8/29/02, #4 lecture

2. Combinational Logic and *Boolean Algebra*

A combinational logic circuit accepts inputs whose values are 0 or 1 and produces outputs, where the output depends only on the present input. It has no memory. Combinational logic is represented by a Boolean algebra function. A number of Boolean algebras can be created. The specific one uses the two values 0 and 1 and is called switching algebra.

Logical Operations

Before we actually get into Boolean Algebra, we will discuss the basic logical operations and their notation. These basic operations are AND, OR, and NOT; associated with each is a graphic symbol.

of Boolean Algebra

The AND Operation

The notation for the AND of A and B is A·B or simply AB.

AND Operation

A	B	L=AB
0	0	0
0	1	0
1	0	0
1	1	1

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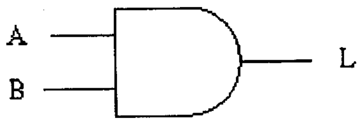

AND Gate

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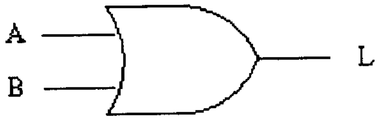
The OR Operation

The notation for the OR of A and B is A+B.

OR Operation

A	B	L=A+B
0	0	0
0	1	1
1	0	1
1	1	1



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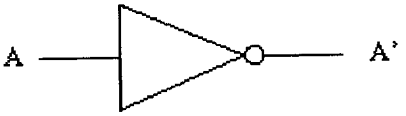
The NOT Operation

The notation for the NOT (Complement) of A is A′.

NOT Operation

A	L=A′ or $\bar{A}$
0	1
1	0

Note: Many textbooks use a bar over a variable or an expression to indicate the complement operation. The prime symbol will be used here because it is more convenient to generate in the html language. Also, the spelling of the word is complement, not compliment.



NOT Gate

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Boolean Algebra

As with any algebra, a definition of the algebra is needed. Please keep in mind that even though the symbols + and · will be used for the specific two-valued Boolean algebra known as switching algebra, Huntington's postulates, given below, apply to all Boolean algebras, independent of the number of values in the algebra. Note that a Boolean algebra can be created for any number of values as long as the number is a power of two, e.g., 4, 8, 16, etc. A 4-valued Boolean algebra might use the symbols 0, 1, alpha, and beta.

Huntington's Postulates

An algebra requires a set, S, one or more operations, in this case the operations are + and ·, and a set of postulates.

1. The algebra is *closed* with respect to + and ·. Closure means that when operating with either symbol on two values from S, no new values are created that are not in S. For example, in the tables for the AND, OR, and NOT operations presented earlier, the set is $S = \{0,1\}$ and the output values are in set S, so the algebra is closed with respect to the operations.
2. The algebra has an *identity* element with respect to + designated by 0 such that $x+0 = 0+x = x$, and an identity element with respect to · suchthat $x\cdot 1 = 1\cdot x = x$. The concept of an identity element is that when operating on any element in the set using the identity element for that operation, the result is that same original element.
3. The operations are *commutative*, i.e., $x+y = y+x$ and $x\cdot y = y\cdot x$. That is, the order is unimportant.
4. · distributes over + and + distributes over ·. Mathematically, these are expressed as:
   $x\cdot(y+z) = (x\cdot y) + (x\cdot z)$
   $x + (yz) = (x+y)\cdot(x+z)$
   Note that the first appears in normal algebra, but the second is not true in normal algebra.
5. Every element x in S has a *complement*, x', also in S such that $x + x' = 1$ and $x\cdot x' = 0$.
6. The set S contains at least two distinct elements.

The symbols + and · are the symbols most commonly used in books on computer logic. However, we also need the symbol + to mean addition. Usually, one can tell from the context which meaning is intended. If the meaning may not be clear, an effort will be made to clearly state which meaning is intended.

Switching Algebra

(Boolean Algebra for two symbols 0,1)

Of course, all the postulates of Boolean algebra apply to ~~switching algebra~~. What follows is directed toward the two-valued algebra called switching algebra. A set of theorems involving only one variable is know as a set of elementary theorems. These follow:

Elementary Theorems

AND	OR	Name	NOT

$A \cdot 0 = 0$ $A+1 = 1$ *(null elements)* $(A')' = A$ *(involution)*

$A \cdot 1 = A$ $A+0 = A$ *(identities)*

$A \cdot A = A$ $A+A = A$ *(idempotent)*

$A \cdot A' = 0$ $A+A' = 1$ *(complements)*

Perfect Induction

An algebra with just postulates is usually not sufficient. Theorems are generally needed, and these need to be proven correct. One way to prove theorems for simple cases is referred to as perfect induction. In perfect induction, one shows that the theorem is true for each possible case. Then it must be true in general.

Consider the case of the idempotent law, $A \cdot A = A$. The only two cases are $A=0$ and $A=1$. Since $0 \cdot 0 = 0$ is true and $1 \cdot 1 = 1$ is true, then $A \cdot A = A$ is true. (See the Operation table for AND.)

Consider another theorem, $A + A'B = A+B$. That this theorem is true is proven below using perfect induction.

Proof of $A + A'B = A+B$

A	B	A + A′B	A+B
0	0	$0+0' \cdot 0 = 0+1 \cdot 0 = 0+0 = 0$	$0+0 = 0$
0	1	$0+0' \cdot 1 = 0+1 \cdot 1 = 0+1 = 1$	$0+1 = 1$
1	0	$1+1' \cdot 0 = 1+0 \cdot 0 = 1+0 = 1$	$1+0 = 1$
1	1	$1+1' \cdot 1 = 1+0 \cdot 1 = 1+0 = 1$	$1+1 = 1$

Switching Algebra obeys some of the laws of ordinary algebra. Note that the precedence of (·) over (+) that exists in ordinary algebra also holds for Boolean algebra. The following are some examples that hold in ordinary algebra:

Commutative

$A \cdot B = B \cdot A$ $A+B = B+A$

Associative

$A \cdot (B \cdot C) = (A \cdot B) \cdot C$ $A+(B+C) = (A+B)+C$

Distributive

$$A\cdot(B+C) = A\cdot B+A\cdot C \qquad A+(B\cdot C) = (A+B)\cdot(A+C)$$

Not correct in ordinar

Duality

The postulates and theorems have been presented in pairs. A reason for this is that each theorem has what is called a dual. The dual is obtained by replacing + by · (· by +) and 1 by 0 (0 by 1). If a proof for some theorem exists, a dual proof in which the above replacements are made, can be used to prove the dual theorem. The existence of the duals results from the duality in the operations. For example, given $0\cdot 0 = 0$, a replacement of all 0's by 1's and the · by + results in $1+1 = 1$, which is true.

Here are a couple more examples:

$$A\cdot 1 = A \qquad A\cdot(B+C) = (A\cdot B)+(A\cdot C)$$

$$A+0 = A \qquad A+(B\cdot C) = (A+B)\cdot(A+C) \quad (d$$

In the example on the right, $(A\cdot B)+(A\cdot C)$ is normally written as simply $AB+AC$, without the dots and without the parentheses. That is what will generally be done in the future. However, if one writes $A+B\cdot A+C$, the evaluation of this expression would first evaluate $B\cdot A$, which is incorrect in this case. Therefore, the parentheses are needed to show how the evaluation is to take place.

Other Theorems

Some important theorems and their proofs are considered next.

DeMorgan's Theorem

$$(A\cdot B)' = A'+B' \qquad (A+B)' = A'\cdot B'$$

This theorem is easily proven by using perfect induction and is left to the student to do. Do not confuse this theorem with the concept of duality. Here, operations are interchanged and variables are complemented.

Absorption

The following theorem is quite useful. There is no counterpart in ordinary algebra.

$$A + AB = A \qquad A(A+B) = A$$

Proof:

$A + AB = \underset{\text{identity}}{A\cdot 1 + AB} = \underset{\text{distributive}}{A\cdot(1+B)} = \underset{\text{null}}{A\cdot 1} = \underset{\text{identity}}{A}$

Consensus

The consensus theorem is quite useful when it is difficult to see where to proceed with algebraic manipulation of a problem. It allows certain terms to be either added to or deleted from an expression.

$AB + BC + A'C = AB + A'C$

$(A+B)(B+C)(A'+C) = (A+B)(A'+C)$ (dual) $\rightarrow$

$(A+B)(B+C+A\bar{A})(\bar{A}+C)$

$= (A+B)(B+C+A)(B+C+\bar{A})$

$= (A+B)(\bar{A}+C)$ $(\bar{A}+C)$

(Absorption)

$A(A+B) = A$

Proof:

$AB + BC + A'C = AB + (A+A')BC + A'C$ (identity,complements)

$= \underline{AB} + \underline{AB}C + \underline{A'BC} + \underline{A'C}$ (distributive)

$= AB + A'C$ (absorption)

An important concept to note is that although the above theorems seem to imply only simple variables, they apply to more complex expressions as well. As a simple example, consider the expression

$(XY + W'Z) + (XY + W'Z)(RS' + T)'$

The above expression is of the same form as the absorption theorem, $A + AB$, where $A = (XY + W'Z)$ and $B = (RS' + T)'$. Thus, the expression can be reduced to $(XY + W'Z)$.

Self-Dual Theorem

The dual of the following theorem is the theorem itself. Therefore, it is referred to as being self-dual.

$(A+B)(A'+C) = A'B + AC$

Proof:

$(A+B)(A'+C) = (A+B)A' + (A+B)C$ (distributive)

$= AA' + A'B + AC + BC$ (distributive)

$= 0 + A'B + AC + BC$ (complements)

$= A'B + AC + BC$ (identity)

$= A'B + AC$ (consensus)

The product of the two terms would normally result in four product terms summed together. In this case, it results in only two product terms. The theorem can be recognized by noting that one term has a variable, in this case A, and the other has its complement, in this case A′. Note carefully which two products are retained in the result. We often will use the commutative law without identifying it. In particular, variables are placed in alphabetical order in each term for clarity. Thus, although one of the products above is actually BA′, it was written as A′B.

Extension Theorem

$A + A'B = A + B$ $\qquad$ $A(A'+B) = AB$

Extension Theorem

This theorem was proven earlier using perfect induction. We will now prove it using Boolean algebra.

Proof:

$$\begin{aligned}
A + A'B &= A(B'+B) + A'B && \text{(complements, identity)}\\
&= AB' + AB + A'B && \text{(distributive)}\\
&= AB' + AB + AB + A'B && \text{(idempotent)}\\
&= A(B'+B) + B(A+A') && \text{(distributive)}\\
&= A + B && \text{(complements, identity)}
\end{aligned}$$

Uniting Theorem

An important theorem and one that we shall be making considerable use of is the uniting theorem.

$AB + AB' = A$ $\qquad$ $(A+B)(A+B') = A$

$$= A(A+\bar{B}) + B(A+\bar{B})$$
$$= A + A\bar{B} + AB$$
$$= A + A(\bar{B}+B)$$
$$= A + A\cdot 1 = A + A = A$$

Proof:

$$\underset{\text{distributive}}{AB + AB' = A(B+B')} \quad \underset{\text{complements}}{= A\cdot 1} \quad \underset{\text{identity}}{= A}$$

Generalized Extension Theorem

The following theorem is a generalization of the extension theorem discussed above. The removal of a literal in one of the two terms amounts to an expansion or extension of that term to cover more minterms; hence the name.

$AB + A'BC = AB + BC$ $\qquad$ $(A+B)(A'+B+C) = (A+B)(B+C)$

Proof:

(1) AB + A′BC = B(A+A′C) = B(A+C) = AB + BC

distributive extension distributive

Alternate Proof:

AB + A′BC = AB+AB′C+BC) = AB+BC

consensus absorption

Thus, another way of looking at the Extension Theorem is that it is consensus followed by absorption.

(2) $(A+B)(\bar{A}+B+C)$ distributive

$= (A+B)(\bar{A}+B) + (A+B)C$.

$= B + A \cdot \bar{A} + (A+B)C$.

↑ distributive. page 5.

→ distri

$= B + 0 + (A+B)C$

$= (B + A + B)(B+C)$

$= (A+B)(B+C)$

Simplification of Expressions

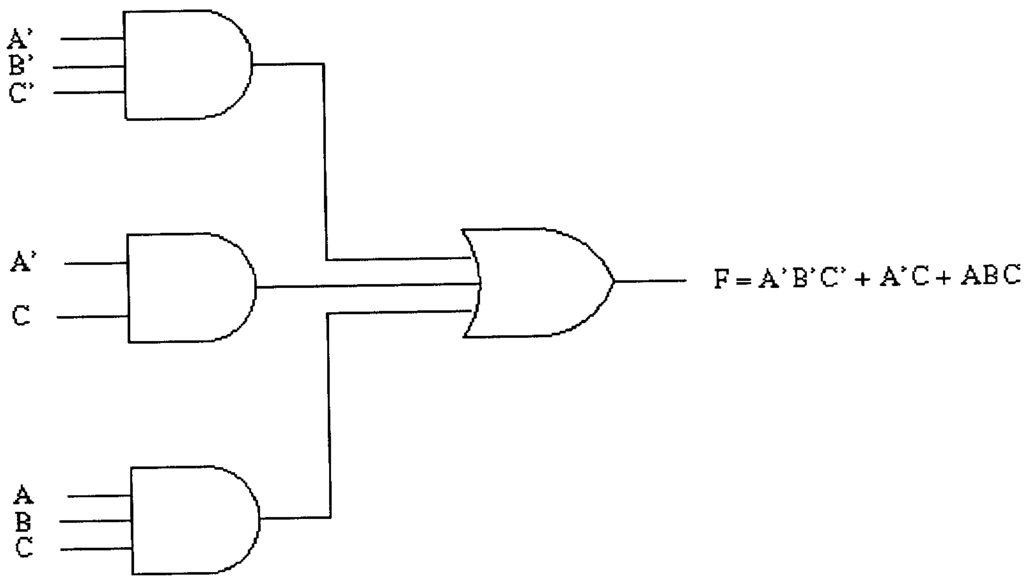
We are interested in simplifying Boolean expressions because the implementation is simpler (and less costly to build.) As an example, consider the following Boolean expression:

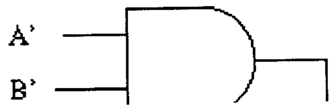
A′B′C′ + A′C + ABC = A′B′C′ + A′C(B′+B) + ABC (identity, complements)

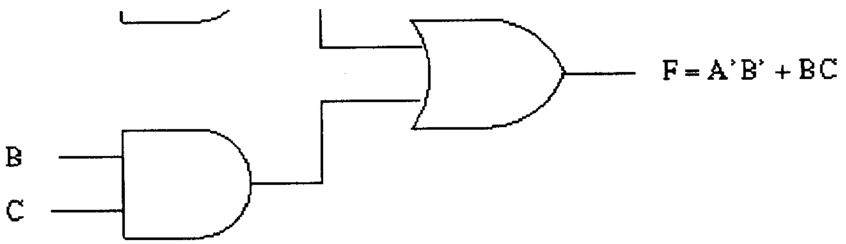
= A′B′C′ + A′B′C + A′BC + ABC (distributive)

= A′B′(C′+C) + (A′+A)BC (distributive)

= A′B′ + BC (complements, identity)







Click here to download the figure. SIMPLE in PDF format.

The original expression requires 4 gates and 11 connections. The simplified version requires 3 gates and only 6 connections. Fewer gates means less hardware (transistors) and fewer connections means fewer wires or conduction paths on an integrated circuit (IC).

simplification

A question that often arises is "How do I know when to stop?" In many cases, knowing when to stop requires that a systematic procedure be used in performing the simplification. However, there is a special case which allows one to determine that simplification can go no further. If a sum-of-products or product-of-sums expression is desired, absorption cannot be used, and there is no consensus between any two terms, then the expression cannot be simplified any further. For example, if the expression is:

A+AB = A(1+B) = A

AB+BC+A'C = AB+A'C

A'B + BC + CD'E

no further reduction can be performed on this sum-of-products expression, since no term can be removed by absorption and there is no consensus between any two terms.

Consensus Revisited

The consensus theorem can be a powerful theorem in the algebraic manipulation of Boolean expressions. The consensus theorem reads:

$$AB + BC + A'C = AB + A'C$$

The way the theorem is written, the BC can be removed from the expression on the left. Used in reverse, the BC can be added to the expression on the right. Note that a necessary condition is that two terms exist, one having a variable that is uncomplemented and the other having the same variable complemented. If this variable is removed from both terms and the product is taken of the remaining terms, the third term is obtained. In the above, removal of A leaves B and removal of A′ leaves C. The product of B and C is the third term. It is this third term, BC, that can be added to or removed from the expression.

An operation called the consensus operation (¢) is defined in order to help with recognizing those terms that can be added or deleted as follows:

$$AB \not{c}\ A'C = BC.$$

The requirement for the consensus to exist is that mentioned earlier, i.e., there must be two terms such that exactly one variable is complemented in one and uncomplemented in the other. Otherwise the consensus does not exist. Suppose that we are given the expression in the section under simplification:

$A'B'C' + A'C + ABC$

Two consensus terms can be formed:

$A'B'C' \mathbin{\text{¢}} A'C = A'B'$
$A'C \mathbin{\text{¢}} ABC = BC$

These may be added (anywhere) to the expression. The expression becomes:

$\underline{A'B'} + A'B'C' + A'C + ABC \underline{+ BC}$

The second and the fourth terms can be removed using absorption yielding:

$A'B' + A'C + BC$

Since the consensus of the first and last term ($A'B' \mathbin{\text{¢}} BC = A'C$) is the middle term, it can be deleted leaving:

$A'B' + BC$.

Approach to Minimization and Theorem Proving

Students sometimes find it difficult to prove a Boolean Theorem or to simplify an expression algebraically. One approach to this problem is to use theorems in somewhat the following order:

Use theorems that involve 0 and 1 to remove the 0's and 1's.

Use the other elementary theorems, e.g., idempotent ($a \cdot a = a$).

Use the distributive law to put the expression into terms that are ORed together.

Use absorption, uniting, and extension.

Use identity and then complements in reverse, e.g., $a = a \cdot 1 = a \cdot (b + b')$

When all else fails and you cannot find a theorem to use, try consensus followed by absortption.

Revised 8/26/98.

Complementing

Remember:

$$\overline{xy} = \bar{x} + \bar{y}$$

$$\overline{(x+y)} = \bar{x} \cdot \bar{y}$$

$$\overline{(x+y+z)} = \bar{x}\bar{y}\bar{z}$$

$$\overline{(xyz)} = \bar{x} + \bar{y} + \bar{z}$$