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Transient Signal Propagation in Lossless, Isotropic Plasmas Volume II

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Abstract

The asymptotic behavior of transient signal propagation in lossless, isotropic plasmas is discussed at length for a typical input signal of a step modulated sine wave. A generalized saddlepoint integration is carried out that gives a continuous solution for the dispersed signal everywhere except at the signal wavefront. The solution for the wavefront is obtained by using a high-frequency expansion technique. Universal curves are presented for the behavior of the distorted signal as a function of the plasma frequency, signal frequency, and propagation distance. The solution is a very good approximation for plasma propagation lengths that are long compared to a wavelength.

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Transient Signal Propagation in Lossless, Isotropic Plasmas

Volume II

1. INTRODUCTION

In a previous report on this subject (Haskell and Case, 1966) numerical solutions were obtained for the propagation of signals in a lossless, isotropic plasma. These numerical integrations could be carried out for propagation distances of a few wavelengths at most. When the distance the wave propagates through the plasma is a very large number of wavelengths, some alternate methods of solution must be found. It is the purpose of this report to describe asymptotic solutions which are valid for long propagation distances.

Early work along these lines was carried out by Sommerfeld (1914) who showed by making a high-frequency expansion that the very first part of a signal, called the signal wavefront, arrives at a given point with the velocity of light, c . Sommerfeld's solution, which is described in detail in Section 2, is valid only for a short time after the arrival of the signal wavefront. By using a saddlepoint method of integration Brillouin (1914) found solutions that are valid in a certain time interval following the Sommerfeld region. The signal in this region is called a precursor since it preceeds the arrival of the main signal. This work of Sommerfeld and Brillouin has been summarized in a book by Brillouin (1960) in which some of the important early papers have been reprinted.

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The arrival of the main signal follows the precursors. However, the standard saddlepoint method of integration must be modified in this region since the saddlepoint is approaching a pole in the complex plane. Methods for appropriately modifying the saddlepoint method under these conditions have been discussed by Cerillo (1950), Clemmow (1950), and Van der Waerden (1950). Cerillo's method involves an expansion about the pole and leads to the so-called coinciding pole solution. This solution is discussed in Appendix C of this report. This method has the disadvantage that it is only valid in the immediate vicinity of the pole. Following the method of Clemmow (1950), Pearson (1953) treated the transient propagation of sound waves in acoustic waveguides. This method leads to a generalized saddlepoint solution which is valid for all times following the arrival of the signal wavefront and is continuous as the saddlepoint crosses the pole.

The method of Van der Waerden (1950) involves a transformation which maps the saddlepoints in the original complex plane into branch points in a new complex plane. The evaluation of the integrals is usually somewhat more cumbersome by using this method. Karbowiak (1957) attempted to use this method to describe the transient propagation of electromagnetic waves in waveguides. However, his original contours of integration violated causality and, as a result, he only included one of the saddlepoints in the problem. His results therefore only include part of the solution. In particular, his prediction of large so-called B-precursors immediately following the arrival of the signal wavefront is incorrect.

The problem of obtaining asymptotic solutions for the transient propagation of electromagnetic waves in waveguides has also been considered by Cerillo (1948) and Namiki and Horiuchi (1952). Both of these treatments make a conformal transformation of a trigonometric nature in the original integral which is unnecessary. In neither case is the problem of the saddlepoint near the pole treated in a manner which is valid over long intervals of time.

In Appendix A of this report the general method of saddlepoint integration is described in detail. This general solution includes the case of a saddlepoint crossing a pole of order n. This case leads to solutions which are related to Fresnel integrals. These results are applied to the problem of the propagation of a turn-on sine wave in a lossless, isotropic plasma. The general asymptotic solution is obtained in Section 3. Approximations to this general solution which are valid over certain intervals of time are discussed in Section 4. It is found that the general solution can be broken down into an anterior transient solution which is valid in the region before the saddlepoint crosses the pole, the main signal build-up which occurs when the saddlepoint is in the neighborhood of the pole, and a posterior transient solution which holds in a region after the saddlepoint has crossed the pole. General curves are presented for the solution in each of these regions and the total response is plotted for a typical case.

The problem of transient signal propagation in a lossless, isotropic plasma may be formulated in the following way. Maxwell's equations which describe the propagation of electromagnetic waves in an isotropic plasma are

$$\text{curl } \underset{\sim}{\mathcal{E}} = -\mu_o \frac{\partial \underset{\sim}{\mathcal{H}}}{\partial t}$$
$$\text{curl } \underset{\sim}{\mathcal{H}} = \underset{-}{J} + \epsilon_o \frac{\partial \underset{\sim}{\mathcal{E}}}{\partial t} \tag{1}$$

where

$$J = -N e \underset{\sim}{v} \quad , \tag{2}$$

N being the electron number density and $\underset{\sim}{v}$ being determined from the equation of motion

$$\frac{\partial \underset{\sim}{v}}{\partial t} = -\frac{e}{m} \underset{\sim}{\mathcal{E}} . \tag{3}$$

Consider the one-dimensional problem in which $\underset{\sim}{\mathcal{E}}$ is linearly polarized in the x_1-direction and is propagating in the x_3-direction. Let the x_1-component of $\underset{\sim}{\mathcal{E}}(t, x_3)$ be written as $\mathcal{E}(t, x_3)$ and the Laplace transform of $\mathcal{E}(t, x_3)$ as $E(s, x_3)$. If one then takes a Laplace transform in time of Eqs. (1) through (3) and solves for $E(s, x_3)$ setting all initial conditions equal to zero, one readily obtains the equation

$$\frac{d^2}{d x_3^2} E(s, x_3) - \left(\frac{s^2 + \Pi^2}{c^2}\right) E(s, x_3) = 0 \tag{4}$$

where

$$\Pi = \left(\frac{Ne^2}{\epsilon_o m}\right)^{1/2}$$

is the plasma frequency.

The problem is to determine the time response $\mathcal{E}(t, x_3)$ in the semi-infinite region $x_3 > 0$ when the time response $\mathcal{E}(t, 0)$ is prescribed at $x_3 = 0$. The solution of Eq. (4) may then be written

$$E(s, x_3) = E(s, 0) \exp \left\{ -\frac{x_3}{c} (s^2 + \Pi^2)^{1/2} \right\} \tag{5}$$

where $E(s,0)$ is the Laplace transform of $\mathcal{E}(t,0)$. The time response $\mathcal{E}(t,x_3)$ is then obtained by taking the inverse of Eq. (5). That is,

$$\mathcal{E}(t,x_3) = \frac{1}{2\pi i} \int_{\gamma_s} E(s,0) \exp\left\{ st - \frac{x_3}{c}\left(s^2 + \Pi^2\right)^{1/2} \right\} ds \tag{6}$$

where the contour γ_s is a line in the complex s plane from $c_o - i\infty$ to $c_o + i\infty$ and C_o is to the right of all singularities. This condition is required by the causal nature of the solutions. This report will deal with the propagation of a turn-on sine wave with a carrier frequency ω_o. It will be convenient to normalize all quantities to ω_o by introducing the following parameters:

$$\begin{aligned} \tau &= \omega_o t \\ \eta &= \frac{\omega_o x_3}{c} \\ P &= \frac{\Pi}{\omega_o} \\ z &= \frac{s}{\omega_o} = x + iy \quad . \end{aligned} \tag{7}$$

Using these definitions, the integral Eq. (6) may be rewritten as

$$\mathcal{E}(\tau,\eta) = \frac{1}{2\pi i} \int_{\gamma_s} \omega_o E(z,0) \exp\left\{ z\tau - \eta\left(z^2 + P^2\right)^{1/2} \right\} dz \tag{8}$$

or, letting

$$\xi = \frac{\tau}{\eta} \tag{9}$$

as

$$\mathcal{E}(\xi,\eta) = \frac{1}{2\pi i} \int_{\gamma_z} \omega_o E(z,0) \exp\left\{ \eta \left[\xi z - \left(z^2 + P^2\right)^{1/2} \right] \right\} dz \quad . \tag{10}$$

2. SOMMERFELD SOLUTION FOR THE ARRIVAL OF THE SIGNAL WAVEFRONT

In order to describe the arrival of the signal wavefront, the integral in Eq. (10) will be evaluated by making a high-frequency expansion. Consider a unit step sine wave input for which

$$E(s,0) = \frac{\omega_o}{s^2 + \omega_o^2}$$

so that

$$\omega_o E(z,0) = \frac{1}{z^2 + 1} \quad .$$

Equation (10) can then be written as

$$\mathcal{E}(\xi, \eta) = \frac{1}{2\pi i} \int_{\gamma_z} \frac{\exp\left\{ z \left[\xi\eta - \eta \left(1 + \frac{P^2}{z^2}\right)^{1/2} \right] \right\}}{z^2 + 1} \, dz \quad . \tag{11}$$

If the radical in the exponent of Eq. (11) is expanded in powers of P^2/z^2, keeping only terms up to first order, then by using Eq. (9) the exponent can be written as

$$\begin{aligned} z \left[\xi\eta - \eta \left(1 + \frac{P^2}{z^2}\right)^{1/2} \right] &= z(\tau - \eta) - \frac{\eta P^2}{2z} \\ &= zr - \frac{q}{r} \end{aligned} \tag{12}$$

where

$$\begin{aligned} r &= \tau - \eta \quad , \\ q &= \frac{\eta P^2}{2} \quad . \end{aligned} \tag{13}$$

Making the further approximation that $z^2 \gg 1$ (that is, $s^2 \gg \omega_o^2$), one can then write Eq. (11) approximately as

$$\mathcal{E}(\tau, \eta) = \frac{1}{2\pi i} \int_{\gamma_z} \frac{\exp\left\{ zr - \frac{q}{z} \right\}}{z^2} \, dz \quad . \tag{14}$$

For $r < 0$, the contour γ_z can be closed in the right half of the z-plane for which case $\mathcal{E}(\tau, \eta) = 0$. That is, no signal arrives prior to $r = 0$ or $\tau = \eta$. For $r > 0$ the contour γ_z can be closed in the left half plane. The integral along the path at $x = -\infty$ vanishes so that Eq. (14) can be written as a closed path for this particular contour. Furthermore, this contour can be deformed into a large circular path so long as we encounter no singularities in the deformation. If the radius of the large circle is selected to be $\sqrt{q/r}$, so that

$$z = \sqrt{\frac{q}{r}}\, e^{i\theta} \tag{15}$$

$$dz = i\sqrt{\frac{q}{r}}\, e^{i\theta}\, d\theta \quad ,$$

then

$$zr - \frac{q}{z} = 2i\sqrt{rq}\, \sin\theta \quad . \tag{16}$$

The integration is to be performed over an interval of 2π in θ . It is convenient to select the interval $-\frac{\pi}{2} < \theta \leq \frac{3\pi}{2}$. Using Eqs. (15) and (16) one may then write Eq. (14) as

$$\mathcal{E}(\tau, \eta) = \frac{1}{2\pi}\sqrt{\frac{r}{q}} \int_{-\pi/2}^{3\pi/2} \exp\left\{ i\left[2\sqrt{rq}\, \sin\theta - \theta\right]\right\} d\theta \quad .$$

Making the further transformation

$$\theta = \psi - \frac{\pi}{2}$$

$$\sin\theta = -\cos\psi \quad ,$$

one obtains

$$\mathcal{E}(\tau, \eta) = \frac{i}{2\pi}\sqrt{\frac{r}{q}} \int_{0}^{2\pi} \exp\left\{ -i\left[2\sqrt{rq}\, \cos\psi + \psi\right]\right\} d\psi \quad . \tag{17}$$

Using the identity (see, for example, McLachlan (1955) p. 192)

$$J_1(2\sqrt{rq}) = \frac{i}{2\pi}\int_0^{2\pi} \exp\left\{-i\left[2\sqrt{rq}\cos\psi + \psi\right]\right\} d\psi$$

one may write Eq. (17) as

$$\mathcal{E}(\tau,\eta) = \sqrt{\frac{r}{q}}\, J_1(2\sqrt{rq}) \; ; \tag{18}$$

or, using Eqs. (13) and (9)

$$\mathcal{E}(\tau,\eta) = \sqrt{2}\,\frac{(\xi-1)^{1/2}}{P}\, J_1\left(\sqrt{2P^2\eta^2(\xi-1)}\right) U(\xi-1) \quad . \tag{19}$$

By letting

$$u = 2P^2\eta(\tau-\eta) \quad , \tag{20}$$

Eq. (19) can be rewritten as

$$P^2\eta\,\mathcal{E}(\tau,\eta) = \sqrt{u}\,J_1(\sqrt{u})\,U(u) \quad . \tag{21}$$

Equation (21) represents a universal curve for the initial arrival of the signal. This curve is plotted in Figure 1.

In order to investigate the range of validity of Eqs. (19) and (21), note from Eq. (15) that

$$|z| = \sqrt{\frac{q}{r}} \quad .$$

The approximations used in the integration of Eq. (11) were

$$|z|^2 \gg P^2$$

and

$$|z|^2 \gg 1 \quad .$$

Since P is normally less than unity (for an underdense plasma) the second condition is the more stringent and is equivalent to

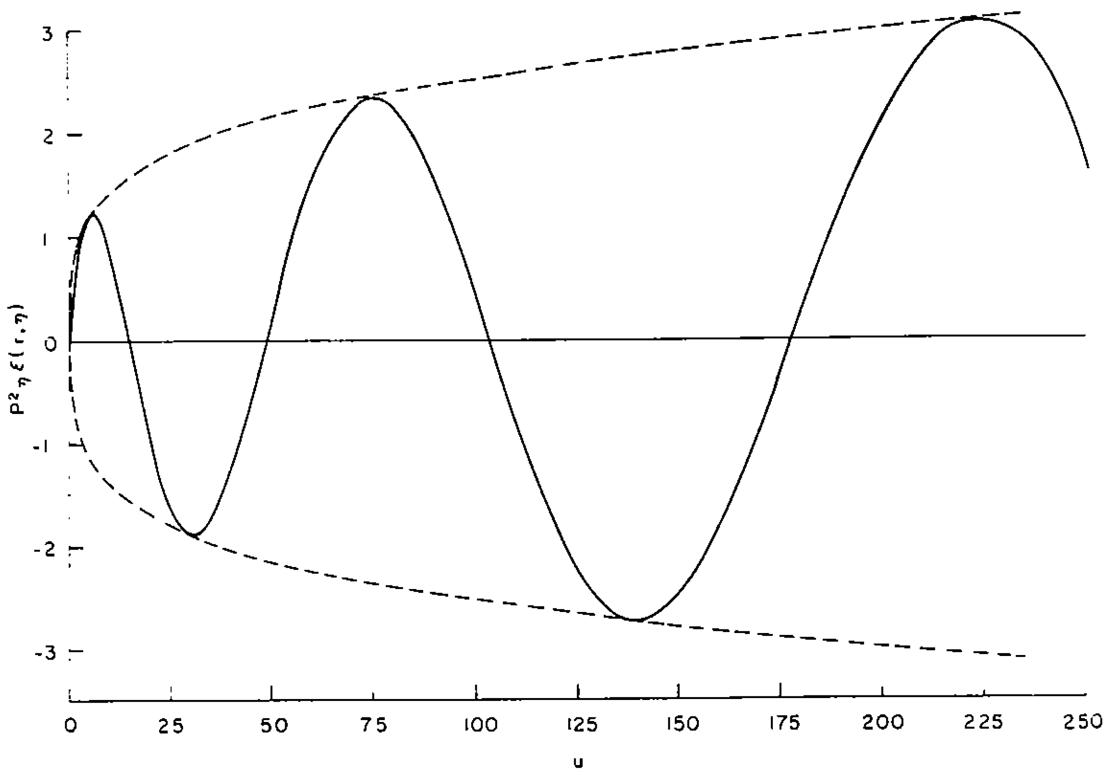

Figure 1. Universal Curve for the Sommerfeld Solution

$$\frac{q}{r} >> 1 \qquad \text{or} \qquad r << q \quad . \tag{22}$$

Using the definitions given by Eq. (13) condition (22) reduces to

$$\tau - \eta << \frac{P^2\eta}{2} \quad . \tag{23}$$

This condition shows that Eqs. (19) and (21) will work well for the front of the signal if η is large. However, as τ increases to the point where condition (23) is no longer valid the expression given by Eq. (19) breaks down. In the following sections the form of the signal for these later times will be determined by using a saddlepoint method of integration.

3. ASYMPTOTIC SOLUTIONS

The electric field intensity $\mathcal{E}(\xi,\eta)$ as a function of normalized time $\xi = \frac{\tau}{\eta}$ and distance $\eta = (\omega_o x_3)/c$ can be written from Eq. (10) in the form

$$\mathcal{E}(\xi,\eta) = \frac{1}{2\pi i} \int_{\gamma_z} F(z) \exp\left[\eta\, w(z)\right] dz \tag{24}$$

where

$$F(z) = \omega_o E(z,0) \tag{25}$$

and

$$\begin{aligned} w(z) &= \xi z - \left(z^2 + P^2\right)^{1/2} \\ &= z\left[\xi - \left(1 + \frac{P^2}{z^2}\right)^{1/2}\right] . \end{aligned} \tag{26}$$

Asymptotic solutions of Eq. (24) can be obtained for large η by using the saddlepoint method of integration. Although ξ is technically a function of η, the results of the previous section show that τ must always be greater than η so that ξ is a time parameter which is equal to one at the time of arrival of the signal wavefront and increases with time thereafter. One can therefore consider $w(z)$ given by Eq. (26) to be independent of η.

The saddlepoint method of integration is described in Appendix A. Differentiating Eq. (26) with respect to z one obtains

$$\begin{aligned} w'(z) &= \xi - z\left(z^2 + P^2\right)^{-1/2} \\ &= \xi - \left(1 + \frac{P^2}{z^2}\right)^{-1/2} . \end{aligned} \tag{27}$$

The saddlepoints are found by setting $w'(z) = 0$ and are given by

$$z_o = \frac{\pm i P \xi}{\sqrt{\xi^2 - 1}} . \tag{28}$$

There are therefore two saddlepoints, both of which lie on the imaginary axis and move with time. At $\xi = 1$ ($\tau = \eta$) the saddlepoints are at $\pm i\infty$ and as time proceeds they move in along the imaginary axis and approach $\pm i P$ for long times. Figure 2a shows the location of the saddlepoints before they have crossed the poles at $\pm i$ while Figure 2b shows the saddlepoints after they have crossed the poles.

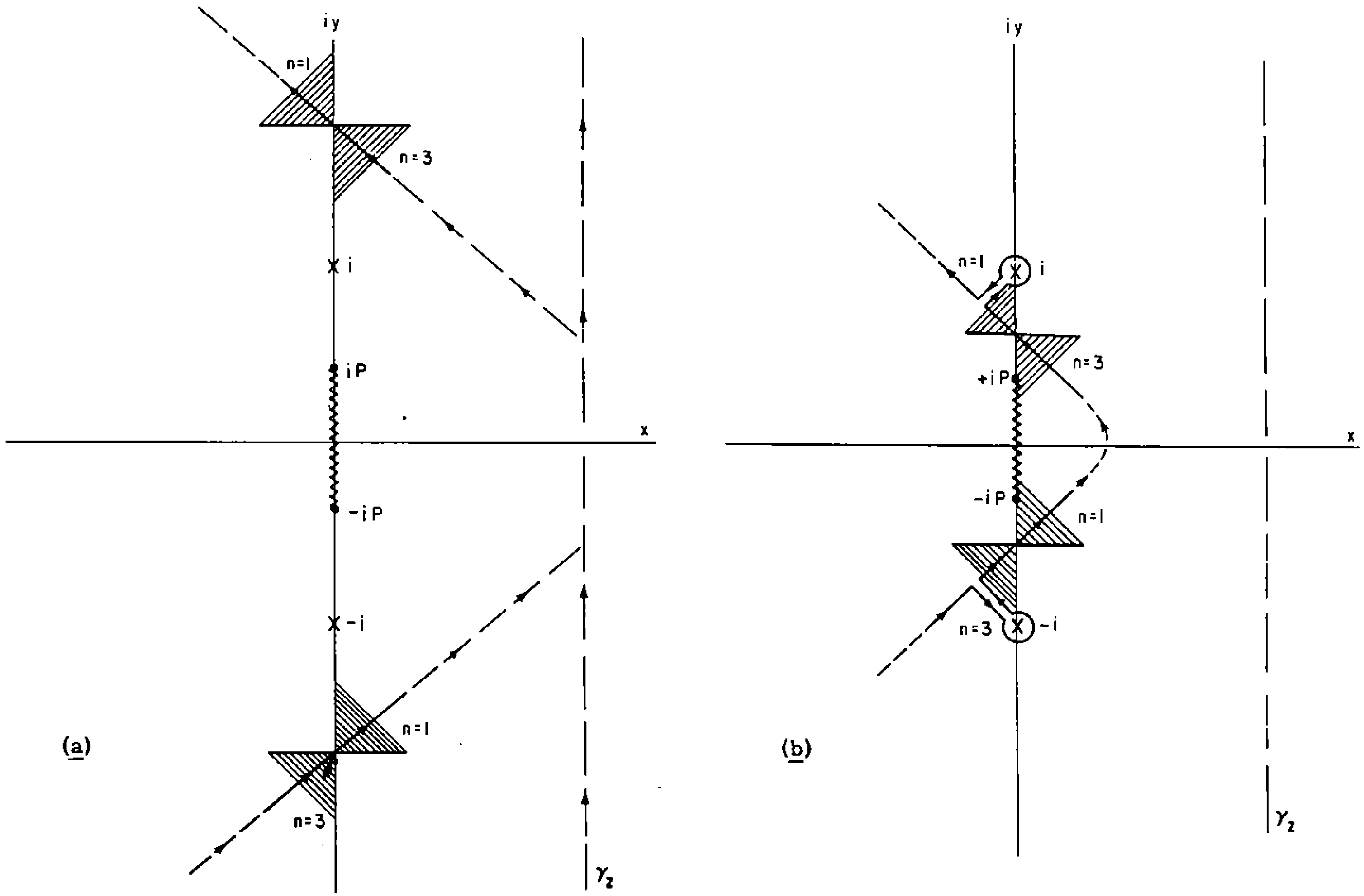

Figure 2. Location of the Saddlepoints and the Paths of Steepest Descent

In order to determine the lines of steepest descent, $w''(z_o)$ must be evaluated. Differentiating Eq. (27) one finds that

$$w''(z) = -P^2 z^{-3}\left(1 + \frac{P^2}{z^2}\right)^{-3/2} \tag{29}$$

At the saddlepoint $z_o^+ = \dfrac{+iP\xi}{\sqrt{\xi^2 - 1}}$

$$w''(z_o^+) = \frac{(\xi^2 - 1)^{3/2}}{P} \exp\left\{-i\frac{\pi}{2}\right\} . \tag{30}$$

Thus referring to Eqs. (A-4) and (A-8) in Appendix A, one sees that the lines of steepest descent are given by $\theta_s = \frac{3\pi}{4}$ (n=1) and $\theta_s = \frac{7\pi}{4}$ (n=3) . Similarly, at the saddlepoint $z_o^- = -iP\xi/\sqrt{\xi^2 - 1}$

$$w''(z_o^-) = \frac{(\xi^2 - 1)^{3/2}}{P} \exp\left\{i\frac{\pi}{2}\right\} \tag{31}$$

from which the lines of steepest descent are given by $\theta_s = \pi/4$ (n=1) and $\theta_s = \frac{5\pi}{4}$ (n=3) . The lines of steepest descent are shown in Figure 2 where the hatched regions correspond to the valleys of the surface $u = u(x,y)$. Along the lines of steepest descent $v(x,y)$ (the imaginary part of $w(z)$) is constant. The straight line segments passing through the saddlepoints in Figure 2 are the lines of steepest descent only in the immediate vicinity of the saddlepoints. Note that if the contour of integration γ_z is deformed to pass through the saddlepoints, the path of integration approaches each saddlepoint along the n=3 path and leaves each saddlepoint along the n=1 path. This was the assumption used in Appendix A, so the results of Appendix A may be used directly.

Consider the time response at $x_3=0$ to be a turn-on sine wave, that is, $\mathcal{E}(t,0) = U(t) \sin \omega_o t$. The Laplace transform is $E(s,0) = \omega_o/(s^2 + \omega_o^2)$ from which, by using Eqs. (25) and (7)

$$F(z) = \frac{1}{z^2 + 1} = \frac{i}{2}\left[\frac{1}{z+i} - \frac{1}{z-i}\right] . \tag{32}$$

Equation (24) then becomes

$$\mathcal{E}(\xi,\eta) = \frac{1}{4\pi}\left\{\int\limits_{\gamma_z} \frac{\exp\left[\eta\left(\xi z - [z^2+P^2]^{1/2}\right)\right]}{z+i}\,dz - \int\limits_{\gamma_z} \frac{\exp\left[\eta\left(\xi z - [z^2+P^2]^{1/2}\right)\right]}{z-i}\,dz\right\} \tag{33}$$

Equation (33) may be written as

$$\mathcal{E}(\xi,\eta) = \frac{1}{4\pi}\left\{ I_a - I_b + I_c - I_d \right\} \tag{34}$$

where

$$I_a = \int_{x_o}^{x_o+i\infty} \frac{\exp[\eta\, w(z)]}{z+i}\,dz \tag{35a}$$

$$I_b = \int_{x_o-i\infty}^{x_o} \frac{\exp[\eta\, w(z)]}{z-i}\,dz \tag{35b}$$

$$I_c = \int_{x_o-i\infty}^{x_o} \frac{\exp[\eta\, w(z)]}{z+i}\,dz \tag{35c}$$

$$I_d = \int_{x_o}^{x_o+i\infty} \frac{\exp[\eta\, w(z)]}{z-i}\,dz \tag{35d}$$

The location of the saddlepoints is given by Eq. (28), so if the paths of integration of the integrals in Eq. (35) are deformed through the saddlepoints, the paths of integration for I_a and I_d go through z_o^+ while the paths of integration for I_b and I_c go through z_o^-.

Note that

$$w(z_o^+) = i\,P\sqrt{\xi^2 - 1}$$
$$w(z_o^-) = -\,i\,P\sqrt{\xi^2 - 1} \quad . \tag{36}$$

Since the integrands in I_a and I_b are always analytic over the range of integration, then I_a and I_b can be evaluated immediately using Eq. (A-19) together with Eqs. (30), (31), and (36). One obtains

$$I_a = \frac{2\pi i}{\sqrt{2\pi}} \frac{(P/\eta)^{1/2}}{(\xi^2-1)^{3/4}} \left(\frac{-i}{1 + \frac{P\xi}{\sqrt{\xi^2-1}}} \right) \exp\left\{ i\left(P\eta\sqrt{\xi^2-1} + \frac{\pi}{4} \right) \right\} \tag{37}$$

$$I_b = \frac{2\pi i}{\sqrt{2\pi}} \frac{(P/\eta)^{1/2}}{(\xi^2-1)^{3/4}} \left(\frac{i}{1 + \frac{P\xi}{\sqrt{\xi^2-1}}} \right) \exp\left\{ -i\left(P\eta\sqrt{\xi^2-1} + \frac{\pi}{4} \right) \right\} \tag{38}$$

which can be combined to give

$$I_a - I_b = 2\sqrt{2\pi}\left(\frac{P}{\eta}\right)^{1/2} \frac{\cos\left(P\eta\sqrt{\xi^2-1} + \pi/4\right)}{(\xi^2-1)^{3/4}\left(1 + \frac{P\xi}{\sqrt{\xi^2-1}}\right)} \quad . \tag{39}$$

Since the saddlepoint crosses a pole in the integration of I_c , this integral can be evaluated by using Eq. (A-30) from Appendix A , which is rewritten as

$$I_1 = \frac{\exp\left[\eta\, w(z_o)\right]}{2\pi i}\, 2\sqrt{\pi}\, \exp\left[-a\beta^2\right] \frac{\beta}{|\beta|} \int_{|\beta|\sqrt{a}}^{\infty} \exp\left\{ \left(\frac{\beta}{|\beta|}\, y \right)^2 \right\} dy \tag{40}$$

where

$$a = \frac{\eta\,|w''(z_o)|}{2}$$

$$\beta = i\, \exp\left(i\frac{\alpha}{2}\right)\,(z_o - z_k) \quad .$$

The path of integration in I_c is deformed through the saddlepoint z_o^- , so from Eqs. (31) and (28):

$$a = \frac{\eta(\xi^2-1)^{3/2}}{2P}$$

$$\beta = \exp\left[i\,\frac{\pi}{4}\right]\left(\frac{P\xi}{\sqrt{\xi^2-1}} - 1\right) \quad . \tag{41}$$

Observe from Eq. (41) that the pole and the saddlepoint coincide (that is, $\beta = 0$) for

$$\xi_g = \frac{1}{\sqrt{1 - P^2}} \quad . \tag{42}$$

Note from Figure 2 for $\xi < \xi_g$, the solution for I_c involves the contribution from the saddlepoint alone; while for $\xi > \xi_g$, I_c has a contribution from a pole residue in addition to the saddlepoint contribution.

Define

$$B = |\beta|\sqrt{a} \quad . \tag{43}$$

For $\xi < \xi_g$, $\frac{P\xi}{\sqrt{\xi^2-1}} > 1$, the following relations hold

$$B = \left(\frac{\eta}{2P}\right)^{1/2}\left(\xi^2-1\right)^{3/4}\left(\frac{P\xi}{\sqrt{\xi^2-1}} - 1\right)$$

$$a\beta^2 = i\,B^2 \tag{44}$$

$$\frac{\beta}{|\beta|} = \exp\left[i\,\frac{\pi}{4}\right] .$$

Using Eqs. (36), (40) and (44), one can write I_c for $\xi < \xi_g$ as

$$I_c = 2\sqrt{\pi}\,\exp\left\{-i\left(P\eta\sqrt{\xi^2-1} + B^2 - \frac{\pi}{4}\right)\right\}\int_B^{\infty}\exp\left[i u^2\right]\,du \quad .$$

Let

$$v = -\sqrt{\frac{2}{\pi}}\,B \quad . \tag{45}$$

Then, using Eqs. (B-8), (B-9), and (B-15), one can write

$$I_c = i\pi \exp\left\{-i\left(P\eta\sqrt{\xi^2-1} + \frac{\pi}{2}v^2\right)\right\}\left[1 + (1-i)\left\{C(v) + i\,S(v)\right\}\right] \quad . \tag{46}$$

For $\xi > \xi_g$, $\frac{P\xi}{\sqrt{\xi^2-1}} < 1$ note that

$$\frac{\beta}{|\beta|} = -\exp\left(i\frac{\pi}{4}\right) \quad .$$

Then the value of I_c for this range of ξ is the sum of the contribution from the pole residue and the saddlepoint contribution by Eq. (40). That is:

$$I_c = 2\pi i \exp\left\{-i\eta\left(\xi - \sqrt{1-P^2}\right)\right\}$$
$$- 2\sqrt{\pi}\exp\left\{-i\left(P\eta\sqrt{\xi^2-1} + B^2 - \frac{\pi}{4}\right)\right\}\int_B^\infty \exp\left[iu^2\right] du \tag{47}$$

where

$$B = \left(\frac{\eta}{2P}\right)^{1/2}\left(\xi^2-1\right)^{3/4}\left(1 - \frac{P\xi}{\sqrt{\xi^2-1}}\right) \quad . \tag{48}$$

Letting

$$v = \sqrt{\frac{2}{\pi}}\, B \tag{49}$$

one obtains

$$I_c = 2\pi i \exp\left\{-i\eta\left(\xi - \sqrt{1-P^2}\right)\right\}$$
$$- i\pi \exp\left\{-i\left(P\eta\sqrt{\xi^2-1} + \frac{\pi}{2}v^2\right)\right\}\left[1 - (1-i)\left\{C(v) + i\,S(v)\right\}\right] \quad . \tag{50}$$

For $\xi = \xi_g$, note that $v = B = 0$ and from Eq. (42)

$$P\eta\sqrt{\xi_g^2 - 1} = \frac{\eta P^2}{\sqrt{1-P^2}} \quad .$$

Thus Eq. (46) becomes

$$I_c(\xi=\xi_g) = i\pi \exp\left\{- i \frac{\eta P^2}{\sqrt{1-P^2}}\right\} . \tag{51}$$

Likewise for $\xi = \xi_g$

$$\eta\left[\xi_g - \sqrt{1-P^2}\right] = \frac{\eta P^2}{\sqrt{1-P^2}}$$

so that Eq. (50) becomes

$$I_c(\xi=\xi_g) = i\pi \exp\left\{- i \frac{\eta P^2}{\sqrt{1-P^2}}\right\} . \tag{52}$$

Thus the solution is continuous when the saddlepoint crosses the pole.

In summary, for $\xi \le \xi_g$

$$I_c = i\pi \exp\left\{- i\left(P\eta\sqrt{\xi^2-1} + \frac{\pi}{2} v^2\right)\right\}\left[1 + (1-i)\left\{C(v) + i\, S(v)\right\}\right] . \tag{53}$$

For $\xi \ge \xi_g$

$$I_c = 2\pi i \exp\left\{- i\eta\left(\xi - \sqrt{1-P^2}\right)\right\}$$
$$- i\pi \exp\left\{- i\left(P\eta\sqrt{\xi^2-1} + \frac{\pi}{2} v^2\right)\right\}\left[1 - (1-i)\left\{C(v) + i\, S(v)\right\}\right] \tag{54}$$

where

$$v = \sqrt{\frac{2}{\pi}}\left(\frac{\eta}{2P}\right)^{1/2}\left(1 - \frac{P\xi}{\sqrt{\xi^2-1}}\right)\left(\xi^2-1\right)^{3/4} . \tag{55}$$

In a manner similar to the above, one may obtain expressions for I_d. These are

For $\xi \le \xi_g$

$$I_d = i\pi \exp\left\{i\left(P\eta\sqrt{\xi^2-1} + \frac{\pi}{2} v^2\right)\right\}\left[1 + (1+i)\left\{C(v) - i\, S(v)\right\}\right] . \tag{56}$$

For $\xi \geq \xi_g$

$$I_d = 2\pi i \exp\left\{ i\eta\left(\xi - \sqrt{1-P^2}\right)\right\}$$
$$- i\pi \exp\left\{ i\left(P\eta\sqrt{\xi^2-1} + \frac{\pi}{2}v^2\right)\right\} \left[1 - (1+1)\left\{C(v) - i\,S(v)\right\}\right] \quad . \tag{57}$$

Combining Eqs. (53), (54), (56), and (57), one finds

For $\xi \leq \xi_g$

$$I_c - I_d = 2\pi\left\{\left[1 + C(v) + S(v)\right]\sin\gamma + \left[C(v) - S(v)\right]\cos\gamma\right\} \tag{53}$$

and for $\xi \geq \xi_g$

$$I_c - I_d = 4\pi \sin\psi - 2\pi\left\{\left[1 - C(v) - S(v)\right]\sin\gamma - \left[C(v) - S(v)\right]\cos\gamma\right\} \tag{59}$$

where

$$\gamma = P\eta\sqrt{\xi^2-1} + \frac{\pi}{2}v^2$$
$$\psi = \eta\left(\xi - \sqrt{1-P^2}\right) \quad .$$

Combining Eqs. (34), (39), (58), and (59), one obtains for the complete asymptotic solution for the electric field:

For $\xi \leq \xi_g$

$$\mathcal{E}(\xi,\eta) = \frac{1}{\sqrt{2\pi}}\left(\frac{P}{\eta}\right)^{1/2} \frac{\cos\left(P\eta\sqrt{\xi^2-1} + \pi/4\right)}{(\xi^2-1)^{3/4}\left(1 + \frac{P\xi}{\sqrt{\xi^2-1}}\right)}$$
$$+ \frac{1}{2}\left\{\left[1 + C(v) + S(v)\right]\sin\gamma + \left[C(v) - S(v)\right]\cos\gamma\right\} \tag{60}$$

and for $\xi \geq \xi_g$

$$\mathcal{E}(\xi,\eta) = \frac{1}{\sqrt{2\pi}}\left(\frac{P}{\eta}\right)^{1/2} \frac{\cos\left(P\eta\sqrt{\xi^2-1} + \pi/4\right)}{(\xi^2-1)^{3/4}\left(1 + \frac{P\xi}{\sqrt{\xi^2-1}}\right)} + \sin\psi$$
$$- \frac{1}{2}\left\{\left[1 - C(v) - S(v)\right]\sin\gamma - \left[C(v) - S(v)\right]\cos\gamma\right\} \tag{61}$$

where

$$\xi_g = \frac{1}{\sqrt{1-P^2}}$$

$$v = \left(\frac{\eta}{\pi P}\right)^{1/2}\left(1 - \frac{P\xi}{\sqrt{\xi^2-1}}\right)\left(\xi^2-1\right)^{3/4} \tag{62}$$

$$\gamma = P\eta\sqrt{\xi^2-1} + \frac{\pi}{2}v^2 \qquad \psi = \eta\left(\xi-\sqrt{1-P^2}\right)$$

Thus Eqs. (60) and (61), together with the definitions in Eq. (62), give the total saddlepoint solution for the turn-on sine wave in a plasma. This solution, coupled with the Sommerfeld solution of Section 2, will give the total transient response for large values of η. The solutions of Eqs. (60) and (61) are somewhat complicated and, under certain conditions, simplified expressions can be obtained. These simplified expressions will be obtained in the following section.

4. REGIONS OF TRANSIENT RESPONSE

In Section 3 the total asymptotic solution for a turn-on sine wave in a plasma has been given. This solution is a good approximation provided η is sufficiently large. The solution given there is also somewhat complicated. Under certain conditions the solution given in Section 3 can be broken down into three regions: a region before the saddlepoint crosses the pole (the anterior transient), a region when the saddlepoint is in the neighborhood of the pole (the main signal build up), and a region after the saddlepoint has traversed the pole (the posterior transient). The expressions for each of these regions will be derived in this section, and the conditions under which these expressions are valid will be given.

4.1 The Anterior Transient Solution

How far must a saddlepoint of $\mathrm{Re}\{w(z)\}$ be away from a pole in order that one could use Eq. (A-19) for (A-24) instead of Eq. (A-30)? To answer this question an asymptotic expansion of Eq. (A-30) can be obtained by integration by parts. Equation (A-30) can be rewritten as

$$I_1 = \frac{1}{2\pi i} \int_{\gamma_z} \frac{\exp[\eta\, w(z)]}{z - z_k}\, dz$$

$$= \frac{\exp[\eta\, w(z_o)]}{2\pi i}\; 2\sqrt{\pi}\; \exp\left[-a\beta^2\right] \frac{\beta}{|\beta|} \int_{|\beta|\sqrt{a}}^{\infty} \exp\left\{\left(y\,\frac{\beta}{|\beta|}\right)^2\right\} dy \qquad (63)$$

where

$$a = \frac{\eta\,|w''(z_o)|}{2}$$

$$\beta = i\, \exp\left[i\,\frac{\alpha}{2}\right](z_o - z_k) \qquad (64)$$

$$\alpha = \arg\left\{w''(z_o)\right\} \quad .$$

Since

$$\exp\left\{\left(y\,\frac{\beta}{|\beta|}\right)^2\right\} dy = \frac{|\beta|^2}{2\beta^2 y}\; d\left(\exp\left\{\left(y\,\frac{\beta}{|\beta|}\right)^2\right\}\right) ,$$

integration of Eq. (63) once by parts gives

$$I_1 = \frac{-\exp[\eta\, w(z_o)]}{2i\sqrt{\pi}\,\sqrt{a}\,\beta} + \frac{\exp[\eta\, w(z_o)]}{2i\sqrt{\pi}}\; \exp(-a\beta^2)\; \frac{|\beta|}{\beta} \int_{|\beta|\sqrt{a}}^{\infty} \frac{\exp\left\{\left(y\,\frac{\beta}{|\beta|}\right)\right\}}{y^2}\, dy \; . \qquad (65)$$

Integration again by parts gives

$$I_1 = \frac{-\exp[\eta\, w(z_o)]}{2i\sqrt{\pi}\,\sqrt{a}\,\beta} - \frac{\exp[\eta\, w(z_o)]}{2i\sqrt{\pi}\;2a^{3/2}\,\beta^3} + (\text{higher order terms}) \qquad (66)$$

which can be written by using Eq. (64) as

$$I_1 = \frac{\exp[\eta\, w(z_o)]}{\sqrt{2\pi}\,[\eta\, w''(z_o)]^{1/2}}\; \frac{1}{z_o - z_k} \left[1 - \frac{1}{\eta\, w''(z_o)\,(z_o - z_k)^2} + \ldots\right] \quad . \qquad (67)$$

Equation (67) is equivalent to the inclusion of the first two terms in Eq. (A-18). Thus Eq. (A-19) may be used as long as the second term in Eq. (67) is small.

Then, for times long before the saddlepoint moves into the vicinity of the pole, Eq. (A-19) may be used for the saddlepoint solution. This solution will be referred to as the anterior transient solution. This solution will be valid as long as the following condition from Eq. (67) is satisfied:

$$\left| \frac{1}{\eta(z_o - z_k)^2 \, w''(z_o)} \right| \ll 1 \quad . \tag{68}$$

From Eq. (32)

$$F(z) = \frac{1}{z^2 + 1} \quad . \tag{69}$$

At the two saddlepoints

$$F(z_o^+) = F(z_o^-) = \frac{\xi^2 - 1}{\xi^2(1-P^2) - 1} \tag{70}$$

and

$$\begin{aligned} w(z_o^+) &= i\,P\sqrt{\xi^2 - 1} \\ w(z_o^-) &= -i\,P\sqrt{\xi^2 - 1} \quad . \end{aligned} \tag{71}$$

The total asymptotic solution to Eq. (24) can therefore be written as

$$\mathcal{E}(\xi,\eta) = \mathcal{E}_+(\xi,\eta) + \mathcal{E}_-(\xi,\eta) \tag{72}$$

where $\mathcal{E}_+(\xi,\eta)$ and $\mathcal{E}_-(\xi,\eta)$ are given by Eq. (A-19) in Appendix A evaluated at the two saddlepoints z_o^+ and z_o^-. Substituting Eqs. (70), (71), (30), and (31) into Eq. (A-19), one obtains for Eq. (72)

$$\begin{aligned} \mathcal{E}(\xi,\eta) &= \left(\frac{P}{2\pi\eta}\right)^{1/2} \frac{\xi^2 - 1}{(\xi^2-1)^{3/4}\left[\xi^2(1-P^2) - 1\right]} \left[\exp\left\{ i\left(P\eta\sqrt{\xi^2-1} + \frac{\pi}{4}\right)\right\} \right. \\ &\qquad \left. + \exp\left\{ -i\left(P\eta\sqrt{\xi^2-1} + \frac{\pi}{4}\right)\right\} \right] \\ &= |\mathcal{E}| \cos\left(\eta P\sqrt{\xi^2-1} + \frac{\pi}{4}\right) \end{aligned} \tag{73}$$

where

$$|\mathcal{E}| = \left| \left(\frac{2P}{\pi\eta}\right)^{1/2} \frac{(\xi^2 - 1)^{1/4}}{\xi^2(1-P^2) - 1} \right| \tag{74}$$

is the amplitude of the asymptotic solution. Equation (74) can be rewritten in the form

$$\frac{|\mathcal{E}|}{M} = \left| \frac{\sqrt{w}}{w^2 - 1} \right| \tag{75}$$

where

$$M = \left(\frac{2}{\pi\eta}\right)^{1/2} \frac{1}{P(1-P^2)^{1/4}} \quad ,$$

$$w^2 = \frac{1-P^2}{P^2} (\xi^2 - 1) \quad . \tag{76}$$

Figure 3 is a plot of $|\mathcal{E}|/M$ vs w . The condition $w = 1$ corresponds to $\xi = 1/\sqrt{1-P^2}$ and represents the time at which the saddlepoint crosses the pole. The above discussion is valid as long as the saddlepoint is not too near the pole. From Eq. (68) this is equivalent to

$$\left| \eta \left(\frac{P\xi}{\sqrt{\xi^2-1}} - 1 \right)^2 \frac{(\xi^2 - 1)^{3/2}}{P} \right| >> 1 \quad . \tag{77}$$

4.2 The Main Signal Build-up

An expression will now be obtained which is valid for the arrival of the main signal. The arrival of the main signal corresponds to times when the saddlepoint is in the vicinity of or coinciding with the pole.

For times when $\xi \sim \xi_g$ the first term in Eqs. (60) and (61) is of the order

$$\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\eta}} \frac{(1-P^2)^{3/4}}{2P} \quad .$$

For large η this term is negligible during the main signal build-up. Now, expand ψ and γ in Eq. (62) in a Taylor series about ξ_g to obtain

$$\psi = \frac{\eta P^2}{\sqrt{1-P^2}} + \eta(\xi - \xi_g) \tag{79}$$

and

$$\gamma = \frac{\eta P^2}{\sqrt{1-P^2}} + \eta(\xi - \xi_g) + \frac{\eta(1-P^2)^2}{2P^4}(\xi - \xi_g)^3 + \dots \quad . \tag{80}$$

Then

$$\psi \simeq \gamma \tag{81}$$

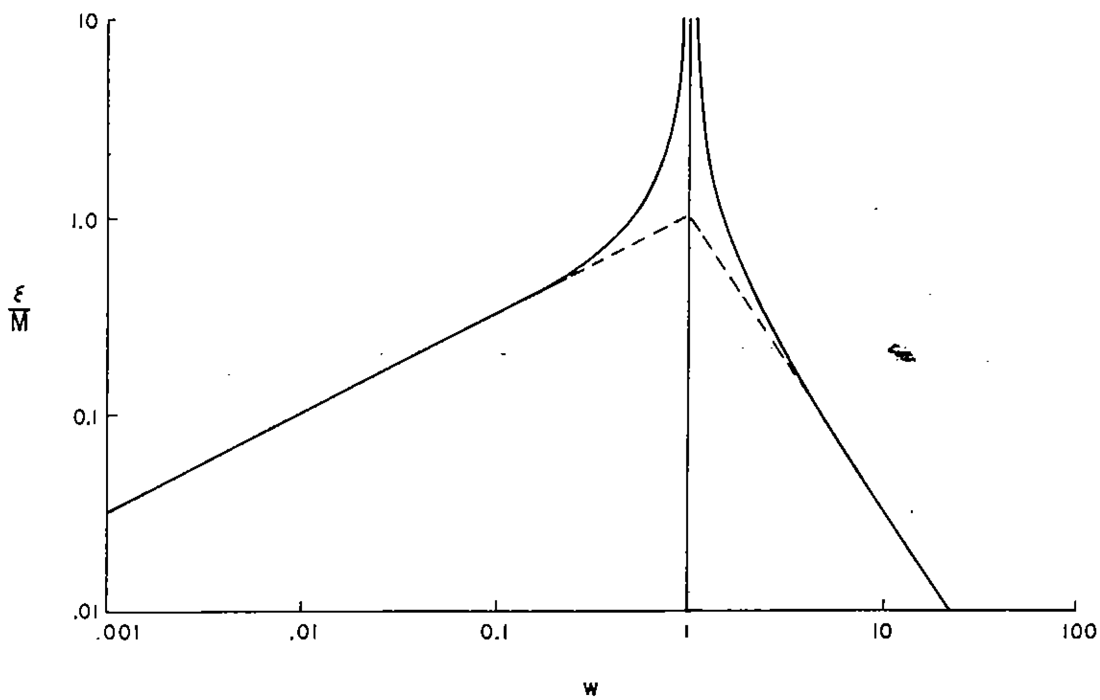

Figure 3. Universal Curve for the Amplitude of the Anterior Transient

for

$$\frac{\eta(1-P^2)^2}{2P^4}(\xi - \xi_g)^3 \ll \eta(\xi - \xi_g) \quad .$$

The condition above is equivalent to

$$(\xi - \xi_g)^2 \ll \frac{2P^4}{(1-P^2)^2} \quad . \tag{82}$$

Thus, from Eqs. (60) and (61), the main signal build-up solution becomes

$$\mathcal{E}(\xi,\eta) \simeq \frac{1}{2}\left\{[1 + C(v) + S(v)]\sin\gamma + [C(v) - S(v)]\cos\gamma\right\} \tag{83}$$

which may be rewritten as

$$\mathcal{E}(\xi,\eta) = A \sin\left[\eta\left(\xi - \sqrt{1-P^2}\right) + \theta_o\right] \tag{84}$$

where

$$A = \frac{1}{\sqrt{2}}\left[\left(\frac{1}{2} + C(v)\right)^2 + \left(\frac{1}{2} + S(v)\right)^2\right]^{1/2} \quad ,$$

$$\theta_o = \tan^{-1}\left[\frac{C(v) - S(v)}{1 + C(v) + S(v)}\right] \quad , \tag{85}$$

$$v = \left(\frac{\eta}{\pi P}\right)^{1/2}\left(1 - \frac{P\xi}{\sqrt{\xi^2-1}}\right)\left(\xi^2 - 1\right)^{3/4} \quad .$$

This is an approximate solution for the main signal build-up for large η . The amplitude function A in Eq. (85) is plotted as a function of v in Figure 4. Table 1 gives the values of v at which the maxima and minima of the amplitude function A occur.

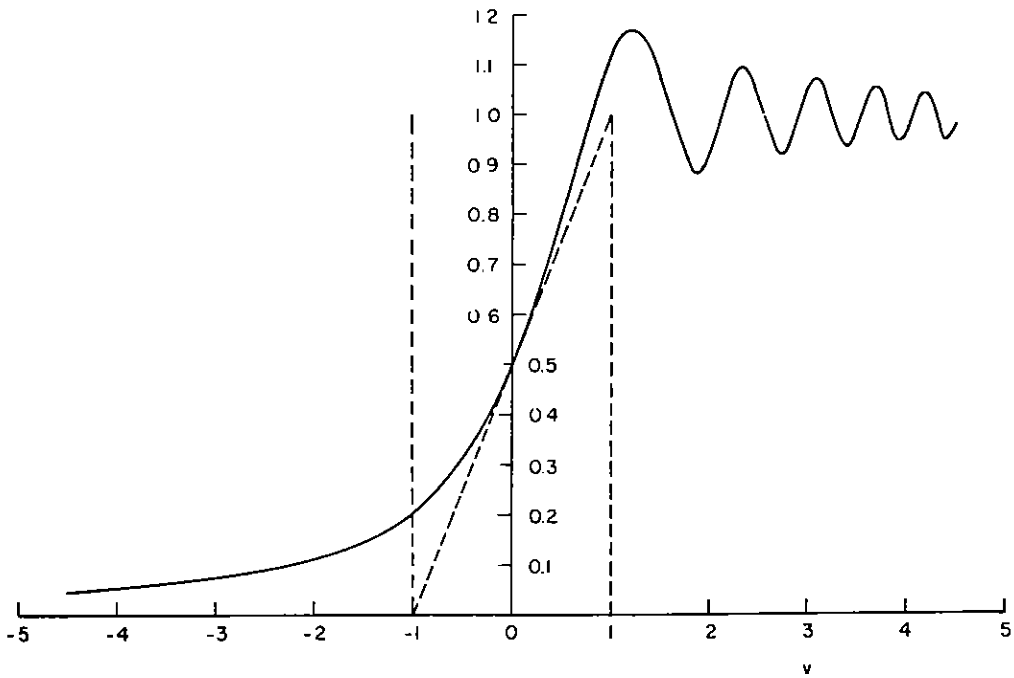

Figure 4. Universal Curve for the Amplitude A of the Main Signal Build-up

Table 1

Value of Extremum	v
1.17 max	1.22
0.88 min	1.87
1.10 max	2.35
0.92 min	2.74
1.07 max	3.08
0.93 min	3.39
1.06 max	3.67
0.94 min	3.93
1.05 max	4.18
0.95 min	4.42

The range of validity of the solution can be determined by requiring that condition (82) be well satisfied and that the first term of Eq. (60) be small. The parameter v in the amplitude function A acts as a stretching factor for time. The stretching factor is a nonlinear function of ξ and thus is a nonlinear function of time. The stretching factor may be expanded in a Taylor series about ξ_g as follows

$$v = \sqrt{\frac{\eta}{\pi}} \frac{(1-P^2)^{3/4}}{P} (\xi - \xi_g) + \sqrt{\frac{\eta}{\pi}} \frac{(1-P^2)^{7/4} (P^2 - \sqrt{1-P^2})}{6P^5} (\xi - \xi_g)^3 + \ldots \quad . \tag{86}$$

Keeping only the first term of this expansion will be a good approximation if

$$(\xi - \xi_g)^2 << \left| \frac{6P^4}{(1-P^2)(P^2 - \sqrt{1-P^2})} \right| \quad . \tag{87}$$

The stretching factor v then becomes a linear function of time and may be written as

$$v \simeq v_L = \sqrt{\frac{\eta}{\pi}} \frac{(1-P^2)^{3/4}}{P} (\xi - \xi_g) \quad . \tag{88}$$

The solution given by Eq. (84) coupled with the linear stretching factor of Eq. (88) may also be derived by making an expansion about the pole. This procedure is carried out in Appendix C.

4.3 The Posterior Transient Solution

After the saddlepoint has crossed the pole and is well away from the pole, one may again use Eq.(A-19) of Appendix A. However, the solution will now include a term arising from the residue of the pole. The solution may be written down immediately by making use of Eqs. (73) and (74). The solution for the posterior transients is then

$$\mathcal{E}(\xi,\eta) = |\mathcal{E}| \cos\left(\eta P \sqrt{\xi^2-1} + \frac{\pi}{4}\right) + \sin\left[\eta\left(\xi - \sqrt{1-P^2}\right)\right]$$

or

$$\mathcal{E}(\xi,\eta) = -|\mathcal{E}| \sin\left(\eta P \sqrt{\xi^2-1} - \frac{\pi}{4}\right) + \sin\left[\eta\left(\xi - \sqrt{1-P^2}\right)\right] \quad . \tag{89}$$

By defining

$$A = \eta P \sqrt{\xi^2-1} - \frac{\pi}{4}$$
$$B = \eta\left(\xi - \sqrt{1-P^2}\right) \tag{90}$$

then assuming $|\mathcal{E}| << 1$ one may use the identity of Appendix D and rewrite Eq. (89) as

$$\mathcal{E}(\xi,\eta) = \left\{1 - |\mathcal{E}| \cos(A-B)\right\} \sin\left(\frac{A+B}{2} + \theta_1\right) \tag{91}$$

where

$$\theta_1 = \tan^{-1}\left[\frac{1 + |\mathcal{E}|}{1 - |\mathcal{E}|} \tan\left(\frac{B-A}{2}\right)\right] .$$

The amplitude of the high frequency signal therefore oscillates about the final value of unity with a decrement, $|\mathcal{E}|$, given by Eq. (74) and represented by that portion of Figure 3 for which $w > 1$. This part of the curve is replotted in Figure 5 as a function of ξ for different values of P.

5. SUMMARY

The results of this report will be summarized by plotting the envelope of the transient response of a unit step-carrier signal which has propagated in a lossless, isotropic plasma under the typical condition $\eta = \omega_o x_3/c = 10^4$ and $P = \Pi/\omega_o = 0.8$. These values were chosen to insure that the approximate solutions given for the four regions of transient response would overlap. The total transient response is shown in Figure 6.

No signal arrives prior to the time $t = x_3/c$ ($\xi = 1$). Even the general asymptotic solution given in Section 3 does not hold at the moment of arrival of the signal wavefront. The Sommerfeld solution given by Eq. (21) and plotted in Figure 1 must therefore be used to describe the very first part of the signal response. The range of applicability of this solution is indicated by region A in Figure 6.

Following the Sommerfeld region the anterior transient solution given by Eq. (74) is indicated as region B in Figure 6. Region C is the main signal build-up solution of Figure 4 with the nonlinear stretching factor v given in Eq. (85). Finally, the posterior transient solution given by Eq. (91) holds for region D of Figure 6.

The main conclusions to be drawn from Figure 6 are the following. The amplitude of the precursor region which preceeds the arrival of the main signal is small and becomes smaller as the propagation distance η increases. Also, the time between the arrival of the signal wavefront and the arrival of the main signal increases with increasing η. The main signal can be considered to arrive at $\xi_g = 1/\sqrt{1-P^2}$, which corresponds to the time of arrival at the group velocity

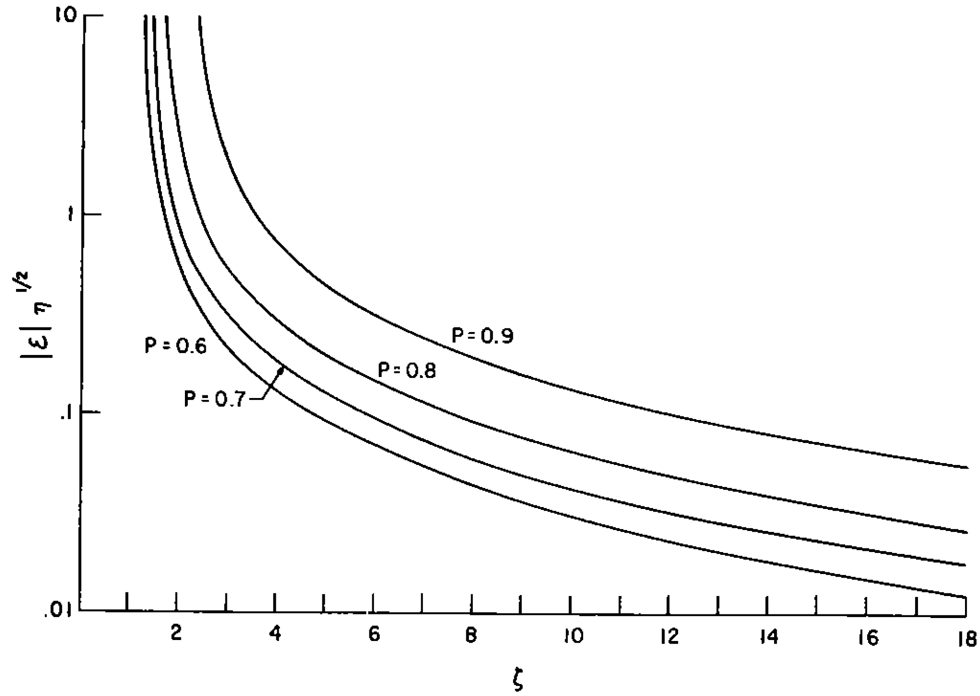

Figure 5. Amplitude Decrement of the Posterior Transient

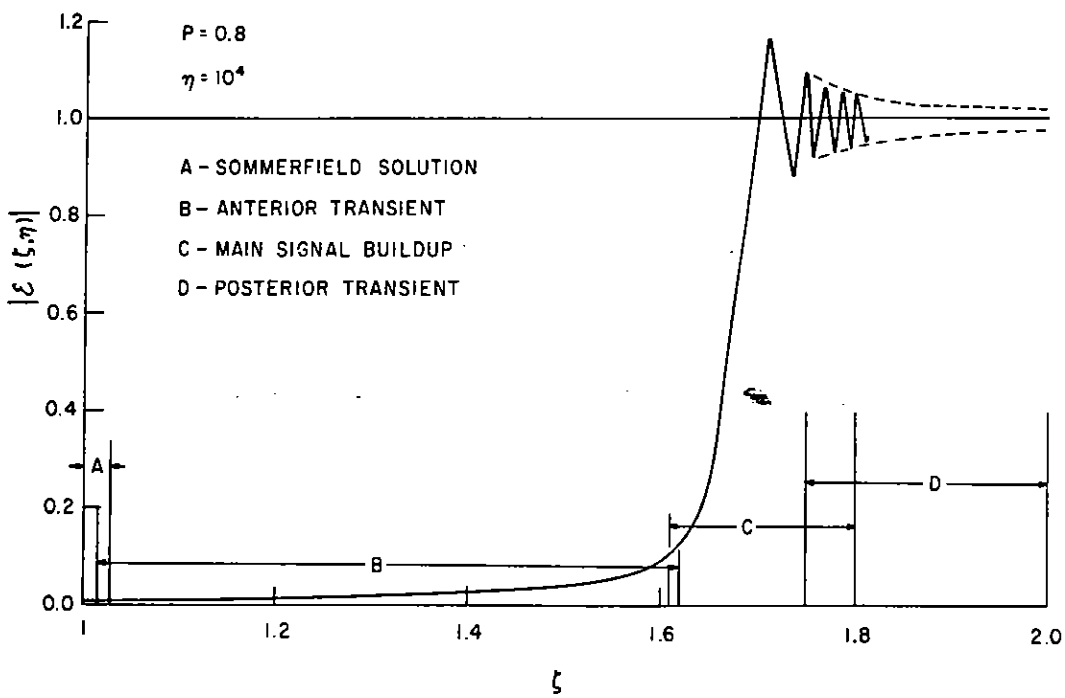

Figure 6. Typical Envelope of the Transient Response of a Step-Carrier Signal Propagating in a Lossless, Isotropic Plasma $\eta = \omega_o x_3/c = 10^4$, $P = \Pi/\omega_o = 0.8$

$t_g = x_3/v_g = x_3/(\partial\omega/\partial k)_{\omega_0}$. At this instant of time the saddlepoint is exactly crossing the pole and the amplitude of the envelope response is equal to 1/2. Once the saddlepoint has crossed the pole, the envelope response becomes oscillatory about its final value of unity.

References

Abramowitz, M. and Stegun, I.A., Eds. (1964) Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematics Series 55, U.S. Department of Commerce.

Brillouin, L. (1914) Ann. Physik 44: 203-240.

Brillouin, L. (1960) Wave Propagation and Group Velocity, Academic Press.

Cerillo, M. (1948) Transient Phenomena in Waveguides, M.I.T. Research Laboratory of Electronics, Technical Report No. 33.

Cerillo, M.V. (1950) An Elementary Introduction to the Theory of the Saddlepoint Method of Integration, M.I.T. Research Laboratory of Electronics, Technical Report No. 55: 2a.

Clemmow, P.C. (1950) Some extensions to the method of integration by steepest descents, Quart. J. Mech. Appl. Math. 3: 241.

Haskell, R.E. and Case, C.T. (1966) Transient Signal Propagation in Lossless, Isotropic Plasmas, Vol. I, AFCRL-66-234.

Karbowiak, A. (1957) Propagation of transients in waveguides, Proc. IEE, Monogram No. 224R, p. 339.

Namiki, M. and Horiuchi, K. (1952) On the transient phenomena in the waveguide, J. Phys. Soc. Japan 7: 190.

Pearson, J.D. (1953) The transient motion of sound waves in tubes, Quart. J. Mech. Appl. Math. 6: 313.

Sommerfeld, A. (1914) Ann. Physik 44: 177-202.

Van der Waerden, B.L. (1950) On the method of saddle points, Appl. Sci. Res. 2B: 33.

Appendix A

Saddlepoint Method of Integration

The method of steepest descent or the saddlepoint method of integration can be used to obtain the asymptotic behavior of integrals of the form

$$I = \frac{1}{2\pi i} \int_{\gamma_z} F(z) \exp\left[\eta\, w(z)\right] dz \tag{A-1}$$

when the real quantity η is large and positive. The complex function $w(z)$ is a function of the complex variable $z = x + iy$. That is

$$w(z) = u(x,y) + i\, v(x,y) \quad . \tag{A-2}$$

If $w(z)$ is analytic then from the Cauchy-Riemann conditions $u_x = v_y$ and $u_y = -v_x$. This implies that u is harmonic, that is, $u_{xx} + u_{yy} = 0$. The stationary point $z_o = (x_o, y_o)$ is the point (or points) where $dw/dz = 0$. At this point, $u_x = 0$ and $u_y = 0$. In the neighborhood of this point on the surface $u = u(x,y)$ one can write

$$\begin{aligned} u(x_o + h, y_o + k) - u(x_o, y_o) &= \frac{1}{2!}\left[u_{xx} h^2 + 2u_{xy} hk + u_{yy} k^2\right] \\ &= \frac{1}{2!} P(h,k) \quad . \end{aligned}$$

Since

$$u_{xx}\, P(h,k) = (u_{xx}\, h + u_{xy}\, k)^2 - (u_{xy}^2 - u_{xx}\, u_{yy})k^2 \quad ,$$

it follows that $P(h,k)$ will have the same sign as u_{xx} for all h and k if

$$(u_{xy})^2 - u_{xx}\, u_{yy} < 0 \quad .$$

For this condition the (x_o, y_o) could therefore be a local maximum or minimum. However, from the Cauchy-Riemann conditions, $u_{xy}^2 - u_{xx}\, u_{yy} = u_{xy}^2 + u_{xx}^2 > 0$ so that the stationary point (x_o, y_o) must be a saddlepoint of $u(x,y)$.

If η is large, a change in v will produce rapid oscillations of the integrand in Eq. (A-1). If the contour can be deformed along a path where v is constant, then the oscillations will disappear and the major contribution to the integral will occur where u is largest. Now, since $\nabla u \cdot \nabla v = u_x v_x + u_y v_y = -u_x u_y + u_x u_y = 0$, the lines of constant u (called level lines) are normal to the lines of constant v which is a line of most rapid change in u and is called a line of steepest descent. The idea of the saddlepoint method of integration is then to deform the contour of integration through the saddlepoints along the lines of steepest descent. If η is large, the major contribution to the integral (A-1) is then obtained from the integration only in the neighborhood of the saddlepoints.

One can expand $w(z)$ in a Taylor series about the saddlepoint z_o as follows

$$w(z) - w(z_o) = \frac{1}{2}(z - z_o)^2\, w''(z_o) \quad . \tag{A-3}$$

Let

$$w''(z_o) = A\, e^{i\alpha} \quad , \tag{A-4}$$

$$(z - z_o) = r\, e^{i\theta} \quad . \tag{A-5}$$

Then

$$\begin{aligned} w(z) - w(z_o) &= \frac{1}{2} A\, r^2 \exp\, i(2\theta + \alpha) \\ &= (u - u_o) + i(v - v_o) \end{aligned} \tag{A-6}$$

where $u_o = u(z_o)$ and $v_o = v(z_o)$.

The level lines are given by $u = u_o$ or, from Eq. (A-6), by $\cos(2\theta + \alpha) = 0$.

Thus the level lines are given by

$$\theta_L = -\frac{\alpha}{2} + \frac{(2n+1)\pi}{4} \qquad n = 0, 1, 2, \ldots \quad . \tag{A-7}$$

Likewise the lines of steepest descent and ascent are given by $v = v_o$ or $\sin(2\theta + \alpha) = 0$, from which

$$\theta_S = -\frac{\alpha}{2} + \frac{n\pi}{2} \qquad n = 0, 1, 2, \ldots \quad . \tag{A-8}$$

Along the lines $v = v_o$, note that $2\theta + \alpha = n\pi$, so that from Eq. (A-6) $u - u_o = \frac{1}{2} A r^2 \cos n\pi$. Therefore, $n = 1, 3$ corresponds to lines of steepest descent while $n = 0, 2$ corresponds to lines of steepest ascent.

Now, let the function $F(z)$ be written as the sum of a holomorphic and meromorphic function in the neighborhood of the saddlepoint. Thus

$$F(z) = G(z) + H(z)$$

where

$G(z)$ is holomorphic in the region of the saddlepoint,

$H(z)$ is meromorphic in the region of the saddlepoint.

Consider first the integral over the analytic part, $G(z)$. If $G(z)$ is expanded in a Taylor series about the saddlepoint

$$G(z) = \sum_{k=0}^{\infty} \frac{G^{(k)}(z_o)}{k!} (z - z_o)^k \quad . \tag{A-9}$$

and substituted into Eq. (A-1), together with Eq. (A-3), one obtains

$$I_G = \frac{1}{2\pi i} \exp\left[\eta\, w(z_o)\right] \sum_{k=0}^{\infty} \frac{G^{(k)}(z_o)}{k!} \int_{\gamma_z} (z - z_o)^k \exp\left\{\frac{\eta\, w''(z_o)}{2} (z - z_o)^2\right\} dz \tag{A-10}$$

where

$$I_G = \frac{1}{2\pi i} \int_{\gamma_z} G(z) \exp\left[\eta\, w(z)\right] dz \quad .$$

Now deform the contour so as to pass through saddlepoint z_o along the lines of steepest descent. Assume that the path of integration approaches the saddlepoint along the steepest descent path corresponding to $n = 3$ and leaves the saddlepoint along the $n = 1$ path. The integration is then made up of two parts. From Eqs. (A-5) and (A-8), one obtains for $n = 1$

$$\begin{aligned} z - z_o &= r \exp[i\theta] = r \exp\left[i\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right] \\ dz &= \exp\left[i\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right] dr \\ (z - z_o)^2 &= -r^2 \exp[-i\alpha] \end{aligned} \tag{A-11}$$

and for $n = 3$

$$\begin{aligned} z - z_o &= r \exp[i\theta] = r \exp\left[i\left(\frac{3\pi}{2} - \frac{\alpha}{2}\right)\right] \\ dz &= \exp\left[i\left(\frac{3\pi}{2} - \frac{\alpha}{2}\right)\right] dr \\ (z - z_o)^2 &= -r^2 \exp[-i\alpha] \quad . \end{aligned} \tag{A-12}$$

From Eq. (A-10) we can then write the prototype integral I_k in the form

$$\begin{aligned} I_k &= \frac{1}{2\pi i} \int_{\gamma_z} (z - z_o)^k \exp\left\{\frac{\eta\, w''(z_o)}{2} (z - z_o)^2\right\} dz \\ &= \frac{1}{2\pi i} \exp\left\{ik\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right\} \int_0^r r^k \exp\left\{-\frac{\eta\, w''(z_o)}{2} r^2 \exp[-i\alpha]\right\} dr \exp\left[i\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right] \\ &+ \frac{1}{2\pi i} \exp\left\{ik\left(\frac{3\pi}{2} - \frac{\alpha}{2}\right)\right\} \int_r^0 r^k \exp\left\{-\frac{\eta\, w''(z_o)}{2} r^2 \exp[-i\alpha]\right\} dr \exp\left[i\left(\frac{3\pi}{2} - \frac{\alpha}{2}\right)\right] . \end{aligned} \tag{A-13}$$

Using Eq. (A-4) one can rewrite Eq. (A-13) as

$$\begin{aligned} I_k = \frac{1}{2\pi i} \Bigg[&\exp\left\{(k+1)\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) i\right\} \int_0^r r^k \exp\left\{-\frac{\eta A}{2} r^2\right\} dr \\ &- \exp\left\{(k+1)\left(\frac{3\pi}{2} - \frac{\alpha}{2}\right) i\right\} \int_0^r r^k \exp\left\{-\frac{\eta A}{2} r^2\right\} dr \Bigg] \quad . \end{aligned} \tag{A-14}$$

Noting that

$$\exp\left\{(k+1)\left(\frac{3\pi}{2}-\frac{\alpha}{2}\right)i\right\} = (-1)^{k+1}\exp\left\{(k+1)\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)i\right\} ,$$

Eq. (A-14) can be rewritten as

$$I_k = \frac{i^{(k+1)}}{2\pi i}\exp\left\{\frac{-i(k+1)\alpha}{2}\right\}\left[1-(-1)^{k+1}\right]\int_0^r r^k \exp\left\{-\frac{\eta A}{2}r^2\right\} dr . \qquad \text{(A-15)}$$

Now if η is sufficiently large, the upper limit of the integration can be extended to infinity without increasing the error significantly. If this is done and use is made of the relation

$$\int_0^\infty r^k \exp\left\{-a\, r^p\right\} dr = \frac{\Gamma\left(\frac{k+1}{p}\right)}{p\, a^{\left(\frac{k+1}{p}\right)}} \qquad k > -1$$

Eq. (A-15) becomes

$$I_k = \frac{i^k}{2\pi}\,\frac{\left[1-(-1)^{k+1}\right]}{2}\,\frac{\Gamma\left(\frac{k+1}{2}\right)}{\left(\frac{\eta A}{2}\right)^{\left(\frac{k+1}{2}\right)}}\exp\left\{\frac{-i(k+1)\alpha}{2}\right\} . \qquad \text{(A-16)}$$

Since I_k is zero when k is odd, it is possible by letting $k = 2m$ and using Eq. (A-4) to rewrite Eq. (A-16) as

$$I_m = \frac{(-1)^m}{2\pi}\,\frac{\Gamma\left(m+\frac{1}{2}\right)}{\left[\frac{\eta\, w''(z_o)}{2}\right]^{\left(m+\frac{1}{2}\right)}} . \qquad \text{(A-17)}$$

If Eq. (A-17) is combined with Eq. (A-10), the asymptotic solution of the integral I_G can be written as

$$I_G = \frac{\exp\left[\eta\, w(z_o)\right]}{\left[2\eta\, w''(z_o)\right]^{1/2}}\sum_{m=0}^{\infty}\frac{(-1)^m\,\Gamma\left(m+\frac{1}{2}\right)G^{(2m)}(z_o)}{\pi\,(2m)!\left[\frac{\eta\, w''(z_o)}{2}\right]^m} . \qquad \text{(A-18)}$$

For sufficiently large values of η the first term in the series of Eq. (A-18) is a good approximation to the integral and is given by

$$I_G \simeq \frac{\exp \ \eta \, w(z_o)}{\sqrt{\pi} \left[2\eta \, w''(z_o)\right]^{1/2}} \; G(z_o) \quad . \tag{A-19}$$

Now consider the evaluation of the integral over the meromorphic part of $F(z)$. Assume that $H(z)$ has a pole z_k of finite order in the neighborhood of the saddle-point and can thus be expanded in a Laurent series about the pole. Then the contribution of the integral over $H(z)$

$$I_H = \frac{1}{2\pi i} \int_{\gamma_z} H(z) \ \exp\left[\eta \, w(z)\right] \, dz \tag{A-20}$$

will involve evaluating integrals of the form

$$I_n = \frac{1}{2\pi i} \int_{\gamma_z} \frac{\exp\left[\eta \, w(z)\right]}{(z - z_k)^n} \, dz \quad . \tag{A-21}$$

Let

$$I_n^{(m)} \equiv \frac{d^m}{d \, z_k^m} I_n \quad .$$

Then the following recursion relations hold:

$$I_{n-1}^{(1)} = (n-1) \, I_n$$

and

$$I_1^{(n-1)} = (n-1)! \; I_n \tag{A-22}$$

or

$$I_n = \frac{1}{(n-1)!} \, \frac{d^{(n-1)}}{d \, z_k^{(n-1)}} \, I_1 \quad . \tag{A-23}$$

Thus, the integrals containing higher order poles can be obtained from I_1 by successive differentiation.

Consider now the evaluation of I_1

$$I_1 = \frac{1}{2\pi i} \int_{\gamma_z} \frac{\exp[\eta\, w(z)]}{z - z_k} \, dz \quad . \tag{A-24}$$

As before, deform the path γ_z along the path of steepest descent through the saddlepoint. Using the relations (A-3), (A-4), (A-11), and (A-12), Eq. (A-24) becomes

$$I_1 = \frac{\exp[\eta\, w(z_o)]}{2\pi} \left[\int_0^R \frac{\exp\left\{-\frac{\eta A}{2} r^2\right\} \exp\left\{-i\alpha/2\right\} dr}{ir \exp\left\{-i\alpha/2\right\} + z_o - z_k} - \int_R^0 \frac{\exp\left\{-\frac{\eta A}{2} r^2\right\} \exp\left\{-i\,\alpha/2\right\} dr}{i\,r \exp\left\{-i\,\alpha/2\right\} + z_o - z_k} \right] .$$

Let $r \to -r$ in the second integral and factor out a $(-i)$ to obtain

$$I_1 = \frac{\exp[\eta\, w(z_o)]}{2\pi i} \int_{-R}^{R} \frac{\exp\left\{-\frac{\eta A}{2} r^2\right\} dr}{r - i \exp\left\{i\,\alpha/2\right\}(z_o - z_k)} \quad .$$

If η is large enough, then one can extend the range of integration from $-\infty$ to $+\infty$ without introducing significant errors. Then

$$I_1 = \frac{\exp[\eta\, w(z_o)]}{2\pi i} \int_{-\infty}^{\infty} \frac{\exp(-ar^2)}{r - \beta} \, dr \tag{A-25}$$

where

$$a = \frac{nA}{2}$$

$$\beta = i \exp\left[i\,\frac{\alpha}{2}\right] (z_o - z_k) \quad . \tag{A-26}$$

Multiplying the integrand of Eq. (A-25) by $\left(\frac{r+\beta}{r+\beta}\right)$ one obtains

$$I_1 = \frac{\exp[\eta\, w(z_o)]}{2\pi i}\left\{\int_{-\infty}^{\infty}\frac{r\,\exp(-ar^2)}{r^2-\beta^2}\,dr + \int_{-\infty}^{\infty}\frac{\beta\,\exp(-ar^2)}{r^2-\beta^2}\,dr\right\} . \qquad \text{(A-27)}$$

The first integral is zero since the integrand is an odd function. In the second integral one can write

$$\frac{1}{r^2-\beta^2} = \int_0^{\infty}\exp\left\{-\chi(r^2-\beta^2)\right\}\,d\chi \quad . \qquad \text{(A-28)}$$

If Eq. (A-28) is substituted into Eq. (A-27) and the order of integration is interchanged, one obtains

$$I_1 = \frac{\exp[\eta\, w(z_o)]}{2\pi i}\,\beta\int_0^{\infty}d\chi\;\exp(\beta^2\chi)\int_{-\infty}^{\infty}\exp\left\{-(a+\chi)r^2\right\}\,dr \quad .$$

Carrying out the integration over r one obtains

$$I_1 = \frac{\exp[\eta\, w(z_o)]}{2\pi i}\,\beta\int_0^{\infty}d\chi\;\exp(\beta^2\chi)\,\frac{\sqrt{\pi}}{(a+\chi)^{1/2}} \quad . \qquad \text{(A-29)}$$

Making the change of variables

$$(a+\chi)^{1/2} = \frac{y}{|\beta|}$$

the integral I_1 in Eq. (A-29) becomes

$$I_1 = \frac{\exp[\eta\, w(z_o)]}{2\pi i}\,2\sqrt{\pi}\;\exp(-a\beta^2)\,\frac{\beta}{|\beta|}\int_{|\beta|\sqrt{a}}^{\infty}\exp\left\{\left(y\,\frac{\beta}{|\beta|}\right)^2\right\}dy \;. \qquad \text{(A-30)}$$

Appendix B

Relations Involving Error Functions and Fresnel Integrals *

Error Function: $$\mathrm{erf}\,(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp\,(-u^2)\, du \qquad \text{(B-1)}$$

Complementary Error Function: $$\mathrm{erfc}\,(z) = 1 - \mathrm{erf}\,(z) \qquad \text{(B-2)}$$

Fresnel Integral: $$F(z) = \int_0^z \exp\left\{ i \frac{\pi u^2}{2} \right\} du \qquad \text{(B-3)}$$

Cosine Fresnel Integral: $$C(z) = \int_0^z \cos\left\{ \frac{\pi u^2}{2} \right\} du \qquad \text{(B-4)}$$

Sine Fresnel Integral: $$S(z) = \int_0^z \sin\left\{ \frac{\pi u^2}{2} \right\} du \qquad \text{(B-5)}$$

*See Abramowitz and Stegun (1964).

Identities

$$\operatorname{erf}(-z) = -\operatorname{erf}(z) \tag{B-6}$$

$$F(-z) = -F(z) \tag{B-7}$$

$$C(-z) = -C(z) \tag{B-8}$$

$$S(-z) = -S(z) \tag{B-9}$$

$$F(z) = C(z) + i\,S(z) \tag{B-10}$$

$$F(z) = \frac{(1+i)}{2}\operatorname{erf}\left[\frac{\sqrt{\pi}}{2}(1-i)z\right] = \frac{\exp\left[i\frac{\pi}{4}\right]}{\sqrt{2}}\operatorname{erf}\left[\sqrt{\frac{\pi}{2}}\,z\,\exp\left\{-i\frac{\pi}{4}\right\}\right]. \tag{B-11}$$

Asymptotic Values

$$C(x) \to \frac{1}{2} \qquad \text{as} \qquad x \to \infty \tag{B-12}$$

$$S(x) \to \frac{1}{2} \qquad \text{as} \qquad x \to \infty \tag{B-13}$$

$$\exp(z^2)\ \operatorname{erfc}(z) \sim \frac{1}{z\sqrt{\pi}} + O\left(\frac{1}{z^2}\right) \quad \left(z \to \infty,\ |\arg z| < \frac{3\pi}{4}\right). \tag{B-14}$$

Other Relations

$$\begin{aligned}\int_B^\infty \exp(i\,u^2)\,du &= \int_0^\infty \exp(i\,u^2)\,du - \int_0^B \exp(i\,u^2)\,du \\ &= \frac{1}{2}\sqrt{\frac{\pi}{2}}(1+i) - \sqrt{\frac{\pi}{2}}\left[C(q) + i\,S(q)\right] \\ &= \frac{\sqrt{\pi}}{2}\exp\left\{i\frac{\pi}{4}\right\}\left[1 - (1-i)\left\{C(q) + i\,S(q)\right\}\right]\end{aligned} \tag{B-15}$$

where $q = \sqrt{2/\pi}\,B$.

Also, in a similar manner

$$\int_B^\infty \exp(-i\,u^2)\,du = \frac{\sqrt{\pi}}{2}\exp\left\{-i\frac{\pi}{4}\right\}\left[1 - (1+i)\left\{C(q) - i\,S(q)\right\}\right]. \tag{B-16}$$

Appendix C

Coinciding Pole Solution for Main Signal Build-up

The solution for the electric field of a turn-on sine wave propagating in a plasma is given by Eq. (34). Neglecting the contributions from I_a and I_b one can write

$$\mathcal{E}(\xi,\eta) = \frac{1}{4\pi}\left(I_c - I_d\right) \tag{C-1}$$

where

$$I_c = \int_{x_o - i\infty}^{x_o} \frac{\exp\left[\eta\, w(z)\right]}{z + i}\, dz \quad ,$$

$$I_d = \int_{x_o}^{x_o + i\infty} \frac{\exp\left[\eta\, w(z)\right]}{z - i}\, dz \quad . \tag{C-2}$$

Consider the prototype integral

$$I_k = \frac{1}{2\pi i} \int_{\gamma_z} \frac{\exp\left[\eta\, w(z)\right]}{z - z_k}\, dz \quad . \tag{C-3}$$

Expand w(z) in a Taylor series about the pole

$$w(z) = w(z_k) + w'(z_k)(z - z_k) + \frac{w''(z_k)}{2}(z - z_k)^2 + \dots \quad . \tag{C-4}$$

Introduce the transformation

$$z - z_k = \frac{Z}{\left(\frac{\eta\, w''(z_k)}{2}\right)^{1/2}} \quad , \qquad X = \frac{\eta\, w'(z_k)}{\left(\frac{\eta\, w''(z_k)}{2}\right)^{1/2}} \quad . \tag{C-5}$$

Then Eq. (C-3) can be written

$$I_k = \frac{\exp\left[\eta\, w(z_k)\right]}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\exp\left[XZ + Z^2\right]}{Z}\, dZ \quad . \tag{C-6}$$

Noting that

$$\frac{\exp(ZX)}{Z} = \frac{1}{Z} + \int_0^X \exp(Z\,u)\, du \quad ,$$

Eq. (C-6) can be rewritten as

$$I_k = \frac{\exp\left[\eta\, w(z_k)\right]}{2\pi i} \int_{-i\infty}^{i\infty} \int_0^X \exp\left[Z^2 + Zu\right] du\, dZ + \int_{-i\infty}^{i\infty} \frac{\exp\left[Z^2\right]}{Z}\, dZ \quad . \tag{C-7}$$

By the method of residues one can readily show that

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\exp\left[Z^2\right]}{Z}\, dZ = \frac{1}{2} \quad . \tag{C-8}$$

The first integral in Eq. (C-7) can be obtained by evaluating

$$I_1 = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \int_0^X \exp\left[Z^2 + Zu\right] du\, dZ \quad . \tag{C-9}$$

Letting Z = ir and reversing the order of integration, one obtains

$$I_1 = \frac{1}{2\pi} \int_0^X du \int_{-\infty}^{\infty} \exp\left\{ - r^2 + i r u \right\} dr \quad .$$

Integrate over r to obtain

$$I_1 = \frac{1}{2\pi} \int_0^X du \, \sqrt{\pi} \quad \exp\left\{ - \frac{u^2}{4} \right\}$$

or

$$I_1 = \frac{1}{\sqrt{\pi}} \int_0^{X/2} \exp(-v^2) \, dv$$

from which (see Appendix B)

$$I_1 = \frac{1}{2} \operatorname{erf}\left(\frac{\eta \, w'(z_k)}{\left[2\eta \, w''(z_k) \right]^{1/2}} \right) \quad . \qquad \text{(C-10)}$$

Using Eqs. (C-8) and (C-10) one obtains for the prototype integral in Eq. (C-7)

$$I_k = \frac{1}{2} \exp\left[\eta \, w(z_k) \right] \left[1 + \operatorname{erf}\left(\frac{\eta \, w'(z_k)}{\left[2\eta \, w''(z_k) \right]^{1/2}} \right) \right] \quad . \qquad \text{(C-11)}$$

The two poles in Figure 2 are at $\pm i$. Thus

$$
\begin{aligned}
w(i) &= i \left[\xi - \sqrt{1-P^2} \right] \quad , \\
w(-i) &= - i \left[\xi - \sqrt{1-P^2} \right] \quad , \\
w'(i) &= w'(-i) = \left[\xi - \frac{1}{\sqrt{1-P^2}} \right] , \qquad \text{(C-12)} \\
w''(i) &= - i \, P^2 (1-P^2)^{-3/2} \quad , \\
w''(-i) &= i \, P^2 (1-P^2)^{-3/2} \quad .
\end{aligned}
$$

Then

$$\frac{\eta\, w'(i)}{\left[2\eta\, w''(i)\right]^{1/2}} = \sqrt{\frac{\pi}{2}}\, v_L \;\exp\left\{i\,\frac{\pi}{4}\right\} \tag{C-13}$$

$$\frac{\eta\, w'(-i)}{\left[2\eta\, w''(-i)\right]^{1/2}} = \sqrt{\frac{\pi}{2}}\, v_L \;\exp\left\{-\,i\,\frac{\pi}{4}\right\} \tag{C-14}$$

where

$$v_L = \sqrt{\frac{\eta}{\pi}}\;\frac{(1-P^2)^{3/4}}{P}\;(\xi - \xi_g)$$

$$\xi_g = \frac{1}{\sqrt{1-P^2}} \quad . \tag{C-15}$$

If the contours of the integrals in Eq. (C-2) are taken along the entire imaginary axis where it is assumed that the major contribution comes from the vicinity of the pole (this is equivalent to neglecting I_a and I_b) then Eq. (C-1) can be written as

$$\mathcal{E}(\xi,\eta) = \frac{2\,\pi\, i}{4\pi}\left[I_k(-i) - I_k(i)\right] \quad . \tag{C-16}$$

Making use of Eqs. (C-11), (C-13) and (C-14), one obtains

$$\begin{aligned}\mathcal{E}(\xi,\eta) = \frac{i}{4}\Bigg\{&\exp(-i\psi)\left[1 + \mathrm{erf}\left(\sqrt{\frac{\pi}{2}}\, v_L \;\exp\left\{-\,i\,\frac{\pi}{4}\right\}\right)\right] \\ &- \exp(i\psi)\left[1 + \mathrm{erf}\left(\sqrt{\frac{\pi}{2}}\, v_L \;\exp\left\{i\,\frac{\pi}{4}\right\}\right)\right]\Bigg\} \quad .\end{aligned} \tag{C-17}$$

where ψ is given in Eq. (62). Using Eq. (B-11) from Appendix B, Eq. (C-17) can be written as

$$\begin{aligned}\mathcal{E}(\xi,\eta) = \frac{i}{4}\left(e^{-i\psi} - e^{i\psi}\right) &+ \frac{i}{4}\, e^{-i\psi}(1-i)\left[C(v_L) + i\, S(v_L)\right] \\ &- \frac{i}{4}\, e^{i\psi}(1+i)\left[C(v_L) - i\, S(v_L)\right]\end{aligned}$$

or

$$\mathcal{E}(\xi,\eta) = \frac{1}{2}\sin\psi + \frac{1}{2}\left[C(v_L) + S(v_L)\right]\sin\psi + \frac{1}{2}\left[C(v_L) - S(v_L)\right]\cos\psi$$

and finally

$$\mathcal{E}(\xi, \eta) = \frac{1}{2} \left\{ \left[1 + C(v_L) + S(v_L) \right] \sin \gamma + \left[C(v_L) - S(v_L) \right] \cos \gamma \right\} \qquad \text{(C-18)}$$

where use has been made of Eq. (81). Note that Eq. (C-17) is equivalent to Eq. (83) provided one uses the linear stretching factor given by Eq. (88).

Appendix D

A Trigonometric Identity for U = X sin A + Y sin B

Consider

$$U = X \sin A + Y \sin B \quad . \tag{D-1}$$

Let

$$A = \alpha + \beta$$
$$B = \alpha - \beta \quad . \tag{D-2}$$

Then

$$\alpha = \frac{A+B}{2}$$
$$\beta = \frac{A-B}{2} \quad . \tag{D-3}$$

Use the identity

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta \quad . \tag{D-4}$$

Substitute Eq. (D-2) into Eq. (D-1) and use Eq. (D-4) to obtain

$$U = (X + Y) \sin\alpha \cos\beta + (X - Y) \cos\alpha \sin\beta$$

or

$$U = \mathrm{Im} \left\{ (X + Y) \cos\beta \, e^{i\alpha} + i(X - Y) \sin\beta \, e^{i\alpha} \right\} \quad . \tag{D-5}$$

Write Eq. (D-5) as

$$U = \left[(X + Y)^2 \cos^2\beta + (X - Y)^2 \sin^2\beta \right]^{1/2} \sin(\alpha + \theta_1) \tag{D-6}$$

where

$$\tan\theta_1 = \frac{X-Y}{X+Y} \tan\beta \quad . \tag{D-7}$$

Rewrite Eq. (D-6) as

$$U = \left[X^2 + Y^2 + 2XY \cos 2\beta \right]^{1/2} \sin(\alpha + \theta_1) \quad . \tag{D-8}$$

If $X \ll 1$ and $Y = 1$, then Eq. (D-8) can be written as

$$\sin B + X \sin A \simeq \left[1 + X \cos 2\beta \right] \sin(\alpha + \theta_1) \tag{D-9}$$

or

$$\sin B + X \sin A \simeq \left[1 + X \cos(A-B) \right] \sin\left[\frac{A+B}{2} + \theta_1 \right] \quad . \tag{D-10}$$