Chapter 7: Sums of Random Variables and Long-Term Averages

7.1 Sums of Random Variables

7.1 $\mathcal{E}[X+Y+Z] = \mathcal{E}[X] + \mathcal{E}[Y] + \mathcal{E}[Z] = 0$

a) From Eqn. 5.3 we have

$$\begin{aligned} VAR(X+Y+Z) &= VAR(X) + VAR(Y) + VAR(Z) \\ &\quad +2COV(X,Y) + 2COV(X,Z) + 2COV(Y,Z) \\ &= 1+1+1+2\left(\frac{1}{4}\right) + 2(0) + 2\left(-\frac{1}{4}\right) = 3 \end{aligned}$$

b) From Eqn. 5.3 we have

$$VAR(X+Y+Z) = VAR(X) + VAR(Y) + VAR(Z) = 3$$

7.2 $\mathcal{E}[S_n] = \mathcal{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathcal{E}[X_i] = n\mu$.

$$VAR(S_n) = \underbrace{\sum_{k=1}^{n} VAR(X_k)}_{\substack{\text{sum of diag.} \\ \text{elements of} \\ \text{covariance matrix } K}} + \underbrace{\sum_{j=1}^{n}\sum_{k=1_{j\neq k}}^{n} COV(X_j, X_k)}_{\substack{\text{sum of off-diag.} \\ \text{element of } K}}$$

$$K = \begin{bmatrix} \sigma^2 & \rho\sigma^2 & 0 & & \cdots & 0 \\ \rho\sigma^2 & \sigma^2 & \rho\sigma^2 & 0 & \cdots & 0 \\ & & & & & 0 \\ & \ddots & \ddots & \ddots & & \rho\sigma^2 \\ & & & \rho\sigma^2 & & \sigma^2 \end{bmatrix}$$

$$\therefore VAR(S_n) = n\sigma^2 + 2(n-1)\rho\sigma^2$$