Theoretical Notes

Note 358

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ON THE DIRECT CALCULATION OF A TRANSIENT PLANE WAVE REFLECTED FROM A FINITELY CONDUCTING HALF–SPACE

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Abstract

This note discusses an approach for computing the electromagnetic field reflected from a lossy half–space directly in the time domain. This approach requires first evaluating the impulse response of the half–space and then convolving it with the specified incident field waveform. To obtain the impulsive reflected field, either for vertical or horizontal polarization, approximations to the Fresnel reflection coefficients are made, thereby permitting an analytical expression in the time domain. Several different numerical examples using this technique are presented to illustrate the use of the method and the error contained in the solution.

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I. Introduction

In many electromagnetic (EM) coupling problems, it is necessary to determine the plane wave reflected field from a finitely conducting earth. For problems analyzed in the frequency domain, such calculations involve the use of the Fresnel reflection coefficients [1], [2]. Transient results, if desired, are typically obtained by performing a numerical Fourier transform of the frequency domain response spectrum.

In some instances, it is desirable to perform EM coupling calculations directly in the time domain. Such is the case for problems involving time–varying media or nonlinear system behavior, or in cases where the lossy earth causes the reflected field response to persist for very long times, thereby causing difficulties with FFT ailing. For this type of analysis, it is useful to have an expression for the transient reflected field from the earth–air interface. As an example of such a problem, Engheta [3] has determined the time–dependent current on an above–ground horizontal cable for an incident transient plane wave excitation. This is done by first computing the the earth reflected field in the frequency domain and then transforming it into the time domain using Fourier techniques for a transient calculation for the induced current. An alternative to this approach would be to compute the plane wave reflected field directly in the time domain and use this in the coupling calculation.

A survey of the available literature on this topic has provided some information on expressions useful for this purpose. Several earlier authors [4], [5] discuss transient fields reflected from a lossy earth, but use the conventional frequency domain approach. Dudley [6] develops a time domain expression for the reflection of a double exponential plane wave in the earth by evaluating the required Fourier transform in the complex frequency plane. This results in the reflected field being represented by a simple pole term, plus a real–valued, singular integral which accounts for a branch cut occurring in the Fourier transform. More recently, Klaasen [7] develops solutions for step–function reflected fields from a lossy earth in terms of two definite integrals involving modified Bessel functions.

In this paper, is is desired to develop approximate analytical expressions which are simpler to those presented in [6] and [7] for the transient earth–reflected fields. Both vertically and horizontally polarized impulsive plane wave fields are considered to be reflected from a conducting earth. As discussed in [8], transient problems involving lossy media typically involve the use of a convolution operator in the temporal domain. Consequently, the form of the earth–reflected field is expected to be that of a convolution operator on the incident field.

II. Earth Reflected Fields

A. Frequency Domain Representation

The earth–reflected electric field spectrum from an incident plane wave $E^{inc}(s)$ which is impinging on a finitely conducting half–space with a vertical angle of incidence ψ is given by [2] as

$$E^{ref}(s) = R(\psi,s)\, E^{inc}(s) \tag{1}$$

In this expression s is the complex frequency Laplace transform variable and $R(\psi,s)$ is the Fresnel reflection coefficient. Figure 1 illustrates the geometry of this problem, where the incident field is decomposed into components having vertical and horizontal polarization of the electric field vectors. For each polarization component of the incident field there is a distinct Fresnel reflection coefficient, denoted by R_v and R_h respectively.

The total E–field at some point above the interface is composed of the incident plus the reflected field, taking into account the vector nature of the fields. For example, the x–directed (horizontal) component of the electric field at a height h for a vertically polarized incident field is given by

$$E_x^{tot}(s) = E^{inc}(s)\, \sin\psi \cos\varphi \,[1 - R_v(\psi,s)e^{-2hs/c\,\sin\psi}] \tag{2}$$

where φ is the azimuthal angle of incidence shown in Figure 1, and c is the propagation velocity of free space, $c = 3.0 \times 10^8$ m/s.

Similarly, the total horizontal component due to a horizontally polarized incident field is expressed as

$$E_x^{tot}(s) = E^{inc}(s)\, \sin\varphi \,[1 + R_h(\psi,s)e^{-2hs/c\,\sin\psi}] \tag{3}$$

The y and z components of the total E–field can be expressed in a similar fashion.

As discussed in [2], the Fresnel reflection coefficients for a conducting half–space having a relative dielectric constant ϵ_r and an electrical conductivity σ_g are

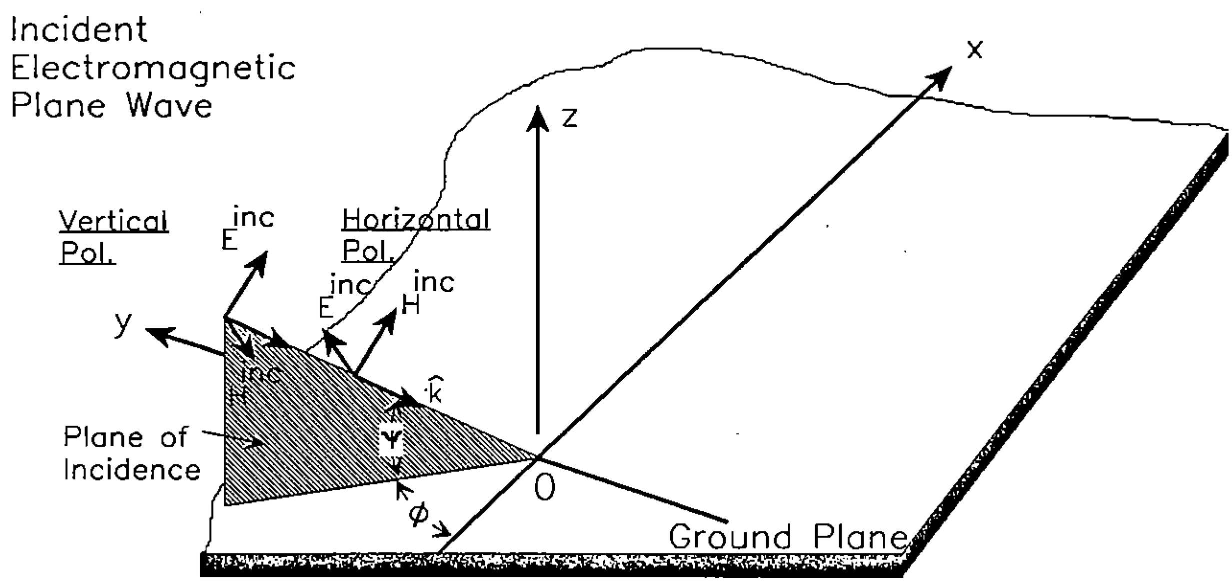


Figure 1. Incident Plane Wave Field on Lossy Half–Space

$$R_v(\psi,s) = \frac{\epsilon_r(1+\frac{\sigma_g}{s\epsilon})\sin\psi - [\epsilon_r(1+\frac{\sigma_g}{s\epsilon}) - \cos^2\psi]^{1/2}}{\epsilon_r(1+\frac{\sigma_g}{s\epsilon})\sin\psi + [\epsilon_r(1+\frac{\sigma_g}{s\epsilon}) - \cos^2\psi]^{1/2}} \tag{4}$$

and

$$R_h(\psi,s) = \frac{\sin\psi - [\epsilon_r(1+\frac{\sigma_g}{s\epsilon}) - \cos^2\psi]^{1/2}}{\sin\psi + [\epsilon_r(1+\frac{\sigma_g}{s\epsilon}) - \cos^2\psi]^{1/2}} \tag{5}$$

where $\epsilon = \epsilon_r\epsilon_o$ and ϵ_o is the permittivity of free space, equal to 8.854×10^{-12} farads/m.

For problems involving a lossy earth typical values of σ_g range from 1 to 30 millimhos/meter [2], although conductivities as high as 100 millimhos/m are sometimes encountered in the literature. Typical values for ϵ_r range from 10 to 30 [2].

B. Time Domain Representation of the Reflected Field

The transient counterpart of Eq.(1) for the reflected field is given as the convolution of the incident field and the inverse Laplace transform of the reflection coefficient r(t) as

$$E^{ref}(t) = \int_{-\infty}^{t} E^{inc}(t-\xi)\, r(\xi)\, d\xi \tag{6}$$

In this manner the transient quantity r(t) is viewed as the impulse response of the conducting half–space.

Thus the total vector E–field at a height h thus is given by

$$\overline{E}(t) = E^{inc}(t)\,\bar{i} + E^{ref}(t^*)\,\bar{k} \tag{7}$$

where the $\bar{i}$ and $\bar{k}$ vectors account for the various vector directions of the fields, and t^* is a retarded time $t^* = t - 2h/c \sin\psi$. The presence of this time shift is evidenced by the exponential functions in Eqs.(2) and (3), and accounts for the propagation delay time experienced by the earth–reflected field.

For the remainder of this paper, the incident field term, the vector nature of the fields, and the propagation time shift in Eq.(7) will be ignored. We will concentrate on developing an expression for the impulse response r(t) for the lossy half–space, for both vertical and horizontal polarizations.

III. Evaluation of the Impulse Response, r(t)

A. Approximations to the Reflection Coefficients

The evaluation of r(t) in Eq.(6) directly in the time domain requires the analytical inverse Laplace transform of the Fresnel reflection coefficients. This is difficult to do without making some approximations. To begin, Eq.(4) can be rewritten as

$$R_v(\psi,s) = \frac{s+\tau \; - \; \beta\sqrt{s(s+\gamma)}}{s+\tau \; + \; \beta\sqrt{s(s+\gamma)}} \tag{8}$$

where $\tau = \sigma/\epsilon$, $\beta = \dfrac{\sqrt{\epsilon_r - \cos^2\psi}}{\epsilon_r \sin\psi}$, and $\gamma = \tau\,(1-\cos^2\psi/\epsilon_r)^{-1}$. For the cases where ϵ_r is on the order of 10 or more, and the angle ψ is large, so that $\cos\psi$ is small, the parameter γ is approximately equal to τ. This allows Eq.(8) to be approximated as

$$R_v(\psi,s) \approx \frac{\sqrt{s+\tau} \; - \; \beta\sqrt{s}}{\sqrt{s+\tau} \; + \; \beta\sqrt{s}} \tag{9}$$

Note that this approximation will limit the applicability of the solution to non–grazing angles of incidence.

The parameter β can range from about 0.25 to ∞ . When $\beta = 1.0$, the angle of incidence $\psi = \cos^{-1}(\sqrt{\epsilon_r/(\epsilon_r+1)}\,)$. This is approximately equal to the Brewster's angle, $\psi_b = \cot^{-1}(\sqrt{\epsilon_r}\,)$. The relative difference between these two values is on the order of 10^{-9}.

For $\psi < \psi_b$, we have $\beta > 1$, and the reflected electric field adds to the incident field at high frequencies, where $|s| > \tau$. For $\beta < 1$, the reflected field subtracts from the incident field.

The horizontally polarized reflection coefficient of Eq.(5) has the form

$$R_h(\psi,s) = \frac{\sqrt{s} - \epsilon_r\beta\sqrt{s+\gamma}}{\sqrt{s} + \epsilon_r\beta\sqrt{s+\gamma}} . \tag{10}$$

Noting again that $\gamma \approx \tau$ for cases involving a large relative dielectric constant and non–grazing angles of incidence, Eq.(10) can be approximated as

$$R_h(\psi,s) \approx -\frac{\sqrt{s+\tau} - (\epsilon_r\beta)^{-1}\sqrt{s}}{\sqrt{s+\tau} + (\epsilon_r\beta)^{-1}\sqrt{s}} \tag{11}$$

Both the vertical and horizontal reflection coefficients now have the form

$$R(\psi,s) \approx \pm\frac{\sqrt{s+2a} - \kappa\sqrt{s}}{\sqrt{s+2a} + \kappa\sqrt{s}} \tag{12}$$

where $a = \tau/2$ and $\kappa = \beta$ for vertical polarization and $\kappa = (\epsilon_r\beta)^{-1}$ for horizontal polarization. The leading + sign is used for vertical polarization, while the – sign is used for horizontal polarization. Hence, to determine the transient reflected field for either polarizations, the same inverse Laplace transformation can be used.

B. Analytical Inversion of the Reflection Coefficients

In order to invert Eq.(12) into the time domain, the Laplace transform pair

$$\mathscr{L}^{-1}[F(s-a)] = e^{at} f(t) \tag{13}$$

can be used. Eq.(12) can be written as a function of (s–a) as

$$R(s-a) = \pm \frac{\sqrt{s+a} - \kappa\sqrt{s-a}}{\sqrt{s+a} + \kappa\sqrt{s-a}} \tag{14}$$

which can be expanded to give

$$R(s-a) = \pm \left[\frac{1-\kappa}{1+\kappa} + \frac{4\kappa}{(1+\kappa)^2} \frac{a}{s+\sqrt{s^2-a^2} + a(1-\kappa)/(1+\kappa)} \right] . \tag{15}$$

Defining $K = \frac{1-\kappa}{1+\kappa}$ and $S = s+\sqrt{s^2-a^2}$, Eq.(15) can be written as

$$R(s-a) = \pm \left[K + \frac{4\kappa}{1-\kappa^2} \sum_{n=1}^{\infty} (-1)^{n+1} (Ka)^n S^{-n} \right] . \tag{16}$$

The inverse Laplace transform of this expression is given as [9]

$$\mathscr{L}^{-1}[R(s-a)] = \pm \left[K\delta(t) + \frac{4\kappa t^{-1}}{1-\kappa^2} \sum_{n=1}^{\infty} (-1)^n nK^n I_n(at) \right] \tag{17}$$

where $\delta(t)$ is the Dirac delta function and $I_n(t)$ is the modified Bessel function of order n.

Using Eq.(13), the resulting transient response for the reflected impulse field $r(t)$ is then given by

$$r(t) = \pm \left[K\delta(t) + \frac{4\kappa}{1-\kappa^2} \frac{e^{-at}}{t} \sum_{n=1}^{\infty} (-1)^{n+1} nK^n I_n(at) \right] . \tag{18}$$

Note that this expression for the impulse response of the ground–plane contains two terms. The first is an impulse which is independent of the ground conductivity, and the second is a response term persisting in time. Inserting this expression into the convolution expression for a general incident E–field in Eq.(6) yields the following expression for the reflected field

$$E^{ref}(t) = \pm \left[K\, E^{inc}(t) + \frac{4\kappa}{1-\kappa^2} \int_{-\infty}^{t} E^{inc}(t-\xi)\, \frac{e^{-a\xi}}{\xi} \sum_{n=1}^{\infty} (-1)^{n+1}\, nK^n\, I_n(a\xi)\, d\xi \right] . \qquad (19)$$

This expression is valid for both vertical and horizontal polarization, depending on the values for κ and K and the $\pm$ sign.

C. Numerical Verification

As a check of the development of Eq.(19) and to see the effects of the approximations used in developing the reflection coefficient expressions in Eqs.(9) and (11), consider the reflection of the double exponential transient electric field given by $E^{inc}(t) = A_o(e^{-\alpha t} - e^{-\beta t})$ with $A_o = 52.5$ (kV/m), $\alpha = 4 \times 10^6$ (1/sec) and $\beta = 4.76 \times 10^8$ (1/sec). This waveform, shown in Figure 2, is the "Bell Laboratory waveform" which is frequently used in electromagnetic pulse (EMP) coupling studies [10]. This incident field is assumed to strike an imperfectly conducting earth, having a conductivity of $\sigma_g =$ 0.01 mhos/meter and a relative dielectric constant $\epsilon_r = 10$, with an angle of incidence of $\psi = 45^\circ$.

Using Eqs.(19) for both the vertical and horizontal polarization cases, the convolution integral was evaluated numerically in the time domain to provide the earth–reflected fields shown in Figures 3a and 3b. Also shown in these figures are the reflected fields computed from the original reflection coefficients of Eqs.(4) and (5) in the frequency domain and converted into the time domain by an FFT. Note that the agreement between the two waveforms is quite good, indicating that Eq.(19) is correct and that the approximations used in deriving the reflection coefficients of Eqs.(9) and (11) are not bad. It is expected, however, that there will be a larger error in the response for smaller angles ψ, due to these approximations.

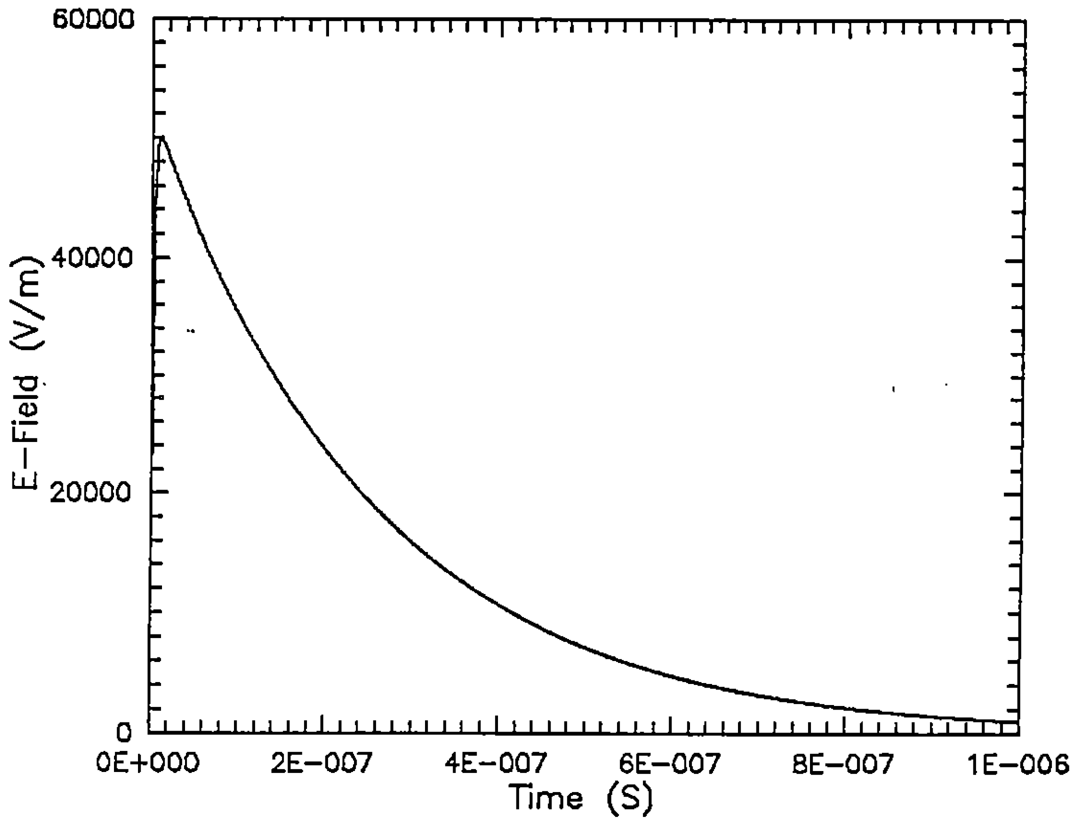
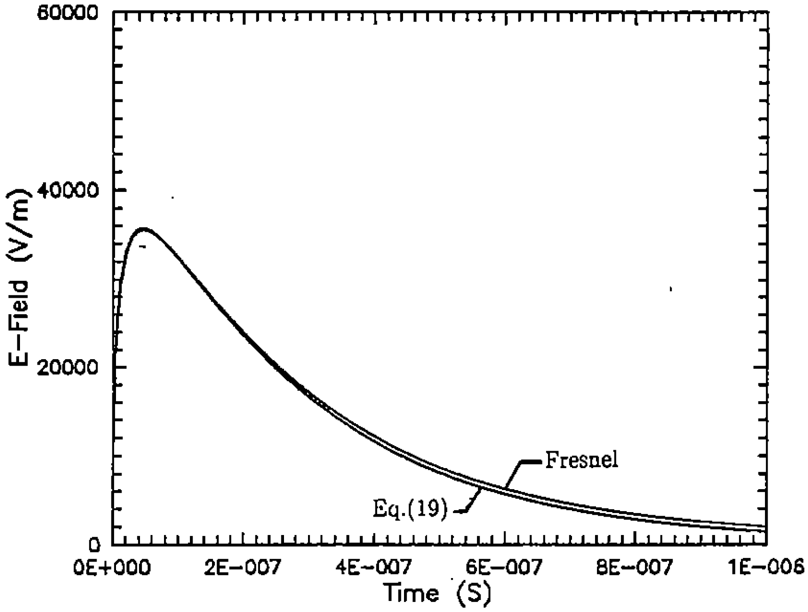
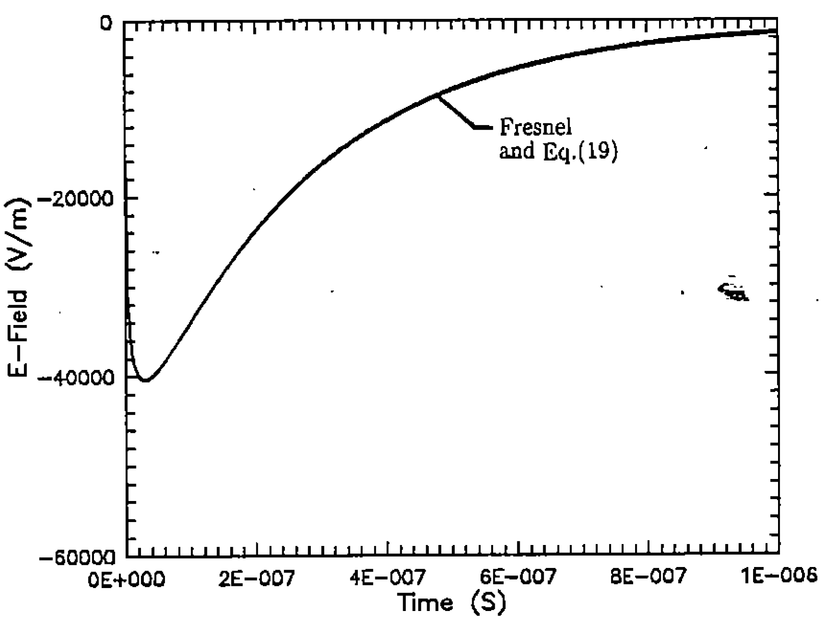


Figure 2. Double Exponential Incident E—Field.



a. Vertical Polarization



b. Horizontal Polarization

Figure 3. Reflected Fields From Fresnel Reflection Coefficients and Eq.(19) for $\psi = 45^{\circ}$, $\epsilon_r = 10$, and $\sigma_g = 0.01$ mhos/m.

IV. Approximations to the Impulse Response

Although the impulse response for the lossy ground in Eq.(18) is seen to be accurate, its application to practical problems is hindered by the infinite series of modified Bessel functions. Such functions can be numerically evaluated [11], but it is tedious to evaluate this sum and then perform the indicated convolution operation. Consequently, it is desired to obtain further approximations to the reflected field which permit a simple evaluation.

In order to simplify the summation expression in Eq.(18), consider re–writing it as

$$r(t) = \pm \left[K\delta(t) + \frac{4\kappa}{1-\kappa^2} e^{-at} \mathcal{S} \right] \tag{20}$$

where the term $\mathcal{S}$ is defined as

$$\mathcal{S} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{nK^{n+1}}{t} I_n(at) \tag{21}$$

Substituting the series expansion for the modified Bessel function [9]

$$I_n(at) = \sum_{k=0}^{\infty} \frac{(at/2)^{2k+1}}{k!\ (n+k)!} \tag{22}$$

into Eq.(22),and interchanging the order of the sums over n and k yields the expression:

$$\mathcal{S} = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{nK^{n+1}}{t} \frac{(at/2)^{2k+1}}{k!\ (n+k)!} \ . \tag{23}$$

Analytically summing each term in n for a fixed k yields

$$\mathcal{S} = \sum_{k=0}^{\infty} f_k \tag{24}$$

where the following terms can be defined after some manipulations:

$$f_o = \frac{K}{2} X$$

$$f_1 = \frac{1}{2K}(1-X) - \frac{at}{4} X$$

$$f_2 = \frac{y}{4K^2}(X-1) + \frac{y^2}{16A}(X+1)$$

$$\vdots$$

with

$$X = e^{-Kat/2}$$

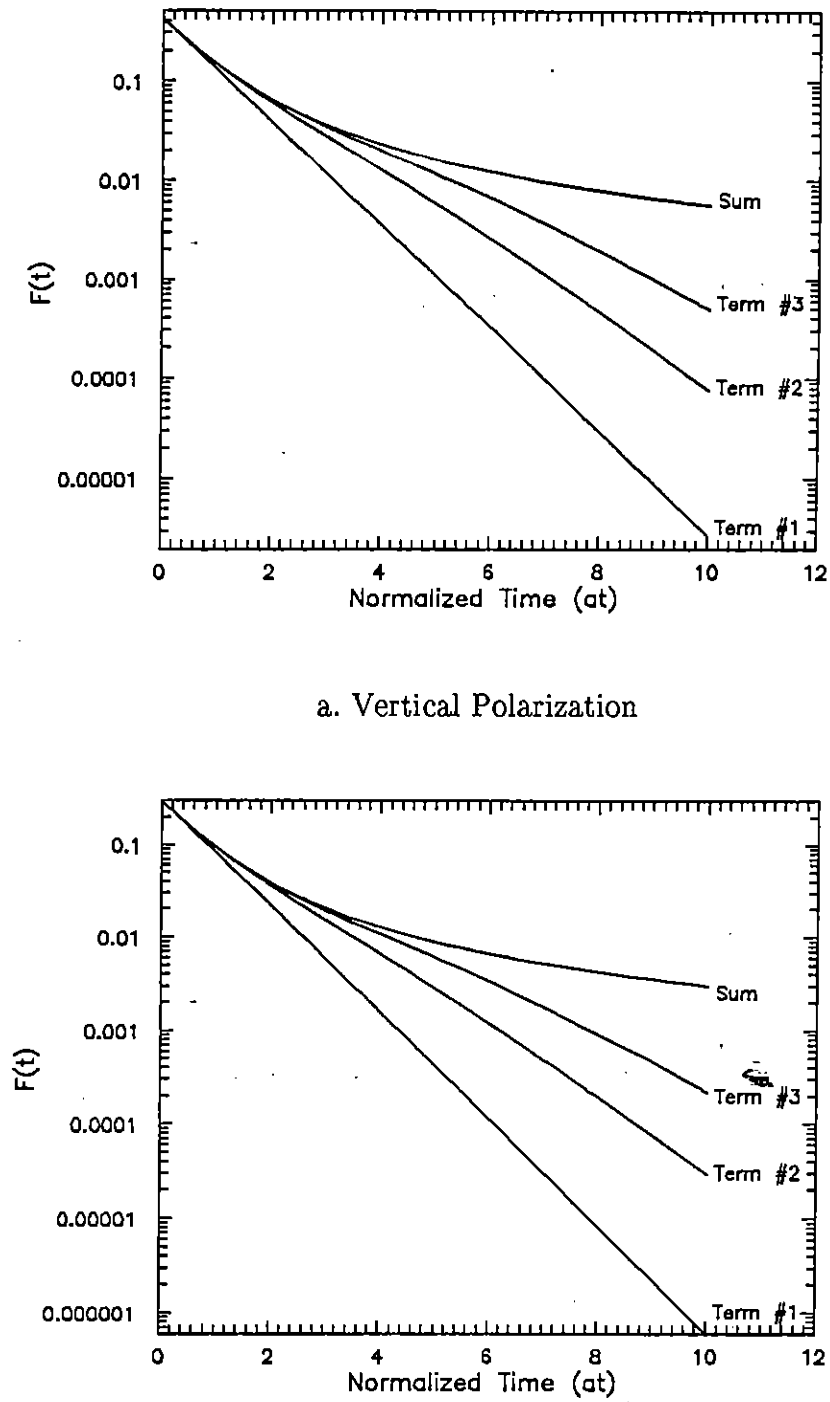
Inserting this result into Eq.(18) gives an approximate expression for r(t) of

$$r(t) \approx \pm \left[K\delta(t) + \frac{4\kappa}{1-\kappa^2} e^{-at} \left[\frac{K}{2} X + \frac{1}{2K}(1-X) - \frac{at}{4} X + \frac{y}{4K^2}(X-1) + \frac{y^2}{16A}(X+1) \right] \right] . \tag{25}$$

As an example of the accuracy of this approximation, Figures 4a and 4b present the actual and approximate functions for r(t), with the δ–function term extracted, for both vertical and horizontal polarizations. The curves labeled "Sum" comes from a direct evaluation of the term $\left(\frac{4\kappa}{1-\kappa^2} e^{-at} \mathscr{T} \right)$ in Eq.(20), with $\mathscr{T}$ being evaluated using the Bessel function sum of Eq.(21). The curves denoted as "Term #1", "Term #2" and "Term #3" result from approximating $\mathscr{T}$ by f_o , f_o+f_1 , and $f_o+f_1+f_3$, respectively. These results are again for the parameters $\psi = 45^o$, $\epsilon_r = 10$, and $\sigma_g = 0.01$ mhos/meter.

Note that for three terms in the series for $\mathscr{T}$, the result are not convergent to the actual solution for $at > 5$. For the specified values of ψ , σ_g and ϵ_r , $a \equiv \tau/2 = 5.6 \times 10^7$. This implies that the approximate transient response will begin to deviate from the actual response at a time of $t \approx 0.9 \times 10^{-7}$ sec. Of course, the total transient response also contains the effect of the δ–function term, as well at the convolution integral, both of which are neglected in Figure 4. This suggests that the deviation of the final transient results may not be as large as that indicated in this figure.

The impulse response of Eq.(25) for both polarizations has been convolved numerically with the double exponential waveform of Figure 2, to provide the transient



a. Vertical Polarization

b. Horizontal Polarization

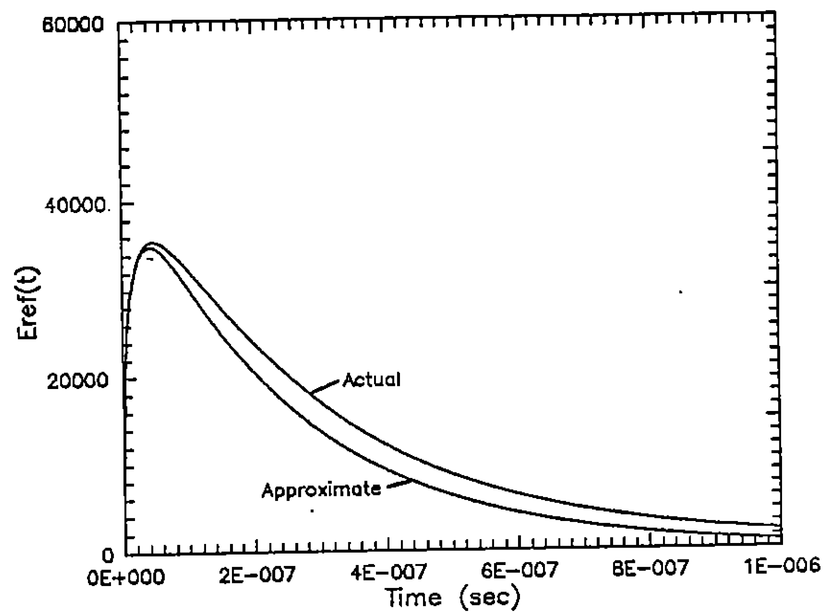
Figure 4. Total and Partial Sums for the Reflected Field Impulse Response for $\psi = 45^\circ$, $\epsilon_r = 10$, and $\sigma_g = 0.01$ mhos/m..

reflected fields shown in Figures 5a and 5b. Because of the relatively simple dependence on the variable t of the terms in Eq.(25), it is possible that for certain incident field waveforms, the convolution integral can be performed analytically. In the present case, however, this integral is evaluated numerically in the time domain. The curves marked "Actual" are the same discussed in Figure 3, and the "Approximate" curves result from only a 2–term approximation for $\mathscr{T}$. There are noticeable differences between these two curves, but the early–time portions of the waveforms agree almost exactly. By taking additional terms in the series representation for $\mathscr{T}$, the agreement at later times will improve.

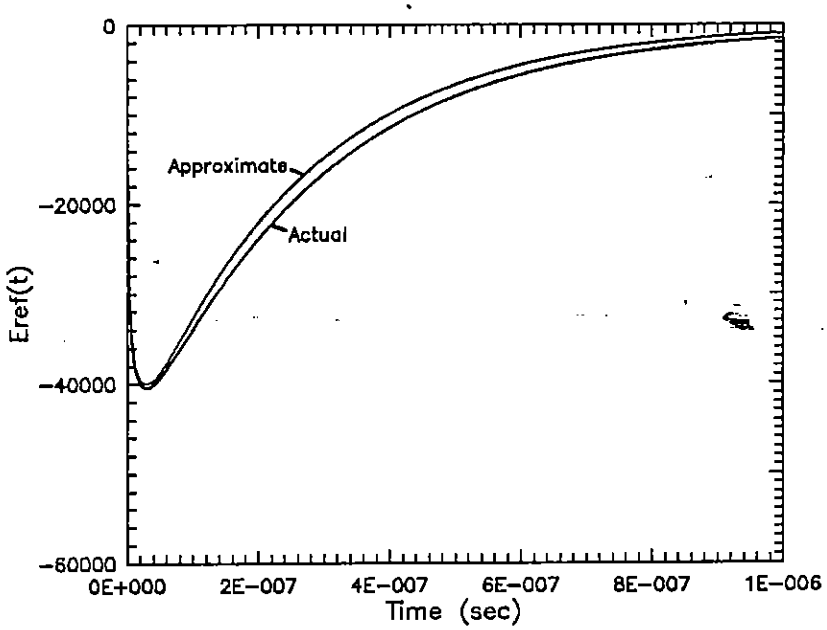
To see the behavior of this approximate 2–term solution for the reflected field, a series of calculations was performed for a number of different angles of incidence ψ and the results plotted as a contour plot. The values of the earth and waveform parameters were the same as for the previous cases, and vertical polarization was first considered. Figure 6a plots constant contours (in kV/m) for the actual transient reflected field, as computed directly from the Fresnel reflection coefficients. This is shown as a function of the angle over the range $0^\circ \leq \psi \leq 90^\circ$ and for $0 \leq t \leq 1$ μs. Similar responses are plotted in Figure 6b for the approximate solution. Agreement between the two appears to be good for large values of ψ, as expected from the approximations made in the analysis. For $\psi < 10^\circ$, however, the agreement is not good.

The difference between the actual and approximate waveform surfaces, expressed as a percent quantity, is one possible way to quantify the error in these calculations. This may be defined as (% Error) = $\frac{E_{actual} - E_{approx}}{E_{actual}}$ x 100 , and also can be plotted as a contour surface, as shown in Figure 6c. Note that for large values of ψ and early times, the error is small. Large errors are seen to appear for late times, but as shown in Figure 4, the amplitudes of the transient waveforms at these times have decreased to low values so this error may not be very important in problems of practical interest. However, for angles less than 10°, or so, the error increases dramatically, indicating that the approximate solution is not valid in this region.

Similar data for the horizontally polarized incident field are presented in Figures 7a–7c. In this case the agreement between the approximate and exact waveforms is much better, even for grazing angles of incidence. Evidently the approximations made for R_h were not as severe as for the vertically polarized case.

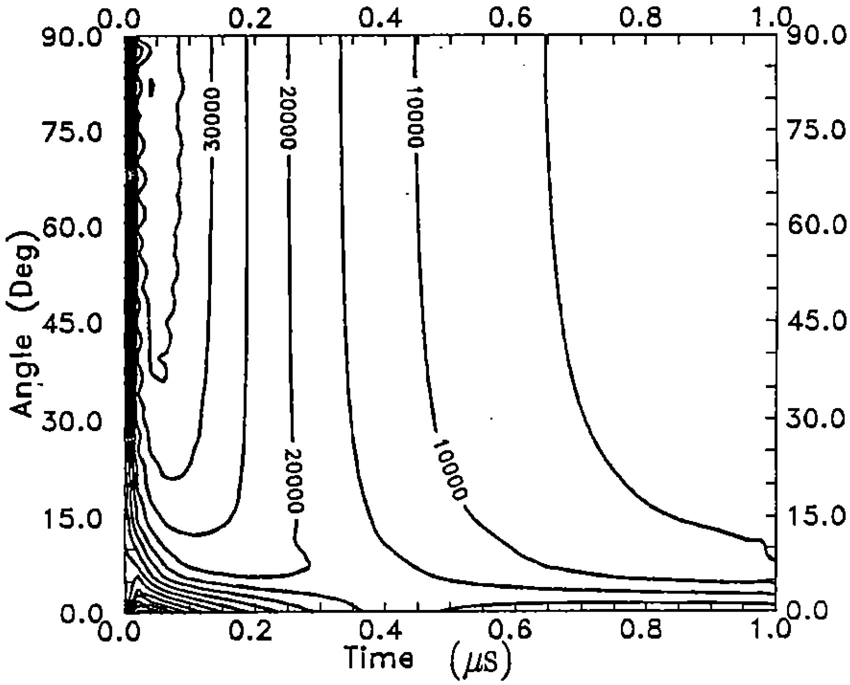


a. Vertical Polarization

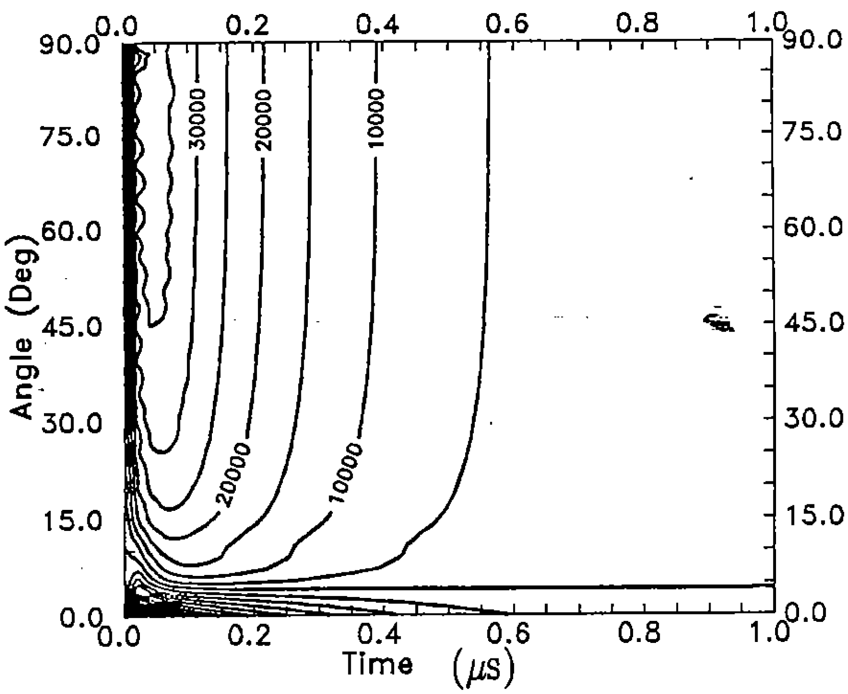


b. Horizontal Polarization

Figure 5. Actual and Approximate Reflected Field Waveforms for $\psi = 45^\circ$, $\sigma_g = 0.1$ mhos/m and $\epsilon_r = 10$, for the Bell Laboratory Waveform Incident Field.

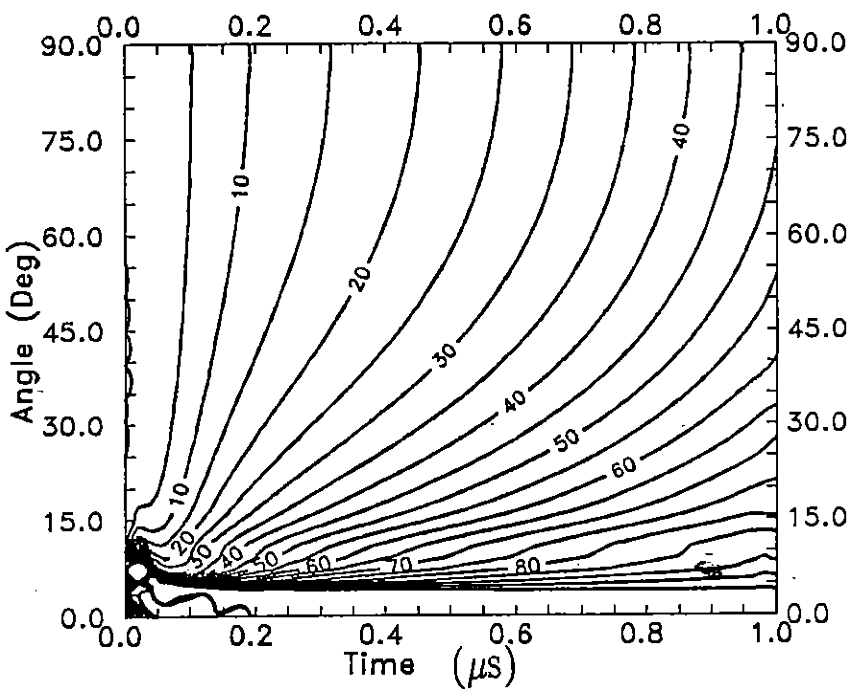


a. Actual Reflected Field



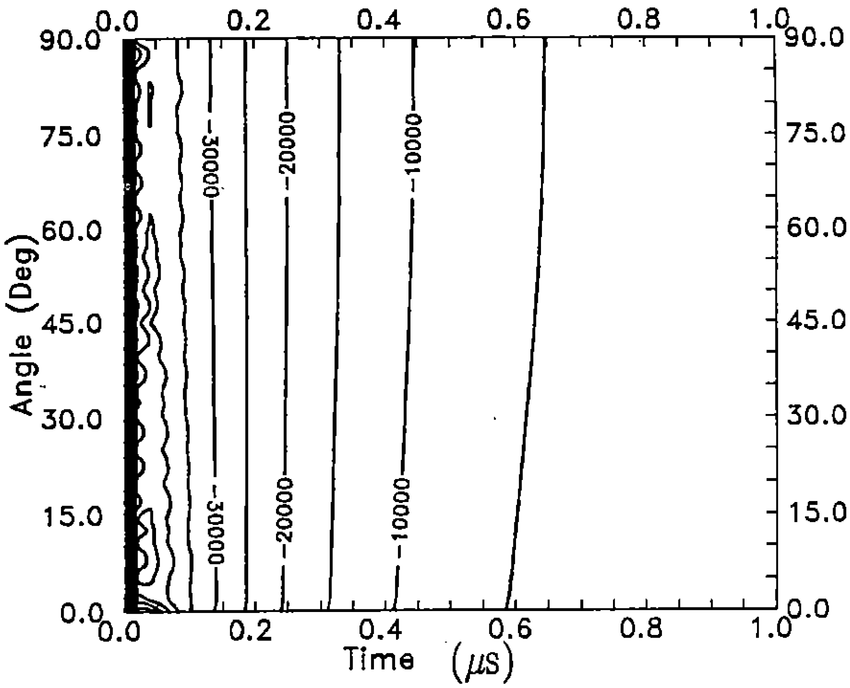
b. Approximate Reflected Field

Figure 6. Contour Plots of the Reflected Fields for Vertical Polarization, $\sigma_g = 0.01$ mhos/m, $\epsilon_r = 10$.

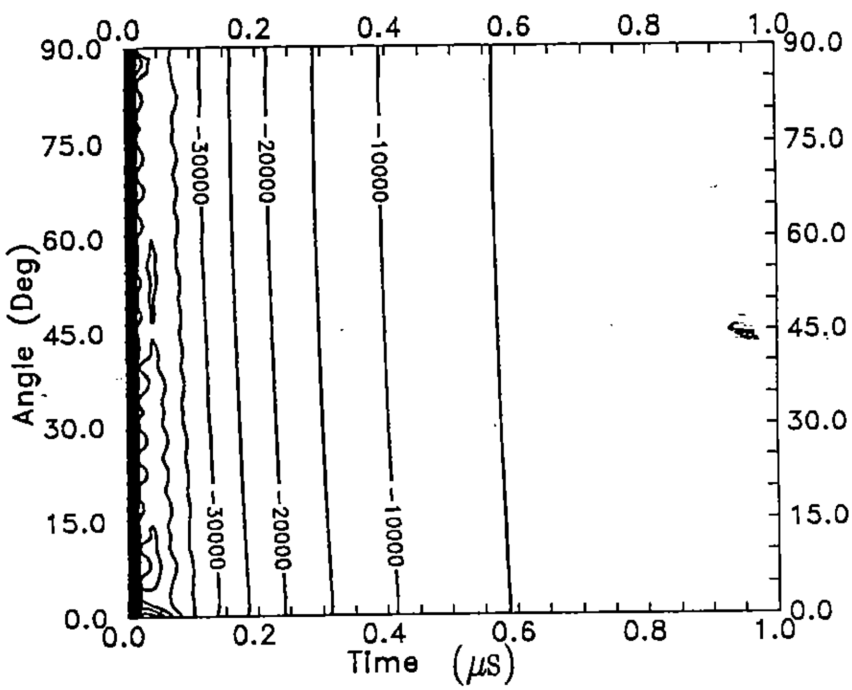


c. Percent Error of the Difference

Figure 6. Contour Plots of the Reflected Fields for Vertical Polarization, $\sigma_g = 0.01$ mhos/m, $\epsilon_r = 10$. (Concluded).

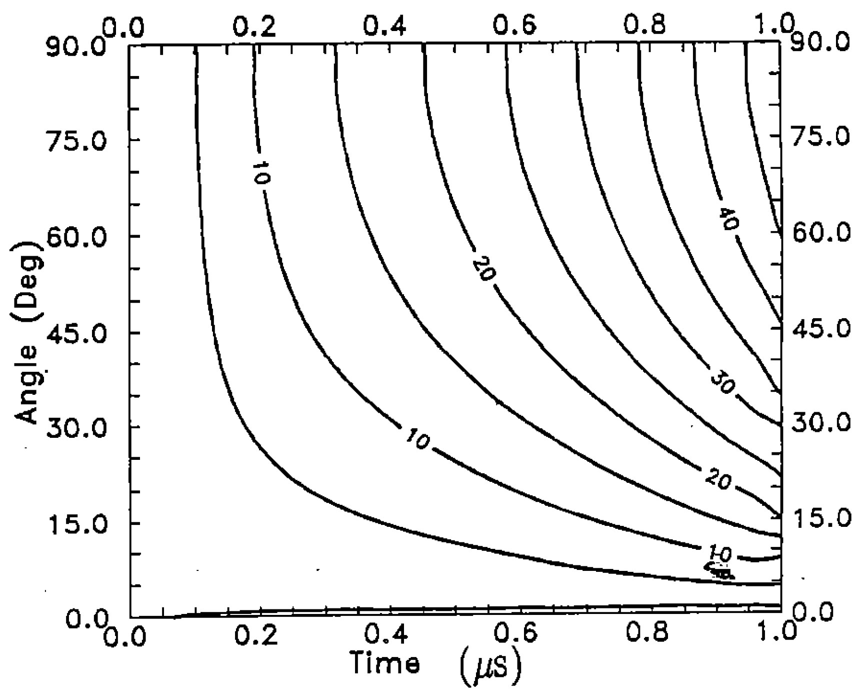


a. Actual Reflected Field



b. Approximate Reflected Field

Figure 7. Contour Plots of the Reflected Fields for Horizontal Polarization, $\sigma_g = 0.01$ mhos/m, $\epsilon_r = 10.$.



c. Percent Error of the Difference

Figure 7. Contour Plots of the Reflected Fields for Horizontal Polarization, $\sigma_g = 0.01$ mhos/m, $\epsilon_r = 10$. (Concluded).

V. Summary and Conclusions

This paper has discussed an analytical approach for computing the electromagnetic field reflected from a lossy half–space directly in the time domain. This approach requires first evaluating the impulse response of the half–space and then convolving it with the specified incident field waveform.

To obtain the impulsive reflected field, both for vertical and horizontal polarization, several approximations to the Fresnel reflection coefficients can be made to permit an analytical representation of this response. As shown in the paper, the form of this response is identical for both polarizations, and involves a Dirac δ–function and an infinite sum of modified Bessel function.

Approximations to this sum can be made and an analytical expression for the impulse response of the half–space results. Several different numerical examples of this technique have been used to illustrate the use of the method and the amount of error contained in the solution. From the results presented in this paper, it is apparent that for non–grazing angles of incidence and early times, this method provides a good approximation to the earth–reflected fields for either vertical or horizontal fields. However, for vertical polarization, the accuracy of the method is questionable for angles of incidence less than about 10°. This is due to the approximations used in simplifying the reflection coefficient. Such restrictions are not particularly evident for the horizontally polarized field. At late times, the accuracy of the method also is degraded, principally due to the truncation of the infinite sum. Additional work in developing higher order early–time terms for representing Eq.(21) is clearly desired, along with the possible development of a suitable late–time approximation to this term.

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