Theoretical Notes

Note 203

Currents Induced on an Impedance Within a Slotted Sphere

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ABSTRACT

The current induced in an impedance connecting the two sections of a slotted hollow sphere when an electron is moving in the vicinity is calculated under the quasistatic approximation.

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FOREWORD

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CONTENTS

ILLUSTRATIONS

TABLES

SECTION I

INTRODUCTION AND GENERAL APPROACH

The problem to be studied is shown schematically in Figure 1. A point charge is moving in the vicinity of a slotted perfectly conducting sphere. The two sections of the sphere are connected internally by a known impedance Z. We wish to calculate the current through the impedance, I, when the electron moves slowly enough for the quasistatic approximation to be valid (This approximation is valid for most of the electrons ejected externally in a system generated EMP situation).

We will take an equivalent circuit approach, calculating the short-circuit current when the impedance Z is equal to zero and the internal admittance of the generator sending current through Z is zero. This generator admittance is due to the capacitance between the two sections of the sphere. When the short-circuit current and capacitance have been calculated, we can use the equivalent circuit of Figure 2 to say that the current through the load Z is just

$$I(\omega) = \frac{I_{sc}(\omega)}{1 - i\omega CZ} \tag{1}$$

where a time variation of the form $e^{-i\omega t}$ has been assumed. If Z is a pure resistance, R, we can transform the above equation into the time domain as

$$I(t) = \int_{-\infty}^{t} e^{\frac{-(t-t')}{RC}} I_{sc}(t')dt' \tag{2}$$

If Z is the impedance of a capacitance C_L, equation (1) transforms into the time domain in the form

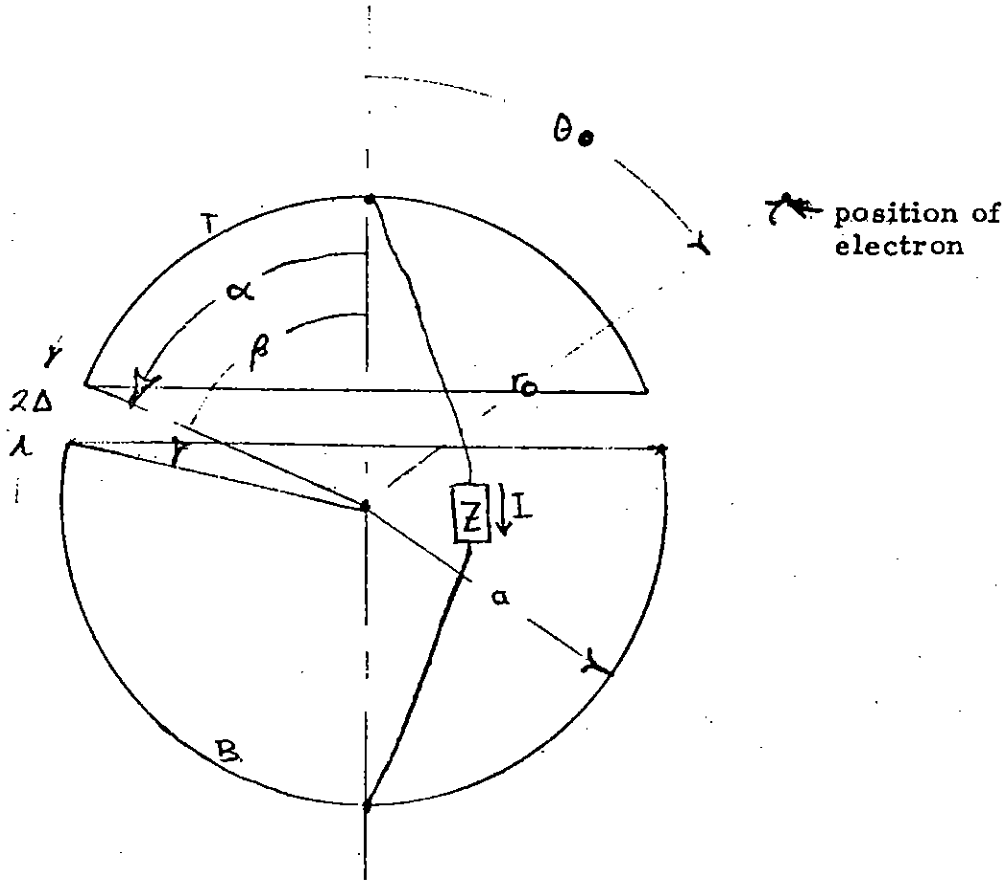

Figure 1: Electron moving around impedance loaded slotted sphere

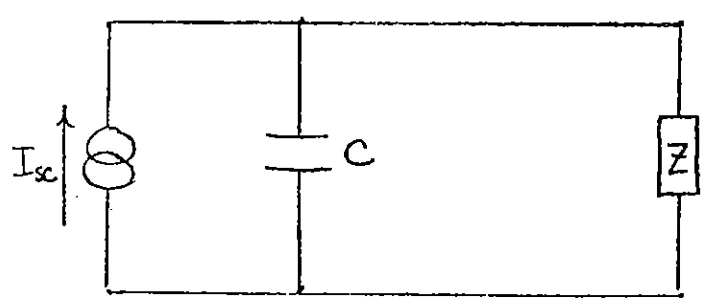

Figure 2: Equivalent circuit of situation in figure 1

$$I(t) = \frac{C_L}{C_L + C} I_{sc}(t) \qquad (3)$$

Other types of Z lead to more complex forms for I in the time domain.

Equations such as (1), (2) or (3) are quite useful in that the effect of the electron generating the current is entirely within the factor I_{sc} (This factor also depends somewhat on the position of the slot, as will be seen in Section III.) while most of the effect of the slot is felt within the factor C. In Section II we will calculate the capacitance C, while in Section III we will compute the short circuit current I_{sc}.

SECTION II

CAPACITANCE

1. GENERAL APPROACH

We can define two pertinent electrostatic problems in the configuration of Figure 1 (with the electron absent). The two problems are

(a) given that the potential on the top segment of the sphere (T) is $+V_o$ and the potential on the bottom segment of the sphere (B) is $-V_o$, what are the total charges on the top and bottom segments of the sphere? We will call these charges Q_T^a and Q_B^a respectively.

(b) the same as (a) except that both potentials are $+V_o$. In this case we will denote the charges by Q_T^b and Q_B^b.

If φ^a is the potential distribution for problem (a) and φ^b is the potential distribution for problem (b) the potential for the capacitance problem (defined by the condition that the total net charge be zero) can clearly be written in the form

$$\varphi = \gamma\varphi^a + \delta\varphi^b \tag{4}$$

where γ and δ are proportionality constants. If the charges in the capacitance problem are denoted by Q_T and Q_B, and the potential on T and B by φ_T and φ_B, we must have

$$Q_T + Q_B = \gamma(Q_T^a + Q_T^b) + \delta(Q_T^a + Q_T^b) = 0 \tag{5}$$

$$\varphi_T = V_o(\gamma + \delta) \tag{6}$$

$$\varphi_B = -V_o(\gamma - \delta) \tag{7}$$

Thus

$$\varphi_T - \varphi_B = 2\gamma V_o \tag{8}$$

and

$$C = \frac{Q_T}{\varphi_T - \varphi_B} = \frac{\gamma Q_T^a + \delta Q_T^b}{2\gamma V_o} = \frac{Q_T^a}{2V_o} + \frac{\delta}{\gamma}\frac{Q_T^b}{2V_o} \tag{9}$$

which, using equation (5), can be written in the form

$$C = \frac{Q_T^a}{2V_o} - \frac{Q_T^a + Q_B^a}{Q_T^b + Q_B^b}\frac{Q_T^b}{2V_o} \tag{10}$$

or, more symmetrically, as

$$C = \frac{Q_T^a Q_B^b - Q_B^a Q_T^b}{Q_T^b + Q_B^b} \cdot \frac{1}{2V_o} \tag{11}$$

Let us solve problem (a) first. That is, find a potential function $\varphi(r, \theta)$ satisfying the boundary conditions of problem (a) and then find the associated total charges on the top and bottom segments. In equations:

$$\varphi^a(r, \theta) = \sum_{n=0}^{\infty} C_n \left(\frac{r}{a}\right)^n P_n(\cos\theta) \qquad r \le a \tag{12}$$

$$= \sum_{n=0}^{\infty} C_n \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta) \qquad r \ge a \tag{13}$$

$$\frac{a\sigma(\theta)}{\epsilon} = \sum_{n=0}^{\infty} (2n+1)\, C_n P_n(\cos\theta) \tag{14}$$

where we must have

$$\sum_{n=0}^{\infty} C_n P_n(\cos\theta) = V_o \qquad 0 \le \theta < \alpha \tag{15}$$

$$= -V_o \qquad \beta < \theta \le \pi \tag{16}$$

$$\sum_{n=0}^{\infty}(2n+1)\, C_n P_n(\cos\theta) = 0 \qquad \alpha < \theta < \beta \tag{17}$$

We will use the approach to the solution of such problems given in Reference 1. Thus setting

$$C_n = A_n + C_n \tag{18}$$

we must have

$$\sum_{n=0}^{\infty} (A_n + B_n) P_n(\cos\theta) = V_o \qquad 0 \le \theta < \alpha \tag{19}$$

$$\sum_{n=0}^{\infty} (2n+1)\, A_n P_n(\cos\theta) = 0 \qquad \alpha < \theta \le \pi \tag{20}$$

$$\sum_{n=0}^{\infty} (2n+1)\, B_n P_n(\cos\theta) = 0 \qquad 0 \le \theta < \beta \tag{21}$$

$$\sum_{n=0}^{\infty} (A_n + B_n)\, P_n(\cos\theta) = -V_o \qquad \beta < \theta \le \pi \tag{22}$$

We can calculate Q_T^a and Q_T^b in terms of these constants as follows, using Equations (14), (20) and (21)

$$\frac{a\sigma_T(\theta)}{\epsilon} = \sum_{n=0}^{\infty} (2n+1) A_n P_n'(\cos\theta) \tag{23}$$

which, using Mehlers integral (Reference 1, equation 2.6.21), becomes

$$\frac{a\sigma_T(\theta)}{\epsilon} = \sum_{n=0}^{\infty} (2n+1)\, A_n \frac{\sqrt{2}}{\pi} \int_{\theta}^{\pi} \frac{\sin(n+\frac{1}{2})u}{\sqrt{\cos\theta - \cos u}} \tag{24}$$

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[1] Ian N. Sneddon, Mixed Boundary Value Problems in Potential Theory, North-Holland, 1966, section (6.5.2).

$$= -\frac{2\sqrt{2}}{\pi} \int_{\theta}^{\pi} \frac{du}{\sqrt{\cos\theta - \cos u}} \frac{d}{du} \sum_{n=0}^{\infty} A_n \cos(n+\tfrac{1}{2})u \tag{25}$$

or, defining

$$j_1(u) = \sum_{n=0}^{\infty} A_n \cos(n+\tfrac{1}{2})u \tag{26}$$

we can write

$$\frac{a\sigma_T(\theta)}{\epsilon} = -\frac{2\sqrt{2}}{\pi} \int_{\theta}^{\pi} \frac{du}{\sqrt{\cos\theta - \cos u}} j_1'(u) \tag{27}$$

and

$$Q_T^a = 2\pi a^2 \int_0^{\alpha} \sigma_T(\theta) \sin\theta \, d\theta \tag{28}$$

i.e.

$$\frac{Q_T}{a\epsilon} = -4\sqrt{2} \int_0^{\alpha} \sin\theta \, d\theta \int_{\theta}^{\pi} \frac{j_1'(u)}{\sqrt{\cos\theta - \cos u}} du \tag{29}$$

but equation (27) has an inverse which can be found, for example, in Reference 2 and can be written in the form

$$j_1'(\theta) = \frac{a}{2\sqrt{2}\,\epsilon} \frac{d}{d\theta} \int_{\theta}^{\pi} \frac{\sin u \; \sigma_T(u) \, du}{\sqrt{\cos\theta - \cos u}} \tag{30}$$

Thus from equation (20) it is clear that $j_1'(u)$ is zero for u greater than α and so the upper limit on the u integration of equation (29) can be written as α. We can now interchange the order of integration on equation (29) and perform the new inner integral to obtain

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[2] R. W. Latham and K. S. H. Lee, "Capacitance and equivalent area of a spherical dipole sensor," AFWL Sensor and Simulation Note 113, July 1970.

$$\frac{Q_T^a}{a\epsilon} = -16 \int_0^{\alpha+} j_1'(u) \sin\frac{u}{2}\, du \tag{31}$$

which, after integrating by parts, and invoking the fact that $j_1(u)$ is zero for $u > \alpha$, can be written as

$$\frac{Q_T}{a\epsilon} = 8 \int_0^{\alpha} j_1(u) \cos\frac{u}{2}\, du \tag{32}$$

In a manner similar to the above, and by defining

$$j_2(u) = \sum_{n=0}^{\infty} B_n \sin\left(n + \tfrac{1}{2}\right) u \tag{33}$$

we can say that

$$\frac{a\sigma_B(\theta)}{\epsilon} = \frac{2\sqrt{2}}{\pi} \int_0^{\theta} \frac{du}{\sqrt{\cos u - \cos\theta}}\, j_2'(u) \tag{34}$$

$$j_2'(\theta) = \frac{a}{2\sqrt{2}\,\epsilon} \frac{d}{d\theta} \int_0^{\theta} \frac{\sin u\ \sigma_B(u)\, du}{\sqrt{\cos u - \cos\theta}} \tag{35}$$

and

$$\frac{Q_B^a}{a\epsilon} = 8 \int_{\beta}^{\pi} j_2(u) \sin\frac{u}{2}\, du \tag{36}$$

To use the formulation of Reference 1 to determine $j_1(x)$ and $j_2(x)$ (which are still unknown) we must first form the functions

$$g_1(x) = \frac{1}{\sqrt{2}} \frac{d}{dx} \int_0^{x} \frac{V_o \sin u\, du}{\sqrt{\cos u - \cos x}}$$

$$= V_o \cos x/2 \tag{37}$$

and

$$g_2(x) = \frac{1}{\sqrt{2}} \frac{d}{dx} \int_x^{\pi} \frac{V_o \sin u}{\sqrt{\cos x - \cos u}} du$$

$$= - V_o \sin x/2 \qquad (38)$$

Using these functions we can now call on the work in Reference 1 to write down a pair of coupled integral equations for $j_1(x)$ and $j_2(x)$ in the form

$$j_1(x) = V_o \cos \frac{x}{2} - \frac{1}{\pi} \int_{\beta}^{\pi} \frac{2 \cos x/2 \sin y/2}{\cos x - \cos y} j_2(y)\, dy$$

$$0 \le x \le \alpha \qquad (39)$$

$$j_2(x) = - V_o \sin x/2 + \frac{1}{\pi} \int_0^{a} \frac{2 \sin x/2 \cos y/2}{\cos x - \cos y} j_1(y)\, dy$$

$$\beta \le x \le \pi \qquad (40)$$

These become more symmetric if we define

$$f_1(x) = \frac{j_1(x)}{V_o} \qquad f_2(x) = \frac{j_2(\pi - x)}{V_o} \qquad (41)$$

for in that case

$$f_2(x) = - \cos x/2 - \frac{2}{\pi} \int_0^{\alpha} \frac{\cos x/2 \cos y/2}{\cos x + \cos y} f_1(y)\, dy \qquad (42)$$

$$f_1(x) = \cos x/2 - \frac{2}{\pi} \int_0^{\pi - \beta} \frac{\cos x/2 \cos y/2}{\cos x + \cos y} f_2(y)\, dy \qquad (43)$$

We can simplify these equations a little by making the further substitutions:

$$z = \tan x/2 \qquad (44)$$

$$z' = \tan y/2 \qquad (45)$$

$$\cos(x/2) f_1(x) = F_1(x) \qquad (46)$$

$$\cos(y/2) f_2(y) = F_2(x) \qquad (47)$$

which reduce our coupled pair to

$$F_1(z) = \frac{1}{1+z^2} - \frac{2}{\pi}\int_0^{\tan\left(\frac{\pi-\beta}{2}\right)} \frac{F_2(z')\,dz'}{1-z^2z'^2} \tag{48}$$

$$F_2(z) = -\frac{1}{1+z^2} - \frac{2}{\pi}\int_0^{\tan\alpha/2} \frac{F_1(z')\,dz'}{1-z^2z'^2} \tag{49}$$

while it follows easily from equation (32), (36) and (44) to (47) that

$$\frac{Q_T^a}{2a\epsilon V_o} = 8\int_0^{\tan\alpha/2} \frac{F_1(z)}{1+z^2}\,dz \tag{50}$$

$$\frac{Q_B^a}{2a\epsilon_o V_o} = 8\int_0^{\tan\left(\frac{\pi-\beta}{2}\right)} \frac{F_2(z)\,dz}{1+z^2} \tag{51}$$

If β is close to π while α is finite we should have Q_T^a/V_o equal to the capacitance of a spherical bowl, i.e. $4\epsilon a(\sin\alpha + \alpha)$, but this follows at once from equation (50) if $F_1(z)$ is $(1+z^2)^{-1}$. Thus we have a check on our equations. So far we have Q_T^a and Q_B^a, but clearly Q_T^b and Q_B^b can be obtained simply be reversing the sign of the free term of equation (49). From this fact and our previous expressions for C we can write the general solution to the capacitance problem in the form

$$\frac{C}{4\epsilon a} = \frac{q_T^a q_B^b - q_B^a q_T^b}{q_T^b + q_B^b} \tag{52}$$

where

$$q_T^a = 2 \int_0^{\tan \alpha/2} \frac{F_1^a(z)}{1+z^2}\, dz \qquad q_B^a = \int_0^{\tan\left(\frac{\pi-\beta}{2}\right)} \frac{F_2^a(z)\, dz}{1+z^2} \tag{53}$$

F_1^a and F_2^a being determined by

$$F_1^a(z) = \frac{1}{1+z^2} - \frac{2}{\pi} \int_0^{\tan\left(\frac{\pi-\beta}{2}\right)} \frac{F_2^a(z')\, dz'}{1 - z^2 z'^2} \tag{54}$$

$$F_2^a(z) = -\frac{1}{1+z^2} - \frac{2}{\pi} \int_0^{\tan \alpha/2} \frac{F_1^a(z')\, dz'}{1 - z^2 z'^2} \tag{55}$$

and where

$$q_T^b = 2 \int_0^{\tan \alpha/2} \frac{F_1^b(z)\, dz}{1+z^2} \qquad q_B^b = 2 \int_0^{\tan\left(\frac{\pi-\beta}{2}\right)} \frac{F_2^b(z)\, dz}{1+z^2} \tag{56}$$

F_1^b and F_2^b being determined by

$$F_1^b(z) = \frac{1}{1+z^2} - \frac{2}{\pi} \int_0^{\tan\left(\frac{\pi-\beta}{2}\right)} \frac{F_2^b(z'')\, dz'}{1 - z^2 z'^2} \tag{57}$$

$$F_2^b(z) = \frac{1}{1+z^2} - \frac{2}{\pi} \int_0^{\tan \alpha/2} \frac{F_1^b(z')\, dz'}{1 - z^2 z'^2} \tag{58}$$

If we denote the angle of the center of the slot by θ_s and the slot width by 2Δ we can write the parameters of the above equations in the form

$$\tan \alpha/2 = \frac{\sin \theta_s - \sin \Delta}{\cos \Delta + \cos \theta_s} \tag{59}$$

$$\tan \frac{\pi - \beta}{2} = \frac{\sin \theta_s - \sin \Delta}{\cos \Delta + \cos \theta_s} \tag{60}$$

So we can write

$$\frac{C}{4\epsilon a} \equiv f(\theta_s, \Delta) \tag{61}$$

where $f(\theta_s, \Delta)$ can be determined numerically by using equations (52) to (58).

We note the special, but important, case of $\theta_s = \pi/2$ which we will use to study the effect of gap width on C. In that case it is clear from equations (52) to (58) that $q_T^a = - q_B^a$ and that $q_T^b = q_B^b$. Thus equation (52) and definition (61) give

$$f(\pi/2, \Delta) = 2 \int_0^{\frac{1 - \sin \Delta}{\cos \Delta}} \frac{F(z)\, dz}{1 + z^2} \tag{62}$$

where

$$F(z) = \frac{1}{1 + z^2} + \frac{2}{\pi} \int_0^{\frac{1 - \sin \Delta}{\cos \Delta}} \frac{F(z')\, dz'}{1 - z^2 z'^2} \tag{63}$$

In other words, for $\theta_s = \pi/2$ we do not have to solve problem (b) and problem (a) simplifies to a single integral equation. We use this special case to study the effect of slot width. The numerical values are given in Table 1 and Figure 3 for values of Δ up to one-tenth radian. In the next section we will derive an approximate formula for the capacitance when Δ is small and θ_s is arbitrary. This expression agrees with the present numerical data, when θ_s is $\pi/2$, to within a quarter of a percent if Δ is less than a tenth. Thus the approximation of the next subsection can be expected to be accurate enough for all practical purposes.

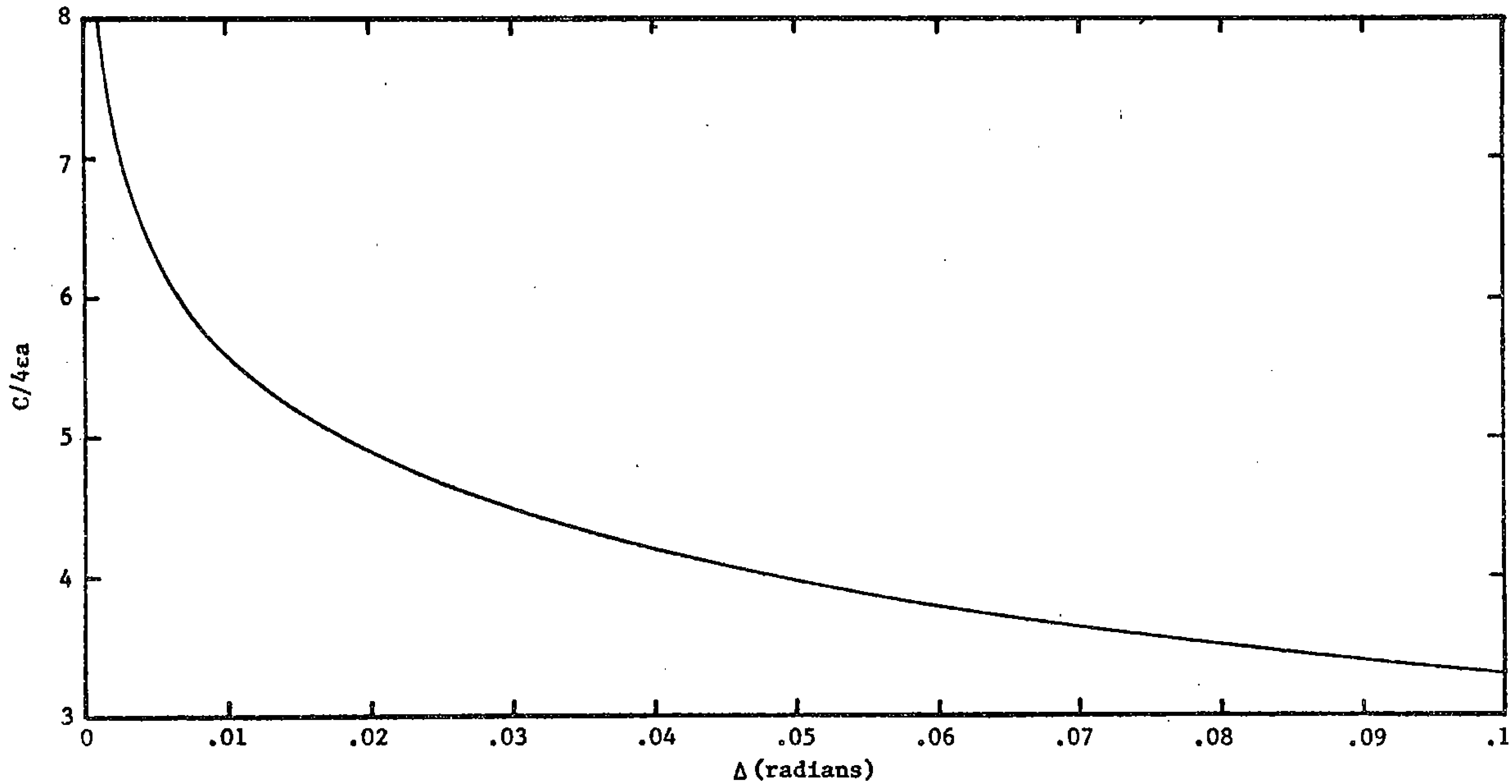

Figure 3: Capacitance across an equatorial gap in a hollow sphere

Table 1

This table gives the capacitance across an equatorial gap in a hollow sphere whose radius is a. The capacitance is normalized to $4\epsilon a$ and is tabulated as a function of the half-angle of the gap, Δ. The first two decimal points of Δ, in radians, are to be read from the left-hand column, while the third decimal point is to be read from the top row.

Δ	0	1	2	3	4	5	6	7	8	9
.00	∞	7.89496	7.20180	6.79632	6.50863	6.28547	6.10313	5.94897	5.81543	5.69765
.01	5.59226	5.49693	5.40988	5.32982	5.25568	5.18665	5.12208	5.06144	5.00424	4.95014
.02	4.89881	4.84997	4.80342	4.75893	4.71633	4.67545	4.63618	4.59840	4.56198	4.52684
.03	4.49289	4.46005	4.42826	4.39742	4.36752	4.33847	4.31025	4.28279	4.25606	4.23001
.04	4.20464	4.17987	4.15571	4.13213	4.10906	4.08652	4.06446	4.04289	4.02177	4.00106
.05	3.98079	3.96092	3.94141	3.92229	3.90353	3.88508	3.86699	3.84920	3.83174	3.81455
.06	3.79766	3.78104	3.76468	3.74859	3.73276	3.71716	3.70181	3.68667	3.67177	3.65706
.07	3.64258	3.62829	3.61422	3.60031	3.58660	3.57308	3.55973	3.54655	3.53355	3.52071
.08	3.50802	3.49549	3.48312	3.47088	3.45880	3.44684	3.43503	3.42336	3.41183	3.40041
.09	3.38912	3.37794	3.36691	3.35597	3.34515	3.33445	3.32385	3.31338	3.30300	3.29271

2. NARROW SLOTS

Any slots that can realistically be expected in the physical situation we are trying to model will be quite narrow. For narrow slots we can make a different approach to the capacitance calculation than the general one given in the previous subsection because for narrow slots we can make a pretty good guess of the field distribution in the slot. Let us begin by writing the surface charge density, in accordance with equation (14), in the form

$$\frac{a\sigma(\theta)}{\epsilon} = \sum_{n=1}^{\infty} (2n+1)\, C_n P_n(\cos\theta) \tag{64}$$

where the $n = 0$ term has been omitted since, for the capacitance problem, we wish the total charge on the sphere to be zero. The charge on the upper segment of the sphere can now be written in the form

$$Q_T = \frac{\epsilon}{a} \cdot 2\pi a^2 \int_0^{\alpha} \sum_{n=1}^{\infty} (2n+1)\, C_n P_n(\cos\theta) \sin\theta \, d\theta \tag{65}$$

or using the little calculation

$$\int_0^{\alpha} P_n(\cos\theta)\sin\theta\, d\theta = \int_{\cos\alpha}^{1} P_n(x)\, dx$$

$$= \frac{1}{2n+1} \int_{\cos\alpha}^{1} [P'_{n+1}(x) - P'_{n-1}(x)] dx$$

$$= \frac{1}{2n+1} [P_{n-1}(\cos\alpha) - P_{n+1}(\cos\alpha)]$$

$$= \frac{\sin^2\alpha}{n(n+1)} P'_n(\cos\alpha)$$

$$= \frac{\sin^2\alpha}{n(n+1)} P_n^1(\cos\alpha) \tag{66}$$

equation (65) can be reduced to

$$\frac{Q_T}{4\epsilon a} = \frac{\pi}{2} \sum_{n=1}^{\infty} \sin^2\alpha \cdot \frac{2n+1}{n(n+1)} \cdot C_n P_n^1 (\cos\alpha) \tag{67}$$

But since the φ corresponding to equation (64) can be written at the surface of the sphere as

$$\varphi = \sum_{n=1}^{\infty} C_n P_n (\cos\theta), \tag{68}$$

we can use the orthogonality theorem for Legendre polynomials to say that

$$\frac{2C_n}{2n+1} = \int_0^{\pi} \varphi(\theta)\, P_n\, (\cos\theta) \sin\theta\, d\theta \tag{69}$$

or, integrating by parts, using equation (66) and noting that φ changes only within the gap

$$\frac{2C_n}{2n+1} = a \int_{\alpha}^{\beta} \frac{E(\theta) \sin^2\theta}{n(n+1)}\, P_n^1 (\cos\theta)\, d\theta \tag{70}$$

where

$$E(\theta) = -\frac{\partial\varphi(\theta)}{a\,\partial\theta} \tag{71}$$

is the electric field within the gap.

Now substituting equation (70) into equation (67) we see that if E is normalized such that

$$\int_{\alpha}^{\beta} E(\theta)\, d\theta = 1. \tag{72}$$

the capacitance can be written as

$$\frac{C}{4\epsilon a} = \frac{\pi}{4} \int_{\alpha}^{\beta} E(\theta)\, d\theta \left\{ \sum_{n=1}^{\infty} \left(\frac{2n+1}{n(n+1)} \right)^2 \sin\alpha \; P_n^1(\cos\alpha) \sin\theta \; P_n^1(\cos\theta) \right\} \tag{73}$$

Making use of the addition theorem for Legendre polynomials (Reference [3], p. 239) we can write

$$\frac{P_n^1(\cos\alpha)\, P_n^1(\cos\theta)}{n(n+1)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_n(\cos\gamma) \cos\varphi \, d\varphi \tag{74}$$

where

$$\cos\gamma = \cos\alpha \cos\theta + \sin\alpha \sin\theta \cos\varphi \tag{75}$$

Thus equation (73) can be rewritten as

$$\frac{C}{4\epsilon a} = \frac{1}{8} \int_{-\pi}^{\pi} \int_{\alpha}^{\beta} E(\theta) \sin\theta \sin\alpha \cos\varphi \, d\varphi \, d\theta \left\{ \sum_{n=1}^{\infty} \frac{(2n+1)^2}{n(n+1)} P_n(\cos\gamma) \right\} \tag{76}$$

The sum in this equation can be exhibited in closed form using equations on page 238 of Reference [3] as follows:

$$\sum_{n=1}^{\infty} \frac{(2n+1)^2}{n(n+1)} P_n(\cos\gamma) = \sum_{n=1}^{\infty} \left(4 + \frac{1}{n(n+1)}\right) P_n(\cos\gamma)$$

$$= \frac{4}{\sqrt{2(1-\cos\gamma)}} + 1 - 2 \ln\left(1 + \sqrt{\frac{1-\cos\gamma}{2}}\right) \tag{77}$$

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[3] W. Magnus, F. Oberhettinger and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag, New York, 1966.

The constant factor makes no difference because it gives zero when the φ integration of equation (76) is performed, so we can say that

$$\frac{C}{4\epsilon a} = I_1 - I_2 \tag{78}$$

where

$$I_1 = \frac{1}{2}\int_{-\pi}^{\pi}\int_{\alpha}^{\beta} \frac{E(\theta)\sin\theta\ \sin\alpha\ \cos\varphi\ d\varphi\ d\theta}{\sqrt{2(1-\cos\gamma)}} \tag{79}$$

$$I_2 = \frac{1}{4}\int_{-\pi}^{\pi}\int_{\alpha}^{\beta} E(\theta)\sin\theta\sin\alpha\ \cos\varphi\ \ln\left(1+\sqrt{\frac{1-\cos\gamma}{2}}\right)d\varphi\ d\theta \tag{80}$$

In integral I_2 we can, for narrow slots, replace θ by α except in $E(\theta)$ and, recalling the normalization for $E(\theta)$, write it in the form

$$I_2 = \frac{1}{4}\int_{-\pi}^{\pi} \sin^2\alpha\ \cos\varphi\ \ln\ (1+\sin\alpha\ \sin\frac{\varphi}{2})\ d\varphi$$

$$= \sin^2\alpha\int_0^{\pi/2} \cos 2\psi\ \ln\ (1+\sin\alpha\ \sin\psi)\ d\psi \tag{81}$$

Integrating by parts we have

$$\frac{I_2}{\sin^2\alpha} = \frac{\sin 2\psi}{2}\ \ln\ (1+\sin\alpha\ \sin\psi)\Bigg|_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin 2\psi}{2}\cdot\frac{\sin\alpha\ \cos\psi}{1+\sin\alpha\ \sin\psi}\ d\psi$$

$$= -\sin\alpha\int_0^{\pi/2} \frac{\sin\psi - \sin^3\psi}{1+\sin\alpha\ \sin\psi}\ d\psi$$

$$= \sin\alpha\int_0^{\pi/2}\left\{\frac{\sin^2\psi}{\sin\alpha} - \frac{\sin\psi}{\sin^2\alpha} + \frac{\cos^2\alpha}{\sin^3\alpha}\left(1-\frac{1}{1+\sin\alpha\sin\psi}\right)\right\}d\psi \tag{82}$$

Thus

$$\frac{I_2}{\sin^2\alpha} = \frac{\pi}{4} - \frac{1}{\sin\alpha} + \frac{\pi}{2}\cot^2\alpha - \cot^2\alpha \int_0^{\pi/2} \frac{d\psi}{1 + \sin\alpha \sin\psi} \qquad (83)$$

The integral in this equation can be performed using no. 307 of Reference [4], the final result being

$$I_2 = \frac{\pi}{4}\sin^2\alpha - \sin\alpha + \frac{\pi}{2}\cos^2\alpha - \cos\alpha\,(\pi/2 - \alpha) \qquad (84)$$

We now return to the evaluation of I_1, defined by equation (79). The φ integral can be performed in terms of elliptic integrals, but we prefer to directly go to the case of small Δ in the following manner. We assume there is an ϵ such that $\Delta << \epsilon << \pi$ and perform the φ integral for $|\varphi| > \epsilon$ by setting $\theta = \alpha$ and obtain

$$I_1' = \int_\alpha^\beta \frac{E(\theta)\sin^2\alpha \cos\varphi\, d\varphi}{2\sin\alpha \sin\varphi/2}$$

$$= \sin\alpha \left\{ \ell n \cot(\epsilon/4) - 2\cos(\epsilon/2) \right\} \qquad (85)$$

We can perform the portion of the I_1 integral for $|\varphi| < \epsilon$ by noting that $\sqrt{2(1 - \cos\gamma)}$ is the distance between the point $0, \varphi$ and the point $\alpha, 0$. We assume the sphere to be flat over the small domain of the integrand and thus reduce the small φ portion of I_1 to

$$I_1'' = \frac{\sin\alpha}{2} \int_{-\epsilon}^{\epsilon} \int_0^{2\Delta} \frac{E(\alpha + y)\, dy\, dx}{\sqrt{x^2 + y^2}} \qquad (86)$$

If we set

$$E(\alpha + y) = \frac{1}{\pi\sqrt{y(2\Delta - y)}} \qquad (87)$$

which has the correct normalization and has the well-known form for the E field in narrow slots we obtain easily

[4] B. O. Peirce, A Short Table of Integrals, Ginn & Co., Boston, 1956.

$$I_1'' = \sin\alpha \, \ln(4\epsilon/\Delta) \qquad (88)$$

Combining this value with I_1' from equation (85) gives us I_1 in the form

$$I_1 = \sin\alpha \left\{ \ln\left(\frac{4\epsilon}{\Delta \tan \epsilon/4}\right) - 2\cos\epsilon \right\} \qquad (89)$$

Since ϵ is small this integral becomes independent of ϵ (to second order) as the following limit

$$I_1 = \sin\alpha \, (\ln 16/\Delta - 2) \qquad (90)$$

Combining this value with I's from equation (84) we obtain from equation (78)

$$\frac{C}{4\epsilon a \sin\alpha} = f(\alpha) - \ln\Delta \qquad (91)$$

where

$$f(\alpha) = (\ln 16 - 1) - \frac{\pi}{4}\left(\frac{1 + \cos^2\alpha}{\sin\alpha}\right) + \frac{\left(\frac{\pi}{2} - \alpha\right)\cos\alpha}{\sin\alpha} \qquad (92)$$

The effect of the slot width is entirely contained in the second factor of equation (91) while the effect of slot position is entirely in the first factor (except for the fact that we are now normalizing C to $4\epsilon a \sin\alpha$ rather than just $4\epsilon a$). The function $f(\alpha)$ is rather slowly varying. It is monotonic and

$$f(0) = \ln 16 - 2 = .772 \qquad (93)$$

while

$$f(\pi/2) = \ln 16 - 1 - (\pi/4) = .9872 \qquad (94)$$

If α is $\pi/2$ equation (91) agrees with the exact results of the previous subsection within $\frac{1}{4}\%$ for Δ less than 1/10. The accuracy for other values of α should be comparable. A table of values of $f(\alpha)$ is given as Table 2 and it is graphed in Figure 4.

Table 2

f(α) for position effect of slot capacitance

α (degrees)	f(α)	α (degrees	f(α)
0	.77259	46	.89549
2	.77299	48	.90257
4	.77415	50	.90951
6	.77602	52	.91629
8	.77856	54	.92288
10	.78172	56	.92925
12	.78545	58	.93539
14	.78971	60	.94126
16	.79444	62	.94686
18	.79968	64	.95215
20	.80519	66	.95713
22	.81111	68	.96177
24	.81734	70	.96606
26	.82384	72	.96999
28	.83057	74	.97354
30	.83749	76	.97670
32	.84457	78	.97945
34	.85177	80	.98180
36	.85905	82	.98373
38	.86637	84	.98524
40	.87371	86	.98632
42	.88103	88	.98697
44	.88830	90	.98719

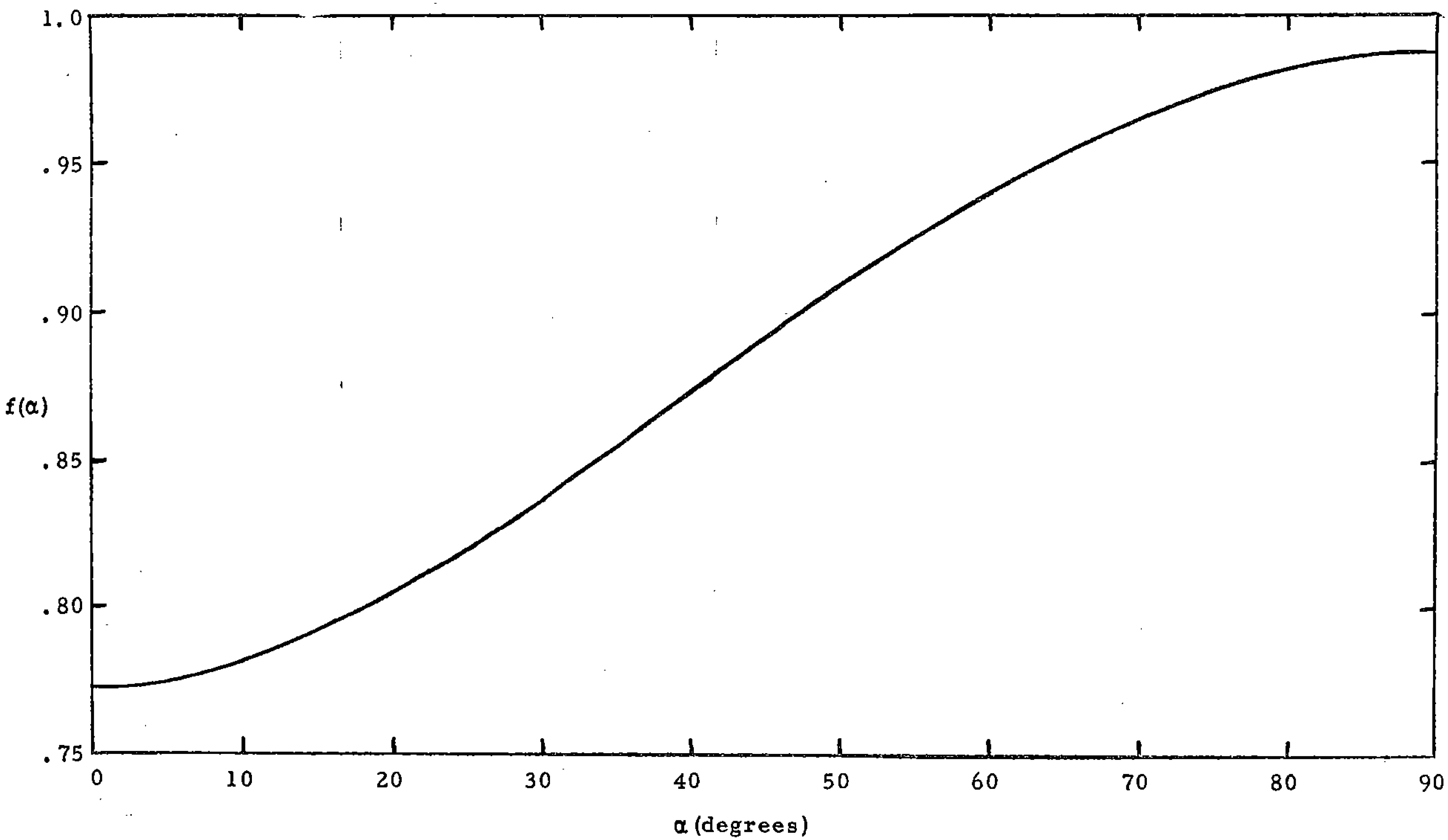

Figure 4: The function f(α) for the position effect of slot capacitance

SECTION III

SHORT CIRCUIT CURRENT

1. GENERAL APPROACH

In this subsection we will indicate a general approach to the calculation of the short-circuit current induced across the circumferential gap in the sphere. Since any realistic gaps can be expected to be quite narrow, and since we have seen in the previous section that narrow gap approximations can be extremely accurate, the present subsection can be considered to be included mainly for purposes of completeness. Any required results can be obtained from the work of the following subsection on narrow gaps.

The short circuit current across the gap is equal to the time derivative of the charge on one of the segments of the sphere (the negative time derivative if we consider the charge on the top segment and the current convention of Figure 1) when both segments are at the same potential. The charge is induced by a point charge traveling in the vicinity of the sphere.

If we set the total charge on the sphere equal to zero (which is adequate as far as final current answers are concerned, the time derivative of any constant net charge on the spherical segments will be zero), the potential of the charges on the sphere must, outside the sphere, be of the form

$$\Phi(r, \theta) = \sum_{n=1}^{\infty} C_n \frac{P_n(\cos\theta)}{r^{n+1}} + \sum_{m=1}^{\infty} \frac{C_{nm} P_n^m(\cos\theta)}{r^{n+1}} \cos m(\varphi - \varphi_o) \tag{95}$$

where symmetry about $\varphi = \varphi_o$ is assumed (φ_o being the φ coordinate of the point charge).

The potential Φ plus the position varying part of the incident potential (i.e. the potential of the point charge) must be a constant on the metallic portion of the spherical surface. Expanding the incident potential in Legendre functions we can therefore say that, on the metallic portions of the spherical surface,

$$\Phi(a, \theta, \varphi) = \frac{-q}{4\pi\epsilon_o r_o} \sum_{n=1}^{\infty} \Big\{ (a/r_o)^n P_n(\cos\theta) P_n(\cos\theta_o)$$

$$+ 2 \sum_{m=1}^{\infty} (a/r_o)^n P_n^m(\cos\theta) P_n^m(\cos\theta_o) \cos(\varphi - \varphi_o) \Big\} \qquad (96)$$

while Φ must be continuous and have a continuous radial derivative through the gap.

Since we are only interested in the total charge on (say) the top segment, we can neglect the φ dependent terms in equation (96) since they give, on integration, zero charge contributions. Thus we are led directly back to a variation of the problem treated in Section II-1. In fact, we can say immediately from equations (32), (37), (38), (39) and (40) that

$$\frac{Q_T}{a\epsilon_o} = 8 \int_0^{\alpha} j_1(u) \cos\frac{u}{2}\, du \qquad (97)$$

where

$$j_1(x) = g_1(x) - \frac{1}{\pi} \int_{\beta}^{\pi} \frac{2\cos x/2 \ \sin y/2}{\cos x - \cos y}\, j_2(y)\, dy \qquad (98)$$

$$j_2(x) = g_2(x) + \frac{1}{\pi} \int_0^{\alpha} \frac{2\sin y/2 \ \cos y/2}{\cos x - \cos y}\, j_1(y)\, dy \qquad (99)$$

where

$$g_1(x) = -\frac{1}{\sqrt{2}} \frac{d}{dx} \int_0^{x} \frac{9}{4\pi\epsilon_o r_o} \sum_{n=1}^{\infty} \frac{P_n(\cos\theta_o) P_n(\cos u)(a/r_o)^n}{\sqrt{\cos x - \cos u}}\, du \qquad (100)$$

$$g_2(x) = -\frac{1}{\sqrt{2}} \frac{d}{dx} \int_x^{\pi} \frac{q}{4\pi\epsilon_o r_o} \sum_{n=1}^{\infty} \frac{P_n(\cos\theta_o) P_n(\cos u)(a/r_o)^n}{\cos x - \cos u} du \tag{101}$$

The integrals defining $g_1(x)$ and $g_2(x)$ can be performed to give

$$g_1(x) = -\frac{q}{4\pi\epsilon_o r_o} \sum_{n=1}^{\infty} (a/r_o)^n P_n(\cos\theta_o) \cos[(n+\tfrac{1}{2})x] \tag{102}$$

$$g_2(x) = -\frac{q}{4\pi\epsilon_o r_o} \sum_{n=1}^{\infty} (a/r_o)^n P_n(\cos\theta_o) \sin[(n+\tfrac{1}{2})x] \tag{103}$$

We can therefore use the transformation leading to equations (42) and (43) to say that

$$\frac{Q_T}{q} = -\frac{2}{\pi} \sum_{n=1}^{\infty} (a/r_o)^{n+1} P_n(\cos\theta_o)\, x_n \tag{104}$$

where

$$x_n = \int_0^{\alpha} f_1^{(n)}(u) \cos\frac{u}{2}\, du \tag{105}$$

and $f_1^{(n)}(u)$ is determined by the coupled integral equations

$$f_1^{(n)}(x) = \cos[(n+\tfrac{1}{2})x] - \frac{2}{\pi} \int_0^{\pi-\beta} \frac{\cos x/2 \cos y/2}{\cos x + \cos y} f_2^{(n)}(y)\, dy \tag{106}$$

$$f_2^{(n)}(x) = (-)^n \cos[(n+\tfrac{1}{2})x] - \frac{2}{\pi} \int_0^{\alpha} \frac{\cos x/2 \cos y/2}{\cos x + \cos y} f_1^{(n)}(y)\, dy \tag{107}$$

or making the substitutions (44) through (47)

$$x_n = \int_0^{\tan\alpha/2} \frac{F_1^{(n)}(z)}{1+z^2}\, dz \tag{108}$$

where

$$F_1^{(n)}(z) = \frac{\cos\left[(2n+1)\tan^{-1} z\right]}{1+z^2} - \frac{2}{\pi} \int_0^{\tan\left(\frac{\pi-\beta}{2}\right)} \frac{F_2^{(n)}(z')}{1 - z^2 z'^2}\, dz' \tag{109}$$

$$F_2^{(n)}(z) = (-)^n \frac{\cos\left[(2n+1)\tan^{-1} z\right]}{1+z^2} - \frac{2}{\pi} \int_0^{\tan \alpha/2} \frac{F_1^{(n)}(z')}{1 - z^2 z'^2}\, dz' \tag{110}$$

Thus the x_n depend only on α and β (or θ_s and Δ) and need be calculated only once for a given slotted sphere.

The short circuit current, by differentiating equation (104), can be written in the form

$$I_{sc} = qv_r F_r(r_o, \theta_o) + qv_\theta F_\theta(r_o, \theta_o) \tag{111}$$

where

$$F_r(r_o, \theta_o) = -\frac{2}{\pi r_o} \sum_{n=1}^{\infty} (n+1)(a/r_o)^{n+1} P_n(\cos\theta_o)\, x_n \tag{112}$$

$$F_\theta(r_o, \theta_o) = -\frac{\;}{\pi r_o} \sum_{n=1}^{\infty} (a/r_o)^{n+1} P_n'(\cos\theta_o) \sin\theta_o\, x_n \tag{113}$$

If an electron is ejected along a radius and travels with uniform velocity V to infinity we have

$$I_{sc}(t) = \frac{2eV}{\pi a} \sum_{n=1}^{\infty} (n+1) \frac{P_n(\cos\theta_o)\, x_n}{(1+Vt/a)^{n+2}}\, U(t) \tag{114}$$

and thus, from equations (2) the current through a resistive load R when the gap capacitance is C would be

$$I(t) = \frac{2eV}{\pi a} \sum_{n=1}^{\infty} (n+1) P_n(\cos\theta_o)\, x_n\, G_n(t) \tag{115}$$

where

$$G_n(t) = \int_0^t e^{-\left(\frac{t-t'}{RC}\right)} (1 + V t'/a)^{-(n+2)} \, dt' \tag{116}$$

2. NARROW SLOTS

For narrow slots we can proceed somewhat differently from the work of the previous subsection just as in the previous section the narrow slot approximation to capacitance was obtained by looking at the problem anew.

If the slot is extremely narrow we may just as well close it altogether and calculate the total current crossing an azimuthal line on a closed sphere by a charge moving in the vicinity.

The current across an azimuthal slot can be obtained by computing the time derivative of the charge on one segment of the sphere (only the electrostatic problem is involved since we are interested in total currents across lines, not current densities).

By solving Laplace's equation one can show easily that the charge density on a closed sphere with zero net charge (which is an adequate assumption from the discussion in the previous subsection) when a point charge q is at position $(r_o, \theta_o, \varphi_o)$ is given by

$$\sigma(\theta, \varphi) = -\frac{q}{4\pi a r_o} \sum_{n=1}^{\infty} \left(\frac{a}{r_o}\right)^n (2n+1)$$

$$\left\{P_n(\cos\theta) P_n(\cos\theta_o) + 2 \sum_{m=1}^{\infty} P_n^m(\cos\theta) P_n^m(\cos\theta_o) \cos m(\varphi - \varphi_1)\right\} \tag{117}$$

and thus the total charge on the top segment $(a < \theta < \alpha)$ is given by

$$Q_T = -\frac{q}{2} \sum_{n=1}^{\infty} (2n+1)(a/r_o)^n P_n(\cos\theta_o) \int_0^{\alpha} P_n(\cos\theta) \sin\theta \, d\theta \tag{118}$$

or, using equation (66)

$$Q_T = -\frac{q}{2} \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} \left(\frac{a}{r_o}\right)^{n+1} \sin^2\alpha \; P_n(\cos\theta_o) P_n'(\cos\alpha) \tag{119}$$

Thus, taking derivatives, we can again write the short-circuit current in the form

$$I_{sc} = qv_r F_r(r_o, \theta_o) + qv_\theta F_\theta(r_o, \theta_o) \tag{120}$$

where now

$$F_r(r_o, \theta_o) = \frac{1}{r_o} \sum_{n=1}^{\infty} \left(1 + \frac{1}{2n}\right) \left(\frac{a}{r_o}\right)^{n+1} \sin^2\alpha \; P_n'(\cos\alpha) \; P_n(\cos\theta_o) \tag{121}$$

$$F_\theta(r_o, \theta_o) = \frac{1}{r_o} \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} \left(\frac{a}{r_o}\right)^{n+1} P_n'(\cos\theta_o) \sin\theta_o \; P_n'(\cos\alpha) \sin^2\alpha \tag{122}$$

and we can identify the x_n of the previous subsection, for narrow slots, to be

$$x_n = \frac{\pi}{4} \frac{(2n+1)}{n(n+1)} \sin^2\alpha \; P_n'(\cos\alpha) \tag{123}$$

This explicit form for x_n can be therefore used in any of the current calculations of the previous subsection, provided the slot is narrow enough.

As a simple explicit example, suppose the electron is ejected radially along the θ_n axis. Then from equation (119) we have

$$\bar{Q}_T = +\frac{q}{2} \sum_{n=1}^{\infty} \left(\frac{a}{r_o}\right)^{n+1} [P_{n+1}(\cos\alpha) - P_{n-1}(\cos\alpha)] \tag{124}$$

$$= \frac{q}{2} \left\{ \left[1 - \frac{a}{r_o}\right]^2 \frac{1}{\sqrt{1 - \frac{2a}{r_o} \cos\alpha + \frac{a}{r_o}^2}} - \left(1 + \frac{a}{r_o} \cos\alpha\right) \right\} \tag{125}$$

or if $\alpha = \pi/2$

$$Q_T = \frac{q}{2}\left\{\frac{1 - (a/r_o)^2}{\sqrt{1 + (a/r_o)^2}} - 1\right\} \tag{126}$$

giving a short-circuit current, in this case, of

$$I_{sc} = \frac{qv_r}{2r_o} \cdot \frac{(a/r_o)^2}{\sqrt{1 + (a/r_o)^2}} \left\{2 + \frac{1 - (a/r_o)^2}{1 + (a/r_o)^2}\right\} U(t) \tag{127)a}$$

or if v_r is constant (= V)

$$I_{sc}(t) = \frac{qV}{2a} \frac{f^3(t)}{\sqrt{1 + f^2(t)}} \left\{\frac{3 + f^2}{1 + f^2}\right\} U(t) \tag{127)b}$$

where

$$f(t) = \frac{1}{1 + \frac{Vt}{a}} \tag{128}$$

Thus

$$I_{sc}(0) = \frac{qV}{\sqrt{2}a} \tag{129}$$

and

$$\frac{I_{sc}(t)}{I_{sc}(0)} = \frac{f^3(t)}{\sqrt{2(1 + f^2(t))}} \left(\frac{3 + f^2(t)}{1 + f^2(t)}\right) \tag{130}$$

or, measuring t in units of a/V

$$\frac{I_{sc}(\tau)}{I_{sc}(0)} = \frac{1}{\sqrt{2[(1 + \tau)^2 + 1]}} \cdot \frac{1}{(1 + \tau)^2} \cdot \frac{3(1 + \tau)^2 + 1}{(1 + \tau)^2 + 1} \tag{131}$$

which is plotted in Figure 5, since this is a typical short-circuit current forcing function. The effect of such a short-circuit current on the current through an impedance load can be easily determined from equations such as (2) and (3) of Section I.

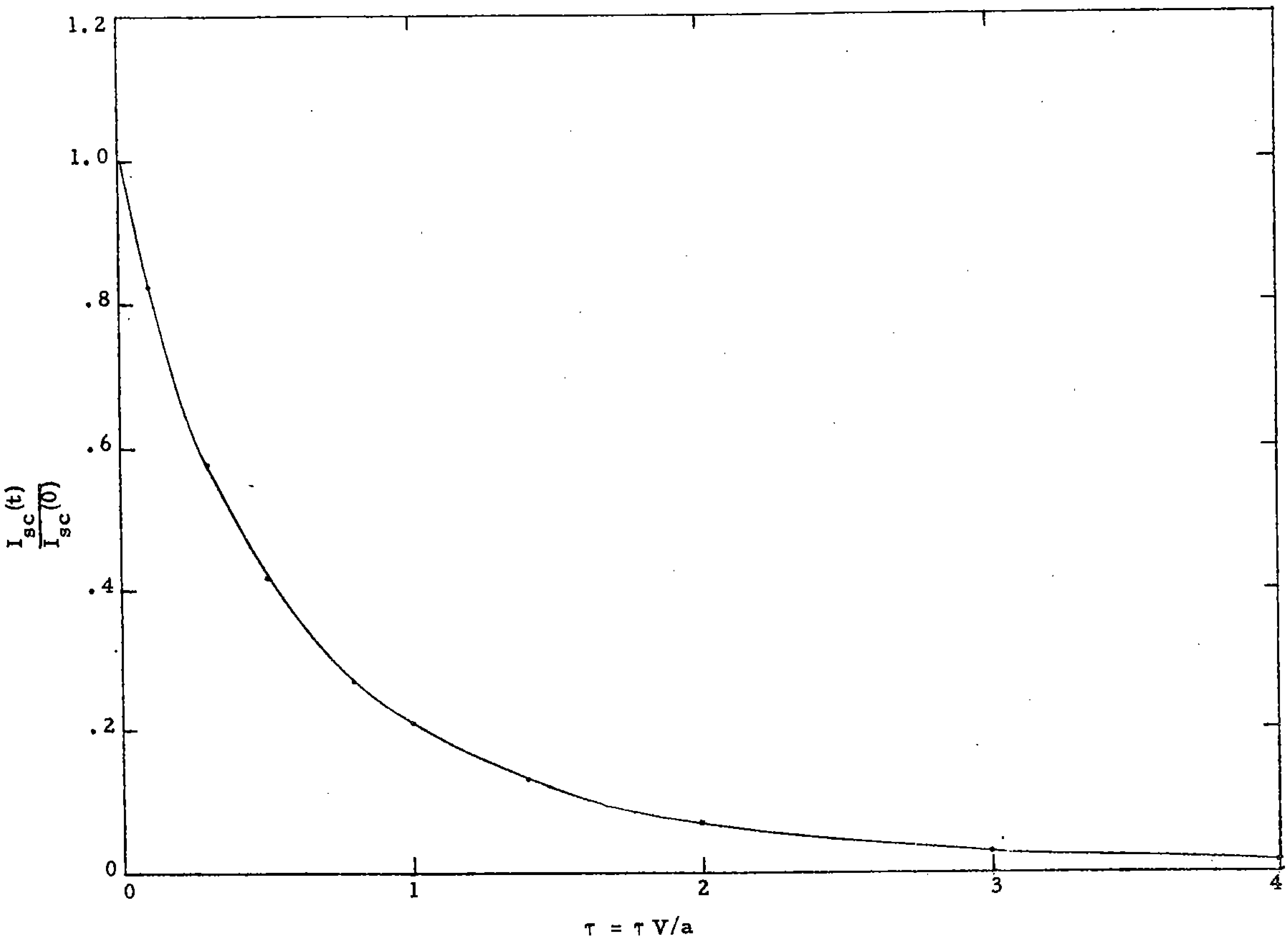

Figure 5: I_{sc} for a charge traveling uniformly along the polar axis

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