Properties of Sinusoids

Relationship:

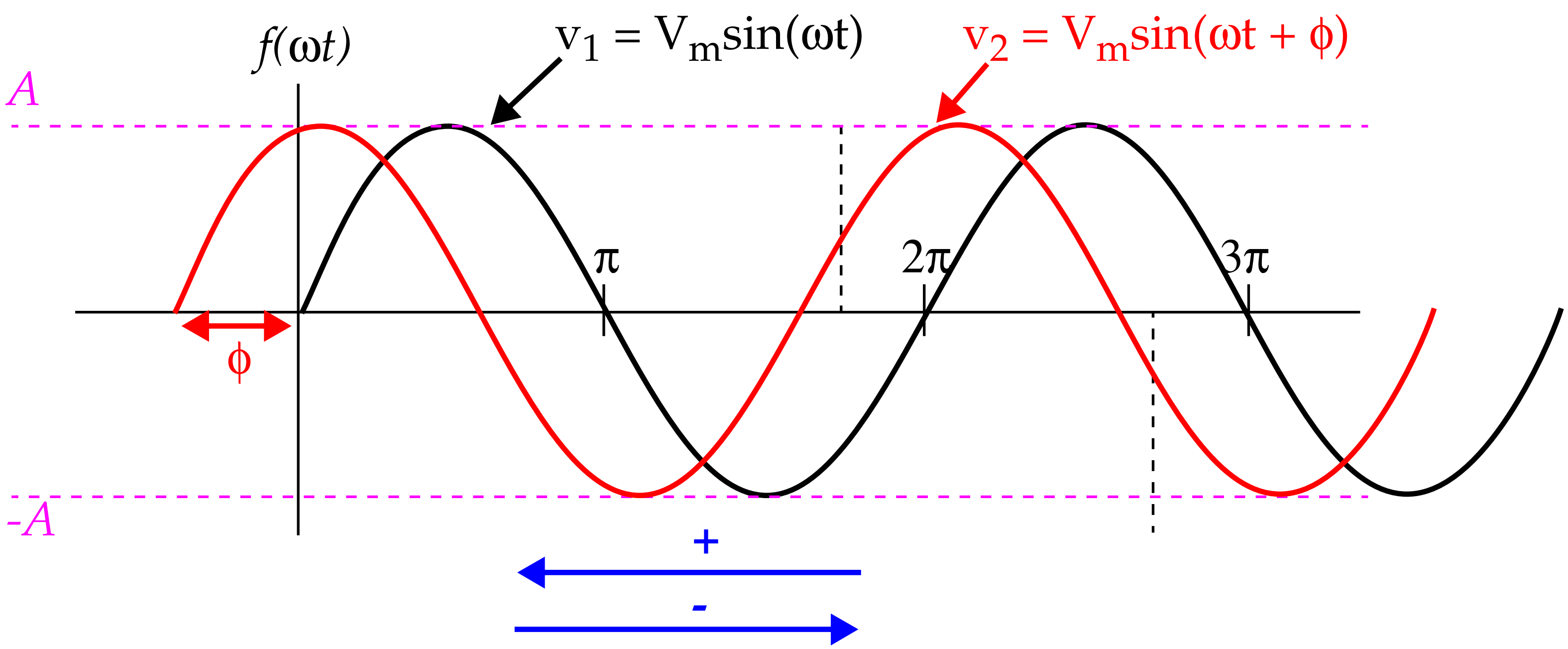
$$\omega = 2\pi f$$

$$f(t) = V_m \cos(\omega t + \phi)$$

ω is the **angular frequency** in radians/sec

ϕ is the **argument** or **phase angle** (in radians)

Period $T = 2\pi/\omega = 1/f$

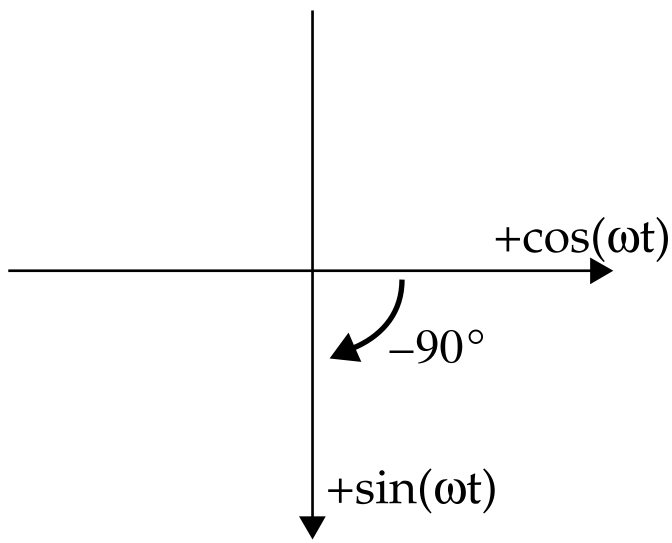


Properties of Sinusoids

Trig identities

$$\sin(\omega t + 90°) = \cos(\omega t)$$
$$\sin(\omega t - 90°) = -\cos(\omega t)$$
$$\cos(\omega t + 90°) = -\sin(\omega t)$$
$$\cos(\omega t - 90°) = \sin(\omega t)$$
$$\sin(\omega t + 180°) = -\sin(\omega t)$$
$$\cos(\omega t + 180°) = -\cos(\omega t)$$
$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

Don't confuse with complex plane described below



Adding two sinusoids of same ω

$$A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \theta)$$

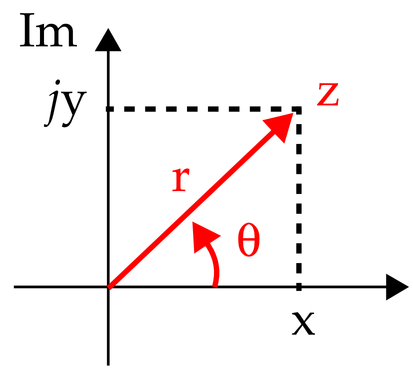
$$C = \sqrt{A^2 + B^2} \qquad \theta = \tan^{-1}\left(\frac{B}{A}\right)$$

Complex Numbers

Complex numbers: rectangular form

$$z = x + jy$$

$$j = \sqrt{-1}$$



$$\frac{1}{j} = -j$$

$$j^2 = -1$$

$$j^3 = -j$$

$$j^4 = 1$$

Polar form

$$z = |z|\angle\theta \longrightarrow z = x + jy = r\angle\theta = \underbrace{r\cos(\theta) + jr\sin(\theta)}$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

rect-to-polar

or

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

polar-to-rect

$$z = x + jy \longrightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = -x + jy \longrightarrow \theta = 180° - \tan^{-1}\left(\frac{y}{x}\right)$$

(-, -) → 180 + (+, -) → 360 -

Complex Numbers

Complex conjugate

$$z^{*} = x - jy = r\angle{-\theta} = e^{-j\theta}$$ (Last form is exponential form)

Sum/difference: use rectangular form. Mult/div: use polar form

$$z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2)$$ $$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

Exponential form

$$z = re^{j\theta}$$ (similar to polar form)

from Euler's formula

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \longrightarrow \cos(\theta) = Re(e^{j\theta})$$

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta) \longrightarrow \sin(\theta) = Im(e^{j\theta})$$

Adding: $$\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

Subtracting: $$\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

Complex Numbers

Useful identities

Assume $z = x + jy = r\angle\theta$

$$zz^* = x^2 + y^2 = r^2$$

$$\sqrt{z} = \sqrt{x + jy} = \sqrt{r}e^{j\theta/2} = \sqrt{r}\angle(\theta/2)$$

$$z^n = (x + jy)^n = r^n\angle n\theta = r^n e^{jn\theta} = r^n(\cos(n\theta) + j\sin(n\theta))$$

$$\ln(re^{j\theta}) = \ln(r) + \ln e^{j\theta} = \ln(r) + j\theta$$

$$e^{j\pi} = -1$$

$$e^{j2\pi} = e^{j0} = 1$$

$$e^{j\pi/2} = e^{j90^\circ} = j$$

$$e^{j\pi/4} = e^{j45^\circ} = \sqrt{j} = \frac{(1 + j)}{\sqrt{2}}$$

$$e^{-j\pi/2} = e^{j270^\circ} = -j$$

(IMPORTANT IDENTITIES FOR US)

$$\sqrt{2j} = (1 + j)$$

$$\sqrt{\frac{1}{j}} = \sqrt{-j} = \frac{(1 - j)}{\sqrt{2}}$$

$$Re(e^{(\alpha + j\omega)t}) = Re(e^{\alpha t}e^{jwt}) = e^{\alpha t}\cos(\omega t)$$

$$Im(e^{(\alpha + j\omega)t}) = Im(e^{\alpha t}e^{jwt}) = e^{\alpha t}\sin(\omega t)$$

Phasors

Sines and cosines can be expressed in terms of *phasors*.

A phasor is a complex number that represents the amplitude and phase of a sinusoid.

Phasors based on Euler's identity

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \quad \text{and} \quad e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

So *cos*(θ) and *sin*(θ) represent the Re and Im parts of $e^{j\theta}$.

We can write a sinusoid

$$v(t) = V_m \cos(\omega t + \phi) = Re(V_m e^{j(\omega t + \phi)})$$

$$v(t) = Re(V_m e^{j\phi} e^{j\omega t})$$

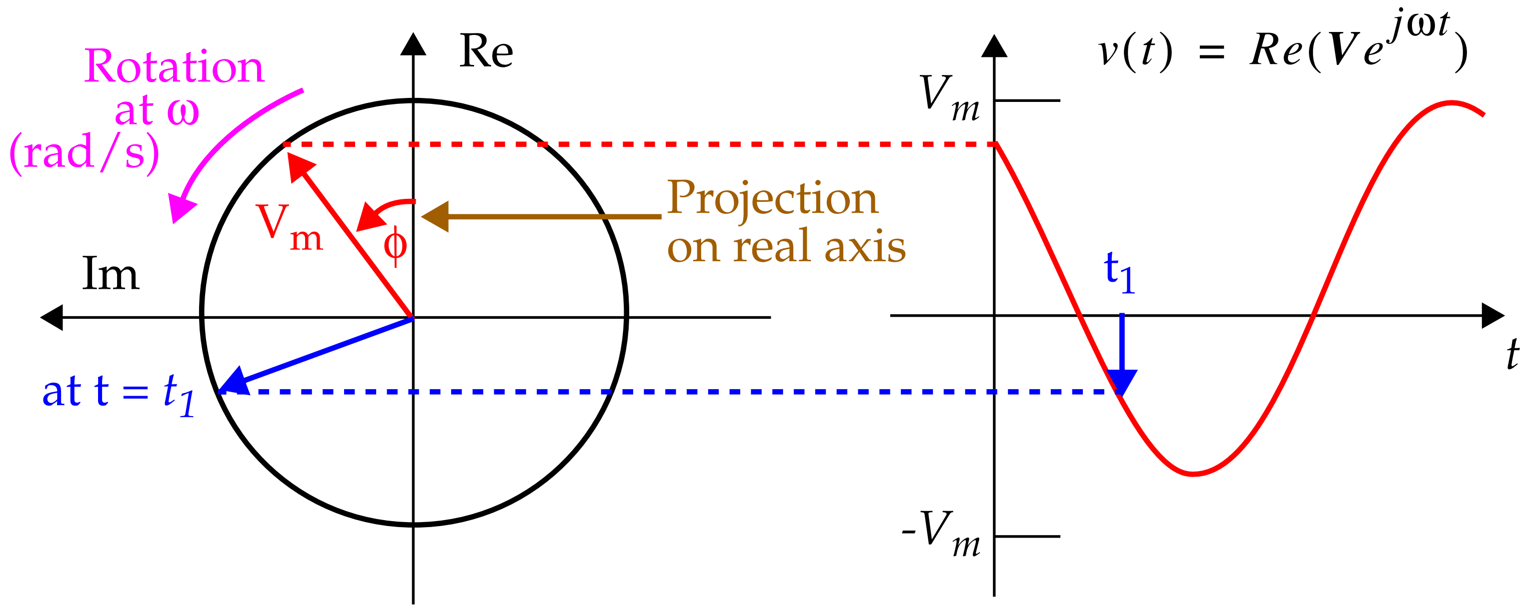
Arg includes phase and time part

Defining the phasor as the magnitude and the non-time part of the arg

$$v(t) = Re(\boldsymbol{V} e^{j\omega t}) \qquad \boldsymbol{V} = V_m e^{j\phi} = V_m \angle\phi$$

Phasors

The plot a rotating phasor: *sinor* $\mathbf{V}e^{j\omega t}$



Note value of sinor at $t = 0$ is phasor **V** of the sinusoid v(t).

NOTE: the term $e^{j\omega t}$ is implicitly present whenever a sinusoid is expressed as a phasor.

This is where the third parameter, ω, is realized -- in the time part.

It is the third constant associated with the sinusoid.

Phasors

To obtain the **sinusoid** associated with a **phasor**, multiply phasor **V** by $e^{j\omega t}$ and take the *Re* part.

$$v(t) = Re(\mathbf{V}e^{j\omega t}) = Re(V_m e^{j(\omega t + \phi)}) = Re(V_m \cos(\omega t + \phi) + jV_m \sin(\omega t + \phi))$$

$$v(t) = V_m \cos(\omega t + \phi)$$

To obtain the **phasor** associated with a **sinusoid**, first write the sinusoid in cosine form (so it can be written as the real part of a complex number), and take out the time factor.

$$v(t) = V_m \cos(\omega t + \phi) = Re(V_m e^{j(\omega t + \phi)}) = Re(V_m e^{j\phi} e^{j\omega t}) = V_m e^{j\phi} = V_m \angle\phi$$

Phasors are complex, and therefore can be represented in rectangular, polar or exponential form.

They are **vectors** (magnitude and direction) and can be plotted as such.

The phasor domain is also known as the frequency domain.

Phasors

Taking the derivative of a sinusoid

$$\frac{dv}{dt}(V_m \cos(\omega t + \phi)) = -\omega V_m \sin(\omega t + \phi) = \omega V_m \cos(\omega t + \phi + 90)$$

$$\frac{dv}{dt}(V_m \cos(\omega t + \phi)) = Re(\omega V_m e^{j\omega t} e^{j\phi} e^{j90°}) = Re(j\omega \boldsymbol{V} e^{j\omega t})$$

The relationships of derivative and integral

$$\frac{dv}{dv} \Leftrightarrow j\omega \boldsymbol{V}$$

$$\int v dt \Leftrightarrow \frac{\boldsymbol{V}}{j\omega}$$

These relationships are useful in computing the steady-state response (where we don't need to know the initial values).

Phasors are also useful for summing sinusoids of the same frequency.

Phasors

Important FACTs to remember:

- *v(t)* is time dependent, while **V** is NOT.
- *v(t)* is ALWAYS real, while **V** is generally complex.
- Phasor analysis applies only when frequency is **constant** and allows two or more sinusoids to be manipulated ONLY if they have the same frequency.

Examples

$$v = -4\sin(30t + 50°)V \longrightarrow -\sin A = \cos(A + 90°)$$

$$v = 4\cos(30t + 50° + 90°) = 4\cos(30t + 140°)$$

$$\boldsymbol{V} = 4\angle 140° V$$

$$\boldsymbol{I} = -3 + j4A$$

(Phasors can be in any of the three forms)

$$\boldsymbol{I} = 5\angle 126.87°$$

$$i(t) = 5\cos(\omega t + 126.87°)A$$

$$\boldsymbol{V} = j8e^{-j20°}V \qquad \text{since} \quad j = 1\angle 90°$$

$$\boldsymbol{V} = j8\angle -20° = (1\angle 90°)(8\angle -20°) = 8\angle 90° - 20° = 8\angle 70°$$

$$v(t) = 8\cos(\omega t + 70°)V$$

Phasors

Used to solve an integrodifferential equation

Solve for *i(t)*

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75°)$$

$$4\boldsymbol{I} + \frac{8\boldsymbol{I}}{j\omega} - 3j\omega\boldsymbol{I} = 50\angle 75°$$

Convert to phasor domain

$$\boldsymbol{I}(4 - j4 - j6) = 50\angle 75°$$

ω is 2 and- 1/j = -j.

$$\boldsymbol{I} = \frac{50\angle 75°}{4 - j10} = \frac{50\angle 75°}{10.77\angle(-68.2)°} = 4.642\angle 143.2°$$

$$m = \sqrt{4^2 + 10^2} = 10.77$$

$$\phi = \tan^{-1}\left(\frac{10}{4}\right) = 360° - 68.2° = 291.8°$$

$$\phi = 75° - 291.8° = -216.8° + 360 = 143.2°$$

So far we’ve shown how to represent a voltage and current in phasors.

What about the passive elements, R, L and C

Phasors Applied to R, L and C circuits

Need to transform the current/voltage relationships for R, L and C into the frequency domain.

For **R**

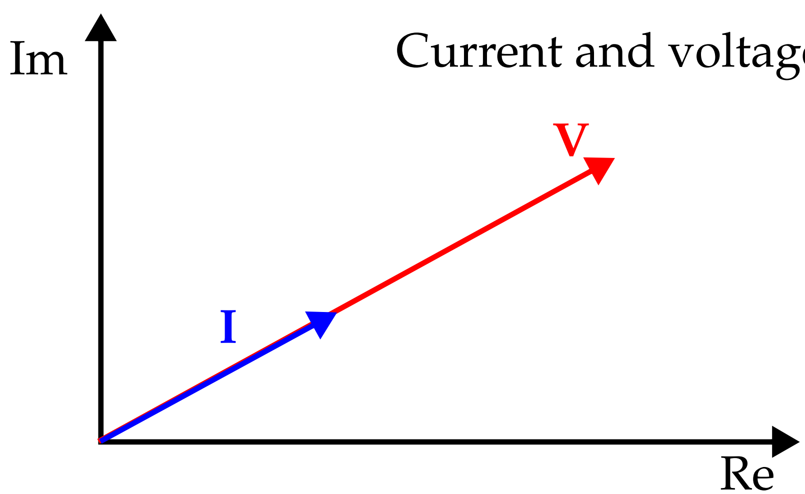
$i = I_m \cos(\omega t + \phi)$ Assume current through a resistance **R**

$v = iR = RI_m \cos(\omega t + \phi)$ And voltage across it

$\boldsymbol{V} = RI_m \angle\phi$ The phasor for this voltage

$\boldsymbol{I} = I_m \angle\phi$ Phasor for the current

$\boldsymbol{V} = R\boldsymbol{I}$



Current and voltage are in phase

Phasors Applied to R, L and C circuits

For **L**

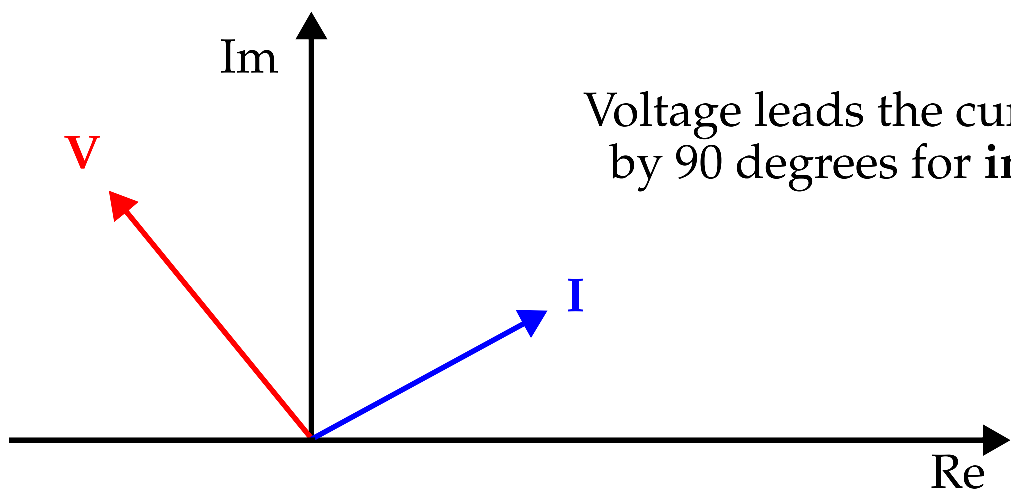
$$i = I_m \cos(\omega t + \phi)$$ Assume current through inductor **L**

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$ Voltage across it

$$v = L\frac{di}{dt} = \omega L I_m \cos(\omega t + \phi + 90°)$$ $$-\sin(A) = \cos(A + 90°)$$

$$\boldsymbol{V} = \omega L I_m e^{j(\phi + 90°)} = \omega L I_m e^{j\phi} e^{j90°}$$ Into a phasor

$$\boldsymbol{V} = j\omega L \boldsymbol{I}$$ $$e^{j90°} = j$$



Voltage leads the current by 90 degrees for **inductor**

The *j* term translates into a phase shift in the time domain.

Impedance

For **C**

$$i = C\frac{dv}{dt} \longrightarrow \boldsymbol{V} = \frac{\boldsymbol{I}}{j\omega C} \quad \text{or} \quad \boldsymbol{I} = j\omega C\boldsymbol{V}$$

Here, current leads voltage by 90 degrees

The first form relates *V* and *I* similar to resistance.

These can be written in terms of their phasor voltage to phasor current

$$\frac{\boldsymbol{V}}{\boldsymbol{I}} = R$$

$$\frac{\boldsymbol{V}}{\boldsymbol{I}} = j\omega L \longrightarrow \boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}}$$

$$\frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{1}{j\omega C} \quad \text{or} \quad \frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{-j}{\omega C}$$

Impedance ***Z*** is the ratio of phasor voltage to current in Ωs

NOTE: the impedance **Z** is NOT a phasor -- it does not correspond to a sinusoidally varying quanity.

Impedance

Example

$$v = 12\cos(60t + 45°)$$

$$L = 0.1 \text{ H}$$

Find steady-state current

$$\boldsymbol{V} = j\omega L\boldsymbol{I}$$

with w = 60 rad/s and $\boldsymbol{V} = 12\angle 45°$

$$\boldsymbol{I} = \frac{\boldsymbol{V}}{j\omega L} = \frac{12\angle 45°}{j60(0.1)} = \frac{12\angle 45°}{6\angle 90°} = 2\angle -45° A$$

So we see the magnitude of the impedance is 6 Ωs and the inductor causes a phase shift in the current (it lags by 90 degrees).

$$Z = R + jX \qquad R = Re(Z) \quad \text{resistance}$$

$$Z = R - jX \qquad X = Im(Z) \quad \text{reactance}$$

Reactance can be *negative* (capacitive, current lags) or *positive* (inductive).

Other forms

$$\boldsymbol{Z} = R + jX = |\boldsymbol{Z}|\angle\theta \qquad |\boldsymbol{Z}| = \sqrt{R^2 + X^2} \qquad \theta = \tan^{-1}\left(\frac{X}{R}\right)$$

$$R = |\boldsymbol{Z}|\cos(\theta) \qquad X = |\boldsymbol{Z}|\sin(\theta)$$

Admittance

Admittance **Y** is the reciprocal of impedance, measured in siemens (S)

$$\boldsymbol{Y} = \frac{1}{\boldsymbol{Z}} = \frac{\boldsymbol{I}}{\boldsymbol{V}}$$

$$\boldsymbol{Y} = G + jB$$

G = Re(**Y**) is called conductance (mohms)
B = Im(**Y**) is called susceptance

$$G + jB = \frac{1}{R + jX}$$

$$G + jB = \frac{1}{R + jX}\frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2}$$

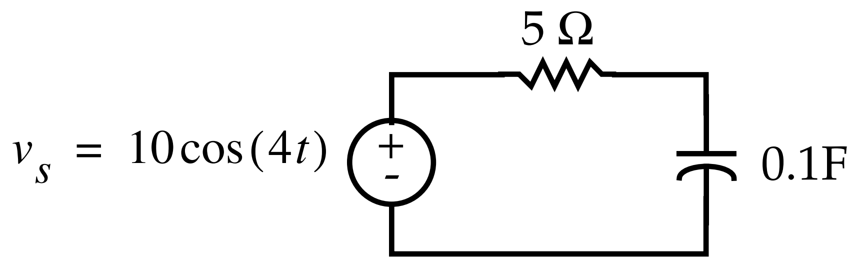
Relating impedance with admittance

$$G = \frac{R}{R^2 + X^2} \qquad B = \frac{-X}{R^2 + X^2}$$

So, G does NOT equal 1/R as was true for resistive circuits.

Examples

$$V_s = 10(\angle 0°)V$$



The impedance is

$$Z = R + \frac{1}{j\omega C} = 5 + \frac{1}{j4(0.1)} = 5 - j2.5\ \Omega$$

And current

$$I = \frac{V_s}{Z} = \frac{10\angle 0°}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2} = 1.6 + j0.8 = 1.789\angle 26.57°$$

OR

$$I = \frac{V_s}{Z} = \frac{10\angle 0°}{5.6\angle -26.57} = 1.789\angle 26.57°$$

Voltage across capacitor

$$V = IZ_C = \frac{I}{j\omega C} = \frac{1.789\angle 26.57°}{j4(0.1)} = \frac{1.789\angle 26.57°}{0.4\angle 90°} = 4.47\angle -63.43°$$

Time domain

$$i(t) = 1.789\cos(4t + 26.57°)$$
$$v(t) = 4.47\cos(4t - 63.43°)$$