On the Capacity of Multi-antenna Systems in the Presence of Rician Fading

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Abstract— **The effects of Rician fading on the capacity of multi-antenna systems designed to be optimal for Rayleigh fading are investigated under the assumption that channel state information (CSI) is available only at the receiver. Capacity bounds for the multi-antenna Rician fading channel are also derived and numerical results for some representative situations are provided.**

I. INTRODUCTION

Prompted by recent results suggesting possible extraordinary capacity gains [5], [13], multiple antenna systems and space-time coding have received considerable attention as a means of providing substantial performance improvement against channel fading in wireless communications systems. In [13], it was established that when the receiver has access to perfect channel state information but not the transmitter, the capacity of a Rayleigh distributed flat fading channel will increase almost linearly with the minimum of the number of transmit and receive antennas.

In this paper we investigate how the capacity predicted with the Rayleigh fading assumption changes when the fading model is replaced by the more general Rician model. The fading coefficients are assumed to be known at the receiver but not at the transmitter. Rician fading is known to be the better model for some fading environments. For example, when there is a direct line of sight (LOS) path in addition to the multiple scattering paths, the natural fading model is Rician.

This paper is organized as follows: In Section II we will review some of the mathematical results that will be needed in the rest of the paper. Section III details the multi-antenna system we are considering and the assumptions on the fading process. Next, in Section IV we will address the general capacity problem for the Rician fading channel and obtain an upper bound for the capacity of multi-antenna systems under Rician fading. Section V will formulate the main problem that we address in this paper, namely the capacity variation of multi-antenna systems designed to achieve Rayleigh fading capacity when the fading process is replaced by the more general Rician fading. In Section V we will first give the general form of the capacity expression under the assumed conditions and then explicitly evaluate the capacity for some interesting special cases. Finally, Section VI will give some concluding remarks.

II. MATHEMATICAL PRELIMINARIES

Before turning to the problem of interest, we first establish some necessary mathematical preliminaries concerning special functions.

The gamma function for $Re\{a\} > 0$ is defined as the following Eulerian integral of the second kind [1],

$$\Gamma(a) = \int_0^\infty e^{-x} x^{a-1} dx. \tag{1}$$

We can also define the complex multivariate gamma function, $\tilde{\Gamma}_m(a)$, in terms of the univariate gamma function as [6], [7],

$$\tilde{\Gamma}_m(a) = \pi^{\frac{1}{2}m(m-1)} \prod_{k=1}^{m} \Gamma\left(a - (k-1)\right). \tag{2}$$

It follows from (2) that, $\tilde{\Gamma}_1(a) = \Gamma(a)$.

The Bessel function of matrix argument, ${}_0\tilde{F}_1$, is defined by the relation [6], [7],

$$ {}_0\tilde{F}_1(m; HH^H) = \int_{\mathcal{U}(m)} e^{\mathrm{t}r(HU+\overline{HU})}(dU) \tag{3}$$

where H is an $n \times m$ complex matrix with $n \leq m$ and $\overline{H}$ denotes the complex conjugate of the matrix H. In the above, $\mathcal{U}(m)$ is the unitary group of all $m \times m$ complex unitary matrices U, i.e. $UU^H = I_m$ and (dU) is the invariant measure on $\mathcal{U}(m)$ normalized to make the total measure unity. Here, I_m denotes the $m \times m$ identity matrix and superscript ^{H} stands for the Hermitian transpose.

It can be shown that for a scalar r, (3) reduces to

$$ {}_0\tilde{F}_1(m; r^2) = \Gamma(m) r^{-(m-1)} I_{m-1}(2r), \tag{4}$$

where the ν-th order modified Bessel function of the first kind, $I_\nu(z)$ (ν an integer) , can be defined as the series [1],

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\nu+2k}}{k!\Gamma(\nu+k+1)}. \tag{5}$$

The Bessel function of two argument matrices, S and T (both $n \times n$), can then be defined via the relation [3], [7],

$$ {}_0\tilde{F}_1(m; S, T) = \int_{\mathcal{U}(n)} {}_0\tilde{F}_1(m; SUTU^H)(dU). \tag{6}$$

III. MODEL DESCRIPTION

Consider a single user communications link in which the transmitter and receiver are equipped with N_T and N_R antennas, respectively. The discrete-time received signal in such a system can be written in matrix form as,

$$\mathbf{y}(i) = \mathbf{H}(i)\mathbf{x}(i) + \mathbf{n}(i) \tag{7}$$

where $\mathbf{y}(i)$, $\mathbf{x}(i)$ and $\mathbf{n}(i)$ are the complex N_R-vector of received signals on the N_R receive antennas, the (possibly) complex N_T-vector of transmit signals on the N_T transmit antennas and the complex N_R-vector of additive receiver noise, respectively, at symbol time i. The components of $\mathbf{n}(i)$ are independent, zero-mean, circularly symmetric complex Gaussian with independent real and imaginary parts having equal variances. The noise is also assumed to be independent with respect to the time index, and $E\{\mathbf{n}(i)\mathbf{n}(i)^H\} = \mathbf{I}_{N_R}$.

The matrix $\mathbf{H}(i)$ in (7) is the $N_R \times N_T$ matrix of complex fading coefficients. The (n_R, n_T)-th element of the matrix $\mathbf{H}(i)$ represents the fading coefficient value at time i between the n_R-th receiver antenna and the n_T-th transmitter antenna. These

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fading coefficients are assumed to be known to the receiver but not to the transmitter. The $\mathbf{H}(i)$ corresponding to each channel use is considered to be independent from that of other channel uses. This gives rise to a memoryless channel, and thus the capacity of the channel can be computed as the maximum mutual information, i.e.

$$C_{N_T,N_R} = \max_{f_{\mathbf{X}}(\mathbf{x})} \mathcal{I}(\mathbf{x}; (\mathbf{y}, \mathbf{H})) \tag{8}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the probability distribution function of the input signal vector $\mathbf{x}$.

The elements of $\mathbf{H}$ are assumed to be complex Gaussian distributed with independent real and imaginary parts each distributed as $\mathcal{N}(\mu/\sqrt{2}, \sigma^2)$. Moreover, the elements of $\mathbf{H}$ are independent of each other. Thus, the elements of $\mathbf{H}$ are independent and identically distributed (i.i.d.) complex Gaussian random variables with, $E\{(\mathbf{H})_{n_R,n_T}\} = \frac{\mu}{\sqrt{2}}(1+j)$ and $\mathtt{V}ar\{(\mathbf{H})_{n_R,n_T}\} = 2\sigma^2$ for $n_R = 1, \ldots, N_R$ and $n_T = 1, \ldots, N_T$. With these assumptions, the distribution of the magnitudes of the elements of $\mathbf{H}$ have the following Rician distribution:

$$f_R(r) = 2(1+\kappa) r e^{-(1+\kappa)r^2 - \kappa} I_0(2\sqrt{\kappa(1+\kappa)}r), \tag{9}$$

where I_0 is the zero'th order modified Bessel function of the first kind and where we have defined the Rice factor

$$\kappa = \frac{\mu^2}{2\sigma^2}, \tag{10}$$

and introduced the normalization, $\mu^2 + 2\sigma^2 = 1$. Note that (9) reduces to the Rayleigh distribution when $\kappa = 0$.

IV. Capacity of the Multi-Antenna Rician Fading Channel

It is shown in [13] that, for the Rayleigh ($\kappa = 0$) flat fading channel under the total power constraint P, i.e. $E\{\mathbf{x}^H\mathbf{x}\} \le P$, the capacity of the channel (7) is achieved when $\mathbf{x}$ is a circularly symmetric complex Gaussian with zero-mean and covariance $Q^0 = (P/N_T)\mathbf{I}_{N_T}$, and that this capacity is given by $C^0_{N_T,N_R} = E\{\log\det(\mathbf{I}_{N_R} + \frac{P}{N_T}\mathbf{H}\mathbf{H}^H)\}$. In the case of deterministic fading where matrix $\mathbf{H}$ has all its elements equal to unity ($\kappa \longrightarrow \infty$), the so called water-filling algorithm specifies the covariance matrix of this capacity achieving Gaussian distribution to be of the form $Q^\infty = \frac{P}{N_T}\Psi_{N_T}$, where Ψ_{N_T} is defined as the $N_T \times N_T$ matrix of all ones.

Moreover, it is easy to show that in a multi-antenna system where the receiver has access to the channel information, but not the transmitter, the capacity achieving transmit signal distribution is in fact Gaussian for any type of fading distribution. This capacity is given by,

$$C_{N_T,N_R} = \max_{\mathbf{Q}\ge 0, \mathrm{t}r\mathbf{Q}\le P} E_{\mathbf{H}}\{\log\det(\mathbf{H}\mathbf{Q}\mathbf{H}^H + I_{N_R})\}. \tag{11}$$

Thus, for a channel with Rician distributed fading, having a general value of κ, one would expect that the covariance matrix Q of the capacity achieving distribution to lie in between these two extremes corresponding to $\kappa = 0$ and $\kappa = \infty$. Specifically, one might expect that the covariance matrix of the capacity achieving distribution under a general power constraint in the Rician case will not necessarily be diagonal.

A. A capacity upper bound for the Rician channel

Observe that for any $\mathbf{Q}$ the matrix $\mathbf{H}\mathbf{Q}\mathbf{H}^H + I_{N_R}$ is positive definite and the function $\log\det$ is concave on the set of positive definite matrices. Thus, applying Jensen's inequality to (11) we have

$$C_{N_T,N_R} \le \max_{\mathbf{Q}\ge 0, \mathrm{t}r\mathbf{Q}\le P} \log\det(N_R\mathbf{Q}\Upsilon + I_{N_T}) \tag{12}$$

where we have used the determinant identity $\det(I + AB) = \det(I + BA)$ and introduced the notation

$$E\{\mathbf{H}^H\mathbf{H}\} = N_R\Upsilon. \tag{13}$$

It is easy to show that the $N_T \times N_T$ matrix Υ is in fact given by,

$$\Upsilon = \frac{1}{1+\kappa}\begin{bmatrix} 1+\kappa & \kappa & \ldots & \kappa \\ \kappa & 1+\kappa & \ldots & \kappa \\ \vdots & \vdots & \ddots & \vdots \\ \kappa & \kappa & \ldots & 1+\kappa \end{bmatrix}. \tag{14}$$

If we denote the eigenvalues of Υ by λ_i for $i = 1, \ldots, N_T$, then it can be shown that

$$\lambda_i = \begin{cases} \frac{1+N_T\kappa}{1+\kappa} & \text{if } i = 1 \\ \frac{1}{1+\kappa} & \text{if } i = 2, \ldots, N_T \end{cases}. \tag{15}$$

We may decompose Υ as $\Upsilon = UDU^T$, where D is the $N_T \times N_T$ diagonal matrix having the eigenvalues in (15) as its diagonal entries and U is an orthogonal matrix. Substituting this in the right hand side of (12) and letting $U^T\mathbf{Q}U = \tilde{\mathbf{Q}}$ we have

$$C_{N_T,N_R} \le \max_{\mathbf{Q}\ge 0, \mathrm{t}r\mathbf{Q}\le P} \log\det(N_R\tilde{\mathbf{Q}}D + I_{N_T}). \tag{16}$$

We observe that for any κ such that $0 \le \kappa < \infty$ the matrix Υ is non-singular and thus all the eigenvalues of Υ are non-zero. Now, it is easy to see that the right hand side is maximized by a diagonal $\tilde{\mathbf{Q}}$ and the maximizing diagonal entries are such that,

$$\tilde{\mathbf{Q}}_{ii} = \begin{cases} \frac{P}{N_T} + \frac{\kappa(1+\kappa)(N_T-1)}{N_R(1+N_T\kappa)} & \text{if } i = 1 \\ \frac{P}{N_T} - \frac{\kappa(1+\kappa)}{N_R(1+N_T\kappa)} & \text{if } i = 2, \ldots, N_T \end{cases}. \tag{17}$$

Hence, the capacity of the Rician channel under total transmit power constraint P is upper bounded, for $0 \le \kappa < \infty$, as

$$\begin{aligned} C_{N_T,N_R} &\le \log\left[1 + (N_T - 1)\kappa + \frac{N_R}{N_T}\frac{1+N_T\kappa}{1+\kappa}P\right] \\ &+ \quad (N_T - 1)\log\left[1 - \frac{\kappa}{1+N_T\kappa}\frac{N_R}{N_T}\frac{1}{1+\kappa}P\right]. \end{aligned} \tag{18}$$

Case 1: $\kappa = 0$

It is easily seen that for $\kappa = 0$, which is the Rayleigh fading case, the above bound reduces to,

$$C_{N_T,N_R} \le N_T \log\left[1 + \frac{N_R}{N_T}P\right]. \tag{19}$$

It is clear from (19) that when $N_R = N_T$ the capacity upper bound is a linear function of N_T. In fact, it was shown in [13] that in this case the capacity can be approximated by a linear function of N_T asymptotically for large numbers of antennas.

Case 2: $N_T = 1$

When $N_T = 1$, the capacity upper bound in (18) becomes,

$$C_{1,N_R} \le \log\left[1 + N_RP\right]. \tag{20}$$

In [13] it was shown that the capacity of the Rayleigh fading channel in this case is asymptotic to $\log(1 + N_RP)$ for large N_R. Thus, we see that again in the Rician channel the capacity is in fact upper bounded by $\log(1 + N_RP)$ for any number of receiver antennas N_R.

V. CAPACITY OF THE RICIAN CHANNEL WITH SIGNALS DESIGNED FOR RAYLEIGH DISTRIBUTED FADING

The aim of this section is to investigate the variation of the capacity results obtained for a Rayleigh distributed fading process in [13], when the fading process is actually replaced by the more general Rician distribution. Thus, henceforth we will restrict attention to the case in which the covariance matrix $\mathbf{Q}$ of $\mathbf{x}$ is a scaled version of the $N_T \times N_T$ identity matrix. As noted earlier this is the form of the covariance matrix of $\mathbf{x}$ that achieves the capacity in the Raleigh fading case under the same input power constraint. Invoking the transmit power constraint, then we have that, $\mathbf{x}$ is circularly symmetric, zero-mean complex Gaussian with $E\{\mathbf{x}\mathbf{x}^H\} = \mathbf{Q} = \frac{P}{N_T}\mathbf{I}_{N_T}$ and the capacity of the channel under Rician fading is given by,

$$C_{N_T,N_R} = E_{\mathbf{H}}\left\{\log\det\left(\frac{P}{N_T}\mathbf{H}\mathbf{H}^H + \mathbf{I}_{N_R}\right)\right\} \tag{21}$$

where the expectation is with respect to the probability distribution function of $\mathbf{H}$, which can be written as [3], [7],

$$f_{\mathbf{H}}(H) = \frac{1}{(\pi)^{N_T N_R}|\Sigma|^{N_T}} e^{-\mathrm{tr}\left[\Sigma^{-1}(H-M)(H-M)^H\right]} \tag{22}$$

where Σ is the Hermitian covariance matrix of the columns (assumed to be the same for all columns) of $\mathbf{H}$ and $M = E\{\mathbf{H}\}$. For the assumed model, we have $\Sigma = 2\sigma^2\mathbf{I}_{N_R}$ and $M = \frac{\mu}{\sqrt{2}}(1+j)\Psi_{N_R,N_T}$ where Ψ_{N_R,N_T} is the $N_R \times N_T$ matrix of ones. Let us define $m = \max\{N_R, N_T\}$, $n = \min\{N_R, N_T\}$ and

$$\mathbf{W} = \left\{ \begin{array}{ll} \mathbf{H}\mathbf{H}^H & \text{if } N_R < N_T \\ \mathbf{H}^H\mathbf{H} & \text{if } N_R \geq N_T \end{array} \right. . \tag{23}$$

Hence $\mathbf{W}$ is always an $n \times n$ square matrix. Then, we have from (21) and (23) that

$$C_{N_T,N_R} = E_{\mathbf{W}}\left\{\log\det\left(\frac{P}{N_T}\mathbf{W} + \mathbf{I}_n\right)\right\}. \tag{24}$$

It is well-known that when $\mathbf{H}$ is distributed as in (22), the distribution of $\mathbf{W}$ is given by the non-central Wishart distribution [3], [7],

$$f_{\mathbf{W}}(W) = e^{-\mathrm{tr}\left[\Sigma^{-1}MM^H\right]} {}_0\tilde{F}_1(m; \Sigma^{-1}MM^H\Sigma^{-1}W) \frac{e^{-\mathrm{tr}\Sigma^{-1}W}|W|^{m-n}}{\tilde{\Gamma}_n(m)|\Sigma|^m} \tag{25}$$

Note that in (25) we have assumed, without loss of generality, that $\mathbf{W} = \mathbf{H}\mathbf{H}^H$. We will continue with this assumption throughout unless stated otherwise.

Since $\mathbf{W}$ in (24) is an $n \times n$ Hermitian matrix, if we denote its (non-negative) eigenvalues as $\lambda_1, \lambda_2, \ldots, \lambda_n$, then we can write $\mathbf{W} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$, where $\mathbf{U}$ is unitary and $\mathbf{\Lambda}$ is the diagonal matrix of eigenvalues. Since $\det\left(\frac{P}{N_T}\mathbf{W} + I_n\right) = \prod_{i=1}^n \left(1 + \frac{P}{N_T}\lambda_i\right)$, the capacity in (24) can be given in terms of the eigenvalue distribution of the non-central Wishart distributed matrix $\mathbf{W}$ as

$$C_{N_T,N_R} = E_{\lambda_1,\ldots,\lambda_n}\left\{\sum_{i=1}^n \log\left(1 + \frac{P}{N_T}\lambda_i\right)\right\}. \tag{26}$$

A. Capacity in the Limit of Large N_T

Before attempting to evaluate the exact capacity in (21), it is instructive to investigate the behavior of the above capacity in the limit as the number of transmit antennas increases without bound. For a fixed N_R, by the strong law of large numbers we have almost surely,

$$\lim_{N_T \longrightarrow \infty} \frac{1}{N_T}\mathbf{H}\mathbf{H}^H = \Upsilon \tag{27}$$

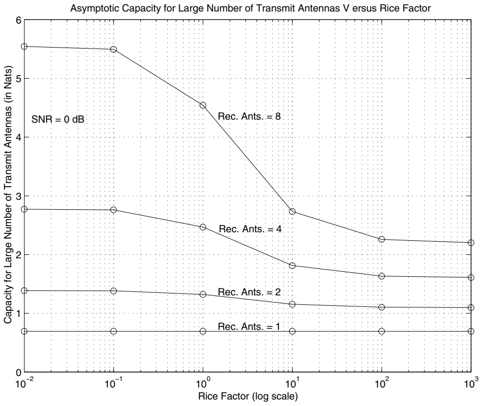

Fig. 1. Asymptotic Capacity for Large Number of Transmit Antennas Versus Rice Factor.

where the matrix Υ is defined in (14) (here it is taken to be an $n \times n$ matrix).

Hence, for a fixed number of receiver antennas when the number of transmit antennas becomes very large, the capacity of the channel in the presence of Rician fading is given by,

$$C_{\infty,N_R} = (N_R - 1)\log\left[1 + \frac{P}{1+\kappa}\right] + \log\left[1 + (N_R\kappa + 1)\frac{P}{1+\kappa}\right]. \tag{28}$$

The following two cases illustrate the dependence of the asymptotic capacity on the Rice factor κ:

$$\lim_{\kappa \longrightarrow 0} C_{\infty,N_R} = N_R\log(1+P) \tag{29}$$

$$\lim_{\kappa \longrightarrow \infty} C_{\infty,N_R} = \log(1+N_R P) \tag{30}$$

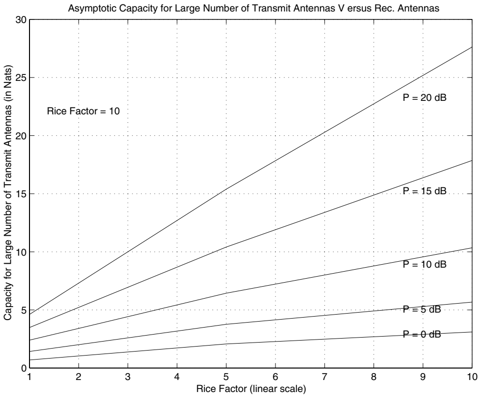

Fig. 2. Asymptotic Capacity for Large Number of Transmit Antennas Versus Number of Receiver Antennas.

Note that (29) is the asymptotic capacity of the Rayleigh channel given by Telatar in [13]. It is easily seen that C_{∞,N_R} in (28) is a monotonically decreasing function in κ for $\kappa > 0$ and $N_R > 1$ and is constant for $N_R = 1$. Thus, we observe that the Rician fading environment will degrade the capacity of a system that is designed to achieve the Rayleigh channel capacity (recall that we have constrained the input signal covariance to be a scaled identity matrix), when the number of transmit antennas is very large. However, this does not necessarily mean that the capacity of the Rician fading channel is less than the Rayleigh fading channel. In fact, in the deterministic case, which is the limiting case of Rician fading with $\kappa \longrightarrow \infty$, the capacity of the channel, as achieved by the water filling algorithm, can easily shown to be $\log(1 + N_R N_T P)$, for any N_R and N_T.

Figure 1 shows the dependence of the asymptotic capacity of the Rician fading channel (in the limit of a large number of transmit antennas) on the Rice factor κ for $P = 0dB$.

Figure 2 shows the asymptotic capacity of the Rician fading channel in the limit of large number of transmit antennas versus number of receiver antennas N_R for $\kappa = 10$. This plot shows the almost linear dependence of the asymptotic Rician capacity on the number of receiver antennas (which is the smaller of N_T and N_R in this case), similar to the previously established linear dependence for the Rayleigh fading environment ([5], [13]).

B. Special Case: $\min\{N_R, N_T\} = 1$

In this section we will evaluate (21) for the special case $n = \min\{N_R, N_T\} = 1$. In this special case, the covariance matrix Σ reduces to a scalar, i.e. $\Sigma = 2\sigma^2$. Thus, the distribution of $\mathbf{W}$ in (25) (which is a scalar) can be written as,

$$f_{\mathbf{W}}(W) = e^{-\kappa m}{}_0\tilde{F}_1\left(m; m\frac{\mu^2}{4\sigma^2}W\right)\frac{e^{-\frac{W}{2\sigma^2}}|W|^{m-1}}{\tilde{\Gamma}_1(m)|2\sigma^2|^m}. \qquad (31)$$

Using (4) in (31), we have that,

$$f_{\mathbf{W}}(W) = \frac{e^{-\kappa m}}{(2\sigma^2)^m}e^{-\frac{W}{2\sigma^2}}|W|^{m-1}\left(\omega\sqrt{W}\right)^{-(m-1)}I_{m-1}(2\omega\sqrt{W}), \qquad (32)$$

where in (32) we have defined $\omega^2 = m\kappa(1+\kappa)$. From (24) the capacity in this special case is,

$$C_{N_T,N_R} = \int_0^\infty \log\left(1 + \frac{P}{N_T}W\right)f_{\mathbf{W}}(W)dW, \qquad (33)$$

where $f_{\mathbf{W}}(W)$ is given in (32). As noted by Telatar in [13] for the Rayleigh case, again from (33) we observe that the capacity is not symmetric in N_R and N_T also in the Rician case. Thus, we have two cases to consider as below.

B.1 $N_R \geq N_T = 1$

From (32) and (33), the capacity of the Rician fading channel in this case is,

$$C_{1,N_R} = \frac{1}{\Gamma(N_R)}\int_0^\infty \log(1 + PW)\, W^{N_R-1}e^{-W}\psi(W, N_R)dW. \qquad (34)$$

where we have introduced the function $\psi(W,m)$,

$$\begin{aligned}\psi(W,m) &= \frac{\Gamma(m)}{[m\kappa(1+\kappa)]^{\frac{1}{2}(m-1)}}\left(\frac{1+\kappa}{e^\kappa}\right)^m e^{-\kappa W} \\ &\times\ W^{-\frac{1}{2}(m-1)}I_{m-1}(2\sqrt{m\kappa(1+\kappa)W}) \qquad (35)\end{aligned}$$

For comparison we note that according to [13], the capacity of the Rayleigh channel in this case is,

$$C^0_{1,N_R} = \frac{1}{\Gamma(N_R)}\int_0^\infty \log(1 + PW)\, W^{N_R-1}e^{-W}dW. \qquad (36)$$

Figure 3 shows the capacity of the Rician fading channel in this special of $N_T = 1$, against the number of receiver antennas for $\kappa = 10$. We have also included on the same graph the corresponding capacity curves for the Rayleigh fading channel ($\kappa = 0$ case). We observe that the capacity of the Rician channel is greater than the capacity of the Rayleigh channel and it can also be shown that this capacity gap increases with the increasing value of κ. We can also observe from Fig. 3 that the capacity gap is prominent for smaller numbers of receiver antennas and, as $N_T \longrightarrow \infty$, the two capacities converge to the same value. However, again we should recall that this higher capacity of the Rician channel was obtained by using a signal distribution which is not necessarily the capacity achieving optimal distribution for this particular channel. These results thus suggest that we may be able to gain even greater capacity improvements by exploiting the *Rician-ness* inherent in the fading process.

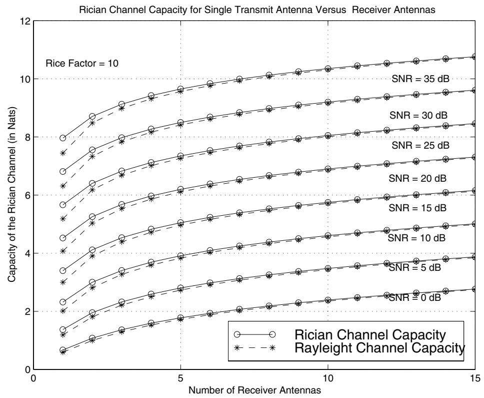

Fig. 3. Rician Channel Capacity for Single Transmit Antenna Versus Receiver Antennas.

B.2 $N_T \geq N_R = 1$

The capacity of the Rician fading channel in this case is,

$$C_{N_T,1} = \frac{1}{\Gamma(N_T)}\int_0^\infty \log\left(1 + \frac{P}{N_T}W\right)W^{N_T-1}e^{-W}\psi(W, N_T)dW \qquad (37)$$

where $\psi(W, N_T)$ is given by (35).

Again, note that the capacity of the Rayleigh channel in this case is [13],

$$C^0_{N_T,1} = \frac{1}{\Gamma(N_T)}\int_0^\infty \log\left(1 + \frac{P}{N_T}W\right)W^{N_T-1}e^{-W}dW. \qquad (38)$$

Figure 4 plots the capacity of the Rician fading channel in the special case of $N_R = 1$, against the number of transmit antennas for $\kappa = 10$. Again, we observe that the capacity of the Rician channel is greater than the capacity of the Rayleigh channel and also we can show that this capacity gap increases with the increasing value of κ before finally converging to the same value for large N_T. From Fig. 4 we note that for a smaller number of transmit antennas the capacity gap is significant. Since in practical communications systems the interest is in gaining as much capacity improvement using smaller numbers of transmit and receive antennas, these results suggest that it is worth

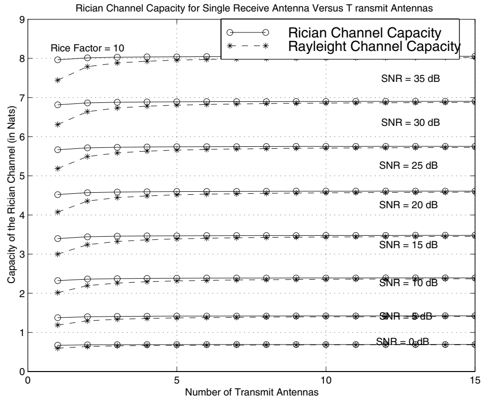

Fig. 4. Rician Channel Capacity for a Single Receiver Antenna Versus Transmit Antennas.

attempting to find optimal signal distributions for the Rician fading channels.

Finally, note that using the series representation (5) of the modified Bessel function, it is straightforward to show that, $\lim_{\kappa \longrightarrow 0} \psi(W, m) = 1$. Thus, in the limit $\kappa \longrightarrow 0$, (34) and (37) reduce to (36) and (38), respectively, as one would expect.

C. General Capacity Expression for the Rician Channel

In order to compute the capacity of the Rician fading channel for an arbitrary number of transmit/receiver antennas, as given in (26), we need to find the latent root distribution of the non-central Wishart distributed matrix $\mathbf{W}$ ([3], [7], [8], [9]). We have the following result from [7].

Theorem 1: If $\mathbf{W}$ has the non-central Wishart distribution given in (25), then the distribution of the latent roots $\hat{\mathbf{\Lambda}} = diag(\hat{\lambda}_1, \ldots, \hat{\lambda}_n)$ of $|W - \hat{\lambda}\Sigma| = 0$ depends only on the latent roots $\hat{\Omega} = diag(\hat{\omega}_1, \ldots, \hat{\omega}_n)$ of $|MM^H - \hat{\omega}\Sigma| = 0$, and is given by

$$\begin{aligned} f_{\hat{\mathbf{\Lambda}}}(\hat{\Lambda}) &= e^{-\mathrm{tr}\hat{\Omega}} {}_0\tilde{F}_1(m; \hat{\Omega}, \hat{\mathbf{\Lambda}}) \frac{\pi^{n(n-1)}}{\tilde{\Gamma}_n(m)\tilde{\Gamma}_n(n)} e^{-\mathrm{tr}\hat{\mathbf{\Lambda}}} \\ &\quad \times |\hat{\mathbf{\Lambda}}|^{m-n} \prod_{i<j}^{n} (\hat{\lambda}_i - \hat{\lambda}_j)^2. \end{aligned} \tag{39}$$

Due to the scaled identity matrix structure of the covariance matrix Σ in our case, it is easily seen that the latent root distribution $\mathbf{\Lambda}$ of the matrix $\mathbf{W}$ can be obtained from the distribution given in (39) by noting that $2\sigma^2\hat{\lambda}_i = \lambda_i$ and $2\sigma^2\hat{\omega}_i = \omega_i$ for $i = 1, \ldots n$, and thus $2\sigma^2\hat{\mathbf{\Lambda}} = \mathbf{\Lambda}$ and $2\sigma^2\hat{\Omega} = \Omega$, where we have denoted the latent root matrix of MM^H by Ω.

From the definition of M above, we see that $MM^H = m\mu^2\Psi_n$, where Ψ_n is the $n \times n$ square matrix of ones. It is easy to show that the only non-zero eigenvalue of the matrix Ψ_n is equal to n. Hence the only non-zero eigenvalue of the matrix MM^H is equal to $mn\mu^2$, and thus $\omega_1 = mn\mu^2$ and $\omega_i = 0$ for $i = 2, \ldots, n$. Hence, by applying the above change of variables to (39) and using the definition of the Rice factor κ, we get the required eigenvalue distribution of the matrix $\mathbf{W}$ as,

$$\begin{aligned} f_{\lambda_1,\ldots,\lambda_n}(\lambda_1, \ldots, \lambda_n) &= (1+\kappa)^{mn} e^{-mn\kappa} {}_0\tilde{F}_1(m; (1+\kappa)\Omega, (1+\kappa)\mathbf{\Lambda}) \\ &\quad \times \frac{\pi^{n(n-1)}}{\tilde{\Gamma}_n(m)\tilde{\Gamma}_n(n)} e^{-(1+\kappa)\sum_{i=1}^{n}\lambda_i} \left(\prod_{i=1}^{n}\lambda_i\right)^{m-n} \prod_{i<j}^{n}\left(\lambda_i - \lambda_j\right)^2 \end{aligned} \tag{40}$$

where the diagonal elements of the matrix Ω are the ω_i's.

Therefore, a general capacity expression for the Rician fading channel for an arbitrary number of transmit/receiver antennas is given, from (26), by

$$C_{N_T,N_R} = \int_{\lambda_1,\ldots,\lambda_n} \sum_{i=1}^{n} \log\left(1 + \frac{P}{N_T}\lambda_i\right) f_{\lambda_1,\ldots,\lambda_n}(\lambda_1, \ldots, \lambda_n) d\lambda_1 \ldots d\lambda_n,$$

where $f_{\lambda_1,\ldots,\lambda_n}(\lambda_1, \ldots, \lambda_n)$ is given in (40) above.

C.1 Special Case: $n = \min\{N_R, N_T\} = 2$

We may simplify the probability distribution function $f_{\lambda_1,\ldots,\lambda_n}(\lambda_1, \ldots, \lambda_n)$ in (40) for the special case of $n = 2$ and thus provide an explicit integral for the capacity of the Rician fading channel. It has been shown that (see Eq. 1.13 of [11]),

$$\begin{aligned} &{}_0\tilde{F}_1\left(m; \begin{bmatrix} s_1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}\right) \\ &= \sum_{k=0}^{\infty} \frac{\left[s_1(r_1 r_2)^{\frac{1}{2}}\right]^k}{(m)_k k!} P_k\left(\frac{r_1 + r_2}{2(r_1 r_2)^{\frac{1}{2}}}\right), \end{aligned} \tag{41}$$

where the Legendre polynomial $P_k(x)$ is defined as [11]

$$P_k(x) = x^k {}_2F_1\left(-\frac{k}{2}, -\frac{k}{2} + \frac{1}{2}; 1; \frac{x^2 - 1}{x^2}\right), \tag{42}$$

and the usual scalar hypergeometric function is [4], [7], [12]

$${}_2F_1(a, b; c; x) = \sum_{q=0}^{\infty} \frac{(a)_q (b)_q}{(c)_q} \frac{x^q}{q!} \tag{43}$$

with the hypergeometric coefficient $(a)_k$ defined to be the product $(a)_k = a(a+1)\ldots(a+k-1)$. Thus,

$${}_2F_1(-\frac{k}{2}, -\frac{k}{2} + \frac{1}{2}; 1; x) = \sum_{q=0}^{\infty} \left(-\frac{k}{2}\right)_q \left(-\frac{k}{2} + \frac{1}{2}\right)_q \frac{x^q}{(q!)^2}. \tag{44}$$

Particularizing (41) for our case with $s_1 = mn\frac{\mu^2}{2\sigma^2} = \kappa mn = 2m\kappa$, $r_1 = (1+\kappa)\lambda_1$ and $r_2 = (1+\kappa)\lambda_2$ we have the required pdf to specify the explicit integral representation for the capacity in this special case of $n = 2$,

$$\begin{aligned} f_{\lambda_1,\lambda_2}(\lambda_1, \lambda_2) &= \frac{\pi^2(1+\kappa)^{2m}}{\tilde{\Gamma}_2(m)\tilde{\Gamma}_2(2)} e^{-(1+\kappa)\sum_{i=1}^{2}\lambda_i} \left(\prod_{i=1}^{2}\lambda_i\right)^{m-2} e^{-2m\kappa} \\ &\quad \times \prod_{i<j}^{2}\left(\lambda_i - \lambda_j\right)^2 \sum_{k=0}^{\infty} \frac{\left[2m\kappa(1+\kappa)(\lambda_1\lambda_2)^{\frac{1}{2}}\right]^k}{(m)_k (k!)^2} P_k\left(\frac{\lambda_1 + \lambda_2}{2(\lambda_1\lambda_2)^{\frac{1}{2}}}\right) \end{aligned} \tag{45}$$

VI. CONCLUSIONS

We have investigated the effect of Rician fading on the capacity of multi-antenna systems designed to be optimal for Rayleigh fading under the assumption of receiver only CSI. We have given an integral expression for the capacity of a general system having an arbitrary number of transmit/receive antennas. In some special cases, we were able to numerically evaluate these capacity expressions. We have also obtained an upper bound for the capacity of the Rician fading channel, in the general case.

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