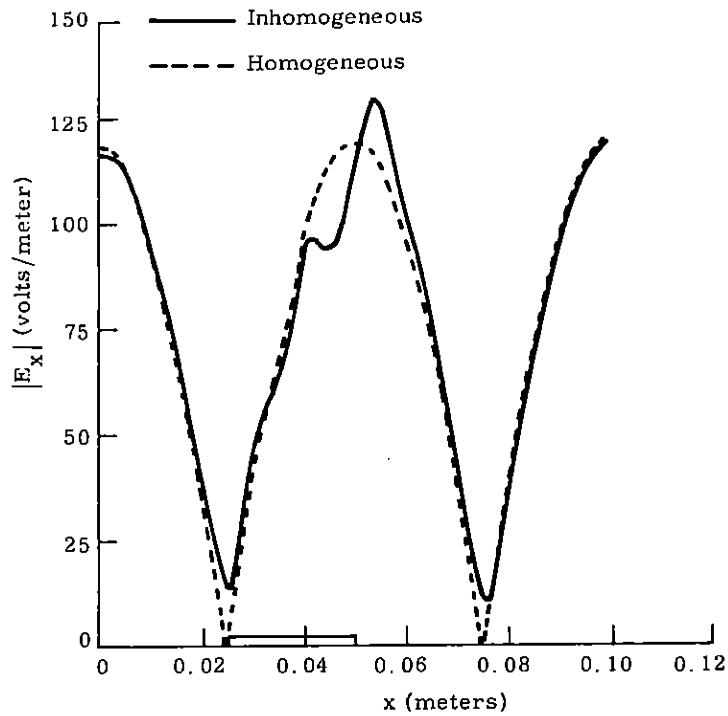

Figure 5. The Magnitude of $E_x(x, y_o)$, $y_o = 0.034$, for a Conductive Inhomogeneity ($\sigma_1 = 5 \times 10^{-3}$ mhos/meter) Extending from x = 0.025 to x = 0.050

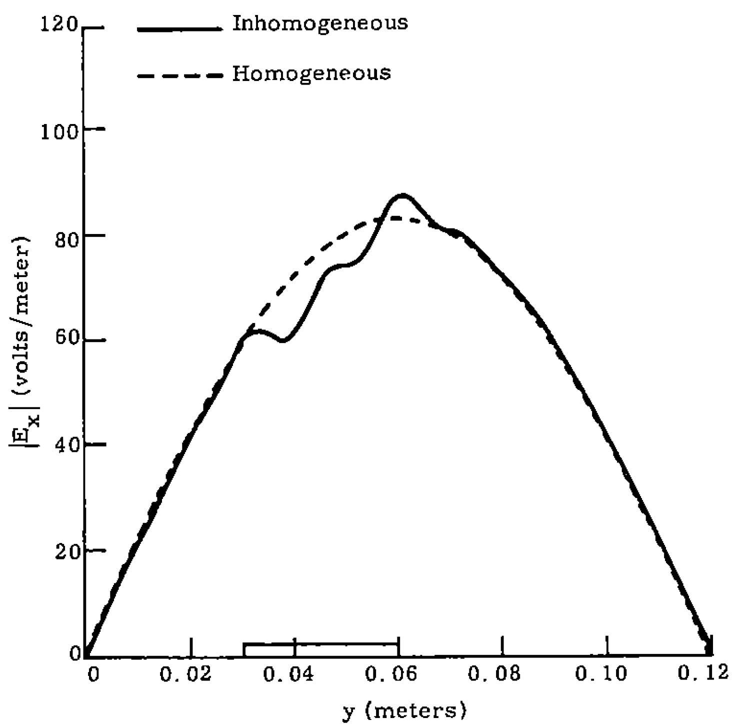

Figure 6. The Magnitude of $E_x(x_o, y)$, $x_o = 0.034$, for a Conductive Inhomogeneity ($\sigma_1 = 5 \times 10^{-3}$ mhos/meter) Extending from y = 0.03 to y = 0.06

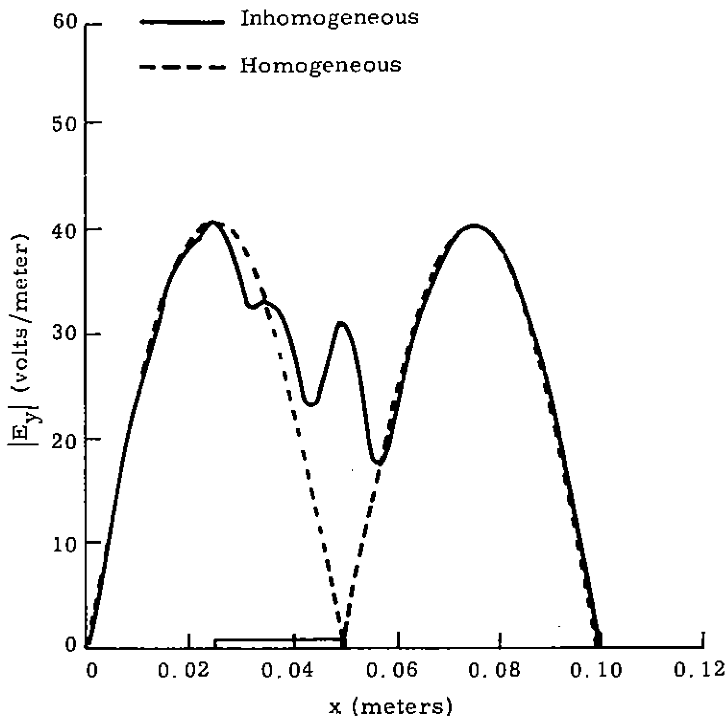

Figure 7. The Magnitude of $E_y(x, y_o)$, $y_o = 0.034$, for a Conductive Inhomogeneity ($\sigma_1 = 5 \times 10^{-3}$ mhos/meter) Extending from x = 0.025 to x = 0.050

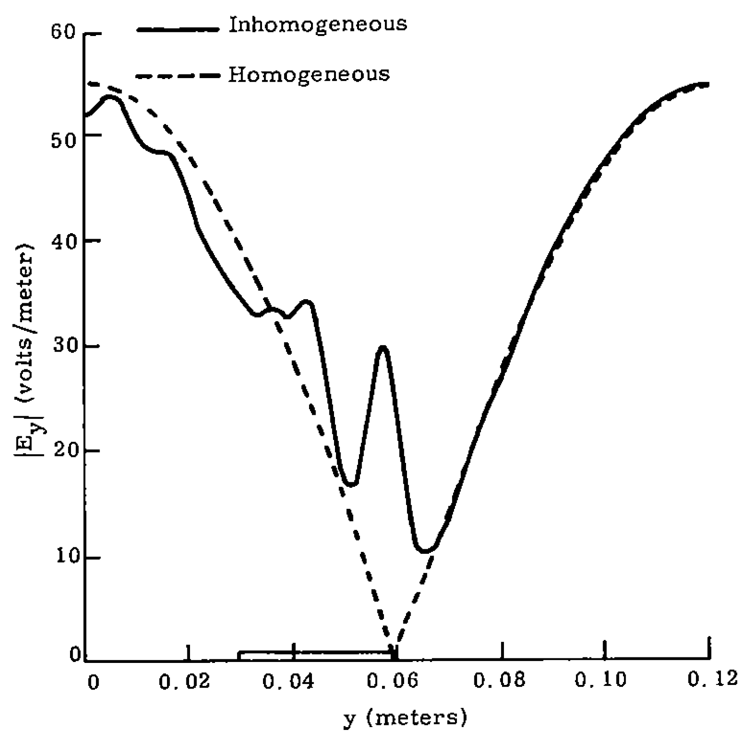

Figure 8. The Magnitude of $E_y(x_o, y)$, $x_o = 0.034$, for a Conductive Inhomogeneity ($\sigma_1 = 5 \times 10^{-3}$ mhos/meter) Extending from y = 0.03 to y = 0.06

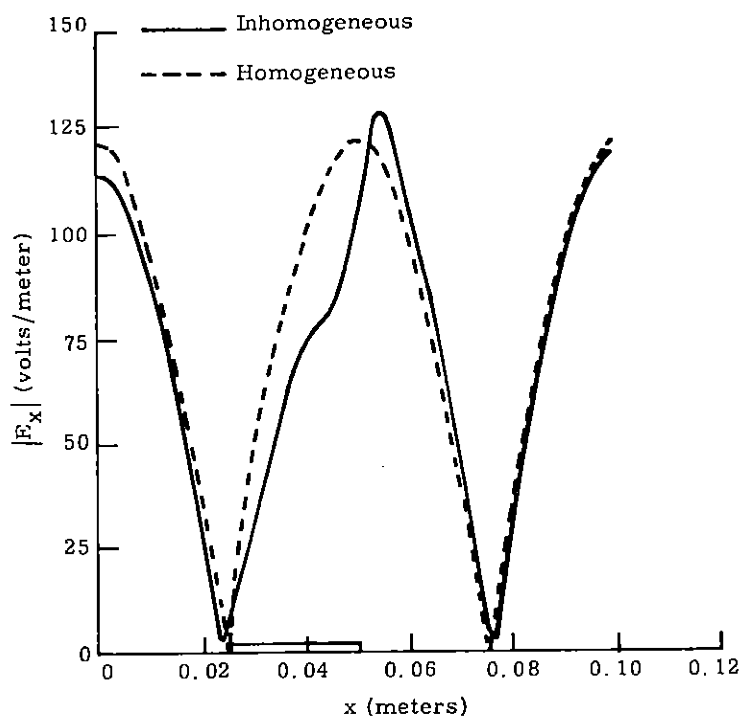

Figure 9. The Magnitude of $E_x(x, y_o)$, $y_o = 0.034$, for a Dielectric Inhomogeneity ($\epsilon_1 = 1.7\ \epsilon_o$) Extending from $x = 0.025$ to $x = 0.050$

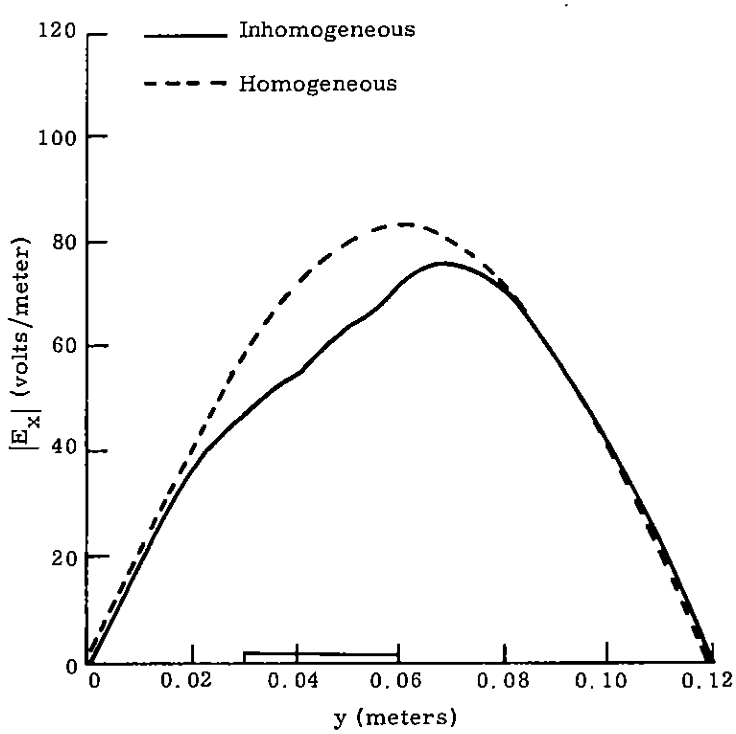

Figure 10. The Magnitude of $E_x(x_o, y)$, $x_o = 0.034$, for a Dielectric Inhomogeneity ($\epsilon_1 = 1.7\ \epsilon_o$) Extending from $y = 0.03$ to $y = 0.06$

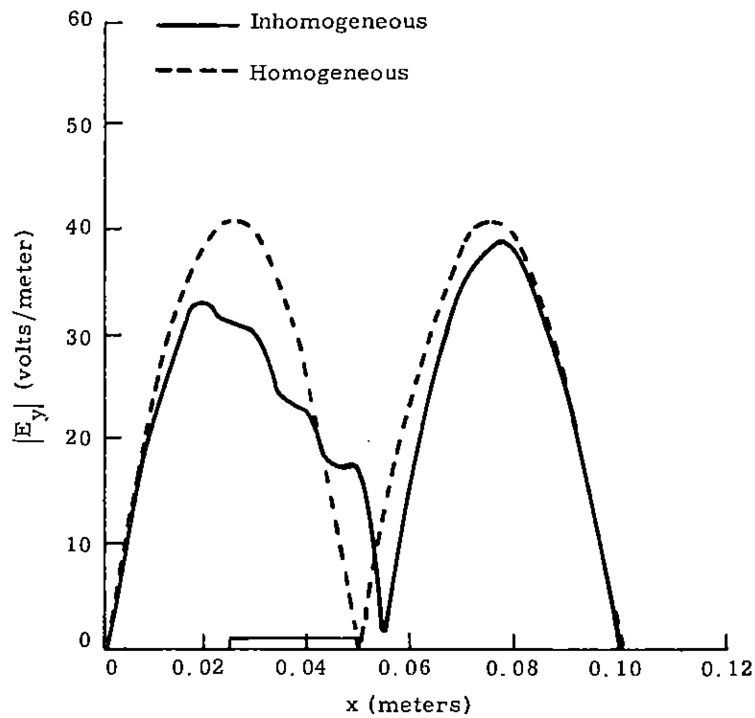

Figure 11. The Magnitude of $E_y(x, y_o)$, $y_o = 0.034$, for a Dielectric Inhomogeneity ($\epsilon_1 = 1.7\ \epsilon_o$) Extending from $x = 0.025$ to $x = 0.05$

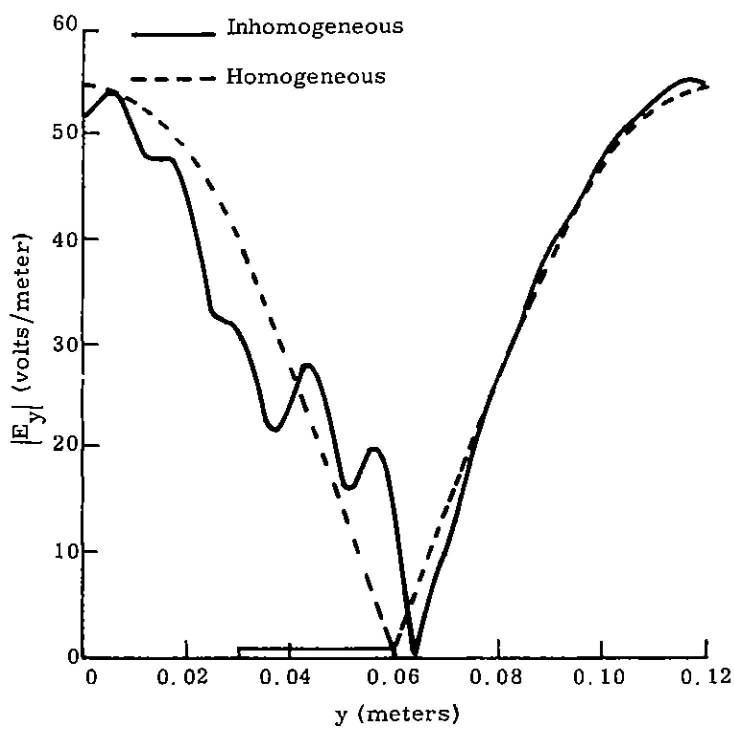

Figure 12. The Magnitude of $E_y(x_o, y)$, $x_o = 0.034$, for a Dielectric Inhomogeneity ($\epsilon_1 = 1.7\ \epsilon_o$) Extending from $y = 0.03$ to $y = 0.06$

CHAPTER VI. CONCLUSIONS

A general theory for the solution of electromagnetic boundary value problems for regions which are not homogeneous was presented in Chapter II. In this theory, the boundary conditions between homogeneous regions inside the cavity appear as sources in an integral equation. The same theory applies to regions for which the parameters vary smoothly rather than discontinuously as in the example problem presented in this study.

The remainder of the paper is devoted to the solution of an example problem illustrating the theory in Chapter II. An important result is the derivation of Equations 51 and 52 which are simply algebraic equations rather than differential or integral equations. As mentioned in Chapter IV, these equations could be solved by several mathematical techniques. An interesting extension of this study would be to solve the matrix equation made up of Equations 84 and 85 to give a more general solution, i.e., wider variation of k_1^2, than presented here.

The technique can be extended in a straightforward way to three dimensions. Instead of two equations, such as Equations 51 and 52, there will be three equations; instead of single series expansions, there will be double series; instead of three field components, there will be six, etc. The difficulties encountered would be primarily practical computational ones, such as the large amounts of both computer storage and computer time required. These problems, however, are always encountered in the solution of three-dimensional problems.

For simplicity, the example problem was worked in rectangular coordinates for a rectangular cavity containing a rectangular inhomogeneity, but the theory holds for any shape of inhomogeneity or for multiple inhomogeneities inside the cavity. The only place where the shape of the inhomogeneities enters in the theory is in η and ξ (Equations 45 and 49) and in the constants K, L, M, and N (Equations 71, 72, 77, and 78). The coordinate system in which a given problem must be treated is determined by the shape of the outer cavity walls and not by the shape of the inhomogeneities. Equation 17 is general for any coordinate system; therefore, the theory applies to any coordinate system for which the Green's function for the electric field for the homogeneous cavity can be found.

REFERENCES

1. Stratton, J. S., Electromagnetic Theory, McGraw-Hill Book Co., Inc., New York, 1941.

2. Panofsky, W. K. H., and M. Phillips, Classical Electricity and Magnetism, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962.

3. Van Bladel, J., Electromagnetic Fields, McGraw-Hill Book Co., Inc., New York, 1964.

4. Harrington, R. F., Time-Harmonic Electromagnetic Fields, McGraw-Hill Book Co., Inc., 1961.

5. Jackson, J. D., Classical Electrodynamics, John Wiley and Sons, Inc., New York, 1962.

6. Stratton, op. cit., p. 482.

7. Ibid., p. 486.

8. Baum, C. E., "Unsaturated Compton Current and Space Charge Fields in Evacuated Cavities," EMP Theoretical Notes, Volume 1, Air Force Weapons Laboratory, Kirtland Air Force Base, New Mexico, April 1967.

9. Mangan, D. L., and G. J. Scrivner, Field Generation Within a Lossy Dielectric Cylinder Excited by a Radiation Pulse, Sandia Laboratories Research Report, SC-RR-71 0870, February 1972.

10. Morse, P. M., and H. Feshbach, Methods of Theoretical Physics, McGraw-Hill Book Co., Inc., New York, 1953.

11. Harrington, R. F., Field Computation by Moment Methods, The Macmillan Company, New York, 1968.

12. Van Bladel, op. cit.

13. James, C. R., Electromagnetic Waves Within Non-Uniform Boundaries and in Inhomogeneous Isotropic Media (PhD dissertation), University of British Columbia, Electrical Engineering, 1964.

14. James, C. R., and Walker, G. B., "Electromagnetic Waves in Media with Finite Discontinuities," Proceedings of the IEEE, Volume 115, Number 3, March 1968.

15. Collin, R. E., Field Theory of Guided Waves, McGraw-Hill Book Co., Inc., New York, 1960, p. 58.

16. Van Bladel, op. cit., p. 509.

17. Morse, op. cit., p. 792.

18. Morse, ibid.

19. Collin, op. cit., p. 52.

20. Collin, ibid., p. 49.

21. Wiley, C. R., Jr., Advanced Engineering Mathematics, McGraw-Hill Book Co., Inc., New York, 1966, p. 194.

22. Indritz, J., Methods in Analysis, The MacMillan Company, New York, 1963, p. 157.

23. Indritz, ibid., p. 309.

24. Indritz, op. cit., p. 179.

APPENDIX

```
C      PROGRAM FIELD
       COMMON/A/A,B,A1,A2,B1,B2
       COMMON/B/AKSQ,K1SQ,IWMU,NF1
       COMMON/C/PIA,PIB
       COMMON/D/ALPH(20),GAMA(20),BETA2(20),DELT2(20)
       COMMON/E/PHI0(20,20),PHI1(20,20),PSI0(20,20),PSI1(20,20),
      1PHI(20,20),PSI(20,20)
       COMMON/FLD/EX,EY,EXH,EYH
       COMMON/G/AK(20,20),AL(20,20),AM(20,20),AN(20,20)
       DIMENSION AJ(20,20),DEN(20,20)
       COMPLEX PHI0,PHI1,PHI,PSI0,PSI1,PSI,K1SQ,IU,IWMU
       COMPLEX EX,EY,EXH,EYH
       REAL LAMMP,LAMPM,MU
       INTEGER P
       X0=0.0
       Y0=.034
       DELTX=2.E-3
       DELTY=.016
       NX=50
       NY=2
       A=.1
       B=.12
       A1=.025
       A2=.05
       B1=.03
       B2=.06
       MSUM=5
       M1=4
       M2=4
       M3=2
       M4=4
       PI=3.14159265
       PIA=PI/A
       PIB=PI/B
       IU=(0.,1.)
       F=1.0E8
       W=2.*PI*F
       MU=4.E-7*PI
       E=1./(36.E9*PI)
       E1=0.
       SIG1=5.E-3
       IWMU=IU*W*MU
       AKSQ=W*W*MU*E
       K1SQ=W*W*MU*E1-IWMU*SIG1
       TYPE 2,AKSQ,K1SQ,SIG1
     2 FORMAT(- KSQ=-,E12.4,- K1SQ=-,2E12.4,-  SIG1=-,E12.4)
       NF1=10
       NTYPE=0
       NOUT=0
    51 CALL ALPHA(MSUM)
       IF(NTYPE.EQ.1)GO TO 52
       TYPE 5
     5 FORMAT(-  I-,8X,-ALPH(I)-,8X,-GAMA(I)-,8X,-BETA2(I)-,8X,
      1-DELT2(I)-)
       DO 11 J=1,M1
        TYPE  9,J,ALPH(J),GAMA(J),BETA2(J),DELT2(J)
     9 FORMAT(I3,2X,4(F12.3,3X))
    11 CONTINUE
```

```
   52  CALL COFF1(MSUM)
      IF(NTYPF.EQ.1)GO TO 53
      TYPF 20
   20 FORMAT(/-    I-,3X,-J -,6X,-AK(I,J)-,6X,-AL(I,J)-,6X,-AM(I,J)-
     1,6X,-AN(I,J)-)
      DO 27 I=1,M2
      DO 27 J=1,M2
      TYPE 28,I,J,AK(I,J),AL(I,J),AM(I,J),AN(I,J)
   27 CONTINUE
   28 FORMAT(2I4,3X,4(E12.4,3X))
   53 CALL SORSJ(AJ,MSUM)
      IF(NTYPE.EQ.1)GO TO 54
      TYPE 32
   32 FORMAT(/,2X,-I-,3X,-J-,4X,-AJ(I,J)-)
      DO 35 I=1,M3
      DO 35 J=1,M3
      TYPE 30,I,J,AJ(I,J)
   30 FORMAT(2I4,2X,E12.5)
   35 CONTINUF
   54 CONTINUF
C     INITIALIZATION OF PHIO AND PSIO
      TYPE 6
    6 FORMAT(/- PHIO AND PSIO-)
      DO 4 M=1,MSUM
      DO 4 P=1,MSUM
    4 DEN(P,M)=1./(AKSQ*(AKSQ-ALPH(P)*ALPH(P)-GAMA(M)*GAMA(M)))
      DO 10 M=1,MSUM
      DO 10 P=1,MSUM
      PHIO(P,M)=IWMU*AJ(P,M)*DELT2(P)*DEN(M,P)
      PSIO(P,M)=-IWMU*ALPH(P)*GAMA(M)*AJ(M,P)*DEN(P,M)
      PHI(P,M)=PHIO(P,M)
      PSI(P,M)=PSIO(P,M)
      IF(M.GT.2)GO TO 10
      IF(P.GT.2)GO TO 10
      TYPE 140,P,M,PHIO(P,M),PSIO(P,M)
  140 FORMAT(1X,2I3,2X,4(E12.5,2X))
   10 CONTINUE
      DO 350 N=1,NF1
      TYPE 225,N
  225 FORMAT(/- N=-,I3)
      CALL CHEK(MSUM,N)
      DO 300 I=1,MSUM
      DO 300 J=1,MSUM
      PHI(I,J)=PHI1(I,J)
      PSI(I,J)=PSI1(I,J)
  300 CONTINUE
  350 CONTINUE
      DO 220 I=1,NX
      DO 220 J=1,NY
      X=(I-1)*DELTX+X0
      Y=(J-1)*DELTY+Y0
      CALL FIELD(X,Y,MSUM,AJ)
      TYPE 160,X,Y,EX,EY
  160 FORMAT(1X,F5.3,2X,F5.3,4(2X,E12.6))
      TYPE 170,EXH,EYH
  170 FORMAT(15X,4(F12.6,2X))
  220 CONTINUE
      END
```

```
C**************************    ALPHA    *****************
      SUBROUTINE ALPHA(M)
      COMMON/B/AKSQ,K1SQ,IWMU,NF1
      COMMON/C/PIA,PIB
      COMMON/D/ALPH(20),GAMA(20),BETA2(20),DELT2(20)
      DO 50 I=1,M
      ALPH(I)=I*PIB
      GAMA(I)=I*PIA
      BETA2(I)=AKSQ-ALPH(I)*ALPH(I)
      DELT2(I)=AKSQ-GAMA(I)*GAMA(I)
   50 CONTINUE
      RETURN
      END
```

```
C***********************    COEF1    ********************
      SUBROUTINE COEF1(M)
      COMMON/A/A,B,A1,A2,B1,B2
      COMMON/D/ALPH(20),GAMA(20),BETA2(20),DELT2(20)
      COMMON/G/AK(20,20),AL(20,20),AM(20,20),AN(20,20)
      DO 50 I=1,M
      DO 50 J=1,M
      IF(I.EQ.J)GO TO 40
C     P=I,M=J
      ALPH1=ALPH(I)-ALPH(J)
      ALPH2=ALPH(I)+ALPH(J)
      TEMP1=(SIN(ALPH1*B2)-SIN(ALPH1*B1))/ALPH1
      TEMP2=(SIN(ALPH2*B2)-SIN(ALPH2*B1))/ALPH2
      AK(I,J)=(TEMP1-TEMP2)/B
      AN(I,J)=(TEMP1+TEMP2)/B
      GAM1=GAMA(I)-GAMA(J)
      GAM2=GAMA(I)+GAMA(J)
      TEMP1=(SIN(GAM1*A2)-SIN(GAM1*A1))/GAM1
      TEMP2=(SIN(GAM2*A2)-SIN(GAM2*A1))/GAM2
      AL(I,J)=(TEMP1-TEMP2)/A
      AM(I,J)=(TEMP1+TEMP2)/A
      GO TO 45
C     I=J OR P=M
   40 ALPHM=2.*ALPH(I)
      AK(I,J)=(1./B)*((B2-B1)-(SIN(ALPHM*B2)-SIN(ALPHM
     1*B1))/(ALPHM))
      AN(I,J)=(1./B)*((B2-B1)+(SIN(ALPHM*B2)-SIN(ALPHM
     1*B1))/(ALPHM))
      GAMN=2.*GAMA(I)
      AL(I,J)=(1./A)*((A2-A1)-(SIN(GAMN*A2)-SIN(GAMN*A1))
     1/(GAMN))
      AM(I,J)=(1./A)*((A2-A1)+(SIN(GAMN*A2)-SIN(GAMN*A1))
     1/(GAMN))
   45 CONTINUE
   50 CONTINUE
      RETURN
      END
```

```
C***********************     SORSJ     ****************
      SUBROUTINE SORSJ(AJ,M)
      DIMENSION AJ(20,20)
      DO 10 I=1,M
      DO 10 J=1,M
   10 AJ(I,J)=0.0
      AJ(2,1)=1.
      RETURN
      END
```

```
C******************************     FIELD     **********************
      SUBROUTINE FIELD(X,Y,M,AJ)
      COMMON/A/A,B,A1,A2,B1,B2
      COMMON/B/AKSQ,K1SQ,IWMU,NF1
      COMMON/D/ALPH(20),GAMA(20),BETA2(20),DELT2(20)
      COMMON/E/PHI0(20,20),PHI1(20,20),PSI0(20,20),PSI1(20,20),
     1PHI(20,20),PSI(20,20)
      COMMON/FLD/EX,EY,EXH,EYH
      COMMON/G/AK(20,20),AL(20,20),AM(20,20),AN(20,20)
      DIMENSION AJ(20,20)
      COMPLEX PHI0,PHI1,PHI,PSI0,PSI1,PSI,K1SQ,IWMU
      COMPLEX EX,EY,EXH,EYH
      EX=(0.,0.)
      EY=(0.,0.)
      EXH=(0.,0.)
      EYH=(0.,0.)
      DO 100 I=1,M
      DO 100 J=1,M
      FXX=COS(GAMA(I)*X)*SIN(ALPH(J)*Y)
      FXY=SIN(GAMA(J)*X)*COS(ALPH(I)*Y)
      EX=EX+PHI(I,J)*FXX
      EY=EY+PSI(I,J)*FXY
   80 EXH=EXH+PHI0(I,J)*FXX
      EYH=EYH+PSI0(I,J)*FXY
  100 CONTINUE
      RETURN
      END
```

```
C*******************   CHEK      *****************
      SUBROUTINE CHEK(MSUM,N)
      COMMON/B/AKSQ,K1SQ,IWMU,NF1
      COMMON/E/PHI0(20,20),PHI1(20,20),PSI0(20,20),PSI1(20,20),
     1PHI(20,20),PSI(20,20)
      COMMON/D/ALPH(20),GAMA(20),BETA2(20),DELT2(20)
      COMMON/G/AK(20,20),AL(20,20),AM(20,20),AN(20,20)
      COMPLEX IWMU,K1SQ,TP(10)
      COMPLEX PHI,PSI,PHI0,PSI0,PHI1,PSI1
      REAL LAMMP,LAMPM
      INTEGER P
      DO 200 P=1,MSUM
      DO 200 M=1,MSUM
      DO 10 L=1,10
      TP(L)=(0.,0.)
   10 CONTINUE
      DO 60 J=1,MSUM
      DO 50 I=1,MSUM
      TP(3)=TP(3)+AM(I,P)*PHI(I,J)
       TP(4)=TP(4)+AN(I,M)*PSI(I,J)
      TP(5)=TP(5)+AM(I,M)*PHI(I,J)
      TP(6)=TP(6)+AN(I,P)*PSI(I,J)
   50 CONTINUE
      TP(7)=TP(7)+TP(3)*AK(J,M)
      TP(8)=TP(8)+TP(4)*AL(J,P)
      TP(9)=TP(9)+TP(5)*AK(J,P)
      TP(10)=TP(10)+TP(6)*AL(J,M)
   60 CONTINUE
      LAMPM=1.0/(AKSQ*(AKSQ-ALPH(M)*ALPH(M)-GAMA(P)*GAMA(P)))
      LAMMP=1.0/(AKSQ*(AKSQ-ALPH(P)*ALPH(P)-GAMA(M)*GAMA(M)))
      PHI1(P,M)=PHI0(P,M)-K1SQ*LAMPM*(TP(7)-TP(8)*ALPH(M)*GAMA(P)/
     1DELT2(P))*DELT2(P)
      PSI1(P,M)=PSI0(P,M)+K1SQ*LAMMP*(TP(9)*(GAMA(M)*ALPH(P)/
     1BETA2(P))-TP(10))*BETA2(P)
      IF(P.GT.4)GO TO 200
      IF(M.GT.4)GO TO 200
      IF(N.LE.3)GO TO 200
      PHIM=CABS(PHI(P,M))
      PHI1M=CABS(PHI1(P,M))
      PSIM=CABS(PSI(P,M))
      PSI1M=CABS(PSI1(P,M))
      DPHI=(PHIM-PHI1M)/PHIM
      DPSI=(PSIM-PSI1M)/PSIM
      IF(P.GT.3)GO TO 200
      IF(M.GT.3)GO TO 200
      TYPE 100,P,M,DPHI,PHIM,PHI1M
      TYPE 110,DPSI,PSIM,PSI1M
  100 FORMAT(2I4,2X,3(E12.5,2X))
  110 FORMAT(10X,3(E12.5,2X))
  200 CONTINUE
      RETURN
      END
```

CURRICULUM VITAE

The author was born in Grandin, Missouri, on October 19, 1940. When he was five years old, he and his family moved to Artesia, New Mexico, where he attended public school; he graduated from the Artesia High School in 1959. He enlisted in the Army Reserve, served on active duty for six months, and completed the remaining requirements of the program. He attended New Mexico State University, from which he graduated summa cum laude with a bachelor's degree in Electrical Engineering in 1964. He received a master's degree in 1966 in Electrical Engineering from the University of New Mexico. He has been employed by Sandia Laboratories since February 1964, except for the school year of 1967-68 when he completed his residency requirements at the University of New Mexico. He married the former Colleen Stroud of Portales, New Mexico, in June 1960. They are the parents of two children, Teri Lynne, age 11, and Kevin Dale, age 7.