Theoretical Notes
Note 300

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THIN WIRE APPROXIMATION IN 2-D FINITE-DIFFERENCE CALCULATION OF EMP RESPONSE

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1. INTRODUCTION

In using the 2-D finite-difference method to solve the EMP response induced on a body of revolution, the cross section of the conductor is required to be electrically thin [1] so that only axial current dominates and a completely symmetrical situation exists. In some applications however, the radius of the antenna is extremely thin and it requires excessive cost to compute the current. The problem is that when the wire is very thin, the minimum Δr is very small so that the time step Δt is limited to be a small number. It then requires a large number of time cycles to compute a curve. This note suggests an approximation to solve the problem. In the following discussion, we shall assume the reader has some knowledge of the existing 2-D finite-difference technique.

2. THE THIN WIRE APPROXIMATION

In order to remove the restriction on the small Δr and small Δt for thin wire and maintain the accuracy of the computation, we need to find an effective Δr for the cell nearest to the antenna. If one knows the relative distribution of the field very near the wire, it is possible to select an average Δr based on this distribution. It is well known that very near the antenna, the electromagnetic fields are dominated by the static field solution in which $B_\theta \sim 1/r$ and $E_r \sim 1/r$. We shall make use of this approximation in the following to derive an equivalent radius Δr for the first cell.

Consider Maxwell's equation for the scattered fields in space,

$$-\frac{\partial B}{\partial t} = \nabla \times E \quad . \tag{1}$$

In a two-dimensional, cylindrical coordinate, Equation (1) becomes (with only B_θ, E_z, E_r components)

$$-\frac{\partial B_\theta}{\partial t} = \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} \quad (2)$$

The integral form of Equation (1) is given by

$$-\frac{\partial}{\partial t}\int_A \vec{B}\cdot d\vec{A} = \oint_C \vec{E}\cdot d\vec{\ell} \quad . \quad (3)$$

Consider a sample cell near the antenna where A is the surface bounded by A,B,C,D in Figure 1 and C is the close loop AB, BC, CD, and DA. Explicitly, Equation (3) can be written as

$$-\frac{\partial}{\partial t}\int_a^{R_z(2)}\int_{z1}^{z_2} B_\theta \; drdz = \int_{AB}\vec{E}\cdot d\vec{\ell}_1 + \int_{BC}\vec{E}\cdot d\vec{\ell}_2 + \int_{CD}\vec{E}\cdot d\vec{\ell}_3 + \int_{DA}\vec{E}\cdot d\vec{\ell}_4 \quad (4)$$

where $R_z(2)$ is the radial distance of E_{z2}; $R_z(1) = 0$ is the axis of the antenna (z-axis) where E_{z1} is located, and a is the radius of the antenna. Assuming B_θ, E_z and E_r is uniform in Δz, Equation (4) becomes (where $B_\theta = \mu_o H_\theta$)

$$-\frac{\partial}{\partial t}\left(\int_a^{R_z(2)} \mu_o H_\theta dr\right)\Delta z = (E_{z1} - E_{z2})\Delta z + \int_a^{R_z(2)} (E_{r2} - E_{r1})dr \quad (5)$$

Very close to the antenna, the magnetic field and the radial electric field is approximately related to the current and charge per unit length on the wire by

$$H_\theta(r,z,t) = \frac{I(z,t)}{2\pi r} \quad (6)$$

$$E_r(r,z,t) = \frac{q(r,t)}{2\pi r} \quad (7)$$

where I and q are functions of z and t, but not functions of r. The current and charge per unit length (and the fields near the wire) can be solved in terms of some known fields at particular locations. This will enable one to evaluate an effective radius relative to the fields at those locations.

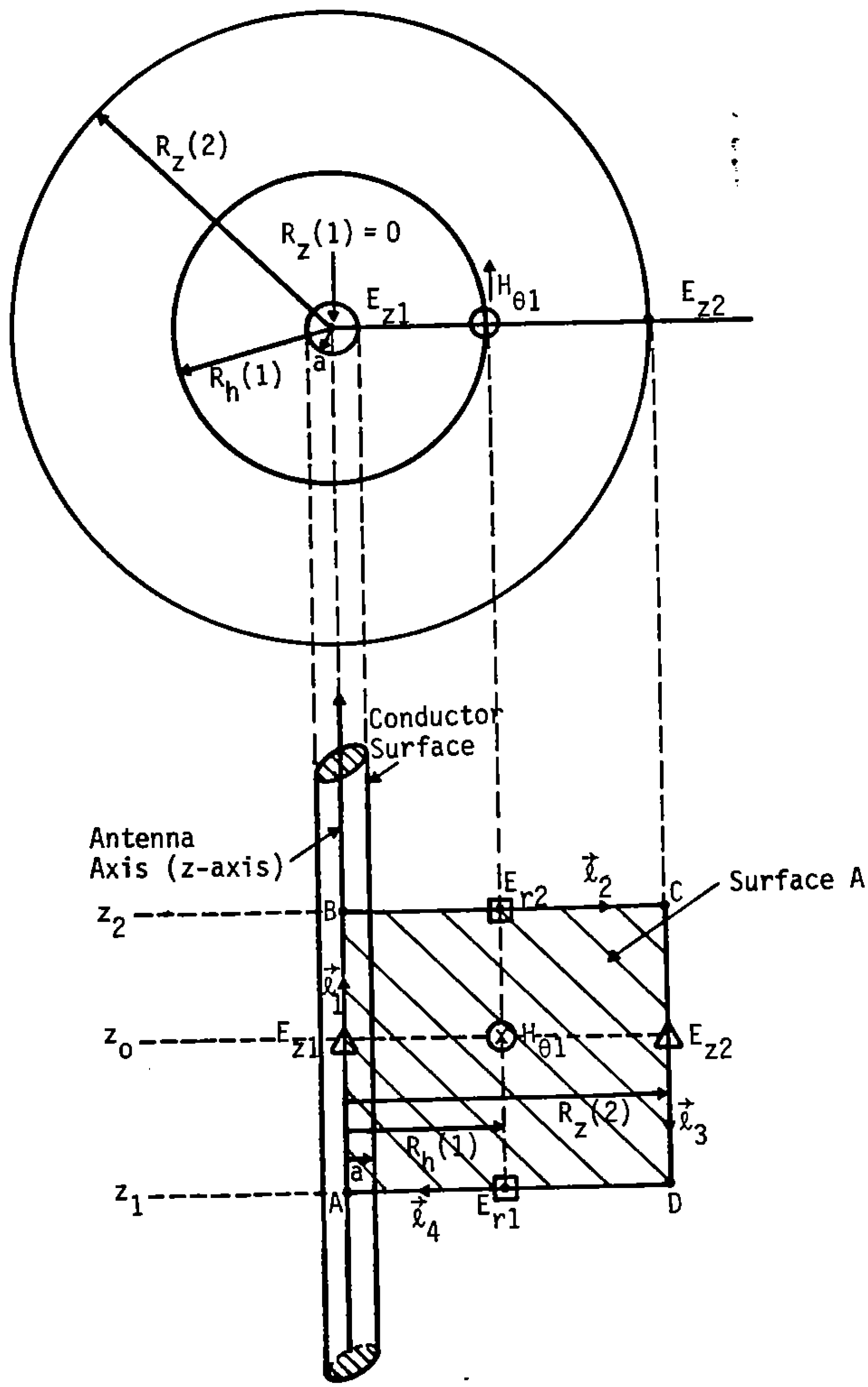

Figure 1. The Integral Path for the Thin Wire Approximation

At $r = R_h(1)$, $z = z_o$ (see Figure 1), from Equation (6) we have

$$H_{\theta 1}(R_h(1), z_o, t) = \frac{I(z_o, t)}{2\pi R_h(1)} \tag{8a}$$

which can be written as,

$$I(z_o, t) = 2\pi R_h(1)\ H_{\theta 1}(R_h(1), z_o, t) \tag{8b}$$

Substituting Equation (8b) into Equation (6), one obtains

$$H_{\theta 1}(r, z_o, t) = \frac{R_h(1)}{r} H_{\theta 1}(R_h(1), z_o, t) \tag{9}$$

where $H_{\theta 1}(r, z_o, t)$ is located at (r, z_o). Similarly, at $R_h(1)$, $z = z_1$ and z_2 (see Figure 1), we have from Equation (7),

$$E_{r1}(R_h(1), z_1, t) = \frac{q(z_1, t)}{2\pi R_h(1)}$$

$$E_{r2}(R_h(1), z_2, t) = \frac{q(z_2, t)}{2\pi R_h(1)} \quad .$$

These equations can be written as

$$q(z_1, t) = 2\pi R_h(1)\ E_{r1}(R_h(1), z_1, t) \tag{10a}$$

$$q(z_2, t) = 2\pi R_h(1)\ E_{r2}(R_h(1), z_2, t) \tag{10b}$$

Substituting Equations (10a) and (10b) into Equation (7), we have

$$E_{r1}(r, z_1, t) = \frac{R_h(1)}{r} E_{r1}(R_h(1), z_1, t) \tag{11a}$$

$$E_{r2}(r, z_2, t) = \frac{R_h(1)}{r} E_{r2}(R_h(1), z_2, t) \tag{11b}$$

where $E_{r1}(r,z_1,t)$ is located at (r,z_1) and $E_{r2}(r,z_2,t)$ is located at (r,z_2). A few comments on the field quantities are in order. In the above manipulation, the fields near the wire are expressed in terms of those field elements commonly used in a unit finite difference cell. These field elements are (see Figure 1):

$$H_{\theta 1}(R_h(1),z_o,t) \text{ located at } (R_h(1),z_o)$$

$$E_{r1}(R_h(1),z_1,t) \text{ located at } (R_h(1),z_1)$$

$$E_{r2}(R_h(1),z_2,t) \text{ located at } (R_h(1),z_2)$$

$$E_{z1}(R_z(1),z_o,t) \text{ located at } (R_z(1),z_o,t)$$

$$E_{z2}(R_z(2),z_o,t) \text{ located at } (R_z(2),z_o,t)$$

where z_o is related to z_1 and z_2 by (assuming uniform grid in z direction),

$$z_o = (z_1 + z_2)/2$$

and $R_z(1)$, $R_z(2)$ and $R_h(1)$ are given by (assuming nonuniform grid in r direction),

$$R_z(1) = 0 \text{ (z-axis)}$$

$$R_h(1) = 0.5\ \Delta r$$

$$R_z(2) = \alpha_1\ \Delta r$$

where Δr is a properly chosen number (usually of several antenna radius) that represents the mesh size in the r direction and α_1 is a constant factor used for expanding mesh (usually $\alpha_1 \leq 1.35$). When $\alpha_1 = 1$, it is a uniform grid in the r direction and $R_h(1) = 0.5(R_z(2) + R_z(1))$. Put Equations (9) and (11) in Equation (5), and we have

$$- \mu_o\ \Delta z\ R_h(1)\ \ell n \frac{R_Z(2)}{a}\ \frac{\partial H_{\theta 1}}{\partial t} = (E_{z1} - E_{z2})\Delta z + R_h(1)\ \ell n \frac{R_Z(2)}{a}\ (E_{r2} - E_{r1}) \quad . \quad (12)$$

Rearranging the above equation and writing $\partial H_{\theta 1}/\partial t$ in finite-difference form, the final result becomes

$$\left(H_{\theta 1}^{n+1} - H_{\theta 1}^{n}\right) = \left(E_{z2}^{n} - E_{z1}^{n}\right)\left(\frac{\Delta t}{\mu_o\ R_h(1)\ \ell n\ \frac{R_Z(2)}{a}}\right) - \left(E_{r2}^{n} - E_{r1}^{n}\right)\left(\frac{\Delta t}{\mu_o \Delta z}\right) \tag{13}$$

where the superscript n is used to indicate that the fields are evaluated at time $t = n\Delta t$. It is instructive to compare this result with the one without thin wire modification. From Equation (2), the ordinary finite-differencing scheme gives

$$\left(H_{\theta 1}^{n+1} - H_{\theta 1}^{n}\right) = \frac{\Delta t}{\mu_o \Delta r}\left(E_{z2} - E_{z1}\right) - \frac{\Delta t}{\mu_o \Delta z}\left(E_{r2} - E_{r1}\right) \quad . \tag{14}$$

A comparison of Equations (13) and (14) shows that Equation (13) is using an effective Δr_e with

$$\Delta r_e = R_h(1)\ \ell n\ \frac{R_z(2)}{a} \quad . \tag{15}$$

This approximation is somewhat different from the one discussed in Reference 2 in two respects: (1) We are solving the scattered fields, not the total fields, and (2) the H_θ and E_r is one-half cell away from the current and charge source. The thin wire approximations in the finite-difference method can be summarized as follows (let $r = (J-1)\Delta r$, $z = I\Delta z$, where I and J are integer indexes):

1. For J = 1 and $1 \leq I \leq IHI$ where IHI is the index denoting the height of the antenna, using Equation (13) to calculate H_θ sequentially in time.

2. For I and J not in the above range, using Equation (14) to calculate H_θ sequentially in time.

3. Calculating E_r using an ordinary finite-difference equation.

4. Calculating E_z using an ordinary finite-difference equation

5. Set $E_z(I,1) = -Einc(t)$ for $1 \leq I \leq IHI$, where Einc(t) is the incident field.

6. Repeat the procedures from (1) to (5).

3. NUMERICAL EXAMPLES

The first example is to calculate the short circuit current at the base of a monopole with a half length of h = 0.58 m and a radius of a = 0.004 m. The incident field is an idealized AURORA field given by [3]:

$$E_{inc}(t) = 7500 \times \sin^2\left(\frac{\pi\tau}{\tau + 2 \cdot \exp(-0.6931 \times \tau^{\beta})}\right)$$

with $\tau = t/120\times10^{-9}$, $\beta = 1.45$.

The air conductivity $\sigma(t)$ is taken to be [3,4]

$$\sigma_e(t) \cong 2\times10^{-12} \times \dot{D}(t)$$

$$\dot{D}(t) = \text{DOSE} \times \text{PEAK} \times F(t/t_p) \times y/R^3$$

$$\text{DOSE} = 4000 \text{ rads}$$

$$\text{PEAK} = 7.5\times10^6$$

$$y = 4.0 \text{ m}$$

$$R = 4.38 \text{ m}$$

$$t_p = 100 \text{ nsec}$$

$$F(\tau) = \sin^2 [\pi\tau/(\tau + 2 \exp(-0.6931 \tau^{\beta}))]$$

$$\beta = 1.77 \ (\text{peak} \times t_p)^{1.9}$$

The resulting base current is shown in Figures 2 and 3. Figure 2 shows the base current calculated without using thin wire approximation (up to 200 nsec). Figure 3 shows the base current for the same problem calculated using the thin wire approximation described in the previous section up to 360 nsec. In this case, $R_z(2) = 0.048$ m, $R_h(1) = 0.02$ m, a = 0.004 m, $\Delta r_e = R_h(1)\ \ell n(R_z(2)/2a) = 0.050$ m which is slightly larger than $R_z(2)$. The actual running time, cost, accuracy, et cetera are compared in Table 1.

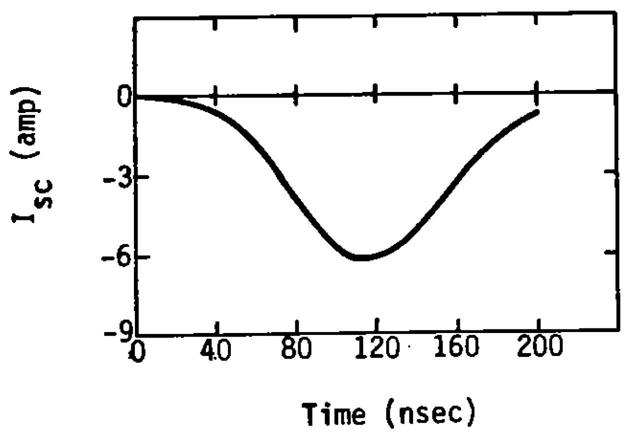

Figure 2. Base Current Calculated Without Using Thin Wire Approximation, $h = 0.58$ m, $a = 0.004$ m, $\sigma = \sigma(t)$.

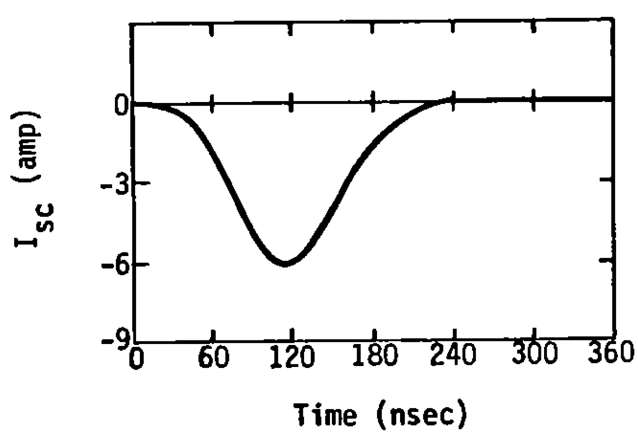

Figure 3. Base Current Calculated With Thin Wire Approximation, $h = 0.58$ m, $a = 0.004$ m, $\sigma = \sigma(t)$ with $R_z(2) = 0.048$ m, $R_h(1) = 0.02$ m.

Table 1. A Comparison of Results Calculated With and Without Thin Wire Approximations for h = 0.58 m, a = 0.004m, and $\sigma = \sigma(t)$

	With Thin Wire Approximations	Without Thin Wire Approximations
Run Time	103.9 sec	545.1 sec
Time Cycle N_{max}	4050	40499
Exit T_{max}	360 nsec	200 nsec
Accuracy	$I_{p1} = 6.038$ at $t = 114.5$ ns $I_{p2} = 4.985$ at $t = 114.5$ ns $I_{p3} = 1.548$ at $t = 114.0$ ns	$I_{p1} = 6.108$ at $t = 114.0$ ns $I_{p2} = 5.026$ at $t = 114.0$ ns $I_{p3} = 1.539$ at $t = 114.6$ ns
Cost	$4.36 on CDC 6600	$22.89 on CDC 6600

Notice that in Table 1, the value calculated without thin approximation is only up to 200 nsec. Assuming the cost and running time are linearly proportional to the exit time T_{max}, then to obtain the result up to 360 nsec, the running time would need to be near 981.2 sec and the cost will be near $41.2 . These are about a factor of 10 times that with thin wire approximation. From Table 1, the accuracy is within 1.5 percent. (I_{p1}, I_{p2} and I_{p3} are peak values at three different locations on the antenna.) This comparison shows the advantage of using thin wire approximation in this case.

A second example is to calculate the short circuit current at the base of an antenna with h = 1 m, a = 0.003 m, using $\sigma = 0.0$ and $E_{inc}(t)$ in the first example. We use $R_h(1) = 0.015$ m, $R_z(2) = 0.036$ m, a = 0.003 m, $\Delta r_e = R_h(1)\ \ell n(R_z(2)/a) = 0.037$ m in this case. The resulting currents are shown in Figures 4 and 5. A comparison of the run time, cost, accuracy is given in Table 2. It is seen that the accuracy if I_{p1} is within 0.5 percent and the peak time is within 0.3 percent when the thin wire approximation is used. The cost and run time are reduced by a factor of 10 using this method.

When the radius a is given and $R_z(2)$ is selected, the effective radius Δr_e is determined from Equation (15). How much difference is introduced in the resulting current if the Δr_e is selected slightly different from what is specified by Equation (15)? In Table 3, a comparison of several calculations is made to study the sensitivity of the result. We use the parameters in example 2 for this purpose. It is seen from Table 3 that the effective radius given by Equation (15) indeed gives the best result (0.37 percent deviation from the one without thin wire approximation). Notice also that, if we use $\Delta r_e = R_z(2)$, the error is only 1.31 percent. This is because, in this case, $R_z(2)$ and $R_h(1)\ell n(R_z(2)/a)$ are quite close to each other so that the difference in the result is small.

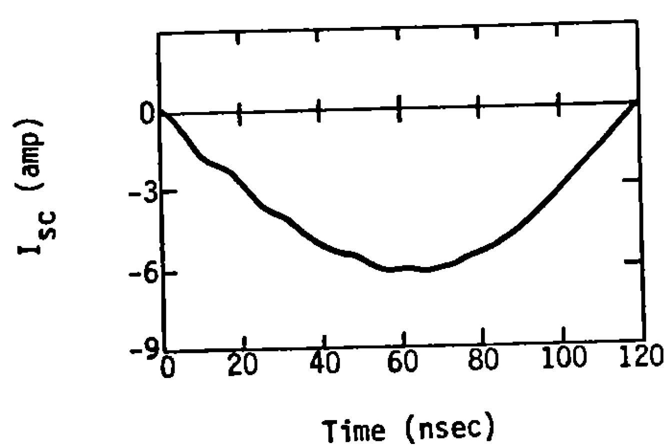

Figure 4. Base Current Calculated Without Using Thin Wire Approximation, h = 1.0 m, a = 0.003 m, σ = 0.

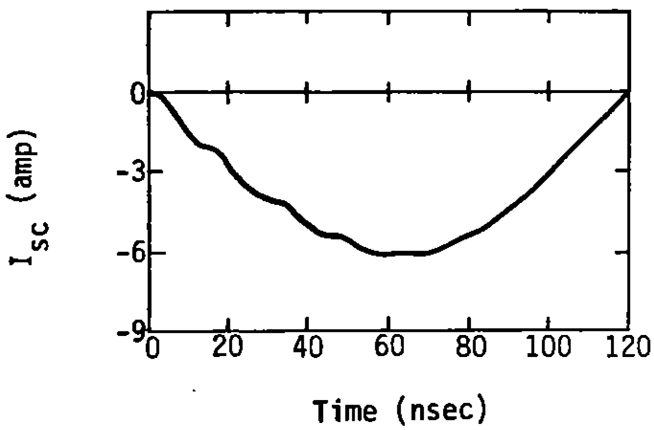

Figure 5. Base Current Calculated With Thin Wire Approximation, $h = 1.0$ m, $a = 0.003$ m, $\sigma = 0$, $R_z(2) = 0.036$ m, $R_h(1) = 0.015$ m.

Table 2. A Comparison of Results for Example 2

	With Thin Wire Approximations	Without Thin Wire Approximations
Run Time	55 sec	483 sec
Time Cycle N_{max}	1800	17999
Exit T_{max}	120 nsec	120 nsec
Accuracy	$I_{p1} = 0.6062$ at $t = 58.0$ ns	$I_{p1} = 0.6085$ at $t = 57.6$ ns
Cost	$2.65 on CDC 6600	$22.31 on CDC 6600

Table 3. Sensitivity Study Using Example 2 Parameters

Case Studied	I_{peak}	t_{peak}	% Deviation I_{peak}	Run Time	Cost
Without Thin Wire Approximation	0.6085	57.6 ns	0.0	483.0 sec	$22.31
$\Delta r_e = 0.025$	0.7239	59.3 ns	18.96	61.0 sec	$ 2.56
$\Delta r_e = R_z(2) = 0.036$	0.6165	58.3 ns	1.31	59.4 sec	$ 2.49
$\Delta r_e = R_H(1) \times \ell n(R_z(2)/z) = 0.0373$	0.6062	58.0 ns	0.37	63.2 sec	$ 2.65
$\Delta r_e = 0.045$	0.5525	57.8 ns	9.2	59.8 sec	$ 2.51

In fact, if we are free to select $R_z(2)$, then it is always possible to find a $R_z(2)$ which satisfies (consider a uniform grid in the following)

$$R_z(2) = \frac{R_z(2)}{2} \ln \frac{R_z(2)}{a} \quad .$$

Solving the above equation, we have

$$R_z(2) = e^2 \cdot a = 7.39\ a \quad .$$

For this value of $R_z(2)$, the effective radius of Equation (15) is equal to the first cell size, and Equation (13) is the same as Equation (14). This may not be important in the 2-D problem discussed here, but in a more complicated 3-D problem, the use of this fact would simplify the formulation.

A question related to the above discussion is that, for a given problem with antenna radius a specified, "What is a good value of $R_z(2)$ to be used?" It is noted that the larger $R_z(2)$ is, the less running time and cost will be needed. Hence we tend to choose a larger $R_z(2)$. However, when the ratio of $R_z(2)/a$ is increased, it is expected that the accuracy of the result decreases since the approximation may not be valid. Thus, it is interesting to examine the results obtained from several calculations with different $R_z(2)/a$ values. Table 4 is a summary of the results calculated for $R_z(2)/a$ = 2, 5, 7.389, 10, 20 and 30 using example 2 parameters. The difference between Table 3 and Table 4 is that in Table 4 the ratio $R_z(2)/a$ is varied, which in turn changes Δr_e accordingly, whereas in Table 3, $R_z(2)/a$ is the same in each case and Δr_e is changed *arbitrarily*.

The second solumn in Table 4 is for the case without using thin wire approximation, where we use $R_z(2) = a$. The rest of the results in the table are all using thin wire approximations. Notice the run time and cost is decreased as the ratio of $R_z(2)/a$ is increased. The accuracy decreases as the ratio becomes large. However, even for the case of $R_z(2) = 30.0\ a$,

Table 4. A Comparison of Results for Different $R_z(2)/a$ Using Example 2 Parameters

Cases	Without Approximation	$R_z(2)/a$ for Cases with Thin Wire Approximation						
	$R_z(2) = a$	2	5	7.389	10	20	30	40
Run time	531.4 sec	234.2 sec	97.1 sec	63.5 sec	63.2 sec	31.7 sec	24.2 sec	22.4 sec
Cost	$22.30	$ 9.83	$ 4.07	$ 2.66	$ 2.65	$ 1.32	$ 1.01	$ 0.94
I_{p1} (% deviation)	0.6085 (0%)	0.6161 (1.2%)	0.6068 (0.3%)	0.6071 (0.2%)	0.6062 (0.4%)	0.6170 (1.4%)	0.632 (3.9%)	0.6445 (5.9%)
t_{p1} (% deviation)	57.8 (0%)	58.8 (1.7%)	57.6 (0.3%)	57.6 (0.3%)	58.1 (0.5%)	58.13 (0.5%)	58.4 (1.0%)	58.6 (1.4%)
N_{max}	17999	8999	3600	2436	1800	900	600	459
T_{max}	120 nsec	120 nsec	120 nsec	120 nsec	120 nsec	120 nsec	120 nsec	120 nsec

the accuracy of I_{p1} is within 4 percent of that without using approximations. The run time is 24 seconds and the cost is only $1.01 for the case with $R_z(2) = 30$ a as compared to the run time of 531 seconds and the cost of $22.30 for the case without approximation. From Table 4, it is seen that when the ratio $R_z(2)/a$ decreases from 7.389 to 2, the run time and cost increases and the accuracy decreases. Thus, there is no advantage in choosing a small value of $R_z(2)/a$ below 5.

Figure 6 presents some sample base currents for different $R_z(2)/a$ ratios. It appears that the difference between these curves are not very significant. Figure 7 is a graphical representation of the data presented in Table 4. It seems that the optimal value of $R_z(2)/a$ is between 5 and 10, where the percentage deviations of I_{p1} and t_{p1} are below one percent, the run time is less than 15 percent of that without using the thin wire approximation and the cost is less than 20 percent of that without using the thin wire approximation.

4. CONCLUSIONS

It is snown that the thin wire approximation cut the running time and cost by more than a factor of 10 when it is applicable, with good accuracy. When one selects $R_z(2) = 7.39$ a, the formulation simplified to ordinary finite-difference formulation. This could be useful for 3-D calculations. From the sensitivity study given in Table 3, it is seen that the choice of the effective radius as given by Equation (15) indeed produces the best result.

Another scheme which may be useful is to use the three-point interpolation formula in Equation (2) and not to introduce the effective radius Δr_e. At this time, the choice of our approximation seems to fit into the antenna and scattering problem with less difficulty. The approximation suggested here is somewhat different from that discussed in Reference 2 although the principle is the same. It would be interesting to compare the results from these different schemes in the future study.

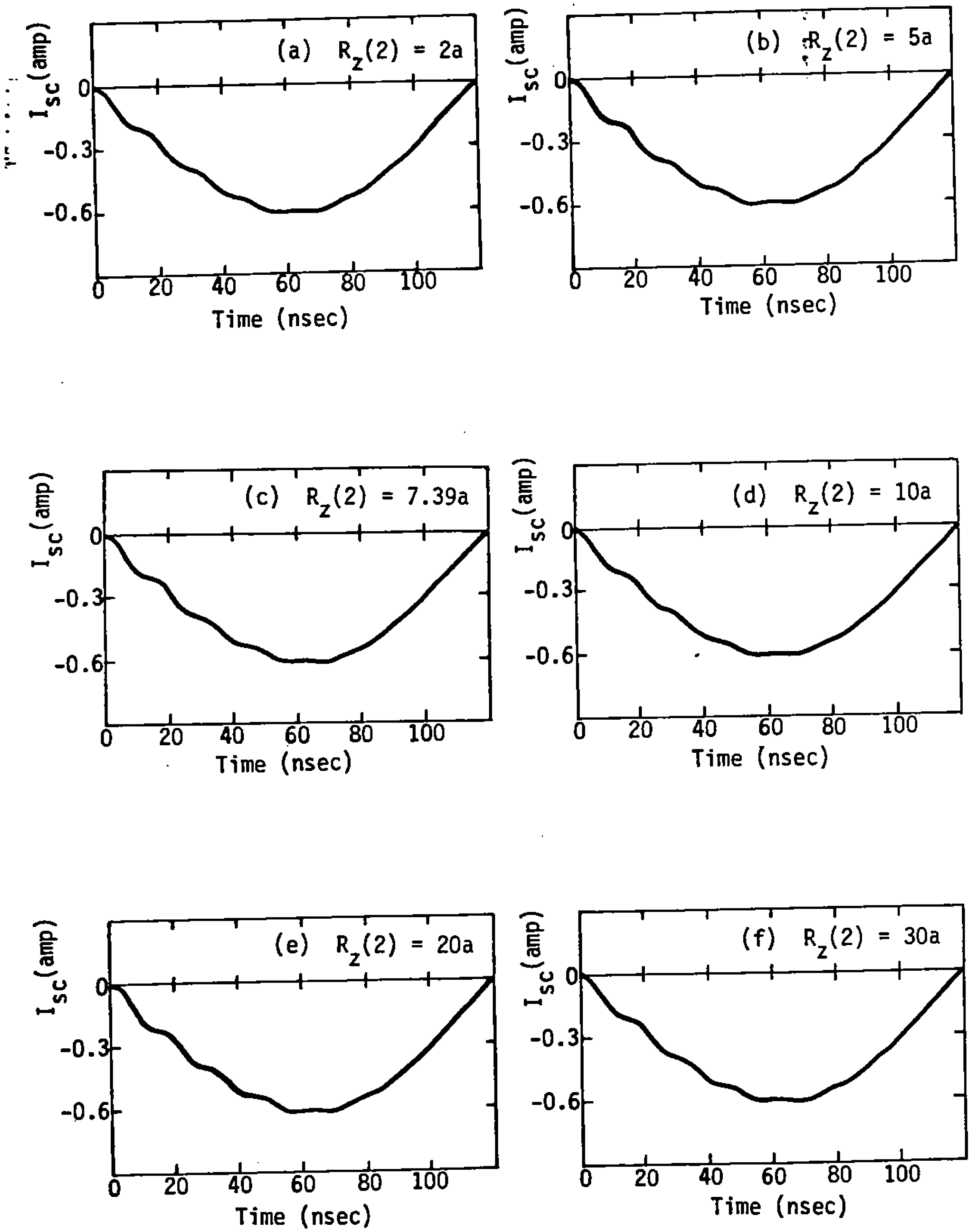

Figure 6. Sample Base Currents for Different $R_z(2)/a$ Using Example 2 Parameters

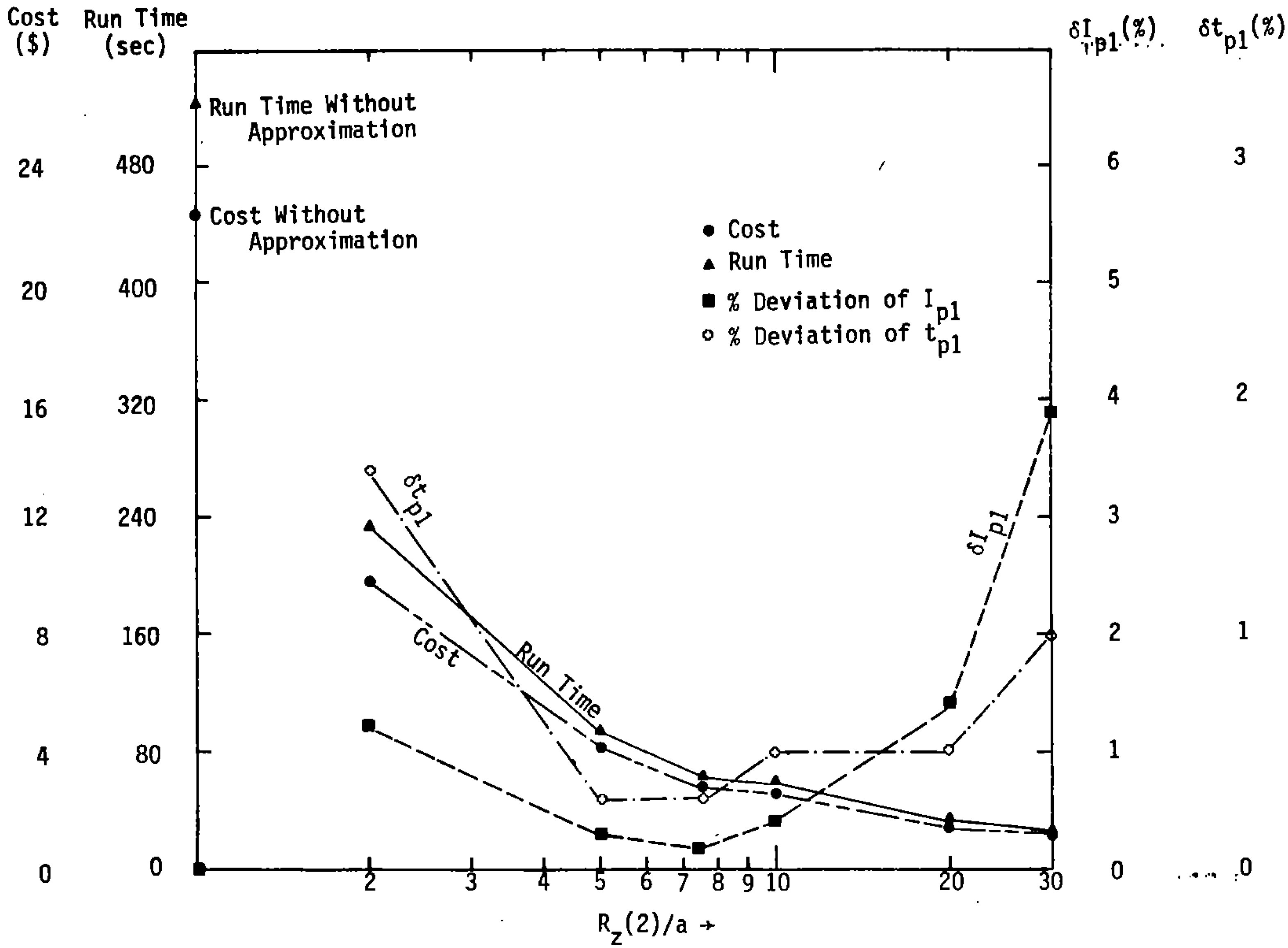

Figure 7. A Comparison of Cost, Run Time, and Percentage Deviation of I_{p1} and t_{p1} as a Function of $R_z(2)/a$ Using Example 2 Parameters

ACKNOWLEDGEMENT

Thanks are due to W. F. Crevier for his useful suggestions and comments.

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