Theoretical Notes

Note 293

January 1978

Magnetic and Electric Contributions to Quasi-Static SGEMP Fields

William J. Karzas
Charles T. C. Mo

R & D Associates
Marina del Rey, CA

ABSTRACT

In this report, we identify and resolve a basic difficulty in finding the system generated electromagnetic pulse (SGEMP) fields that has caused ambiguities of the same order of magnitude as the fields' leading terms. The difficulty is the hitherto unrecognized absence of one condition for determining the quasi-static electric contribution to the surface SGEMP fields and its full reconciliation with the magnetostatic contribution. The results apply as well to any low-frequency electromagnetic scattering problems involving highly conducting scatterers.

In resolving the difficulty, after exhibiting its existence and delineating its significance, we conceive and make use of two mathematical conjectures. One is the decomposition of a tangential vector field on a two-dimensional closed simple surface into a "surface-divergenceless", or magnetostatic, part and a "surface-curlless", or electrostatic, part; the other is the requirement of "surface-curllessness" of a surface current were it to generate a magnetostatic field parallel to the surface on that surface. The first conjecture is critical to the validity of our methodology and solution. Its proof is reduced to the solvability of a generalized wave equation on a curved surface, unexpectedly the same formal equation obeyed by the relativistic electromagnetic four-potentials in a curved spacetime. The second conjecture simplifies the calculation greatly but is not critical. Except for some special cases, it is not proved, but neither is any counter example found.

Abstract (continued)

With these conjectures, we show that the electric and magnetic contributions to the quasi-static fields fall out separately from the general formalism and reconcile fully with each other to yield the whole fields. The results are that the electrostatic part of the surface current is driven by the driving electrostatic field and is "surface-curlless"; the magnetostatic part is driven by the driving magnetostatic field component perpendicular to the highly conducting surface and is "surface-divergenceless"; furthermore, only scalar field mechanisms are needed to exploit away-from-surface conditions in determining and reconciling the two parts. Finally, the application and significance of the results are illustrated by several simple but practically interesting problems. The illustrations show that the relative amplitudes of the electrostatic and the magnetostatic parts of surface currents (and thus the ambiguities removed by obtaining them correctly) vary from zero to infinity in the cases often encountered.

PREFACE

This effort was performed under the auspices of the Air Force Weapons Laboratory for the Computer Sciences Corporation PO-50647 under the AFWL contract F29601-76-C-0034. The authors acknowledge the technical discussions with G. Schlegel and T. Wilcox during the investigation. Also, brief but helpful discussions with C. E. Baum, J. D. Cole, C. Greifinger, P. S. Greifinger, K. S. H. Lee, L. Marin, J. Mathews, C. H. Papas and G. B. Whitham are acknowledged. Further, the review comments from D. Holliday are acknowledged.

CONTENTS

ILLUSTRATIONS

SECTION I

INTRODUCTION AND SUMMARY

1. INTRODUCTION

System-generated electromagnetic pulse (SGEMP) is the electromagnetic field created by energetic photons impinging on a material body, such as by the X-rays and γ-rays from a nuclear detonation illuminating a satellite or an electronic system (ref. 1). That such SGEMP could severely interfere with the functional performance of an electronic system has been recognized for a long time. Thus, the generation mechanisms and the effects of SGEMP have become the subject of extensive investigations in recent years (ref. 2).

The investigations in the SGEMP phenomena are very involved and are divided naturally into several areas according to the particular aspect of the problem being dealt with and the technology being used (ref. 2). These include (a) the characterization of the impinging photon environments; (b) the determination of the charges and currents (mostly electrons) that are created by the photons via photoemissions from and Compton scatterings with the atoms of the system and serve as the direct driving source of the SGEMP; (c) the treatment of the ambient medium to obtain its electromagnetic properties such as the conductivity induced in the medium by the process of ionizing collisions made by energetic electrons; (d) the determination of the SGEMP itself by setting up and solving (may lead to equivalent

---

1. Karzas, W. J. and R. Latter, "Electromagnetic Radiation from a Nuclear Explosion in Space," Physics Review, Vol. 126, June 1962, pp. 1919-1926 and also Theoretical Note 27, October 1961.
2. IEEE Transactions on Nuclear Science: Annual Conference on Nuclear and Space Radiation Effects, Vol. NS-23, No. 6, Session G, December 1976.

circuit network model) the classical electromagnetic boundary value problem with prescribed sources and medium properties; and (e) the final assessment of component impairments and performance degradation on the system due to a given SGEMP. Roughly speaking, in order to arrive at the final assessment, the results obtained in each of the areas just outlined are needed as the inputs to the next area. Of course, dividing the problem into separate areas is only approximately correct because interactive mechanisms among different areas do exist and must be considered simultaneously to render more accurate results. In any case, however, the mechanism and the methodology in each of these separate areas must be firmly understood and established, whether or not the areas are considered simultaneously or iteratively in the evaluation of a final assessment.

In this report, we devote our entire effort to the determination of the SGEMP from prescribed driving sources. We treat the problem as a classical electromagnetic boundary value problem driven by the photon-induced charges and currents as specified input quantities (refs. 3,4) and subjected to suitable boundary conditions. In particular, we shall consider only the case for a system made of conductors and shall emphasize the quasi-static fields.

The quasi-static fields are the dominant SGEMP fields if the motions of the charges are "slow" and the spatial extent of the problem under consideration is "small"--precisely the conditions often encountered in reality by SGEMP problems. However, the seemingly simple problem of determining these quasi-static fields contains a fundamental difficulty regarding the portions

---

3. Stratton, J. A., Electromagnetic Theory, McGraw-Hill Book Company, 1941.

4. Jackson, J. D., Classical Electrodynamics, John Wiley and Sons, 1962.

of fields contributed electrically and magnetically and their reconciliation. This difficulty heretofore has not been resolved to the best knowledge of the authors, and indeed has caused substantial (of the order of one) ambiguities in the dominant SGEMP magnetic field. In fact, we must point out that this difficulty, severe of consequence as it is, is so un-obvious that it not only has failed to draw enough attention to get itself resolved earlier but even its existence has evaded the recognition of most researchers in the related community.

This report presents the difficulty and its resolution, as the result of a rigorous investigation. To clarify both the existence of the problem and its resolution, we have adopted a tutorial style, in which ideas and their development are presented in explicit detail, even at the expense of conciseness. In the following, Subsection I.2 briefly summarizes the findings of the report, Section II describes the problem, Section III presents its solution, and Section IV shows some examples.

2. SUMMARY

In this report, we identify and resolve a basic difficulty in finding the SGEMP fields that has caused ambiguities of the same order of magnitude as the fields' leading terms. The difficulty is the hitherto unrecognized absence of one condition for determining the quasi-static electric contribution to the surface SGEMP fields and its full reconciliation with the magnetostatic contribution. The results apply as well to any low-frequency electromagnetic scattering problems involving highly conducting scatterers.

Specifying carefully the physics of a typical SGEMP problem in terms of its geometry, driving sources, driving fields, temporal and spatial ranges of practical interest, and material boundary conditions for fields, we establish, first the quasi-static nature of the dominant fields. Then, following a clear-cut separation of the known but often confusing quasi-static electric field and its effects on charges, we exhibit the difficulty and delineate its significance. Viewed mathematically, the difficulty reflects the insufficiency of conditions on the surface alone to determine the fields on the surface--we have to invoke conditions away from the surface. Viewed physically, the surface current contains contributions of the same general order of magnitude but with different surface distributions from both the quasi-static electric and magnetic fields--and the difficulty is to find and combine these currents correctly on the surface. Further discussed are some "brute-force" or "hand-waving" methods that have been used in the past and conceal the difficulty. These methods, in addition to being much more complicated and expensive to compute, not only obscure the whole physical mechanism of the field generation but are either of questionable validity or simply incorrect.

We resolve the difficulty and present the solution explicitly with the help of two mathematical conjectures we have conceived: one being the decomposition of a tangential vector field on a two-dimensional, closed, simple surface into a "surface-divergenceless", or magnetostatic, part and a "surface-curlless", or electrostatic, part; the other being the requirement for the "surface-curllessness" of a surface current were it to generate a magnetostatic field parallel to the surface on that surface. The first conjecture is critical to the validity of our methodology and solution. We have reduced its proof to the solvability of a generalized wave equation on a curved surface, unexpectedly the same formal equation obeyed by the relativistic electromagnetic four-potentials in a curved spacetime. The second conjecture simplifies the calculation greatly but is not critical.

Except for some special cases, we have not proved it; but neither have we found counter examples. Although we have not exhibited mathematically rigorous proofs, based on physical intuitions and experiences we are quite convinced of the truths of the two conjecturs.

With these conjectures, we show that the electric and magnetic contributions to the quasi-static fields fall out separately from the general formalism and reconcile fully with each other to yield the whole fields. The results are that the electrostatic part of the surface current is driven by the driving electrostatic field and is "surface-curlless"; the magnetostatic part is driven by the driving magnetostatic field component perpendicular to the highly conducting surface and is "surface-divergenceless"; furthermore, only scalar field mechanisms are needed to exploit away-from-surface conditions in determining and reconciling the two parts. In presenting these results, we have made clear distinctions among assumptions and implications, physics-imposed requirements and mathematical-model-imposed ones, and approximations and exact expressions.

Finally, we illustrate the application and significance of the results by several simple but practically interesting problems. The illustrations show that the relative amplitudes of the electrostatic and the magnetostatic parts of surface currents (and thus the ambiguities removed by obtaining them correctly) vary from zero to infinity in the cases often encountered.

SECTION II

THE BASIC SGEMP GENERATION PROBLEM

1. THE DRIVING FIELDS

Consider a simple SGEMP generation problem as depicted by Figure 1. The charge density $\rho^{dr}(\underset{\sim}{x}, t)$ and the current density $\underset{\sim}{J}^{dr}(\underset{\sim}{x}, t)$, themselves created by the illuminating energetic photons (ref. 5) (not shown in Figure 1 because they are not further treated in this report), are specified and serve as the driving sources for the electromagnetic fields.* These driving charges, moving in the volume V_{out} that surrounds the object, could be emitted either by the closed surface S of the otherwise electrically isolated object itself or by some other nearby objects, but they are totally absent in the volume V_{in} inside the object, i.e.,

$$\rho^{dr}(\underset{\sim}{x}, t) \equiv \rho^{dr}(\underset{\sim}{x}, t) = \sum_i \rho^{dr(i)}(\underset{\sim}{x}, t), \quad \underset{\sim}{x} \varepsilon S \text{ and } V_{out}$$

$$\equiv 0, \quad \underset{\sim}{x} \varepsilon V_{in} \qquad (1)$$

$$\underset{\sim}{J}^{dr}(\underset{\sim}{x}, t) \equiv \underset{\sim}{J}^{dr}(\underset{\sim}{x}, t) = \sum_i \rho^{dr(i)}(\underset{\sim}{x}, t)\ \underset{\sim}{v}^{(i)}(\underset{\sim}{x}, t), \quad \underset{\sim}{x} \varepsilon S \text{ and } V_{out}$$

$$\equiv 0, \quad \underset{\sim}{x} \varepsilon V_{in} \qquad (2)$$

---

5. Evans, R. D., The Atomic Nucleus, McGraw-Hill Book Company, 1955, Chapters 22-25.

*Notice that if the motions of the driving charges are not influenced significantly by the fields generated by these charges, the $\rho^{dr}(\underset{\sim}{x}, t)$ and $\underset{\sim}{J}^{dr}(\underset{\sim}{x}, t)$ can be specified and then used to obtain the fields via a formalism based on the Maxwell equations and the boundary conditions. If the fields generated do influence significantly the motions of the driving charges, that same formalism must still be used except that the equations of motion of the charges, with the fields entering in the Lorentz force terms to influence the motions of these charges themselves, must be solved simultaneously. That is, that same formalism and the charges' equations of motion make an enlarged total system.

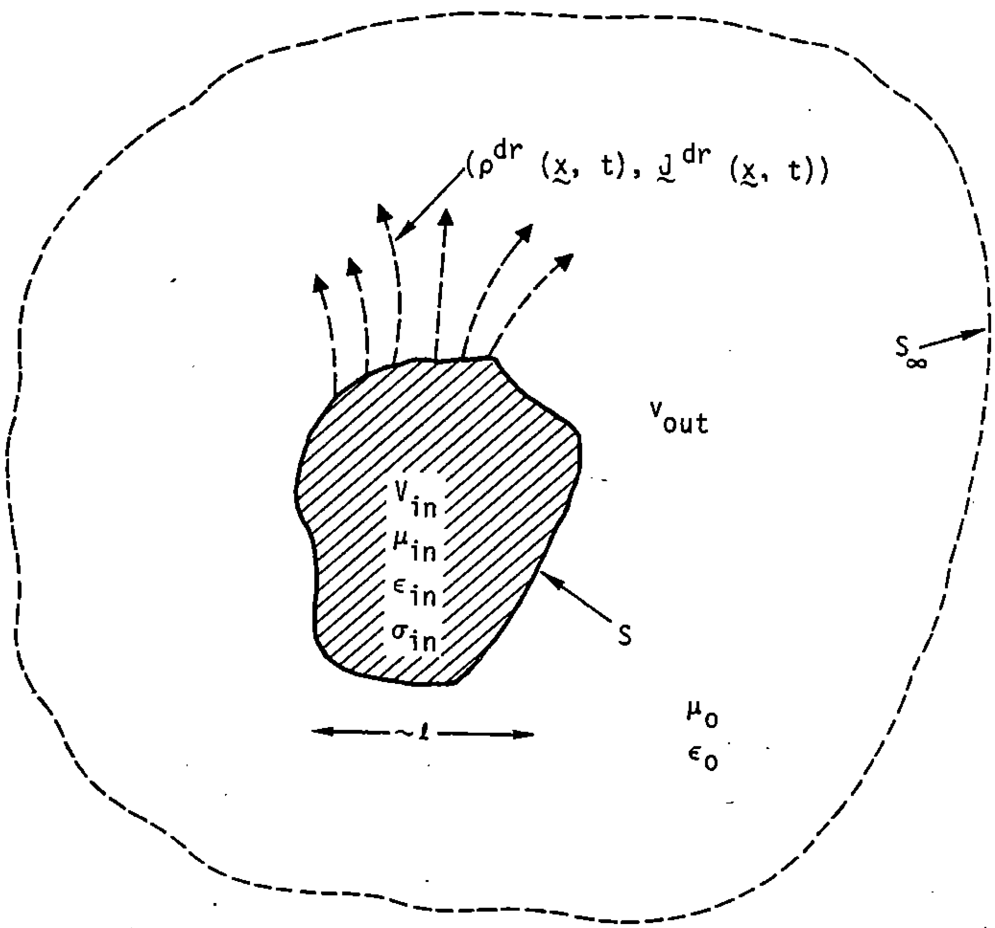


Figure 1. Configuration of the SGEMP Generation Problem

Here $v^{(i)}(s, t)$ is the velocity of the i^{th} particle whose charge density is

$$\rho^{dr(i)}(\underset{\sim}{x}, t) \equiv q_i \delta^3[\underset{\sim}{x} - \underset{\sim}{f}^{(i)}(t)] \tag{3}$$

where $\underset{\sim}{f}^{(i)}(t)$ is the position of that charge.

As a macroscopic description of the charged fluid, we shall hereafter represent the source by its macroscopically averaged charge density $\rho^{dr}(\underset{\sim}{x}, t)$ and current density $\underset{\sim}{J}^{dr}(\underset{\sim}{x}, t)$. Further, we emphasize that the $\rho^{dr}(\underset{\sim}{x}, t)$ may and often does include a layer of surface charges "clamped in" on S as a result of these charges leaving S, i.e.,

$$\rho^{dr}(\underset{\sim}{x}, t) = \rho_0^{dr}(\underset{\sim}{x}, t) + \sigma^{dr}(\underset{\sim}{x}, t)\ \delta\ (\underset{\sim}{x} - \underset{\sim}{x}_s),\ \underset{\sim}{x} \varepsilon\ V_{out} \text{ and } S,$$

$$\underset{\sim}{x}_s \varepsilon \quad S \tag{4}$$

where $\rho_0^{dr}(\underset{\sim}{x}, t)$ is purely in V_{out}. However, there is no surface current in the driving current density (we shall come back to remark on relaxing this constraint in Subsections III-2 and III-3C). For the simple SGEMP case that interests us, we model the object occupying the volume V_{in} and surface S as a "good" conductor (discussed and defined in Subsection II-2) with a conductivity, σ_{in}, a dielectric constant ε_{in}, and a permeability μ_{in}, and model the ambient medium in V_{out} as a uniform simple one with constant ε_0 and μ_0 but no conductivity. The typical approximate dimension of the object is the length ℓ.

The basic problem concerning the generation of SGEMP is to solve for the overall electromagnetic (EM) fields in the above model, a simplified and yet adequate model for most SGEMP situations when the values of those characteristic quantities just described are appropriately specified (ref. 6).* Of special interest are the fields in V_{out} near S and the fields on S. This deceptively simple problem does contain a basic difficulty that could make the determination of the dominant magnetic field ambiguous. We shall proceed to describe the situation in what follows.

If the object had the same EM properties as the ambient medium or if it were absent, i.e., $\mu_{in} = \mu_o$, $\varepsilon_{in} = \varepsilon_o$, $\sigma_{in} = 0$, then the EM fields generated by the specified driving source $\rho^{dr}(\underset{\sim}{x}, t)$ and $J^{dr}(\underset{\sim}{x}, t)$, when expressed in the Coulomb gauge, would simply be (Appendix A)

$$\underset{\sim}{E}^{dr}(\underset{\sim}{x}, t) = -\underset{\sim}{\nabla}\phi^{dr}(\underset{\sim}{x}, t) - \frac{\partial}{\partial t}\underset{\sim}{A}^{dr}(\underset{\sim}{x}, t) \tag{5}$$

$$\underset{\sim}{B}^{dr}(\underset{\sim}{x}, t) = \underset{\sim}{\nabla} \times \underset{\sim}{A}^{dr}(\underset{\sim}{x}, t). \tag{6}$$

Here,

$$\phi^{dr}(\underset{\sim}{x}, t) \equiv \frac{1}{4\pi\varepsilon_o}\int \frac{\rho^{dr}(\underset{\sim}{x}', t)}{R}\, d^3x' \tag{7}$$

$$\underset{\sim}{A}^{dr}(\underset{\sim}{x}, t) \equiv \frac{\mu_o}{4\pi}\int \frac{\underset{\sim}{J}_t^{dr}\left(\underset{\sim}{x}', t-\frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c}\right)}{R}\, d^3x' \tag{8}$$

and $c \equiv (\mu_o\varepsilon_o)^{-1/2}$, $R \equiv |\underset{\sim}{R}| \equiv |\underset{\sim}{x}-\underset{\sim}{x}'|$, and $\underset{\sim}{J}_t^{dr}(\underset{\sim}{x}, t)$ is the "transverse" (solenoidal) part of the driving current given by

---

6. Carron, N.J. and C. L. Longmire, On the Structure of the Steady State Space-Charge-Limited Boundary Layer on One Dimension, Theoretical Note 281, November 1975.

*See the many references in Ref. 2 and the AFWL Theoretical Notes edited by C. E. Baum. These references contain many interesting results in different aspects of the SGEMP problem.

$$J_t^{dr}(\underset{\sim}{x}, t) \equiv \nabla x \nabla x \int \frac{\underset{\sim}{J}^{dr}(\underset{\sim}{x}, t)\, d^3x}{4\pi R} \tag{9}$$

Throughout this report, unrestricted volume integrations are carried over all spatial volumes.

In the expression (5) and (6), the Coulomb gauge has been chosen purposefully. This choice explicitly separates the driving electric field into two parts (Appendix A). One part, $-\underset{\sim}{\nabla}\phi^{dr}(\underset{\sim}{x}, t)$, is the quasi-static instantaneous Coulomb field and the other, $-\frac{\partial}{\partial t} \underset{\sim}{A}^{dr}(\underset{\sim}{x}, t)$, is the "transverse" field including both retardation and radiation effects. One can easily see that at positions near the driving source and near the object, when these two are near each other as they are in most typical SGEMP problems, the quasi-static part is the dominant part of the driving electric field while the remaining part is of the order of $\beta \equiv v/c$ or ξ_R smaller. Here the v is the typical speed of the source charges and is quite smaller than c (Appendices A and B) and the ξ_R parameterizing the spatial nearness will be defined soon (see (13b)). However, as a result of the $\underset{\sim}{\nabla}$x operator, the inclusion or not of the transverse subscript for the driving current density in the expression for the driving magnetic field is immaterial. The dominant part of this driving magnetic field, under the conditions of being near the object and source and satisfying the slowness in the source time rate, is merely (Appendix B) its instantaneous magnetostatic part. That is (when not used under integration, R represents the typical distance from the location of source to the position of field observation), under the condition

$$\frac{\left(\begin{array}{c}\text{changes of the driving}\\ \text{source in } \Delta t\end{array}\right)}{(\text{the driving sources})} \cdot \frac{R}{c\Delta t} << 1 \tag{10}$$

the dominant driving fields are (Appendix C)

$$E^{dr,o}(x, t) \equiv -\nabla\phi^{dr}(x, t) \qquad (11)$$

$$B^{dr,1}(x, t) \equiv \nabla \times A^{dr,1}(x, t) \equiv \nabla \times \int \frac{\mu_o J^{dr}(x', t)\, d^3x'}{4\pi R} \quad . \qquad (12)$$

The superscripts o and 1 indicate that the labeled quantities are, respectively, of the order of zero and one in β and ξ_R. More specifically, we notice that the physical constraint (10) implies

$$\beta \equiv v/c << 1 \quad . \qquad (13a)$$

and

$$\xi_R \equiv \tau_R/\tau_x \equiv R/(c\tau_x) << 1 \qquad (13b)$$

where τ_x is the smaller of the typical rise time and the time width of the illuminating X-ray pulse, and therefore the smaller of those of the driving current pulse. Thus, the driving fields near the object are predominantly quasi-static in nature if both charges are relatively slow. In most real SGEMP problems of interest with typical electron emission speeds of $\beta \sim 0.2$ to 0.5 (for ~10 to ~100 keV X-rays), and $R \sim \ell \sim 1$ meter, this quasi-static situation holds appproximately. This is the situation we shall investigate.

2. CONDITIONS FOR THE SCATTERED FIELDS

The driving fields described above, $E^{dr}(x, t)$ and $B^{dr}(x, t)$, are the EM fields created directly by the specified driving sources $\rho^{dr}(x, t)$ and $J^{dr}(x, t)$ in a homogeneous infinite medium characterized by the same medium properties as those in V_{out}. These driving fields would have existed everywhere had it not been for the presence of the object. In the presence of the object, the driving fields excite additional pieces of EM fields $\underset{\sim}{E}^{sc}(\underset{\sim}{x}, t)$ and $\underset{\sim}{B}^{sc}(\underset{\sim}{x}, t)$ such that the combined total fields

$$\underset{\sim}{E}^{total}(\underset{\sim}{x}, t) \equiv \underset{\sim}{E}^{dr}(\underset{\sim}{x}, t) + \underset{\sim}{E}^{sc}(\underset{\sim}{x}, t) \tag{14a}$$

$$\underset{\sim}{B}^{total}(\underset{\sim}{x}, t) \equiv \underset{\sim}{B}^{dr}(\underset{\sim}{x}, t) + \underset{\sim}{B}^{sc}(\underset{\sim}{x}, t) \tag{14b}$$

satisfy the Maxwell equations, with the driving sources and possible induced ones, everywhere in the true inhomogeneous medium and have physics-dictated mathematical behaviors appropriate to geometrical boundaries and medium inhomogeneities. Although one can always decompose the total fields according to (14), this decomposition is a useful one only when each of the driving fields and the scattered fields obeys separate equations and couples to the other only through boundary conditions, such as in regions beyond the spatial extent of the medium inhomogeneities of the scatter or in regions where the total fields vanish. The simple SGEMP problem does exhibit only such regions, as we shall see.

a. In the Outside Volume V_{out}

In the region outside the scattering conductor, V_{out}, the scattered fields obey the same Maxwell equations as do the driving fields, except without any driving source. Since the driving fields are predominantly quasi-static near and on the conductor and they excite the scattered fields via only their effects there, the scattered fields must also be predominantly quasi-static there. Thus, total SGEMP fields under condition (10) are approximately

$$\underset{\sim}{E}^{total}(\underset{\sim}{x}, t) \sim \underset{\sim}{E}^{total,o}(\underset{\sim}{x}, t) \equiv \underset{\sim}{E}^{dr,o}(\underset{\sim}{x}, t) + \underset{\sim}{E}^{sc,o}(\underset{\sim}{x}, t) \tag{15a}$$

$$\underset{\sim}{B}^{total}(\underset{\sim}{x}, t) \simeq \underset{\sim}{B}^{total,1}(\underset{\sim}{x}, t) \equiv \underset{\sim}{B}^{dr,1}(\underset{\sim}{x}, t) + B^{sc,1}(x, t) \tag{15b}$$

where the scattered fields obey

$$\begin{cases} \underset{\sim}{\nabla}\cdot\underset{\sim}{E}^{sc,o}_{out}(\underset{\sim}{x}, \underset{\sim}{t}) = 0 & (16a) \\ \underset{\sim}{\nabla}\mathrm{x}\underset{\sim}{E}^{sc,o}_{out}(\underset{\sim}{x}, \underset{\sim}{t}) = -\dfrac{\partial \underset{\sim}{B}^{sc,1}_{out}(\underset{\sim}{x}, t)}{\partial t} = \end{cases}$$

$$0(\beta^2, \beta\xi, \xi^2) \approx 0 \tag{16b}$$

and

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{B}^{sc,1}_{out}(\underset{\sim}{x}, t) = 0 \tag{17a}$$

$$\underset{\sim}{\nabla}\mathrm{x}\underset{\sim}{B}^{sc,1}_{out}(\underset{\sim}{x}, t) = \mu_o\varepsilon_o \frac{\partial \underset{\sim}{E}^{sc,o}_{out}(\underset{\sim}{x}, t)}{\partial t} . \tag{17b}$$

Here, the subscript "out" indicates field solutions for the region V_{out} in which these solutions must not have any singularity. The subscript "in", when used later, should be similarly understood. Also, (16b) is always true for quasi-static fields, and will be understood and omitted hereafter. In this way, the "quasi-electrostatic" field is determined by (16) and by the boundary conditions to be obtained in the following; then it is used to determine the quasi-magnetic field.

b. In the Region V_{in} and on the Surface S

(1) The Electric Field

In the region V_{in} inside the conductor, the "good" conductivity σ_{in} dictates a quick dissipation of electric field with an e-folding time $\tau_\sigma \equiv \varepsilon_{in}/\sigma_{in}$ (refs. 3,4). This implies that the electric field inside the conductor cannot be accumulatively piling up beyond the time interval τ_o whatever source generates that field. Typically, for a "good" conductor of $\sigma_{in} \sim 10^7$ mho/meter and $\varepsilon_{in} \sim 10^{-11}$ farad/meter, we have

$$\tau_\sigma \equiv \frac{\varepsilon_{in}}{\sigma_{in}} \sim 10^{-18} \text{sec}, \tag{18}$$

an extremely short time. Thus, if there exists no driving source that acts fast enough in such a short time scale comparable to τ_σ to keep replenishing the electric field, the electric field is always negligibly small in the conductor. Accordingly, the SGEMP electric field in V_{in} is negligibly small compared to that in V_{out} and can be set to zero <u>if</u>

$$\tau_x >> \tau_\sigma. \tag{19}$$

In reality, the condition (19) is <u>amply</u> satisfied and the conductor is termed a "good" conductor. Consequently, by further invoking the continuity of tangential electric field across any material boundary surface--a result always implied by the Maxwell equations--we are well justified to require at all times that*

$$\underset{\sim}{n}(\underset{\sim}{x}_s) \times \underset{\sim}{E}^{total} (\underset{\sim}{x}_{s\pm}, t) = 0 \quad x_s \varepsilon S \tag{20a}$$

and thus

$$\underset{\sim}{n}(\underset{\sim}{x}_s) \times \underset{\sim}{E}^{total,o} (\underset{\sim}{x}_{s\pm}, t) = 0 \quad \underset{\sim}{x}_s \varepsilon S \;. \tag{21a}$$

---

*Notice that $\sigma \to \infty$ gives the "perfect conductor" condition (20), as the limiting implication of (18) and (19).

Here, $n(x_s)$ is the unit normal vector of the surface S and points into V_{out}. Also, here and hereafter, the notations $\underset{\sim}{x}_{s+}$ and $\underset{\sim}{x}_{s-}$ indicate limiting positions onto S taken respectively from V_{out} and V_{in}. So indicated are the values of the functions dependent upon these positions. The limiting indicators, + or −, will be dropped when both limits given the same value for the function under consideration and when such a practice gives rise to ambiguity. The boundary condition (20) connects the electric fields in regions V_{in} and V_{out}, as can be seen clearly by rewriting (20b) as

$$\underset{\sim}{n}(\underset{\sim}{x}_s) \times \underset{\sim}{E}^{sc,o}_{out}(\underset{\sim}{x}_{s+}, t) = -\underset{\sim}{n}(\underset{\sim}{x}_s) \times \underset{\sim}{E}^{dr,o}(\underset{\sim}{x}_{s+}, t) \tag{21}$$

$$\underset{\sim}{n}(\underset{\sim}{x}_s) \times \underset{\sim}{E}^{sc,o}_{in}(\underset{\sim}{x}_{s-}, t) = -\underset{\sim}{n}(\underset{\sim}{x}_s) \times \underset{\sim}{E}^{dr,o}(\underset{\sim}{x}_{s-}, t) \tag{22}$$

Notice, of course, that in V_{in} the $\underset{\sim}{E}^{sc,o}_{in}(\underset{\sim}{x}, t)$ satisfies

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{E}^{sc,o}_{in}(\underset{\sim}{x}, t) = 0, \tag{23}$$

a condition similar to (16a).

Finally, we make the remark that for the electric field in the above and for the magnetic field to be discussed, we insist that the same boundary conditions on S are also satisfied on S_∞. This is motivated by the requirement of simple uniqueness for the mathematical solution and is justified since both the scattered and the driving fields vanish on S_∞. With such a condition on S_∞, (16), (20), and (23) uniquely determine $\underset{\sim}{E}^{sc,o}(\underset{\sim}{x}, t)$ everywhere. In particular, these equations imply $\underset{\sim}{E}^{sc,o}_{in}(\underset{\sim}{x}, t) = -\underset{\sim}{E}^{dr,o}(\underset{\sim}{x}, t)$ in V_{in}, since both are homogeneous Neumann problems with the same boundary condition (ref. 4). This result is consistent with setting the $\underset{\sim}{E}^{total}(\underset{\sim}{x}, t) \equiv 0$ in V_{in}, as it should be.

(2) The Magnetic Field

The magnetic field in V_{in} and its boundary condition on S encompass further complications than do the electric ones. First, the magnetic field in a good conductor does not dissipate away with the electric decay time τ_σ. In fact, a magnetic field constant in time can exist indefinitely in a conductor described by μ_{in}, ε_{in}, and σ_{in} no matter how large σ_{in} is, provided it has diffused there already. Second, the time it takes for the magnetic field to diffuse from the outside into a good conductor of typical dimension ℓ is (ref. 7):

$$\tau_{df}(\ell) \sim \mu_{in}\sigma_{in}\ell^2. \tag{24}$$

This magnetic diffusion time does play an important role in determining the magnetic boundary condition on S, as we shall show in the following.

The time t_p at which most of the interesting SGEMP phenomena occur lies in the interval I_p:

$$t_p \varepsilon I_p \equiv (0, \tau_p) \tag{25}$$

where t = 0 is the starting time for the pulse of the driving source $\rho^{dr}(\underset{\sim}{x}, t)$ and $\underset{\sim}{J}^{dr}(\underset{\sim}{x}, t)$, and

---

7. Karzas, W. J., and T. C. Mo, Linear and Non-Linear EMP Diffusion Through a Ferromagentic Conducting Slab, RDA-TR-9900-001, R & D Associates, July 1975; also published as AFWL EMP Interaction Note No. 291; and to be published in the EMP Special Issue of the IEEE Transactions on Antennas and Propagation (scheduled January 1978).

$$\tau_p \equiv \begin{pmatrix}\text{maximum time length of interest} \\ \text{for the SGEMP problem}\end{pmatrix}$$

$$\sim \max(\tau_x, \tau_\beta) \tag{26}$$

$$\tau_\beta \equiv \begin{pmatrix}\text{typical time duration that} \\ \text{the charges travel near the} \\ \text{object and contribute signi-} \\ \text{ficantly to the SGEMP fields} \\ \text{of interest}\end{pmatrix}$$

$$\sim \frac{R}{v} \equiv \frac{\tau_R}{\beta} \ . \tag{27}$$

Now, depending on τ_p, the t_p may or may not exceed $\tau_{df}(\ell)$, and the magnetic boundary conditions that should be imposed on S, contrary to the electric ones, are different for those two cases (see Figure 2 depicting the various time parameters).

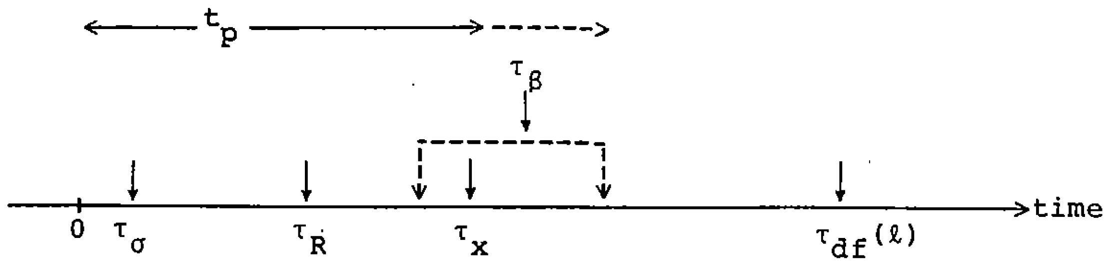
For the case in which the charges do move slowly and the current pulse shape does change gently so that the quasi-static condition (10) or (13) is satisfied, yet fast and swift enough respectively to make the interesting time duration of the problem τ_p short compared to the magnetic diffusion time, i.e.,

$$\tau_{df}(\ell) >> \tau_p \tag{28}$$

the magnetic field has hardly had time to diffuse into the conductor at any t_p. In such a case, the total magnetic field in V_{in} is negligible compared to that in V_{out} and can be set to zero. Accordingly, we are well justified to require

$$\underset{\sim}{n}(\underset{\sim}{x}_s) \cdot \underset{\sim}{B}^{total}(\underset{\sim}{x}_{s\pm}, t) = 0, \quad x_s \varepsilon S \ . \tag{29}$$

Typical SGEMP problem: $\tau_o << \tau_R < \tau_x < \tau_\beta << \tau_{dr(\ell)}$



$\tau_\sigma \equiv \frac{\varepsilon_{in}}{\sigma_{in}}$ - electric relaxation time for the conductor.

$\tau_R \equiv \frac{R}{C}$ - typical EM retardation time over the distance of interest for the SGEMP problem ($R \sim \ell$).

τ_x ≡ typical time scale for the x-ray pulse (~ the smaller of the rise time and pulse duration).

$\tau_\beta \equiv \frac{R}{V} \equiv \frac{\tau_r}{\beta}$ - typical time duration that the charges travel near the object and contribute significantly to the SGEMP fields near the object.

$t_p \equiv \frac{2\pi}{\omega_p}$ - typical times at which the SGEMP fields are of interest (ω_p therefore is the main frequency contents).

~ $\tau_{df}(\delta_p)$ where δ_p is the skin depth of wave at frequency ω_p.

$\tau_{df}(\ell) \equiv \mu_{in}\sigma_{in}\ell^2$ ~ magnetic diffusion time into object of typical dimension ℓ.

Figure 2. Various Time Parameters for the SGEMP Problem

This is obtained by further invoking the condition of continuous magnetic field component normal to any discontinuous surface, a condition implied by the Maxwell equations themselves. For most SGEMP problems of interest, the no-diffusion condition (28) is well met. Thus, the boundary condition (29), which has already invoked the property that the magnetic field vanishes in V_{in}, should be imposed on S. On the one hand, it helps (17) to determine the $\underset{\sim}{B}_{out}^{sc,1}(\underset{\sim}{x}, t)$ in V_{out}. On the other hand, it helps to define mathematically the $\underset{\sim}{B}_{in}^{sc,1}(\underset{\sim}{x}, t)$ in V_{in}, via

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{B}_{in}^{sc,1}(\underset{\sim}{x}, t) = 0, \qquad \underset{\sim}{x}\varepsilon V_{in} \tag{30a}$$

$$\underset{\sim}{\nabla}\times\underset{\sim}{B}_{in}^{sc,1}(\underset{\sim}{x}, t) = \mu_o\varepsilon_o \frac{\partial \underset{\sim}{E}_{in}^{sc,o}(\underset{\sim}{x}, t)}{\partial t}, \quad \underset{\sim}{x}\varepsilon V_{in} \tag{30b}$$

$$\underset{\sim}{B}_{in}^{sc,1}(\underset{\sim}{x}_{s-}, t) \cdot \underset{\sim}{n}(\underset{\sim}{x}_s) = -\underset{\sim}{B}^{dr,1}(\underset{\sim}{x}_s, t) \cdot \underset{\sim}{n}(\underset{\sim}{x}_s) \tag{30c}$$

Equation (30) uniquely determines the $\underset{\sim}{B}_{in}^{sc,1}(\underset{\sim}{x}, t)$ in V_{in} to be $-\underset{\sim}{B}^{dr,1}(\underset{\sim}{x}, t)$ there, consistent as it should be with the physical requirement of zero magnetic field in V_{in} that leads to (29) in the first place. Finally, we must emphasize here that for this quasi-static good-conductor no-diffusion case, the internal medium parameters of the object, μ_{in}, ε_{in}, and σ_{in}, do not enter explicitly in the mathematical formulation in determining the scattered fields, but rather they enter only implicitly via (18), (19), (24), and (28) to justify the formulation.

Now, we turn our attention to the other case in which the time duration of interest τ_p is not always smaller than the magnetic diffusion time $\tau_{df}(\ell)$ (see Figure 2 for time parameters). This happens even under the quasi-static restrictions of (10) or (11), in any one of the following situations.

First, still under condition (28), if one is simply looking for the fields at times much later than the $\tau_{df}(\ell)$, then the magnetic field has substantially diffused into the conductor and the magnetic boundary condition (29) is not valid. Although by this time the SGEMP magnetic field has become so small and thus commands no practical interest, the only correct way to obtain it is by solving the diffusion equation in V_{in}, connected by the relation of continuous tangential electric and magnetic fields on S to the fields in V_{out} (ref. 7). At such times, the electric field near the object is virtually electrostatic and is given by the electrostatic surface charge distribution on the conductor, and the continuous magnetic field dictates that there is no surface current flowing on the surface of the conductor. Second, if (18) is violated because the charges move too slowly, i.e.,

$$\tau_{df}(\ell) \lesssim \frac{R}{c\beta} \tag{31a}$$

or because the pulse rises too gently and lasts too long, i.e.,

$$\tau_{df}(\ell) \lesssim \tau_x \tag{31b}$$

then in the time of interest the diffusion has happened and (29) becomes invalid. Again, the whole magnetic field for this case is very small and should be found via the diffusion equation and the tangential continuity of fields as was in the first situation. In any case, these well-diffused situations do not arise in realistic SGEMP problems of interest. In view of these diffusion processes, we can conclude that the boundary condition (29) and the SGEMP magnetic fields near the conductor and at times t_p obtained by invoking that condition together with equations (17) and (30) are accurate to the order of $t_p/\tau_{df}(\ell)$.

SECTION III

THE DIFFICULTY AND ITS SOLUTION FOR THE BASIC SGEMP GENERATION PROBLEM

1. THE ELECTRIC FIELD AND THE SURFACE CHARGE DENSITY

The determination of the quasi-static electric SGEMP field is straightforward and well known. However, the surface part of the driving charge, $\sigma_o^{dr}(\underset{\sim}{x}_s, t)$, may sometime cause ambiguity as to its double inclusion in the scattered field and deserves some clarification. This point can be treated most clearly by observing two facts. First, the $\sigma_o^{dr}(\underset{\sim}{x}_s, t)$ on S generates an $\underset{\sim}{E}^{dr,o,S}(\underset{\sim}{x}, t)$, and the $\rho_o^{dr}(\underset{\sim}{x}, t)$ in V_{out} generates an electrostatic field $\underset{\sim}{E}^{dr,o,V}(\underset{\sim}{x}, t)$ (see (14)). Thus, the quasi-static electric field of the whole SGEMP problem can be split correspondingly and fully into these two parts. In particular, the scattered field can be split as

$$\underset{\sim}{E}^{sc,o}(\underset{\sim}{x}, t) = \underset{\sim}{E}^{sc,o,S}(\underset{\sim}{x}, t) + \underset{\sim}{E}^{sc,o,V}(\underset{\sim}{x}, t). \qquad (32)$$

Second, in order for the surface-driven part of the total quasi-static electric field to satisfy

$$\underset{\sim}{E}^{dr,o,S}(\underset{\sim}{x}, t) + \underset{\sim}{E}^{sc,o,S}(\underset{\sim}{x}, t) \equiv 0, \ \underset{\sim}{x}\varepsilon V_{in} \qquad (33a)$$

where both the $\underset{\sim}{E}^{dr,o,S}(\underset{\sim}{x}, t)$ and the $\underset{\sim}{E}^{sc,o,S}(\underset{\sim}{x}, t)$ are generated purely by surface charge densities on the same S, these two surface charge densities must be negative and lie on top of each other. Consequently, we have

$$\underset{\sim}{E}^{dr,o,S}(\underset{\sim}{x}, t) + \underset{\sim}{E}^{sc,o,S}(\underset{\sim}{x}, t) \equiv 0, \ \underset{\sim}{x} \text{ everywhere} \qquad (33b)$$

and can hereafter completely avoid the participation of the surface-charge-driven electric field in the problem except in determining an integration constant for the $\underset{\sim}{E}^{sc,o,V}(\underset{\sim}{x}, t)$.

Making use of these observations, we immediately get:

$$\underset{\sim}{E}^{total,o}(\underset{\sim}{x}, t) = \underset{\sim}{E}^{sc,o,V}(\underset{\sim}{x}, t) + \underset{\sim}{E}^{dr,o,V}(\underset{\sim}{x}, t) \tag{34a}$$

where

$$\underset{\sim}{E}^{sc,o,V}(\underset{\sim}{x}, t) \equiv \underset{\sim}{E}^{sc,o,V}_{out}(\underset{\sim}{x}, t) \equiv -\underset{\sim}{\nabla}\phi^{sc,V}_{out}(\underset{\sim}{x}, t), \quad \underset{\sim}{x}\varepsilon V_{out} \tag{34b}$$

$$\underset{\sim}{E}^{sc,o,V}(\underset{\sim}{x}, t) \equiv \underset{\sim}{E}^{sc,o,V}_{in}(\underset{\sim}{x}, t) \equiv -\underset{\sim}{\nabla}\phi^{sc,V}_{in}(\underset{\sim}{x}, t), \quad \underset{\sim}{x}\varepsilon V_{in} \tag{34c}$$

and

$$\left\{\begin{array}{ll}
\underset{\sim}{\nabla}^2\phi^{sc,V}_{out}(\underset{\sim}{x}, t) = 0, \qquad \underset{\sim}{x}\varepsilon V_{out} & (35a) \\
\underset{\sim}{n}(\underset{\sim}{x}_s) \times \left[\underset{\sim}{\nabla}\phi^{sc,V}_{out}(\underset{\sim}{x}, t)\right]_{\underset{\sim}{x}=\underset{\sim}{x}_{s+}} = \underset{\sim}{n}(\underset{\sim}{x}_s) \times \underset{\sim}{E}^{dr,o,V}(\underset{\sim}{x}_s, t) & (35b) \\
-\varepsilon_o \oint_S \left[\underset{\sim}{\nabla}\phi^{sc,V}_{out}(\underset{\sim}{x}, t)\right]_{\underset{\sim}{x}=\underset{\sim}{x}_{s+}} \cdot \underset{\sim}{n}(\underset{\sim}{x}_s)\, dA = \oint_S \sigma^{dr}_o(\underset{\sim}{x}_s, t)\, dA & (35c)
\end{array}\right.$$

$$\left\{\begin{array}{ll} \nabla^2 \Phi_{in}^{sc,V}(\underset{\sim}{x}, t) = 0 \qquad \underset{\sim}{x} \varepsilon V_{in} & (36a) \\ \underset{\sim}{n}(\underset{\sim}{x}_s) \times \left[\nabla \Phi_{in}^{sc,V}(\underset{\sim}{x}, t)\right]_{\underset{\sim}{x}=\underset{\sim}{x}_{s-}} = \underset{\sim}{n}(\underset{\sim}{x}_s) \times \underset{\sim}{E}^{dr,o,V}(\underset{\sim}{x}_s, t) & (36b) \\ \oint_S \left[\nabla \Phi_{in}^{sc,V}(\underset{\sim}{x}, t)\right]_{\underset{\sim}{x}=\underset{\sim}{x}_{s-}} \cdot \underset{\sim}{n}(\underset{\sim}{x}_s)\, dA = 0 & (36c) \end{array}\right.$$

Notice that (35c) and (36c) are consistent with (34a) because the $\underset{\sim}{E}^{dr,o,V}(\underset{\sim}{x}, t)$ is generated by sources in V_{out} only and thus contributes no net charge in the closed surface integration on S.

Now, by the very definition of $\underset{\sim}{E}^{total,o}(\underset{\sim}{x}, t)$ its tangential components are always continuous across S. However, its normal component is not, and cannot be, continuous across S.* Invoking the well-known facts that specifying the fields everywhere determines the source and that a surface charge density on S implies a jump in the normal electric field across S, there must exist a total surface charge density on the conducting surface S associated with the total electric field and it must be given by

$$\sigma^{total}(\underset{\sim}{x}_s, t) \equiv \varepsilon_o \left[\underset{\sim}{E}^{total,o}(\underset{\sim}{x}_{s+}, t) - \underset{\sim}{E}^{total,o}(\underset{\sim}{x}_{s-}, t)\right] \cdot \underset{\sim}{n}(\underset{\sim}{x}_s) \tag{37a}$$

$$= \varepsilon_o \left[\underset{\sim}{E}^{sc,o,V}(\underset{\sim}{x}_{s+}, t) + \underset{\sim}{E}^{dr,o,V}(\underset{\sim}{x}_{s+}, t)\right] \cdot \underset{\sim}{n}(\underset{\sim}{x}_s) \tag{37b}$$

*Imposing the condition of continuous normal electric field component, in addition to the tangential ones, makes the problem of solving for $\Phi^{sc,V}(\underset{\sim}{x}, t)$ overdetermined, which in general has no solution.

Physically, this $\sigma^{total}(\underset{\sim}{x}_s, t)$ on S can be thought of as a byproduct of the electric field in the process of adjusting itself to meet the boundary condition (20).

We emphasize that the $\sigma^{total}(\underset{\sim}{x}_s, t)$ given by (37) is the overall total surface charge density physically existing on S. It is the sum of $\sigma_o^{dr}(\underset{\sim}{x}_s, t)$, the "clamped in" surface driving charges, and $\sigma^{sc}(\underset{\sim}{x}_s, t)$, the "scattering" surface charge density as given by

$$\sigma^{total}(\underset{\sim}{x}_s, t) = \sigma_o^{dr}(\underset{\sim}{x}_s, t) + \sigma^{sc}(\underset{\sim}{x}_s, t). \tag{38}$$

Here, the $\sigma^{sc}(\underset{\sim}{x}_s, t)$ is associated with the scattered electric field only and is defined by

$$\sigma^{sc}(\underset{\sim}{x}_s, t) \equiv \epsilon_o \left[\underset{\sim}{E}_{out}^{sc,o}(\underset{\sim}{x}_{s+}, t) - \underset{\sim}{E}_{in}^{sc,o}(\underset{\sim}{x}_{s-}, t)\right] \cdot \underset{\sim}{n}(\underset{\sim}{x}_s) \tag{39a}$$

and carries no net amount of charge on S:

$$\oint_S \sigma^{sc}(\underset{\sim}{x}_s, t)\, dA = 0. \tag{39b}$$

Results (38) and (39) suggest an alternative physical interpretation: the scattered electric field is generated by the $\sigma^{sc}(\underset{\sim}{x}_s, t)$ which has precisely rearranged the charges on the surface (of total net amount zero) in such a way as to create an electric field to counter the driving one according to (21) and (22).

Finally, we notice that the "scattering" surface charge density $\sigma^{sc}(\underset{\sim}{x}_s, t)$ can be further decomposed into

$$\sigma^{sc}(\underset{\sim}{x}_s, t) = \sigma^{sc,S}(\underset{\sim}{x}_s, t) + \sigma^{sc,V}(\underset{\sim}{x}_s, t) \qquad (40)$$

Here the surface-driven part $\sigma^{sc,S}(\underset{\sim}{x}_s, t)$ and the volume-driven part $\sigma^{sc,V}(\underset{\sim}{x}_s, t)$ are defined in a manner similar to (39a) but with the $\underset{\sim}{E}^{sc,o}(\underset{\sim}{x}_s, t)$ replaced, respectively, by $\underset{\sim}{E}^{sc,o,S}(\underset{\sim}{x}, t)$ and $\underset{\sim}{E}^{sc,o,V}(\underset{\sim}{x}, t)$ (see (32) through (36)). From the foregoing discussions, under such a further decomposition we clearly have $\sigma^{sc,S}(\underset{\sim}{x}_s, t) = -\sigma_o^{dr}(\underset{\sim}{x}_s, t)$ and $\sigma^{sc,V}(\underset{\sim}{x}_s, t) = \sigma^{total}(\underset{\sim}{x}_s, t)$. Although this does provide a further analytical insight, we should not allow ourselves to be diverted from the two essential physical points: The overall observable surface charge density is $\sigma^{total}(\underset{\sim}{x}_s, t)$ and the surface charge density that generates and is generated by the scattered electric field is $\sigma^{sc}(\underset{\sim}{x}_s, t)$. Of course, when the driving source contains no charge emission from the isolated conductor itself, $\sigma_o^{dr}(\underset{\sim}{x}_s, t) \equiv 0 \equiv \sigma^{sc,S}(\underset{\sim}{x}_s, t)$ and there is no distinction among $\sigma^{total}(\underset{\sim}{x}_s, t)$, $\sigma^{sc}(\underset{\sim}{x}_s, t)$, and $\sigma^{sc,V}(\underset{\sim}{x}_s, t)$.

2. A DIFFICULTY IN FINDING THE SURFACE MAGNETIC FIELD

After the electrostatic field has been determined, it can be used to determine the quasi-static SGEMP magnetic field everywhere uniquely via (15b), (17), (29) and (30). Being so determined, the magnetostatic field can be evaluated in particular at x_{s+} on the conductor surface S. But such a process calling for a full solution of the magnetic field everywhere in the three-dimensional space provides more information than we need and demands more effort than we have to expend, if our primary interest is in the magnetic field, or the current flows, on S only. One "well-known" alternative that bypasses the ordeal of solving for the full magnetic field in three-dimensions and that was frequently used to obtain the surface magnetic field directly is to use the magnetic surface integral equation method. However, this method reveals little physical insight into the generation mechanism of the surface magnetic field such as how it is excited by and varies with the driving charges. Further, it basically invokes only one necessary property of the required magnetic field, and its full sufficiency has not been explicitly established.* Thus, the question is: is there a simple method, and if so what is it, to directly solve for the quasi-static magnetic field on S which provides both physical insights into the field generation process and gives values of the field on S that could be used as starting values for further obtaining the fields in V_{out} via a simple free-space propagation?

To start answering the questions, we proceed as follows: First, if we had solved the problem, on S there must exist an overall total surface current density $\underset{\sim}{K}^{total}(\underset{\sim}{x}_s, t)$ given by

$$\underset{\sim}{K}^{total}(\underset{\sim}{x}_s, t) \equiv \frac{1}{\mu_o} \underset{\sim}{n}(\underset{\sim}{x}_s) \times \left[\underset{\sim}{B}_{out}^{total,1}(\underset{\sim}{x}_{s+}, t) - \underset{\sim}{B}_{in}^{total}(\underset{\sim}{x}_{s-}, t)\right] \tag{41}$$

whose physical validity is implied by the Maxwell equations, similar and analogous to the case for surface charge density given by (37a). Second, from and for the scattered magnetic field we can define, similar to (41), a scattering surface current density

$$\underset{\sim}{K}^{sc}(\underset{\sim}{x}_s, t) \equiv \frac{1}{\mu o} \underset{\sim}{n}(\underset{\sim}{x}_s) \times \left[\underset{\sim}{B}_{out}^{sc,1}(\underset{\sim}{x}_{s+}, t) - \underset{\sim}{B}_{in}^{sc,1}(\underset{\sim}{x}_{s-}, t)\right]. \tag{42}$$

Third, from (41) and (42) we immediately have

---

*The electric integral equation has divergence problems associated with it, and the magnetic one was not explicitly proved to possess a unique and correct solution (see K.S.H. Lee and L. Marin, "Limitations of Wire Grid Modeling of a Closed Surface," AFWL EMP Interaction Note 231, May 1975).

$$\underset{\sim}{K}^{total}(\underset{\sim}{x}_s, t) = \underset{\sim}{K}^{sc}(\underset{\sim}{x}_s, t) + \underset{\sim}{K}^{dr}(\underset{\sim}{x}_s, t) \tag{43}$$

where $\underset{\sim}{K}^{dr}(\underset{\sim}{x}_s, t)$ is the surface part, if any, in the driving source current density. For the present problem, we have let $\underset{\sim}{K}^{dr}(\underset{\sim}{x}_s, t) \equiv 0$, and there is no distinction between $\underset{\sim}{K}^{total}(\underset{\sim}{x}_s, t)$ and $\underset{\sim}{K}^{sc}(\underset{\sim}{x}_s, t)$. Fourth, an examination of (17b), (30b), and (39b) reveals that

$$-\underset{\sim}{\nabla}_{(s)} \cdot \underset{\sim}{K}^{sc}(\underset{\sim}{x}_s, t) = \frac{\partial \sigma^{sc}(\underset{\sim}{x}_s, t)}{\partial t} \tag{44a}$$

$$= \frac{\partial \sigma^{total}(\underset{\sim}{x}_s, t)}{\partial t} - \frac{\partial \sigma_o^{dr}(\underset{\sim}{x}_s, t)}{\partial t} \tag{44b}$$

$$= \frac{\partial}{\partial t} \sigma^{total}(\underset{\sim}{x}_s, t) + \underset{\sim}{n}(\underset{\sim}{x}_s) \cdot \underset{\sim}{J}^{dr}(\underset{\sim}{x}_s, t) + \underset{\sim}{\nabla}_{(s)} \cdot \underset{\sim}{K}^{dr}(\underset{\sim}{x}_s, t) \tag{44c}$$

Here the surface divergence on S, $\underset{\sim}{\nabla}_{(s)} \cdot$, is defined for the surface components of a vector field $\underset{\sim}{V}(\underset{\sim}{x})$ on S by

$$\underset{\sim}{\nabla}_{(s)} \cdot \left[\underset{\sim}{n}(\underset{\sim}{x}_s) \times \underset{\sim}{V}(\underset{\sim}{x}_s) \right] \equiv -\underset{\sim}{n}(\underset{\sim}{x}_s) \cdot \left[\underset{\sim}{\nabla} \times \underset{\sim}{V}(\underset{\sim}{x}) \right]_{\underset{\sim}{x}=\underset{\sim}{x}_s} . \tag{45}$$

Equation (44) states the conservation of surface charges. It is well-known and can also be obtained intuitively by extremely simple considerations. Fifth, in view of (29) and (41), a knowledge of $\underset{\sim}{K}^{total}(\underset{\sim}{x}_s, t)$ on S is obviously a knowledge of $\underset{\sim}{B}^{total,1}(\underset{\sim}{x}_s, t)$ except for a trivial rotation of 90° on S. Thus, (44) is one condition on S that can be used to determine the two-component surface vector field $\underset{\sim}{K}^{total}(\underset{\sim}{x}_s, t)$ or $\underset{\sim}{B}^{total,1}(\underset{\sim}{x}_s, t)$.

The difficulty lies in finding the one more condition needed to solve for the surface current $\underset{\sim}{K}^{total}(\underset{\sim}{x}_s, t)$ or the magnetic field $\underset{\sim}{B}^{total,1}(\underset{\sim}{x}_s, t)$, on S. To put it differently, we see that on S there are five independent conditions: one in (17a), three in (17b) from which (44) is a derived but not independent condition, and one in (29). But there are six unknown quantities involved: the three magnetic field components and their three normal (to S) derivatives. Clearly, we are short of one independent condition on S to determine these six field quantities on S and cannot hope to obtain it by manipulating vector or differential identities among these five conditions.

Further, a closer examination of those five equations on S shows that starting with any one of the infinite sets of the six under-determined field components and their normal derivatives will result in fields in V_{out} that always satisfy the five field equations there and are determined uniquely. This, of course, is just another way of exhibiting the difficulty.

The lack of one condition for the surface current is a basic difficulty in the understanding of the SGEMP phenomena. It has heretofore defied solution, and even evaded wide recognition. In the next section we shall search for and obtain a solution for it.

3. A SOLUTION OF THE BASIC DIFFICULTY

To determine uniquely the magnetostatic field in V_{out}, and thus on S+, in addition to the five equations mentioned at the end of the previous section, we must invoke the field behavior on S_∞. This is the one more condition that we have not used. In searching for the one condition needed to determine the magnetic field on S directly, we must use it in some way.

Guided by physical intuitions, special case studies, and detailed examinations of the defining equations, our searching effort results in two mathematical conjectures. The conjectures are physically plausible, and even obvious to some, but in this report we have not been able to demonstrate rigorous and complete mathematical proofs for them. Partly based upon these conjectures, we have resolved the basic difficulty in obtaining the magnetic field on S directly.

a. The Two Conjectures

Conjecture one is as follows: In a simply connected, closed, smooth, two-dimensional surface S, a smooth tangential vector field $\underset{\sim}{K}(\underset{\sim}{x}_s)$ can be decomposed uniquely as

$$\underset{\sim}{K}(\underset{\sim}{x}_s) = \underset{\sim}{K}_I(\underset{\sim}{x}_s) + \underset{\sim}{K}_{II}(\underset{\sim}{x}_s) \qquad (46)$$

such that

$$\underset{\sim}{\nabla}_{(s)} \cdot \underset{\sim}{K}_I(\underset{\sim}{x}_s) \equiv 0 \qquad (47)$$

$$\underset{\sim}{\nabla}_{(s)} \cdot \left[\underset{\sim}{n}(\underset{\sim}{x}_s) \times \underset{\sim}{K}_{II}(\underset{\sim}{x}_s)\right] \equiv 0. \qquad (48)$$

Here the $\underset{\sim}{\nabla}_{(s)} \cdot$ is defined by (45) and the $\underset{\sim}{n}(\underset{\sim}{x}_s)$ is the unit vector normal to S. The required properties of $\underset{\sim}{K}_I(\underset{\sim}{x}_s)$ and $\underset{\sim}{K}_{II}(\underset{\sim}{x}_s)$ can be stated alternatively but equivalently as

$$\oint_C \underset{\sim}{K}_I(\underset{\sim}{x}_s) \times \underset{\sim}{n}(\underset{\sim}{x}_s) \cdot d\underset{\sim}{\ell} \equiv 0 \qquad (47')$$

$$\oint_C \underset{\sim}{K}_{II}(\underset{\sim}{x}_s) \cdot d\underset{\sim}{\ell} \equiv 0 \qquad (48')$$

where C is any closed contour on S.

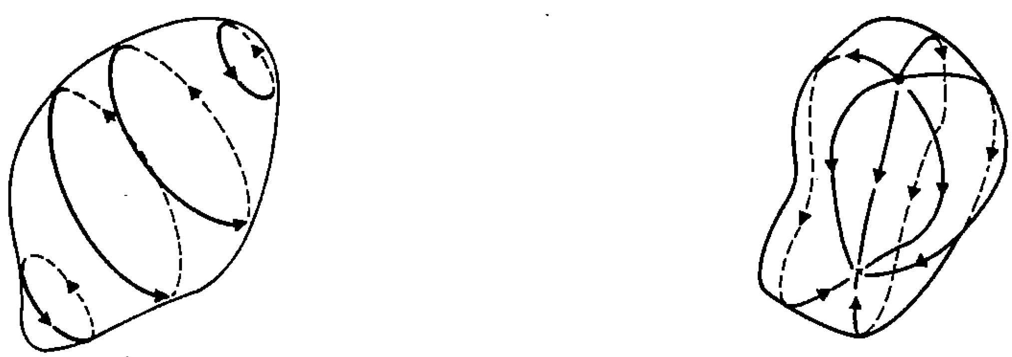
The part $\underset{\sim}{K}_I(\underset{\sim}{x}_s)$ is "surface-divergenceless" (see (47)); or, equivalently, it has no net emergence from within any closed loop on S (see (47')). This is a clear analog to a solenoidal vector field in a three-dimensional space, such as the magnetic field or the velocity field of an incompressible fluid of which neither has net flux emerging from any closed-box surface. The part $\underset{\sim}{K}_{II}(\underset{\sim}{x}_s)$ is "surface-curlless" (see (48)); or, equivalently, it has zero circulation around any closed loop on S. It is an obvious analog to a gradient vector field in three dimensions such as the electrostatic field or the irrotational flow velocity field that commands zero amount after being integrated along any closed line contour. From these analogs, the $\underset{\sim}{K}_I(\underset{\sim}{x}_s)$ and $\underset{\sim}{K}_{II}(\underset{\sim}{x}_s)$ will be referred to as the "magnetostatic" type and "electrostatic" type, respectively. Figure 3 depicts their behaviors graphically. Perceived from its three-dimensional analog, for which a decomposition of this kind is well-known (ref. 8), the conjecture seems plausible; it may even be trivial to prove. However, the effort we spent showed that the latter was not the case. The following proof we obtained is not rigorously complete, but reduces the validity of the conjecture to the solvability of a partial differential equation.

---

8. Morse, P. M. and H. Fishback, Methods of Theoretical Physics, Part II, McGraw-Hill, 1953, p. 1763.

Magnetostatic Type
Surface Field $\underset{\sim}{K}_I(\underset{\sim}{x}_s)$

Electrostatic Type
Surface Field $\underset{\sim}{K}_{II}(\underset{\sim}{x}_s)$



$$\underset{\sim}{\nabla}(s) \cdot \underset{\sim}{K}_I(\underset{\sim}{x}_s) \equiv 0$$

$$\updownarrow$$

$$\oint_{C \text{ on } S} \underset{\sim}{n}(\underset{\sim}{x}_s) \times \underset{\sim}{K}_I(\underset{\sim}{x}_s) \cdot d\underset{\sim}{\ell} \equiv 0$$

$$\underset{\sim}{\nabla}(s) \cdot [\underset{\sim}{n}(\underset{\sim}{x}_s) \times \underset{\sim}{K}_{II}(\underset{\sim}{x}_s)] \equiv 0$$

$$\updownarrow$$

$$\oint_{C \text{ on } S} \underset{\sim}{K}_{II}(\underset{\sim}{x}_s) \cdot d\underset{\sim}{\ell} \equiv 0$$

Figure 3. The "Magnetostatic" and the "Electrostatic Type Surface Fields

Consider the tensor equation (Appendix D):

$$K^{\mu} = f^{\mu;\alpha}{}_{;\alpha} - R^{\mu}{}_{\lambda} f^{\lambda} \quad (49)$$

where the semicolon (;) indicates covariant or contravariant derivatives and the $R^{\mu}{}_{\lambda}$ is the Ricci curvature tensor (ref. 9).* This equation in two dimensions reduces to

$$K^{\mu} = f^{\mu;\alpha}{}_{;\alpha} - C_g f^{\mu} \quad (50)$$

where C_g is the Gaussian curvature scalar for the two-dimension surface. If (49) or (50) possesses a unique solution for f^{μ}, with whatever integrability and regularity conditions on geometry and field as required to ensure that solution, then we can decompose K^{μ} in terms of f^{μ}. The results are (Appendix D)

$$K^{\mu} = K^{\mu}_{I} + K^{\mu}_{II} \quad (51)$$

where the

$$K^{\mu}_{I} \equiv (f^{\alpha;\mu} - f^{\alpha;\mu})_{;\alpha} \quad (52)$$

---

9. Sokolnikoff, I. S., Tensor Analysis, John Wiley and Sons, 1951, Capter 3.

*Also, notice that (50) resembles a Klein-Gordon equation for vector fields generalized in curved space. Of course, if the manifold itself is flat, when and only when $R_{\mu\mu\alpha\beta} = 0$, then (50) can be reduced to the Cartesian version of the Poisson equation. Equation (49) is exactly the same as the equation obeyed by the EM four-potential in a curved spacetime required by general relativity; see any textbook on general relativity, e.g., C. W. Misner, D. S. Thorne, J. A. Wheeler, Gravitation, Freeman & Co., 1978, p. 569.

satisfies the generalized version of condition (47):

$$K^{\mu}_{I;\mu} = 0 \tag{53}$$

and the

$$K^{\mu}_{II} \equiv f^{\alpha}{}_{;\alpha}{}^{;\mu} \tag{54}$$

satisfies the generalized version of condition (48):

$$\left(\frac{\varepsilon^{\mu\nu}}{\sqrt{|g|}} K_{II;\nu}\right)_{;\mu} = 0. \tag{55}$$

Here, the $\varepsilon^{\mu\nu}/\sqrt{|g|}$ is the antisymmetric unit pseudo-tensor and the g is the determinant of the metric coefficient $g_{\mu\nu}$. In this way, the validity of conjecture one is reduced to the unique solvability of (50), with whatever constraints accordingly imposed. We leave the proof just here.

Conjecture two is: Let $\underset{\sim}{B}(\underset{\sim}{x})$ be the magnetostatic field generated by a smooth surface current density $\underset{\sim}{K}(\underset{\sim}{x}_s)$ on a simply connected, closed, smooth surface S. Then $\underset{\sim}{B}(\underset{\sim}{x}_s) \cdot \underset{\sim}{n}(\underset{\sim}{x}_s) \equiv 0$ if and only if that $\underset{\sim}{K}(\underset{\sim}{x}_s)$ satisfies (48).

The physical plausibility of this conjecture can be argued loosely by considering an isolated, highly conductive object, with or without net charge on it, excited by some incident EM field. The incident field excites on the surface some current flow which in turn creates its own EM field. If the magnetic field created is purely tangential to the surface, then we have

$$0 \equiv \frac{-\partial}{\partial t} \int_A \underset{\sim}{B}^c(\underset{\sim}{x}_s) \cdot d\underset{\sim}{A} = \oint_C \underset{\sim}{E}^c(\underset{\sim}{x}_s) \cdot d\underset{\sim}{\ell}$$

$$\sim \oint_C \underset{\sim}{K}^c(\underset{\sim}{x}_s) \cdot d\underset{\sim}{\ell} \quad .$$

Here the argument is loose in that we used the assumption, although plausible, that the current flows in the direction of and is proportional to the incident, and thus also the created, electric field on S. This, of course, does not constitute a proof.

We have not found rigorous mathematical proofs within this investigation for either of the two conjectures, especially the second, and have purposefully left the smoothness requirements in the conjectures undefined. A clear delineation of the smoothness conditions, and of other possible conditions, can be obtained only when the conjectures are being established with mathematical rigor. However, we have explicitly established that both conjectures are true for planar and spherical surfaces. Before making use of these two conjectures, we would like to emphasize that based on this investigation and our past experiences and intuitions we are quite convinced of the truths of the conjectures. Further, they, especially conjecture one which is critical to the validity of the solution, have immediate and profound implications in reality for SGEMP problems, and for all EM scattering problems involving "perfect conductor" boundary conditions, especially at low frequencies.

b. The Decomposition of the Electrostatic Contribution to the Surface Current Density

By use of conjecture one, the scattering surface current density $\underset{\sim}{K}^{sc}(\underset{\sim}{x}_s, t)$ defined on S by (42), if and after it were found, can be decomposed as

$$\underset{\sim}{K}^{sc} = \underset{\sim}{K}_I^{sc}(\underset{\sim}{x}_s, t) + \underset{\sim}{K}_{II}^{sc}(\underset{\sim}{x}_s, t). \tag{56}$$

Here, the "magnetostatic" part $\underset{\sim}{K}_I(\underset{\sim}{x}_s, t)$ satisfies

$$\underset{\sim}{\nabla}_{(s)} \cdot \underset{\sim}{K}_I^{sc}(\underset{\sim}{x}_s, t) = 0 \tag{57}$$

and the "electrostatic" part $\underset{\sim}{K}_{II}^{sc}(\underset{\sim}{x}_s, t)$ satisfies

$$\underset{\sim}{\nabla}_{(s)} \cdot \left[\underset{\sim}{n}(\underset{\sim}{x}_s) \times \underset{\sim}{K}_{II}^{sc}(\underset{\sim}{x}_s, t)\right] = 0. \tag{58}$$

Next, the scattered fields can be decomposed correspondingly into a part I, generated by the surface charge-current $(\sigma_I^{sc}(\underset{\sim}{x}_s, t) \equiv 0, \underset{\sim}{K}_I^{sc}(\underset{\sim}{x}_s, t))$ and a part II generated by the $(\sigma_{II}^{sc}(\underset{\sim}{x}_s, t) \equiv \sigma^{sc}(\underset{\sim}{x}_s, t), \underset{\sim}{K}_{II}^{sc}(\underset{\sim}{x}_s, t))$. We shall investigate these two parts separately.

Consider first the type II scattered surface charge-current and the part II scattered fields generated by it via equations:

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B}_{II}^{sc,1}(\underset{\sim}{x}, t) = 0 \tag{59}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B}_{II}^{sc,1}(\underset{\sim}{x}, t) = -\mu_o \varepsilon_o \frac{\partial}{\partial t} \underset{\sim}{\nabla}\Phi^{sc,V}(\underset{\sim}{x}_s, t) + \mu_o \underset{\sim}{K}_{II}^{sc}(\underset{\sim}{x}_s; t)\, \delta(\underset{\sim}{x}-\underset{\sim}{x}_s) \tag{60}$$

Integrating (60) across S immediately gives

$$\underset{\sim}{K}_{II}^{sc}(\underset{\sim}{x}_s, t) = \frac{1}{\mu_0} \underset{\sim}{n}(\underset{\sim}{x}_s) \times \left[\underset{\sim}{B}_{II}^{sc,1}(\underset{\sim}{x}_{s+}, t) - \underset{\sim}{B}_{II}^{sc,1}(\underset{\sim}{x}_{s-}, t)\right], \tag{61}$$

a result similar to, and actually motivating the definition in, (42). Taking $n(x_s) \cdot$ to (60) on S, and making use of (45), (58), and (61), yields

$$-\underset{\sim}{\nabla}_{(s)} \cdot \underset{\sim}{K}_{II}^{sc}(\underset{\sim}{x}_s, t) = \frac{\partial}{\partial t} \sigma^{sc}(\underset{\sim}{x}_s, t) \qquad (62)$$

Thus, the "electrostatic" type II surface current density, $K_{II}(x_s, t)$, is completely determined by the two equation (58) and (62). It is solely driven by the electrostatic scattered surface charge and is indeed electrostatic. This type II surface charge-current density $(\sigma_{II}^{sc}(\underset{\sim}{x}_s, t), \underset{\sim}{K}_{II}^{sc}(\underset{\sim}{x}_s, t))$ generates the quasi-static fields: the $\underset{\sim}{E}^{sc,o}(\underset{\sim}{x}, t) = -\underset{\sim}{\nabla}\Phi^{sc}(\underset{\sim}{x}, t)$ and the $\underset{\sim}{B}_{II}^{sc,1}(\underset{\sim}{x}, t)$ by a free-space Green's function. These fields so generated satisfy the total quasi-static electric problem fully, including meeting its boundary condition and making its nullness in V_{in}. However, the $\underset{\sim}{B}_{II}^{sc,1}(\underset{\sim}{x}_s, t)$ on S may have a perpendicular component. This makes the total magnetostatic problem not yet fully satisfied. The next subsection considers just that.

c. The Conditions for and the Magnetostatic Contribution to the Surface Current Density

Since the $\sigma_I^{sc}(\underset{\sim}{x}_s, t) \equiv 0$, the remaining surface current density of type I generates only a magnetostatic field, $\underset{\sim}{B}_I^{sc,1}(\underset{\sim}{x}, t)$, but no electrostatic field. The equations are the same as (59) and (60) except with the subscript II there replaced by I and the $\Phi^{sc,V}(\underset{\sim}{x}, t)$ term deleted. As a result, the $\underset{\sim}{B}_{II}^{sc,1}(\underset{\sim}{x}, t)$ can be represented by

$$
\underset{\sim}{B}_I^{sc,1}(\underset{\sim}{x},t) \equiv \begin{cases} \underset{\sim}{B}_{I,out}^{sc,1}(\underset{\sim}{x}, t) = -\underset{\sim}{\nabla}\psi_{out}^{sc}(\underset{\sim}{x}, t), \; \underset{\sim}{\nabla}^2\psi_{out}^{sc}(\underset{\sim}{x}, t) = 0 \\ \qquad \underset{\sim}{x}\varepsilon V_{out} \\ \underset{\sim}{B}_{I,in}^{sc,1}(\underset{\sim}{x}, t) = -\underset{\sim}{\nabla}\psi_{in}^{sc}(\underset{\sim}{x}, t), \; \underset{\sim}{\nabla}^2\psi_{in}^{sc}(\underset{\sim}{x}, t) = 0 \\ \qquad \underset{\sim}{x}\varepsilon V_{in} \end{cases} \tag{63}
$$

Here, the fields are required, of course, to be finite in their respective regions and have continuous normal components across S.

Now, as solutions of the Laplace equations the $\psi_{out}^{sc}(\underset{\sim}{x}, t)$ and $\psi_{in}^{sc}(\underset{\sim}{x}, t)$ are not uniquely determined yet. In fact, any pair of such solutions with their normal gradients on S equal to each other can represent a legitimate magnetostatic field. Therefore, we can exploit this one degree of freedom and require the vanishing of the total magnetic field:

$$
\begin{aligned}
-\underset{\sim}{n}(\underset{\sim}{x}_s) \cdot \left[\underset{\sim}{\nabla}\psi_{out}^{sc}(\underset{\sim}{x}, t)\right]_{\underset{\sim}{x}=\underset{\sim}{x}_s+} &= -\underset{\sim}{n}(\underset{\sim}{x}_s) \cdot \left[\underset{\sim}{\nabla}\psi_{in}^{sc}(\underset{\sim}{x}, t)\right]_{\underset{\sim}{x}=\underset{\sim}{x}_s-} \\
&= \underset{\sim}{n}(\underset{\sim}{x}_s) \cdot \left[\underset{\sim}{B}^{dr,1}(\underset{\sim}{x}_s, t) \right. \\
&\qquad \left. + \underset{\sim}{B}_{II}^{sc,1}(\underset{\sim}{x}_s, t)\right]
\end{aligned} \tag{64}
$$

Thus, the magnetic fields $-\underset{\sim}{\nabla}\psi_{out}^{sc}(x, t)$ and $-\underset{\sim}{\nabla}\psi_{in}^{sc}(x, t)$ are uniquely determined as the solution of the scalar Neumann problem. In particular, since none of sources that generate the magnetic fields exist in V_{in}, the uniqueness of the Neumann problem implies that the total magnetostatic field in V_{in} vanishes identically.

After obtaining the $\underset{\sim}{B}^{sc,1}_{out}(\underset{\sim}{x}, t)$, the $\underset{\sim}{K}^{sc}_{I}(\underset{\sim}{x}, t)$ is simply given by

$$\underset{\sim}{K}^{sc}_{I}(\underset{\sim}{x}, t) = \frac{-1}{\mu_0} \underset{\sim}{n}(\underset{\sim}{x}_s) \times \left[\underset{\sim}{\nabla}\psi^{sc}_{out}(\underset{\sim}{x}, t) - \underset{\sim}{\nabla}\psi^{sc}_{in}(\underset{\sim}{x}, t) \right]_{\underset{\sim}{x} \to \underset{\sim}{x}_s} \qquad (65)$$

which, as a result of (45), satisfies of course (57) as it should. The "magnetostatic" surface current $\underset{\sim}{K}^{sc}_{I}(\underset{\sim}{x}, t)$ is driven by the perpendicular component of the sum of the driving and the part II scattered magnetostatic fields. It does so via a purely magnetostatic formalism. It is thus indeed magnetostatic.

Before proceeding further, we must make several important remarks. First, in determining the $\underset{\sim}{\psi}^{sc}_{out}(\underset{\sim}{x}, t)$ in V_{out}, we did make use of its boundary condition on S_∞, just as pointed out in the beginning of Subsection IV-3. The condition used on S_∞ is a zero normal derivative. Second, the value of $\underset{\sim}{K}^{sc}_{I}(\underset{\sim}{x}_s, t)$ is determined not only by the magnetic field in V_{out}, but also that in V_{in}. The remarks show that we do have to go off the surface S to find the surface current, although merely through a simple mechanism of scalar field. Third, after the decomposition of the surface current, the successive decomposition of fields actually split the problem into two pieces: one being driven solely electrostatically via inhomogeneous differential equations (see (60)) and the other being driven solely magnetostatically via homogeneous differential equations with inhomogeneous boundary conditions (see (64)).

Fourth, we must point out that any decomposition of the surface current, after proper recombination, gives the same $\underset{\sim}{K}^{sc}(\underset{\sim}{x}_s, t)$ and thus the same $\underset{\sim}{K}^{total}(\underset{\sim}{x}_s, t)$

on S. This can be easily seen by invoking the uniqueness of the Neumann problem. The motivation of decomposing the $\underset{\sim}{K}^{sc}(\underset{\sim}{x}_s, t)$ as does (56), in addition to the latter's rendering a clear procedure that removes ambiguity in deriving the currents and then the fields and providing insight into the physical current-generating mechanism, is calculational simplicity.

The simplicity becomes particularly appealing if conjecture two is true. In this case, the $\underset{\sim}{n}(\underset{\sim}{x}_s)\cdot\underset{\sim}{B}_{II}^{sc,1}(\underset{\sim}{x}_s, t) = 0$ and no evaluation of the three dimensional $\underset{\sim}{B}_{II}^{sc,1}(\underset{\sim}{x}_s, t)$ is needed to find its surface amount; and the $\underset{\sim}{K}_I^{sc}(\underset{\sim}{x}, t)$ is driven <u>solely</u> by the perpendicular component of the driving magnetic field on S via (64). In this sense, conjecture two is a convenience but not an absolute necessity. If it is generally true, we have found an extremely convenient short cut for evaluating surface current. If it is not, we just complete the story by including the additional driver $\underset{\sim}{B}_{II}^{sc,1}(\underset{\sim}{x}_s, t)\cdot\underset{\sim}{n}(\underset{\sim}{x}_s)$ for $\underset{\sim}{B}_I^{sc,1}(\underset{\sim}{x}, t)$ in (64), without changing any of the interpretations or procedures. For the cases of plane and spherical geometry, conjecture two is explicitly proved (Appendix D).

Finally, we emphasize that conjecture one is essential for the whole procedure of decomposition to be valid. Further, it is important to remind ourselves that the non-circulation condition on S applies only to the <u>electrostatic part</u> of the <u>scattering</u> surface current density. It <u>does not</u> apply to the total surface current density, nor to the whole scattering one, nor to the driving one if it exists. Of course, under the special circumstances that both the magnetostatic part of the surface scattering current $\underset{\sim}{K}_I^{sc}(\underset{\sim}{x}_s, t)$ and the driving source surface current $\underset{\sim}{K}^{dr}(\underset{\sim}{x}_s, t)$ are zero, the $\underset{\sim}{K}_{II}^{sc}(\underset{\sim}{x}_s, t)$ on S becomes the sole and the total surface current. In such cases, the no-circulation condition on S applies to all of these currents. In general, the two types of scattering currents for SGEMP problems are of the same order of magnitude, but with different spatial distributions on S, and of equal importance. Examples in the next section will further illustrate this point.

SECTION IV

SPECIAL CASE EXAMPLES

In this section, we shall give several very simple examples to illustrate the application of our theoretical results and the severeness of their implications.

1. THE PURE MAGNETOSTATIC CASE

If the driving source is such that there is only a $\tilde{B}^{dr,1}(\tilde{x}, t)$ but no $\tilde{E}^{dr,0}(\tilde{x}, t)$, then the electrostatic contribution to the scattered field is zero and the magnetostatic contribution, the part II contribution, to the magnetostatic field constitutes the whole scattered field. In particular, from (43) and (56), the current on S is given by

$$\tilde{K}^{total}(\tilde{x}_s, t) = \tilde{K}_I^{sc}(\tilde{x}_s, t) \tag{66}$$

where $\tilde{K}_I^{sc}(\tilde{x}_s, t)$ is given by (61).

As an example of such a special case, consider a superconducting metallic sphere immersed in a static uniform magnetic field whose component normal to the sphere serves as the driving field. The result is well known. It has a surface current

$$\tilde{K}^{total}(\tilde{x}_s, t) = \tilde{K}_I^{sc}(\tilde{x}_s, t) = +\frac{3}{2} H_o \sin\theta \cdot \tilde{e}_\phi \tag{67}$$

and a magnetostatic field (see Figure 4)

$$\begin{aligned} \tilde{H}^{total}(\tilde{x}, t) &= 0, \; r < a \\ &= -\tilde{e}_r H_o \cos\theta \left(1 - \frac{a^3}{r^3}\right) \\ &\quad + \tilde{e}_\theta H_o \sin\theta \cdot \left(1 + \frac{a^3}{2r^3}\right), \; r>a . \end{aligned} \tag{68}$$

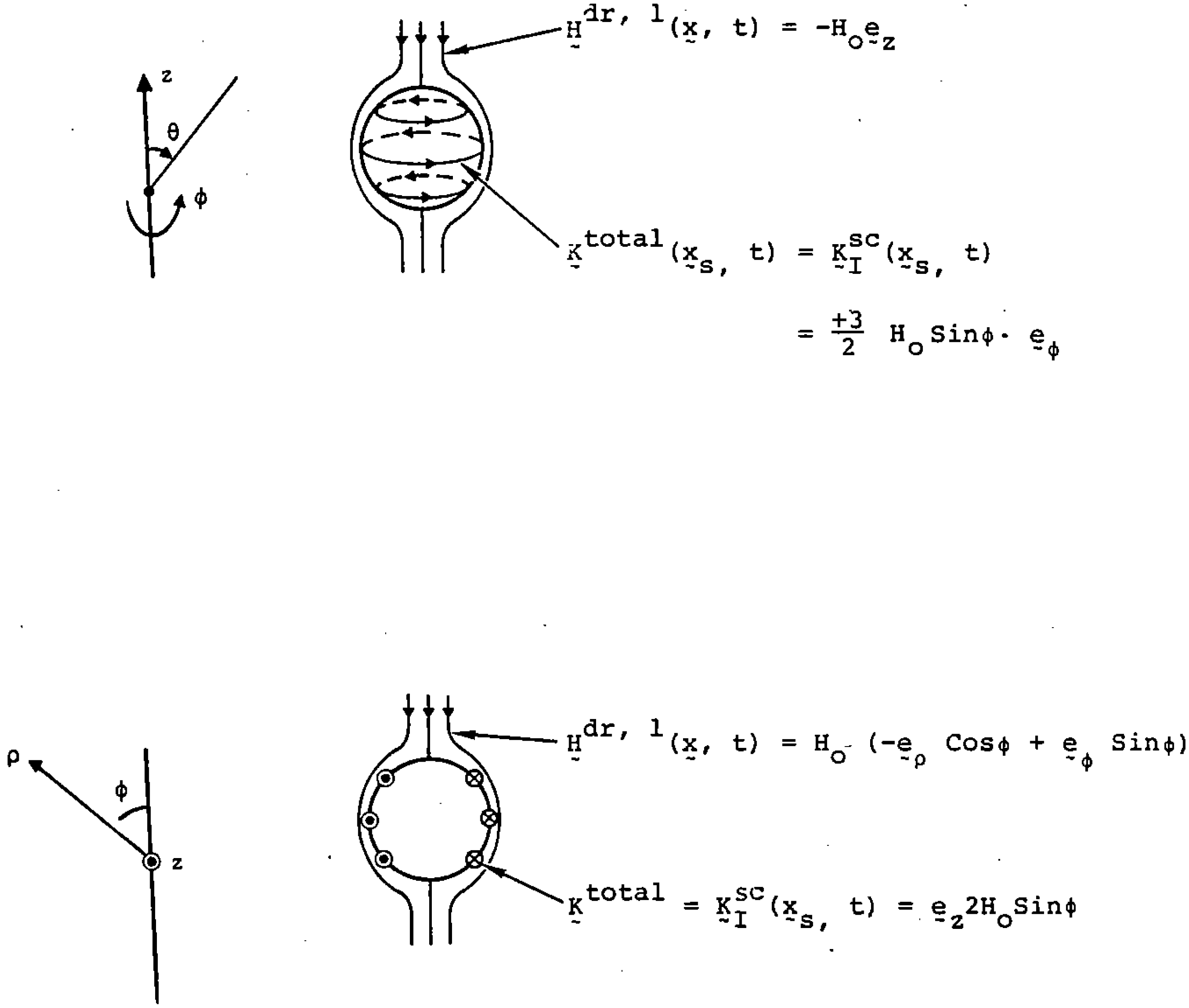


Figure 4. The Surface Currents in the Two Pure Magnetostatic Cases

If the superconducting sphere is replaced by an infinitely long superconducting circular cylinder pointing out of the plane of the paper, the well-known result is

$$\underset{\sim}{K}^{total}(\underset{\sim}{x}_s, t) = \underset{\sim}{K}_I^{sc}(\underset{\sim}{x}_s, t) = \underset{\sim}{e}_z 2H_o \sin\theta \tag{69}$$

and

$$\begin{aligned} \underset{\sim}{H}^{total}(\underset{\sim}{x}, t) &= 0, \quad \rho < a \\ &= -\underset{\sim}{e}_\rho H_o \cos\phi \left(1 - \frac{a^2}{\rho^2}\right) \\ &\quad + \underset{\sim}{e}_\phi H_o \sin\phi \left(1 + \frac{a^2}{\rho^2}\right) \quad \rho > a . \end{aligned} \tag{70}$$

Notice that for such cases, the electrostatic $\underset{\sim}{K}_{II}^{sc}(\underset{\sim}{x}_s, t) \equiv 0$ and the magnetostatic $\underset{\sim}{K}_I^{sc}(\underset{\sim}{x}_s, t)$ make up the total surface current which <u>does not</u> satisfy the zero-surface-circulation condition (58).

2. THE PURE ELECTROSTATIC CASE

If the driving magnetic field on S is parallel to the surface S, then $\underset{\sim}{K}_I^{sc}(\underset{\sim}{x}_s, t) \equiv 0$ and the only scattered quasi-static field is contributed by the electrostatic surface current $\underset{\sim}{K}_{II}^{sc}(\underset{\sim}{x}_s, t)$. The $\underset{\sim}{K}_{II}^{sc}(\underset{\sim}{x}_s, t)$ is determined by (64) and (65) and becomes the total surface current:

$$\underset{\sim}{K}^{total}(\underset{\sim}{x}_s, t) = \underset{\sim}{K}_{II}^{sc}(\underset{\sim}{x}_s, t). \tag{71}$$

This makes $K^{total}(x_s, t)$ also satisfy the no-surface circulation condition for $K_{II}^{sc}(x_s, t)$.

An example of such a case is the radial and azimuthally symmetric emission of charges from a conducting sphere. In this case the symmetry dictates that $\tilde{H}^{dr,1}(\tilde{x}_s, t)$ has only an azimuthally symmetric azimuthal component, and so does $\tilde{H}^{sc,1}(\tilde{x}, t) = \tilde{H}_{II}^{sc,1}(\tilde{x}, t)$. This leaves the condition (65) identically satisfied. Consequently, (64) alone is sufficient to determine that sole θ-direction of the $\tilde{K}_{II}^{sc}(\tilde{x}_s, t)$, which in this case is also the total current.

3. MIXED CASES

The examples above show that the relative magnitude of the magnetostatic surface current $\tilde{K}_I^{sc}(\tilde{x}_s, t)$ and the electrostatic $\tilde{K}_{II}^{sc}(\tilde{x}_s, t)$ can vary from 0 to ∞, depending on the driving field and the conductor geometry. In general, they both contribute significantly and comparably to the total surface current.

Although we will not present detailed applicational computations, we shall elaborate more for the spherical case. In that case, one can always represent the scattered fields by two decoupled scalar Debye potentials, one of the electric type and one of the magnetic type. The corresponding scattering surface currents, for and from each of these fields, indeed obey, respectively, the electrostatic and magnetostatic surface current requirements (58) and (57) reduced to that spherical occasion. In the spherical case, detailed results for the particular problem of a charge circulating the sphere have been obtained using this Debye potential approach (ref. 10). The resulting surface current consists of an electric

10. Lee, K.S.H., and L. Marin, "Interaction of External SGEMP with Space Systems," AFWL EMP Theoretical Note 179, August 1973; also Higgins, D. F., K.S.H. Lee, and L. Marin, "System-Generated EMP," to be published in the EMP Special Issue of the IEEE Transactions on Antennas and Propagation, scheduled for January 1978.

part and a magnetic part. They are precisely the spherical case version of the electrostatic and magnetostatic parts delineated in this report, and they are of the same order of magnitude in the particle velocity. This is but one example for the case of mixed and comparable contributions.

Finally, we emphasize again that in finding the magnetostatic and electrostatic pieces of the scattering current, one must be very careful in applying the correct conditions to each. Namely, (53) and (55) apply to the $\underset{\sim}{K}_{I}^{sc}(\underset{\sim}{x}_s, t)$ and (54) and (58) to the $\underset{\sim}{K}_{II}^{sc}(\underset{\sim}{x}_s, t)$. A mishandling will in general result in errors in the magnetic field of the same order of magnitude as its leading term.

APPENDIX A

The electromagnetic fields generated by the source $\rho(\mathbf{x}, t)$ and $\mathbf{J}(\mathbf{x},t)$ in an unbounded uniform medium of dielectric constant ε_o and permeability μ_o are

$$\mathbf{E}(\mathbf{x}, t) = -\nabla\Phi(\mathbf{x}, t) - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{x}, t) \qquad \text{(A-1)}$$

$$\mathbf{B}(\mathbf{x}, t) = \nabla \times \mathbf{A}(\mathbf{x}, t) \qquad \text{(A-2)}$$

where the scalar potential $\Phi(\mathbf{x}, t)$ and vector potential $\mathbf{A}(\mathbf{x}, t)$ satisfy

$$\nabla^2\Phi(\mathbf{x}, t) = \frac{-\rho(\mathbf{x},t)}{\varepsilon_o} - \frac{\partial}{\partial t}\nabla\cdot\mathbf{A}(\mathbf{x}, t) \qquad \text{(A-3)}$$

$$\nabla^2\mathbf{A}(\mathbf{x}, t) - \mu_o\varepsilon_o\frac{\partial^2}{\partial t^2}\mathbf{A}(\mathbf{x}, t) = -\mu_o\mathbf{J}(\mathbf{x}, t)$$

$$+ \nabla\left[\nabla\cdot\mathbf{A}(\mathbf{x}, t) + \mu_o\varepsilon_o\frac{\partial\Phi(\mathbf{x}, t)}{\partial t}\right] \qquad \text{(A-4)}$$

Since the value of the $\nabla\cdot\mathbf{A}(\mathbf{x}, t)$ does not influence the EM fields, its value can be chosen to be anything. If we choose the Coulomb gauge condition

$$\nabla\cdot\mathbf{A}^{(c)}(\mathbf{x}, t) \equiv 0 \qquad \text{(A-5)}$$

the resulting expression for the scalar potential generated by $\rho(x, t)$ is

$$\Phi^{(c)}(\underset{\sim}{x},\ t) \equiv \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\underset{\sim}{x}',\ t)}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d^3x' \qquad \text{(A-6)}$$

where the unrestricted integration is carried over all the spatial volume. With this $\Phi^{(c)}(\underset{\sim}{x},\ t)$, we can solve (A-4) for the vector potential $\underset{\sim}{A}^{(c)}(\underset{\sim}{x},\ t)$ in the Coulomb gauge.

To get the solution for $\underset{\sim}{A}^{(c)}(\underset{\sim}{x},\ t)$ from (A-4), we first examine the right-hand side of (A-4). Assuming the $\underset{\sim}{J}(\underset{\sim}{x},\ t)$ is a regularly behaved vector field in <u>all</u> spatial volume, an assumption readily justifiable on physical grounds, we immediately can make use of the Helmholtz theorem which uniquely decomposes the regularly behaved $\underset{\sim}{J}(\underset{\sim}{x},\ t)$ in <u>all</u> spatial volume into a longitudinal (irrotational) part $\underset{\sim}{J}_\ell(\underset{\sim}{x},\ t)$ and a transverse (solenoidal) part $\underset{\sim}{J}_t(\underset{\sim}{x},\ t)$:

$$\underset{\sim}{J}(\underset{\sim}{x},\ t) = \underset{\sim}{J}_\ell(\underset{\sim}{x},\ t) + \underset{\sim}{J}_t(\underset{\sim}{x},\ t) \qquad \text{(A-7)}$$

where

$$\underset{\sim}{J}_\ell(\underset{\sim}{x},\ t) \equiv -\underset{\sim}{\nabla}\underset{\sim}{\nabla}\cdot\int \frac{\underset{\sim}{J}(\underset{\sim}{x}',\ t)}{4\pi|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d^3x' \qquad \text{(A-8)}$$

$$\underset{\sim}{J}_t(\underset{\sim}{x},\ t) \equiv \underset{\sim}{\nabla}\times\underset{\sim}{\nabla}\times\int \frac{\underset{\sim}{J}(\underset{\sim}{x}',\ t)}{4\pi|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d^3x' \quad . \qquad \text{(A-9)}$$

Obviously, $\underset{\sim}{\nabla}\times\underset{\sim}{J}_\ell(\underset{\sim}{x},\ t) \equiv 0$ and $\underset{\sim}{\nabla}\cdot\underset{\sim}{J}_t(\underset{\sim}{x},\ t) \equiv 0$.

Now, from the solution (A-6) for the $\Phi^{(c)}(\underset{\sim}{x},\ t)$, we have

$$\mu_0 \varepsilon_0 \underset{\sim}{\nabla} \frac{\partial}{\partial t} \Phi^{(c)}(\underset{\sim}{x}, t) = \mu_0 \varepsilon_0 \underset{\sim}{\nabla} \frac{\partial}{\partial t} \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\underset{\sim}{x}', t)}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d^3x'$$

$$= \mu_0 \underset{\sim}{\nabla} \frac{\partial}{\partial t} \int \frac{\rho(\underset{\sim}{x}', t)}{4\pi|\underset{\sim}{x}-\underset{\sim}{x}'|} d^3x'$$

$$= -\mu_0 \underset{\sim}{\nabla} \int \frac{\underset{\sim}{\nabla} \cdot \underset{\sim}{J}(\underset{\sim}{x}', t)}{4\pi|x-x'|} d^3x' \qquad \text{(A-10)}$$

where for the last equality we have used the relation of charge conservation which is implied by the Maxwell equations. From (A-10), we can further proceed to obtain

$$\mu_0 \varepsilon_0 \underset{\sim}{\nabla} \frac{\partial}{\partial t} \Phi^{(c)}(\underset{\sim}{x}, t) = -\mu_0 \underset{\sim}{\nabla} \int d^3x' \left[\underset{\sim}{\nabla}' \cdot \frac{\underset{\sim}{J}(\underset{\sim}{x}', t)}{4\pi|x-x'|} - \left(\underset{\sim}{\nabla}' \frac{1}{4\pi|x-x'|} \right) \cdot \underset{\sim}{J}(\underset{\sim}{x}', t) \right]$$

$$= -\mu_0 \underset{\sim}{\nabla} \left\{ \oint_{S_\infty} \frac{\underset{\sim}{J}(\underset{\sim}{x}', t)}{4\pi|\underset{\sim}{x}-\underset{\sim}{x}'|} \cdot d\underset{\sim}{a}' + \int d^3x' \left(\underset{\sim}{\nabla} \frac{1}{4\pi|\underset{\sim}{x}-\underset{\sim}{x}'|} \right) \cdot \right.$$

$$\left. \underset{\sim}{J}(\underset{\sim}{x}', t) \right\}$$

$$= -\mu_0 \underset{\sim}{\nabla} \underset{\sim}{\nabla} \cdot \int d^3x' \frac{\underset{\sim}{J}(\underset{\sim}{x}', t)}{4\pi|\underset{\sim}{x}-\underset{\sim}{x}'|}$$

$$= \mu_0 \underset{\sim}{J}_\ell(\underset{\sim}{x}, t) \qquad \text{(A-11)}$$

Using the gauge condition (A-5), the decomposition relation (A-7), and the above relation (A-11), we can rewrite (A-4), the equation for $\underset{\sim}{A}^{(c)}(\underset{\sim}{x}, t)$, as

$$\nabla^2 A^{(c)}(x, t) - \mu_o \varepsilon_o \frac{\partial^2}{\partial t^2} A^{(c)}(x, t) = -\mu_o J_t(x, t) \quad \text{(A-12)}$$

Thus, the casuality restricted vector potential $A^{(c)}(x, t)$ generated by the $\rho(x, t)$ and $J(x, t)$ is

$$A^{(c)}(x, t) = \frac{\mu_o}{4\pi} \int \frac{J_t\left(x', t - \frac{|x-x'|}{c}\right)}{|x-x'|} d^3x' \quad \text{(A-13)}$$

where $c \equiv (\mu_o \varepsilon_o)^{-1/2}$.

With the $\Phi^{(c)}(x, t)$ and $A^{(c)}(x, t)$ given, respectively, by (A-6) and (A-13), we have the expressions for the fields

$$E(x, t) = -\nabla \Phi^{(c)}(x, t) - \frac{\partial}{\partial t} A^{(c)}(x, t) \quad \text{(A-14)}$$

$$B(x, t) = \nabla x A^{(c)}(x, t) \quad \text{(A-15)}$$

Now, an inspection of (A-6) reveals that the $\Phi^{(c)}(x, t)$ is obtained from the charge density $\rho(x, t)$ instantaneously without retardation precisely the way an electrostatic potential is obtained from a time-independent charge distribution. Thus, as expressed in (A-14), the part of the electric field associated with this $\Phi^{(c)}(x, t)$, $-\nabla\Phi^{(c)}(x, t)$, is precisely the instantaneous Coulomb field generated by the charge distribution $\rho(x, t)$ at the same time as if it were not changing with time at all. This piece of the electric field, $-\nabla\Phi^{(c)}(x, t)$, is called the quasi-static electric field, for obvious reasons. Further, because the instantaneous Coulomb field far awav from the charges is directed purely along the radial direction from the charge's present position to the observation point, this $-\nabla\Phi^{(c)}(x, t)$ is also called the longitudinal electric field. Moreover, since the expression (A-14) indeed decomposes thè

$\underset{\sim}{E}(\underset{\sim}{x}, t)$ into a curlless part, $-\nabla\Phi^{(c)}(\underset{\sim}{x}, t)$, and a divergence-less part, $\frac{-\partial}{\partial t}\underset{\sim}{A}^{(c)}(\underset{\sim}{x}, t)$, they are respectively the irrotational and the solenoidal parts of the $\underset{\sim}{E}(\underset{\sim}{x}, t)$ as decomposed in the Helmholtz theorem described by (A-7) to (A-9). We would also like to point out that far away from the sources the component of electric field that does contribute to radiation is transverse to the direction to the sources and comes solely from the part $\frac{-\partial}{\partial t}\underset{\sim}{A}^{(c)}(x, t)$. Finally, we remark that the divergence-less part, $[-\partial\underset{\sim}{A}^{(c)}(\underset{\sim}{x}, t)]/\partial t$, although often called the "transverse" part, is not purely transverse. It in fact contains a longitudinal component which exactly cancels the instantaneous Coulomb field outside the causality region, and only approaches purely transverse at locations of long retarded time.

For the $\underset{\sim}{B}(\underset{\sim}{x}, t)$, the gauge condition has no effect to its expressions at all since the expression for $\underset{\sim}{B}(\underset{\sim}{x}, t)$ is totally insensitive to the choice of $\underset{\sim}{\nabla}\cdot\underset{\sim}{A}(\underset{\sim}{x}, t)$. In fact, we have

$$\underset{\sim}{\nabla}\times\int\frac{\underset{\sim}{J}_\ell\left(\underset{\sim}{x}', t-\frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c}\right)}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\,d^3x'$$

$$=\int\left[\left(\underset{\sim}{\nabla}\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\right)\times\underset{\sim}{J}_\ell\left(\underset{\sim}{x}', t-\frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c}\right)+\frac{\underset{\sim}{\nabla}\times\underset{\sim}{J}_\ell\left(\underset{\sim}{x}', t-\frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c}\right)}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\right]d^3x'$$

$$=\int\left[-\underset{\sim}{\nabla}'\times\frac{\underset{\sim}{J}_\ell(\underset{\sim}{x}', t-\frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c})}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\right.$$

$$\left.+\frac{\underset{\sim}{\nabla}'\times\underset{\sim}{J}_\ell\left(\underset{\sim}{x}',\frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c}\right)+\underset{\sim}{\nabla}\times\underset{\sim}{J}_\ell\left(\underset{\sim}{x}',\frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c}\right)}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\right]d^3x'$$

$$= -\oint_{S_\infty} d\underset{\sim}{a}' \times \frac{\underset{\sim}{J}_\ell\left(\underset{\sim}{x}', t - \frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c}\right)}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$$

$$+ \int \frac{\left(\underset{\sim}{\nabla}'_f + \underset{\sim}{\nabla}'_b + \underset{\sim}{\nabla}\right) \times \underset{\sim}{J}_\ell\left(\underset{\sim}{x}', \frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c}\right)}{|\underset{\sim}{x}-\underset{\sim}{x}'|} d^3x' \equiv 0 \qquad \text{(A-17)}$$

where in the last integrand the $\underset{\sim}{\nabla}'_f$ operates on the $\underset{\sim}{x}'$ that appears as an individual argument of the $\underset{\sim}{J}_\ell(\underset{\sim}{x}', \frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c})$ and the $\underset{\sim}{\nabla}'_b$ operates on the $\underset{\sim}{x}'$ that appears in the $|\underset{\sim}{x}-\underset{\sim}{x}'|$ part of the argument of the $\underset{\sim}{J}_\ell\left(\underset{\sim}{x}', \frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c}\right)$. Thus, the expression for $\underset{\sim}{B}(\underset{\sim}{x}, t)$ is blind to the transverse subscript t in (A-13):

$$\underset{\sim}{B}(\underset{\sim}{x}, t) = \underset{\sim}{\nabla} \times \underset{\sim}{A}^{(c)}(\underset{\sim}{x}, t) = \underset{\sim}{\nabla} \times \int \frac{\mu_o \underset{\sim}{J}_t\left(\underset{\sim}{x}', t - \frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c}\right)}{4\pi|\underset{\sim}{x}-\underset{\sim}{x}'|} d^3x'$$

$$= \underset{\sim}{\nabla} \times \int \frac{\mu_o \underset{\sim}{J}\left(\underset{\sim}{x}, t - \frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c}\right)}{4\pi|\underset{\sim}{x}-\underset{\sim}{x}'|} d^3x \qquad \text{(A-18)}$$

APPENDIX B

From Appendix A, in the Coulomb gauge we have

$$\underset{\sim}{E}(\underset{\sim}{x}, t) = -\underset{\sim}{\nabla} \int \frac{\rho(\underset{\sim}{x}', t)\, d^3x'}{4\pi\varepsilon_o R} - \frac{\partial}{\partial t} \int \frac{\mu_o \underset{\sim}{J}_t(\underset{\sim}{x}', t-\frac{R}{C})}{4\pi R}\, d^3x'$$

$$= -\nabla \int \frac{\rho(x', t)d^3x'}{4\pi\varepsilon_o R} - \frac{\mu_o}{4\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{c^n} \frac{\partial}{\partial t}$$

$$\int \frac{\partial^n}{\partial t^n} R^{n-1} \underset{\sim}{J}_t(\underset{\sim}{x}', t)\, d^3x'$$

$$= -\nabla \int \frac{\rho(\underset{\sim}{x}', t)}{4\pi\varepsilon_o R}\, d^3x' - \sum_{n=0}^{\infty} \frac{\mu_o}{4\pi} \frac{(-1)^n}{c^n} \frac{\partial^{n+1}}{\partial t^{n+1}}$$

$$\int R^{n-1} \underset{\sim}{J}_t(\underset{\sim}{x}', t)\, d^3x' \qquad \text{(B-1)}$$

$$\underset{\sim}{B}(\underset{\sim}{x}, t) = \underset{\sim}{\nabla} \times \int \frac{\mu_o \underset{\sim}{J}_t\left(\underset{\sim}{x}', t-\frac{R}{C}\right)}{4\pi R}\, d^3x'$$

$$= \underset{\sim}{\nabla} \times \int \frac{\mu_o \underset{\sim}{J}\left(\underset{\sim}{x}', t-\frac{R}{C}\right)}{4\pi R}\, d^3x'$$

$$= \underset{\sim}{\nabla} \times \int \frac{\mu_o}{4\pi R} \sum_{n=0}^{\infty} \frac{(-1)^n}{c^n} R^n$$

$$\frac{\partial^n}{\partial t^n} \underset{\sim}{J}(\underset{\sim}{x}', t)\, d^3x'$$

$$= \sum_{n=0}^{\infty} \frac{\mu_o}{4\pi} \frac{(-1)^n}{c^n} \frac{\partial^n}{\partial t^n} \nabla \times$$

$$\int R^{n-1} \underset{\sim}{J}(\underset{\sim}{x}', t)\, d^3x'$$

$$= \frac{\mu_o}{4\pi} \nabla \times \int \frac{J(x', t)\, d^3x'}{R}$$

$$+ \sum_{n=1}^{\infty} \frac{\mu_o}{4\pi} \frac{(-1)^n}{c^n} \frac{\partial^n}{\partial t^n} \nabla \times$$

$$\int R^{n-1} J(x', t)\, d^3x' \qquad \text{(B-2)}$$

For the electric field, the ratio of successive terms in the summation is of the order

$$\left| \frac{R^n \frac{\partial^{n+1}}{\partial t^{n+1}} J_t(x', t)}{R^{n-1} \frac{\partial^n}{\partial t^n} J_t(x', t)} \right| \lesssim \left| \frac{R}{C} \cdot \frac{\begin{pmatrix}\text{change rate of the driv-} \\ \text{ing source amplitude}\end{pmatrix}}{\begin{pmatrix}\text{the driving source} \\ \text{amplitude}\end{pmatrix}} \right|$$

$$\sim \frac{R}{C\tau_x} \equiv \xi_R \qquad \text{(B-3)}$$

Thus, $\xi_R << 1$ implies that the leading term, of n=0, is the dominant term of the summation. Further, the ratio of this leading term to the Coulomb term is of the order of

$$\left| \frac{\mu o R^{-1} \max\left(\frac{\partial J_t(x', t)}{\partial t}, J_t(x', t) \frac{v}{R} \right)}{\frac{\rho(x', t)}{\varepsilon_o R^2}} \right| \sim$$

$$\left| \frac{Rv \max\left(1, \frac{\tau_x v}{R}\right)}{C^2 \tau_x} \right| \sim \max\left(\beta\xi_R, \beta^2\right) \qquad \text{(B-4)}$$

Thus, the dominant driving electric field is the instantaneous Coulomb field when evaluated at locations near to a slowly changing driving source, i.e., when $\xi_R << 1$ and $\beta << 1$. This established (10), (13) and (11).

The justification for the dominant driving magnetic field being the instantaneous magnetostatic one is similar to the one that leads to (B-2). It demanded only the condition of nearness in location and slowness in current time rate, $\xi R \ll 1$, but imposed no restriction on the particle speed. This is clearly as it should be, because magnetic field is generated by current but not by charge.

APPENDIX C

SOME PROPERTIES OF $\underset{\sim}{E}^{dr,o}(\underset{\sim}{x}, t)$ and $\underset{\sim}{B}^{dr,1}(\underset{\sim}{x}, t)$

The quasi-static driving fields $\underset{\sim}{E}^{dr,o}(\underset{\sim}{x}, t)$ and $\underset{\sim}{B}^{dr,1}(\underset{\sim}{x}, t)$ do not satisfy the Maxwell equations exactly. They satisfy these equations approximately as follows:

$$\underset{\sim}{\nabla} \times \underset{\sim}{H}^{dr,1}(\underset{\sim}{x}, t) = \frac{\varepsilon_0 \partial \underset{\sim}{E}^{dr,o}(\underset{\sim}{x}, t)}{\partial t} + \underset{\sim}{J}^{dr}(\underset{\sim}{x}, t)$$

$$\equiv \underset{\sim}{J}_t^{dr}(\underset{\sim}{x}, t) \qquad \text{(C-1)}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{H}^{dr,1}(\underset{\sim}{x}, t) = 0 \qquad \text{(C-2)}$$

$$\frac{\partial \underset{\sim}{B}^{dr,1}(\underset{\sim}{x}, t)}{\partial t} = 0 \ (\beta^2) \qquad \text{(C-3)}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E}^{dr,o}(\underset{\sim}{x}, t) = 0 \qquad \text{(C-3')}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E}^{dr,o}(\underset{\sim}{x}, t) = \rho^{dr}(\underset{\sim}{x}, t) \qquad \text{(C-4)}$$

The approximate properties are important in keeping track of fields to the right order of magnitude in computations.

APPENDIX D

REDUCTION PROOF OF CONJECTURE #1 AND SOME SPECIAL CASE VALIDITIES OF BOTH CONJECTURES #1 AND #2

Using tensor notations in the two-dimensional differentiable manifold defined by the intrinsic geometry of the closed two-dimensional surface S with metric tensor $g_{\mu\nu}$ for the infinitesimal length element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{D-1}$$

the decomposition of conjecture #1, (46), is rewritten as

$$K^\mu = K_I^\mu + K_{II}^\mu . \tag{D-2}$$

Now, if the generalized vector Poisson equation

$$S^\mu + f^{\mu;\alpha}{}_{;\alpha} = K^\mu \tag{D-3}$$

in this two-dimensional Riemannian space possesses a unique solution, where some needed regularity condition on K^μ and the solution f^μ had been imposed, then we can rewrite, or decompose, K^μ in terms of f^μ. Here, standard notations of tensor analysis are used and the ";" indicates covariant or contravariant derivatives, respectively, in the sub- and superscript positions. S^μ is a vector to be determined as needed. Further, we emphasize that the existence of solution to (D-3) may need some integrability conditions that impose restrictions on the property of the space itself.

In any case, if the solution f^μ exists and can be found for (D-3), we would have

$$K^{\mu} = \left(f^{\mu;\alpha} - f^{\alpha;\mu}\right)_{;\alpha} + f^{\alpha;\mu}{}_{;\alpha} + S^{\mu} \tag{D-4}$$

Now in two dimension any second rank antisymmetric tensor must be expressible as the product of a scalar and the antisymmetric unit pseudotensor $\varepsilon^{\mu\nu}/\sqrt{|g|}$. Thus,

$$f^{\mu;\alpha} - f^{\alpha;\mu} = \frac{\varepsilon^{\mu\alpha}}{\sqrt{|g|}} f \tag{D-5}$$

where the scalar

$$f \equiv \sqrt{|g|}\left[f^{1;2} - f^{2;1}\right] \tag{D-6}$$

and the $g \equiv \det(g_{\mu\nu})$. Then (D-4) becomes

$$K^{\mu} = \left(\frac{\varepsilon^{\mu\alpha}}{\sqrt{|g|}} f\right)_{;\alpha} + f^{\alpha;\mu}{}_{;\alpha} + S^{\mu} \tag{D-7}$$

Next, we shall show that the terms in (D-7) are sufficient (but are not necessary) for the decomposition purpose required by Conjecture #1.

First, the "surface-divergenceless" condition (47), re-written in tensor notation in the two-dimensional Riemann space, is

$$K_I^{\mu}{}_{;\mu} = 0. \tag{D-8}$$

But the first term in (D-7) obviously satisfies the same:

$$\left[\left(\frac{\varepsilon^{\mu\alpha}}{\sqrt{|g|}} f\right)_{;\alpha}\right]_{;\mu} = \frac{1}{\sqrt{|g|}}\left[\varepsilon^{\mu\alpha} f\right]_{,\alpha\mu} \equiv 0 \tag{D-9}$$

where the "," denotes plain partial differentiation.

Second, the "surface-curlless" condition (48), when similarly rewritten in tensor notation, is

$$\left[\frac{\varepsilon^{\mu\nu}}{\sqrt{|g|}}(K_{II})_{\nu}\right]_{;\mu} = 0 \qquad \text{(D-10)}$$

However, the second term in (D-7), when inserted into the left-hand side of (D-10) in place of $(K_{II})^{\mu}$, gives

$$\left[\frac{\varepsilon^{\mu\nu}}{\sqrt{|g|}} f^{\alpha}{}_{;\nu;\alpha}\right]_{;\mu} = \left[\frac{\varepsilon^{\mu\nu}}{\sqrt{|g|}}\left(f^{\alpha}{}_{;\nu;\alpha} + f^{\beta}R_{\beta\nu}\right)\right]_{;\mu}$$

$$= \frac{1}{\sqrt{|g|}}\left[\varepsilon^{\mu\nu}f^{\alpha}{}_{;\alpha;\nu}\right]_{,\mu} + \left[\frac{\varepsilon^{\mu\nu}}{\sqrt{|g|}} f^{\beta}R_{\beta\nu}\right]_{;\mu}$$

$$= \frac{\varepsilon^{\mu\nu}}{\sqrt{|g|}}\left(f^{\alpha}{}_{;\alpha}\right)_{,\nu\mu} + \left[\frac{\varepsilon^{\mu\nu}}{\sqrt{|g|}} f^{\beta}R_{\beta\nu}\right]_{;\mu} \qquad \text{(D-11)}$$

Thus, if we define

$$S^{\mu} \equiv -f^{\beta}R_{\beta}{}^{\mu} \qquad \text{(D-12)}$$

and use both the second and the third, S^{μ}, terms of (D-7) as the $(K_{II})^{\mu}$, then the requirement (D-10) and the conjecture #1 are satisfied. Of course, the unique solvability of (D-3), with its S^{μ} defined by (D-12), may require some integrability conditions. These conditions, whatever they are, impose constraints to the general validity of conjecture #1. This establishes (49) to (55) in the text of the report.

Now, we turn our attention to some special cases for which we shall explicitly show that the conjectures are valid.

First, for a plane surface the validity of conjecture #1 is obvious--everything is similar to the three-dimensional Euclidean space. For conjecture #2, it demands that

$$0 = \left[\underset{\sim}{n}(\underset{\sim}{x}) \cdot \underset{\sim}{B}(\underset{\sim}{x}) \right]_{\underset{\sim}{x} \to \underset{\sim}{x}_s} = \left[\oint \frac{\underset{\sim}{n}(\underset{\sim}{x}) \cdot \underset{\sim}{e}_R(\underset{\sim}{x}, \underset{\sim}{x}') \times \underset{\sim}{K}(\underset{\sim}{x}')\, d^2x'}{R^2} \right]_{\underset{\sim}{x} \to \underset{\sim}{x}_s}$$

$$= \left[\oint \frac{\underset{\sim}{n}(\underset{\sim}{x}) \times \underset{\sim}{e}_R(\underset{\sim}{x}, \underset{\sim}{x}') \cdot \underset{\sim}{K}(\underset{\sim}{x}')\, d^2x'}{R^2} \right]_{\underset{\sim}{x} \to \underset{\sim}{x}_s}$$

$$= \left[\int \frac{\sin\theta}{R^2}\, d\rho \oint \underset{\sim}{K}(\underset{\sim}{x}') \cdot d\underset{\sim}{\ell}' \right]_{\underset{\sim}{x} \to \underset{\sim}{x}_s} \qquad \text{(D-13)}$$

Thus, we do obtain the required property

$$\oint_{C \text{ on } S} \underset{\sim}{K}(\underset{\sim}{x}') \cdot d\underset{\sim}{\ell}' = 0 \qquad \text{(D-14)}$$

for the surface current that generates the parallel-to-the-surface magnetostatic field.

Second, for a spherical surface, we shall show that the conjecture #1 is true for any vector field $\underset{\sim}{K}(\underset{\sim}{x}, t)$ that represents a surface electric current. This is so because the electromagnetic field generated by the $\underset{\sim}{K}(\underset{\sim}{x}_s, t)$ can be expressed entirely by the two well-known decoupled scalar Debye potentials, ψ and χ, and in particular

$$B_\theta = \frac{1}{r} \frac{\partial^2 (\gamma\psi)}{\partial r \partial \theta} + \frac{1}{r \sin\theta} \frac{\partial^2}{\partial t \partial \phi} (r\chi) \qquad \text{(D-15)}$$

$$B_\phi = \frac{1}{r\sin\theta}\frac{\partial^2(r\psi)}{\partial r\partial\phi} - \frac{1}{r}\frac{\partial^2(r\chi)}{\partial t\partial\theta} \qquad \text{(D-16)}$$

Clearly, the ψ-part satisfies (47) and is caused by the "magnetostatic type" surface current $\underset{\sim}{K}_I(\underset{\sim}{x}_s, t)$; and the χ-part satisfies (48) and is caused by the "electrostatic type" surface current $\underset{\sim}{K}_{II}(\underset{\sim}{x}_s, t)$. As to the validity of conjecture #2, one can easily see it in two different ways. One is that the χ-part, just established of type II, produces no radial magnetic field at all. The other is that we can proceed and prove the validity similar to the planar case, since we can always choose a surface line integration on contour of constant θ and R just as in (D-13) and (D-14).