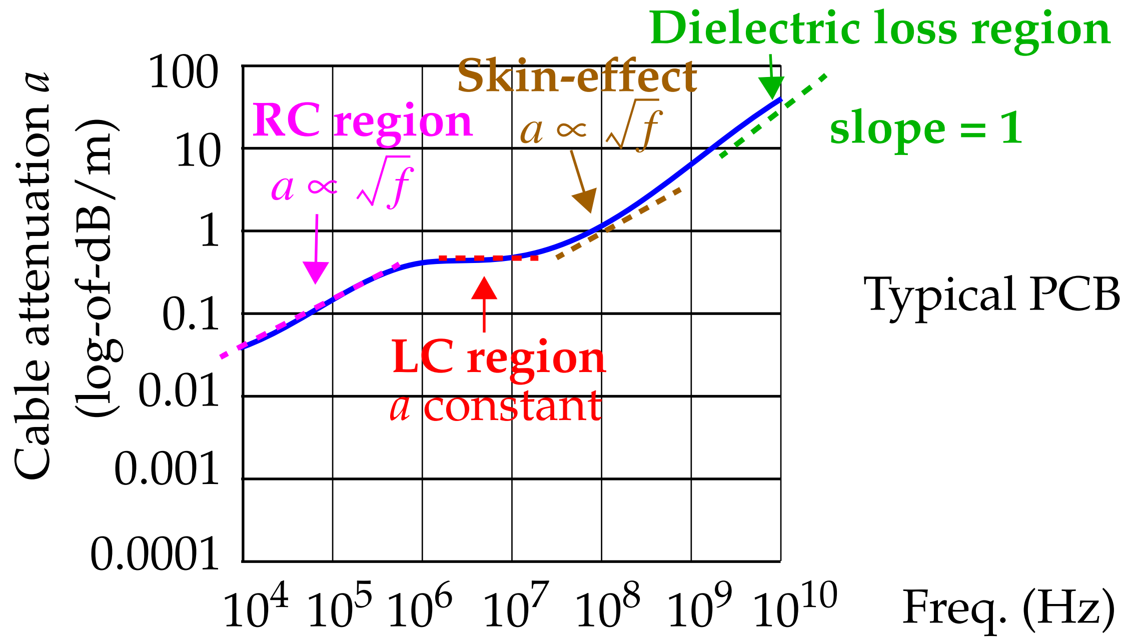
Performance Regions

The propagation function for all copper media has the following regions

Since dB is already a logarithmic unit, this is a double log



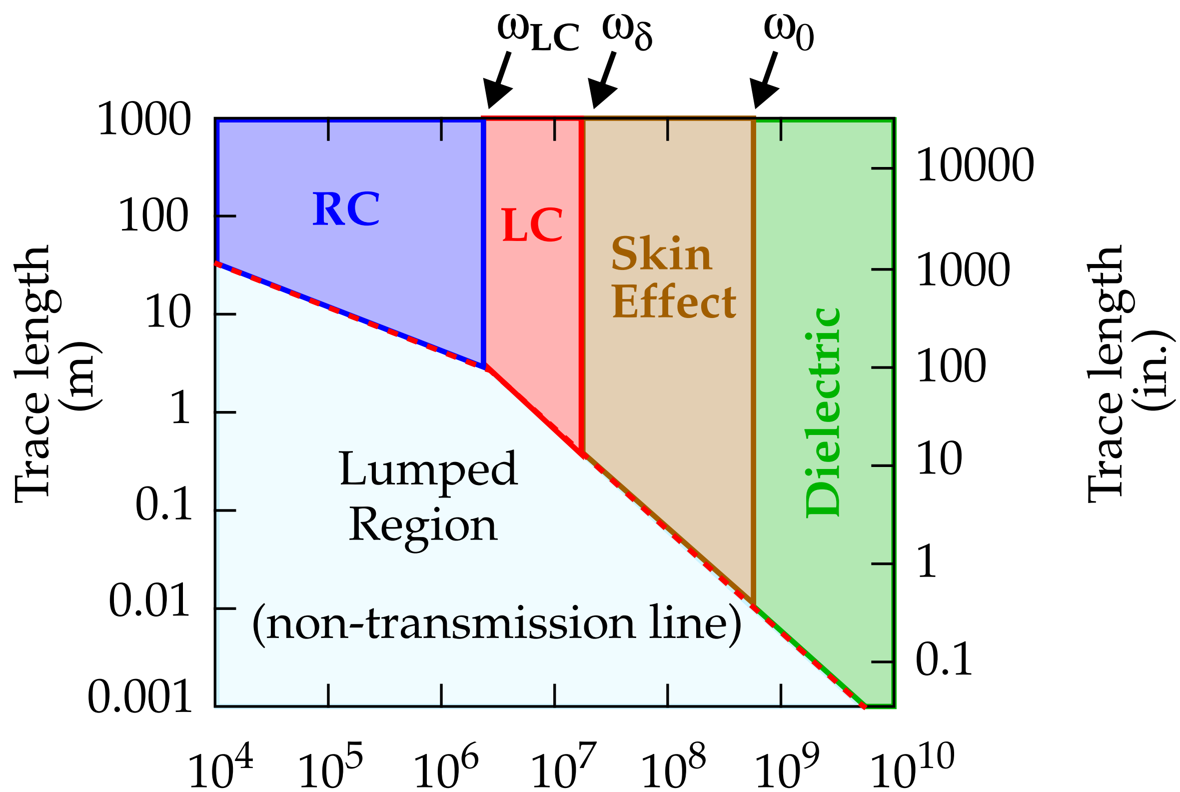
Typical PCB trace

In order of increasing frequency

- RC region
- LC region
- Skin-effect region
- Dielectric loss region
- Waveguide dispersion region

Performance Regions

Regions as a function of trace length



6-mil (150 μm), 50-Ω, FR-4 PCB stripline

In lumped region, transmission line is short enough that it acts as a simple lumped circuit element.

Performance Regions

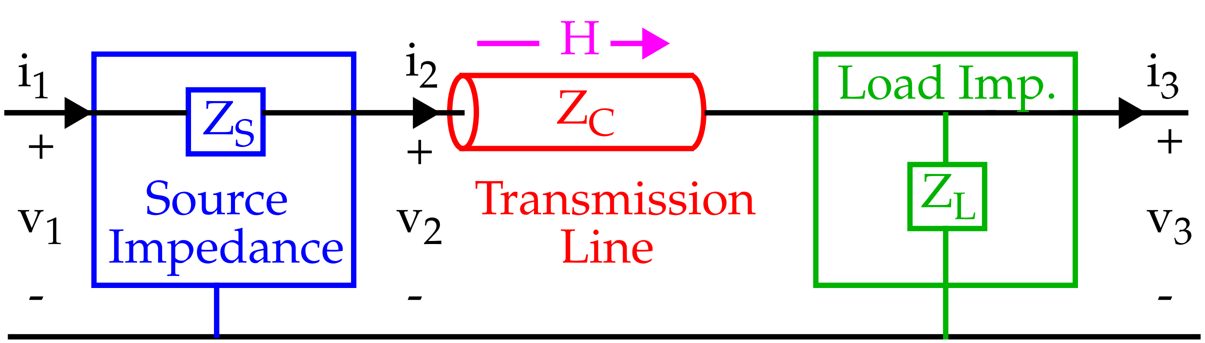
In the RC region, the inductive reactance, ωL, dwindles to insignificance in comparison to DC resistance -- only R and C matter in this region.

In LC region, inductive reactance exceeds DC resistance.

In skin effect region, *internal inductance* of conductors becomes significant compared to DC resistance, and forces a redistribution of current.

Dielectric-loss-limited region entered when dielectric losses become comparible to resistive losses.

The performance of transmission line depend on four factors



Two-Port Analysis

The factors are Z_C (characteristic impedance), H (the raw, **unloaded** one-way propagation function), Z_S (source impedance) and Z_L (load impedance).

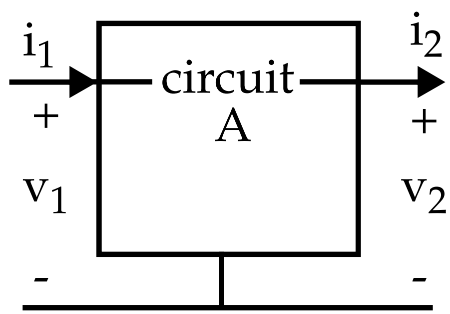
We need to derive two parameters:

- The input impedance of the transmission line (with load), e.g., v_2/i_2
- The gain v_3/v_1

We need to derive these using a two-port analysis.

transmission matrix **A**

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{bmatrix} \begin{bmatrix} v_2 \\ i_2 \end{bmatrix}$$



Shared common reference terminal

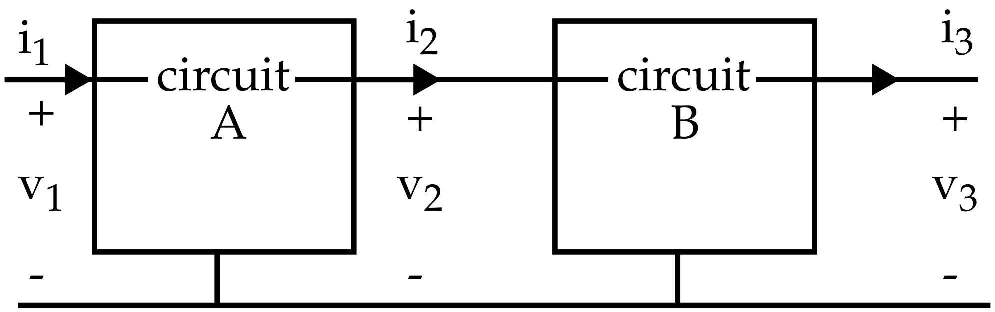
Transmission matrix A prescribe the actions of the circuit.

Each of the 4 elements is a function of frequency.

Two-Port Analysis

This form of the transmission matrix is amenable to *cascaded* systems.

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} v_2 \\ i_2 \end{bmatrix} \qquad \begin{bmatrix} v_2 \\ i_2 \end{bmatrix} = \boldsymbol{B} \begin{bmatrix} v_3 \\ i_3 \end{bmatrix} \longrightarrow \begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \boldsymbol{AB} \begin{bmatrix} v_3 \\ i_3 \end{bmatrix}$$



Let's derive the **input impedance** and **gain** from the transmission matrix.

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{bmatrix} \begin{bmatrix} v_2 \\ i_2 \end{bmatrix} \xrightarrow[i_2 = 0]{\text{open circuit}} \begin{matrix} v_1 = a_{0,0} v_2 \\ i_1 = a_{1,0} v_2 \end{matrix} \rightarrow Z_{\text{in, open}} = \frac{a_{0,0}}{a_{1,0}}$$

Input impedance is calculated first assuming the output circuit (v_2, i_2) is open-circuited, i.e., $i_2 = 0$.

Two-Port Analysis

Input impedance with the output shorted to ground, i.e., $v_2 = 0$.

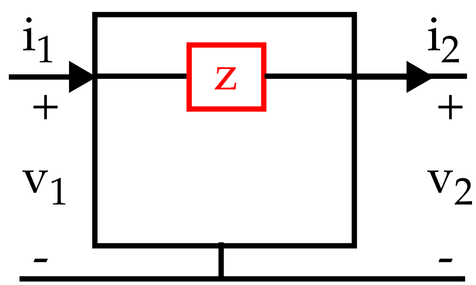
$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{bmatrix} \begin{bmatrix} v_2 \\ i_2 \end{bmatrix} \xrightarrow[v_2 = 0]{\text{short circuit}} \begin{aligned} v_1 &= a_{0,1} i_2 \\ i_1 &= a_{1,1} i_2 \end{aligned} \rightarrow Z_{\text{in, short}} = \frac{a_{0,1}}{a_{1,1}}$$

For the first case (open-circuit), the **voltage transfer function** is

$$\text{Voltage gain} = \frac{v_2}{v_1} = \frac{1}{a_{0,0}}$$

There are three *classic* forms of transmission matrices used for transmission lines: a **series impedance**.

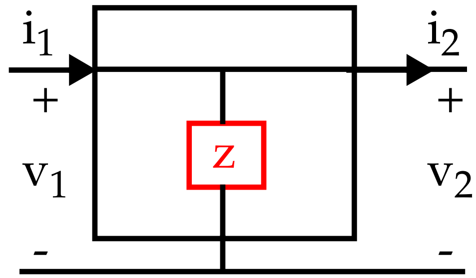
$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_2 \\ i_2 \end{bmatrix}$$



Two-Port Analysis

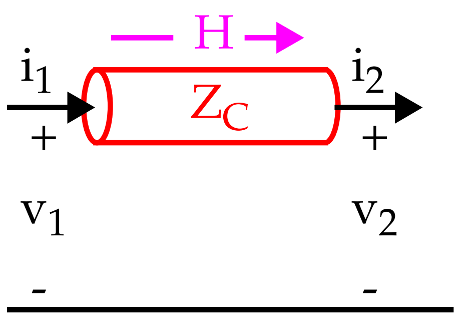
A **shunt impedance**

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z^{-1} & 1 \end{bmatrix} \begin{bmatrix} v_2 \\ i_2 \end{bmatrix}$$



A **transmission line**

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} \left(\frac{H^{-1}+H}{2}\right) & \left(Z_C\frac{H^{-1}-H}{2}\right) \\ \left(\frac{1}{Z_C}\left(\frac{H^{-1}-H}{2}\right)\right) & \left(\frac{H^{-1}+H}{2}\right) \end{bmatrix} \begin{bmatrix} v_2 \\ i_2 \end{bmatrix}$$



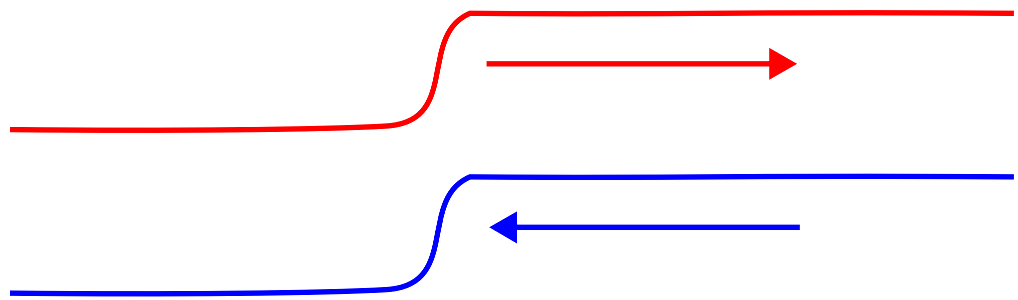
The series impedance and shunt impedance forms are easily derived from Kirchoff's laws.

The transmission line is derived in the following way.

Two-Port Analysis

The general form of solution for signals on a transmission line is composed of two traveling waves

One propagating to the right and one moving to the left.



One the right side, let the signal amplitudes be denoted by *a* and *b*.

The currents associated with these 2 wfms at that point must be +a/Z_C and -b/Z_C.

The superposition of these waves represent the excursion of voltage and current

$$v_2 = a + b \quad \text{and} \quad i_2 = \frac{a}{Z_C} - \frac{b}{Z_C}$$

Two-Port Analysis

Solving for *a* and *b*

$$a = \frac{v_2 + Z_C i_2}{2}$$

$$b = \frac{v_2 - Z_C i_2}{2}$$

At the left end of the line, the same conditions hold, except that the amplitudes of the right- and left-traveling wfms must be adjusted to account for their propagation through the medium.

The amplitude of the left-traveling wfm is diminished by H, while the right-traveling wfm is *increased* by H^{-1}.

This ensures that it arrives at the right end at the correct amplitude *a*.

$$v_1 = aH^{-1} + bH$$

$$i_1 = \frac{a}{Z_C}H^{-1} - \frac{b}{Z_C}H$$

Summing the voltages and currents at the left end of the line

Two-Port Analysis

Substituting a and b into these equations to relate to v_2 and i_2

$$v_1 = \left(\frac{v_2 + Z_C i_2}{2}\right)H^{-1} + \left(\frac{v_2 - Z_C i_2}{2}\right)H$$

$$i_2 = \left(\frac{\frac{v_2 + Z_C i_2}{2}}{Z_C}\right)H^{-1} - \left(\frac{\frac{v_2 - Z_C i_2}{2}}{Z_C}\right)H$$

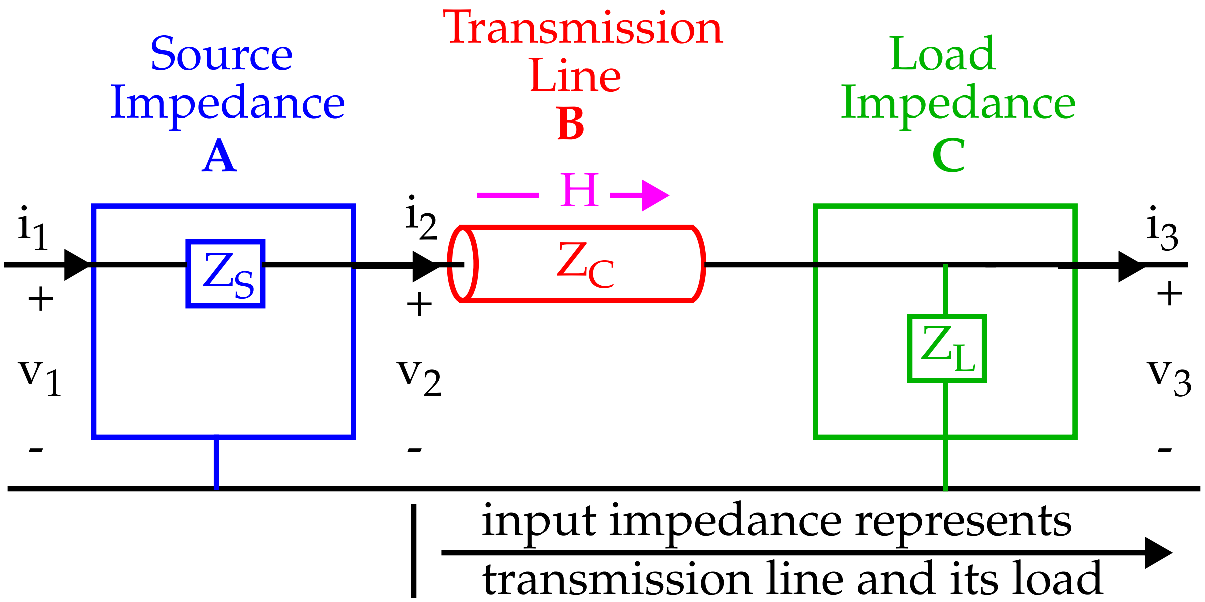
Collecting terms associated with v_2 and i_2 yields the form that we showed earlier.

$$v_1 = \left(\frac{H^{-1} + H}{2}\right)v_2 + Z_C\left(\frac{H^{-1} - H}{2}\right)i_2$$

$$i_1 = \frac{1}{Z_C}\left(\frac{H^{-1} - H}{2}\right)v_2 + \left(\frac{H^{-1} + H}{2}\right)i_2$$

Input Impedance

The transmission line is modeled as a cascade of three 2-port circuits



The input impedance of the loaded transmission line is obtained from the **cascaded combination** of **BC**

$$\boldsymbol{BC} = \begin{bmatrix} \left(\frac{H^{-1}+H}{2}\right) & \left(Z_C\frac{H^{-1}-H}{2}\right) \\ \left(\frac{1}{Z_C}\frac{H^{-1}-H}{2}\right) & \left(\frac{H^{-1}+H}{2}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{Z_L} & 1 \end{bmatrix}$$

Input Impedance

Multiplying

$$\begin{bmatrix} v_2 \\ i_2 \end{bmatrix} = \begin{bmatrix} \left(\frac{H^{-1}+H}{2}\right) + \frac{Z_C}{Z_L}\left(\frac{H^{-1}-H}{2}\right) & Z_C\left(\frac{H^{-1}-H}{2}\right) \\ \frac{1}{Z_C}\left(\frac{H^{-1}-H}{2}\right) + \frac{1}{Z_L}\left(\frac{H^{-1}+H}{2}\right) & \left(\frac{H^{-1}+H}{2}\right) \end{bmatrix} \begin{bmatrix} v_3 \\ i_3 \end{bmatrix}$$

The input impedance v_2/i_2 equals the ratio $\mathbf{BC_{0,0}}/\mathbf{BC_{1,0}}$ (with $i_3 = 0$).

$$Z_{\text{in, loaded}} = \frac{\left(\frac{H^{-1}+H}{2}\right) + \frac{Z_C}{Z_L}\left(\frac{H^{-1}-H}{2}\right)}{\frac{1}{Z_C}\left(\frac{H^{-1}-H}{2}\right) + \frac{1}{Z_L}\left(\frac{H^{-1}+H}{2}\right)}$$

Input Impedance

Input Impedance (after simplifying)

$$Z_{\text{in, loaded}} = Z_C\left(\frac{\left(\frac{H^{-1}+H}{2}\right)+\frac{Z_C}{Z_L}\left(\frac{H^{-1}-H}{2}\right)}{\left(\frac{H^{-1}-H}{2}\right)+\frac{Z_C}{Z_L}\left(\frac{H^{-1}+H}{2}\right)}\right)$$

Some interesting simplifications can be derived for special situations.

For $Z_L >> Z_C$ ⟶ $Z_{\text{in, open-circuit}} = Z_C\left(\frac{H^{-1}+H}{H^{-1}-H}\right)$

Right-hand terms get small.

For $Z_L = Z_C$ ⟶ $Z_{\text{in, end-terminated}} = Z_C$

Right-hand terms go to unity.

For $Z_L << Z_C$ ⟶ $Z_{\text{in, short-circuit}} = Z_C\left(\frac{H^{-1}-H}{H^{-1}+H}\right)$

Left-hand terms get small.

Transfer Function

Gain or voltage transfer function, v_3/v_1 of the *loaded* transmission line is obtained from the cascaded combination of all three parts **ABC**.

$$\boldsymbol{ABC} = \begin{bmatrix} 1 & Z_S \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{H^{-1}+H}{2}\right) & \left(Z_C\frac{H^{-1}-H}{2}\right) \\ \left(\frac{1}{Z_C}\frac{H^{-1}-H}{2}\right) & \left(\frac{H^{-1}+H}{2}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{Z_L} & 1 \end{bmatrix}$$

The **voltage gain** $G_{FWD} = v_3/v_1$ is the inverse of the first element of **ABC**

$$\frac{v_3}{v_1} = [\boldsymbol{ABC}_{0,0}]^{-1}$$

$$\frac{v_3}{v_1} = \left[\left(\frac{H^{-1}+H}{2}\right) + \frac{Z_C}{Z_L}\left(\frac{H^{-1}-H}{2}\right) + \frac{Z_S}{Z_C}\left(\frac{H^{-1}-H}{2}\right) + \frac{Z_S}{Z_L}\left(\frac{H^{-1}+H}{2}\right)\right]^{-1}$$

$$G_{FWD} = \frac{v_3}{v_1} = \frac{1}{\left[\left(\frac{H^{-1}+H}{2}\right)\left(1+\frac{Z_S}{Z_L}\right) + \left(\frac{H^{-1}-H}{2}\right)\left(\frac{Z_S}{Z_C}+\frac{Z_C}{Z_L}\right)\right]} \quad \text{(Simplified)}$$

Transfer Function

This gain expression is most useful for analyzing *lumped-element* and *RC-mode* transmission lines.

For the *LC-mode* and *skin-effect* regions, we'll use the following variation

$$G = \frac{1}{2}\frac{\left(1 - \left(\frac{Z_S - Z_C}{Z_S + Z_C}\right)\right)\left(1 + \left(\frac{Z_L - Z_C}{Z_L + Z_C}\right)\right)}{1 - H^2\left(\frac{Z_S - Z_C}{Z_S + Z_C}\right)\left(\frac{Z_L - Z_C}{Z_L + Z_C}\right)} = \frac{1}{2}\frac{(1 - \Gamma_1)(1 + \Gamma_2)}{1 - H^2\Gamma_1\Gamma_2}$$

Γ_1 is the **reflection coefficient** at the source end while Γ_2 is the reflection coefficient at the load end.

Our goal is to transmit signals between source and load *undistorted*.

Therefore, we must ensure the propagation functions (G's above) remain flat over the **band of frequencies** covering the bulk of the spectral content of the signal.

Transfer Function

The bulk of the useful spectral content of a random data sequence spans a range from DC up to

$$f_{knee} = \frac{0.5}{t_r} \text{ Hz} \qquad \omega_{knee} = 2\pi\frac{0.5}{t_r} \text{ rad/s}$$

An estimate of the **midpoint** of the spectral content associated with rising and falling edges is slightly less at

$$f_{edge} = \frac{0.35}{t_r} \text{ Hz}$$