

ECE 340

Lecture 6: Random variable

Jean Liu
Email: Jliu@mrn.org

Reading

- This class: Section 3.1, 3.2, 3.3
- Next class: Section 3.3, 3.4

Outline

- Random variable
 - Definition
 - Notation
- Probability mass function (pmf)
- Expected value

- Samples

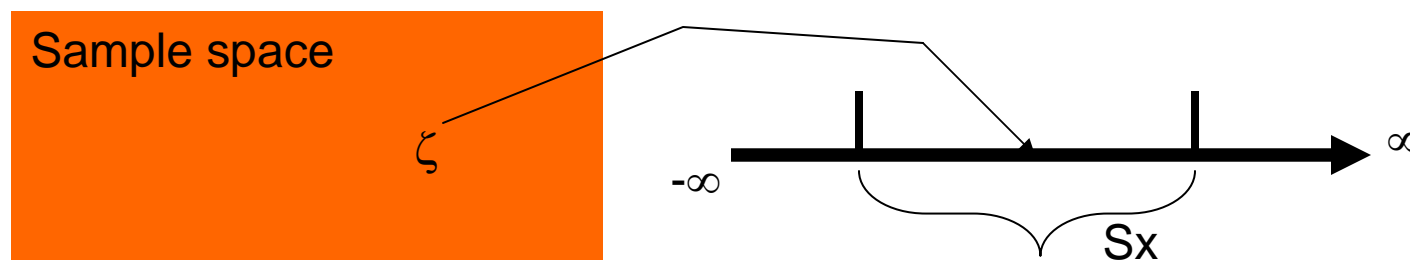
Random experiment

- Temperature in ABQ on Feb.
- Length of queue at a movie theater
- The number of words in your emails

- The color of a car in the street
- The side of coin tossed N time
- The mood of me today

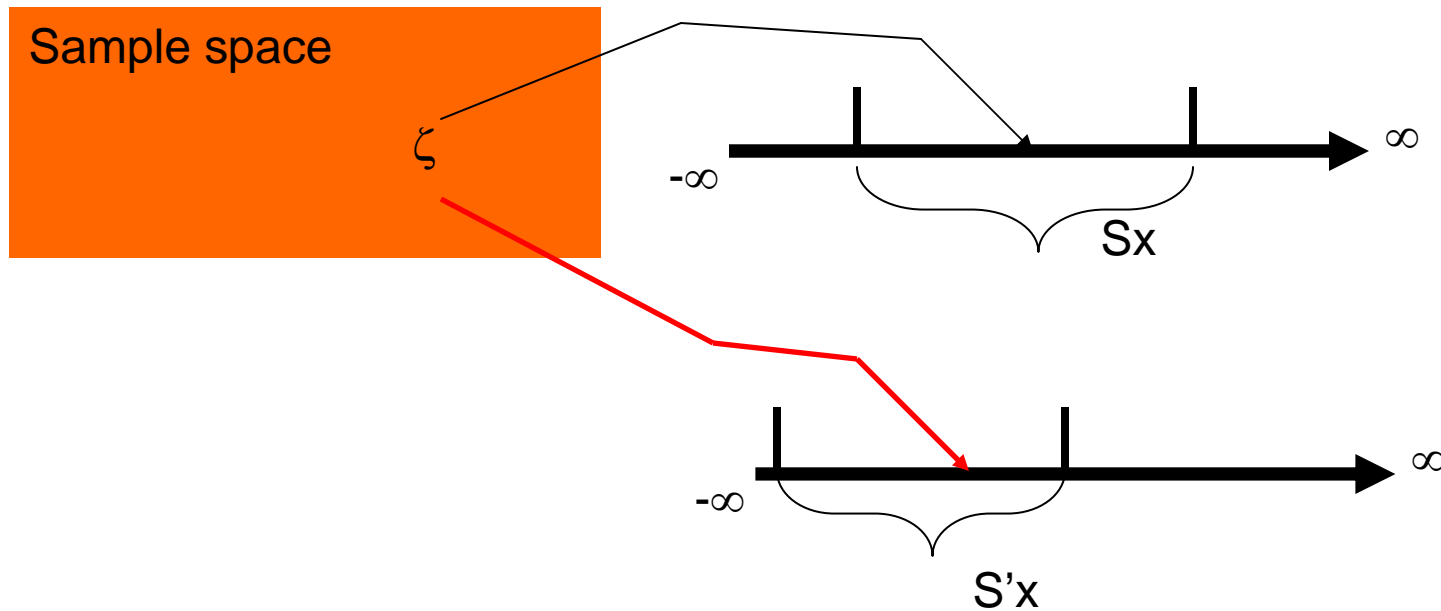
An assignment of a value (number) to every possible outcome

Mathematically: A function from the sample space to the real numbers



– discrete or continuous

Can have several random variables defined on the same sample space



- Notation:
- – random variable X, Y
- – numerical possible value $x, y,$

- $X(\zeta)=x, Y(\zeta)=y,$

Example 1

- A coin tossed 3 times. The sequence of heads and tails are noted.
- Let X be number of heads

ζ	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
$X(\zeta)$	3	2	2	2	1	1	1	0

Example 1

- A coin tossed 3 times. The sequence of heads and tails are noted.
- Let X be number of tails
- Other functions?

ζ	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
$X(\zeta)$	0	1	1	1	2	2	2	3

Example 1

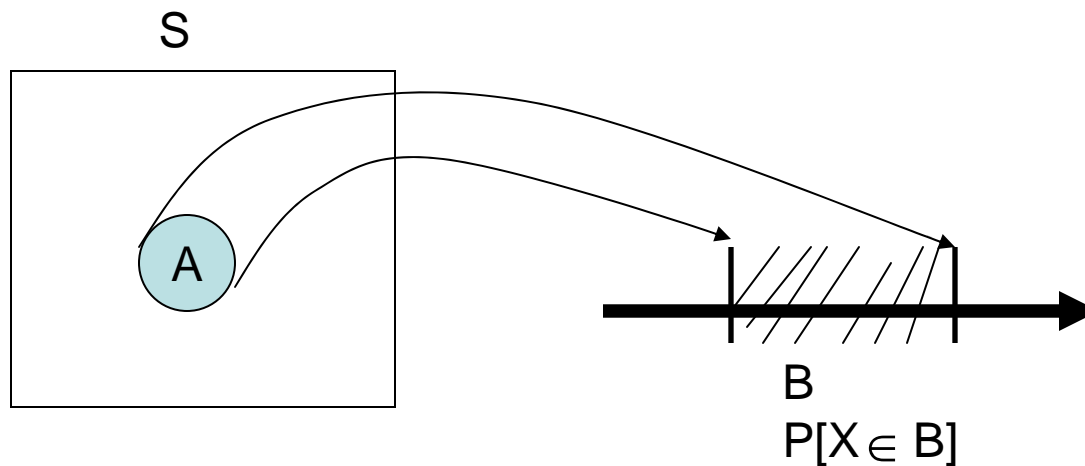
- A coin tossed 3 times. The sequence of heads and tails are noted.
- Let X be number of heads
- Let Y be rewards {\$8 for 3 Heads, \$1 for 2 heads; \$0 for others}

ζ	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
$X(\zeta)$	3	2	2	2	1	1	1	0
$Y(\zeta)$	8	1	1	1	0	0	0	0

ζ	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
$X(\zeta)$	3	2	2	2	1	1	1	0
$Y(\zeta)$	8	1	1	1	0	0	0	0

$$P[Y=1]=P[X=2] = P[\{HHT,HTH,THH\}]=3/8$$

$$P[Y=8] = P[X=3]=P[HHH]=1/8$$

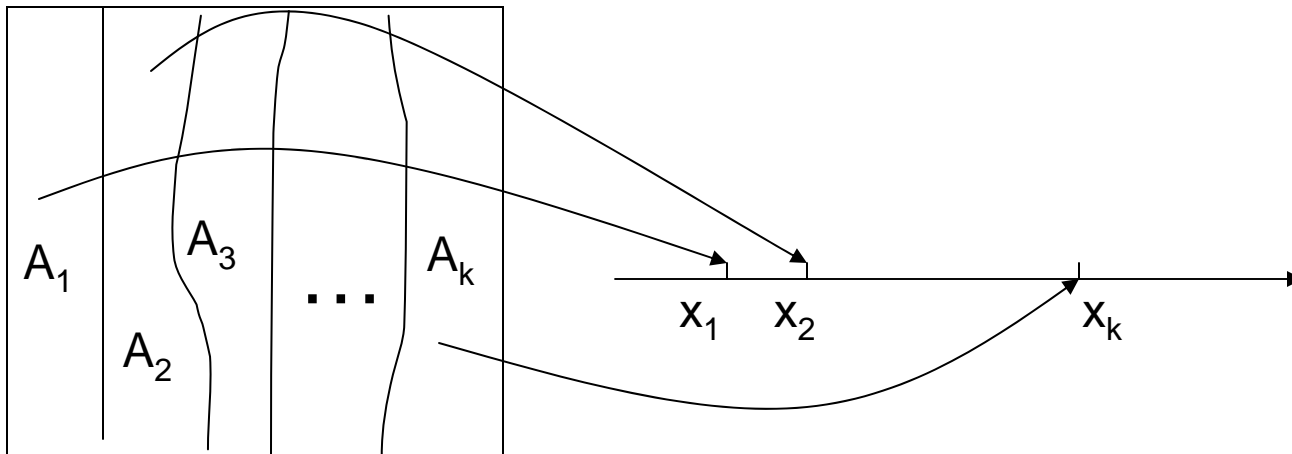


Discrete random variable

- A X is defined as a random variable that assumes values from a countable set.
- $S_X = \{ x_1, x_2, x_3, \dots \}$

Probability mass function

$$p_X(x) = P[X = x] = P[\{\zeta : X(\zeta) = x\}]$$



$$p_X(x_1) = P[A_1]; \quad p_X(x_2) = P[A_2];$$

Probability mass function

$$(I) \quad p_X(x) \geq 0 \quad \text{for all } x$$

$$(II) \quad \sum_{x \in S_X} p_X(x) = 1$$

$$(III) \quad P[X \text{ in } B] = \sum_{x \in B} p_X(x)$$

Example 2

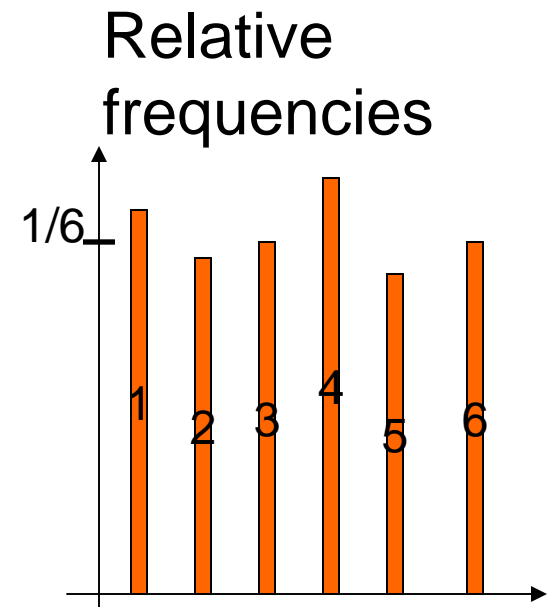
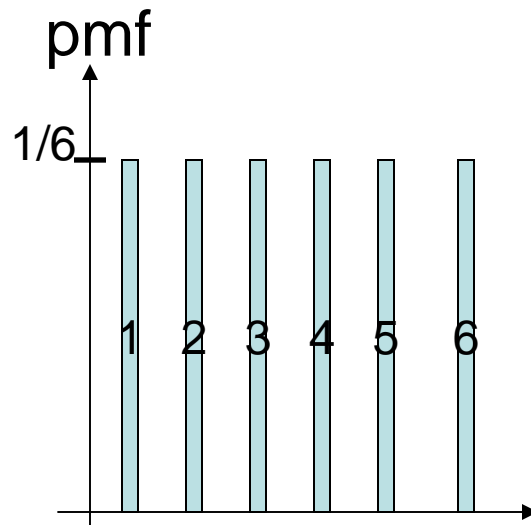
- Toss a six-side dice
- Outcomes: $\{1,2,3,4,5,6\}$
- X is the number of the side.
- $S_x: \{1,2,3,4,5,6\}$

$p_x\{x\}$

$p_x\{1\}; p_x\{2\}; \dots p_x\{6\};$

$$p_X\{1\} = p_X\{2\} = \dots = p_X\{6\} = 1/6$$

$$p_X\{1\} + p_X\{2\} + \dots + p_X\{6\} = 1$$



Uniform random variable

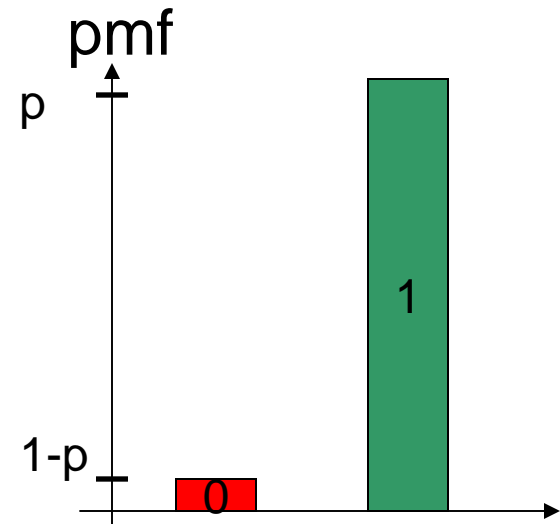
Bernoulli random variable

Success or failure experiment

$$I(\zeta) = \begin{cases} 0 & \text{if } \zeta \text{ failure} \\ 1 & \text{if } \zeta \text{ success} \end{cases}$$

$$p_I(0) = 1 - \rho$$

$$p_I(1) = \rho$$

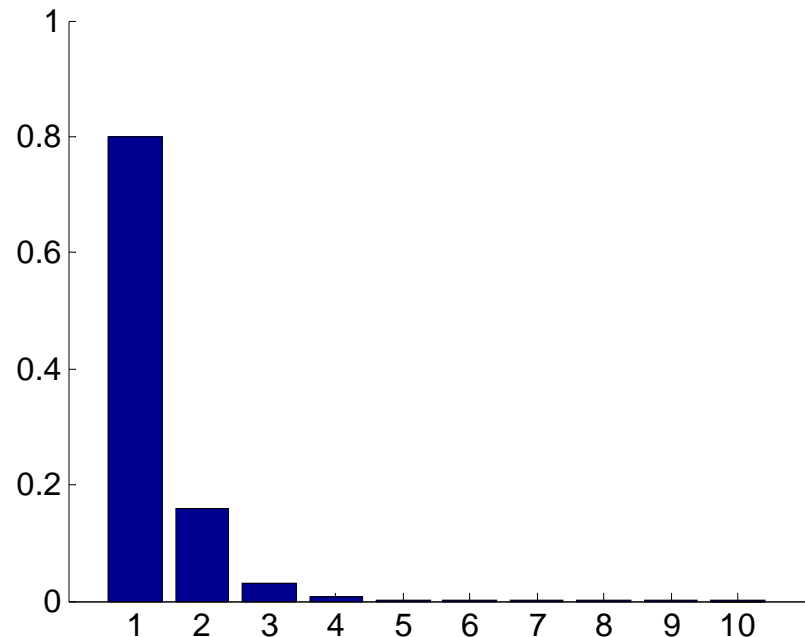


Geometric random variable

- Let X be the number of times a message needs to be transmitted until it arrives correctly.

$$p_X(k) = P[X = k] = P[000\dots 1] = (1 - \rho)^{k-1} \cdot \rho,$$

$$k = 1, 2, \dots$$



Expected value (mean)

- Definition:

$$E[X] = \sum_{x \in S_X} x \cdot p_X(x) = \sum_k x_k \cdot p_X(x_k)$$

- Interpretation
 - Center of gravity
 - Average in a large number of repetitions (infinite)

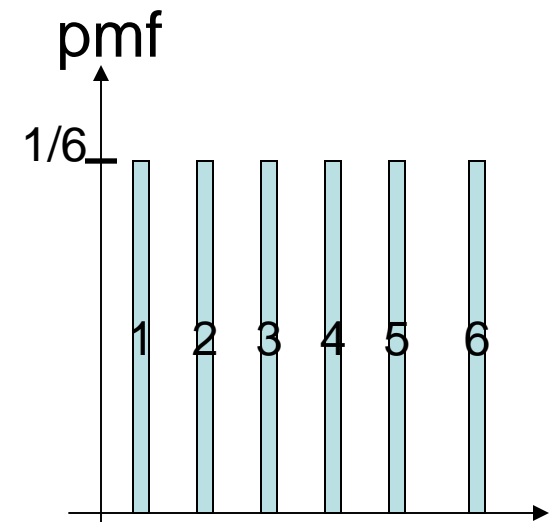
Uniform random variable

- Toss a fair dice

$$E[X] = \sum_k x_k p(x_k)$$

$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$= 3.5$$

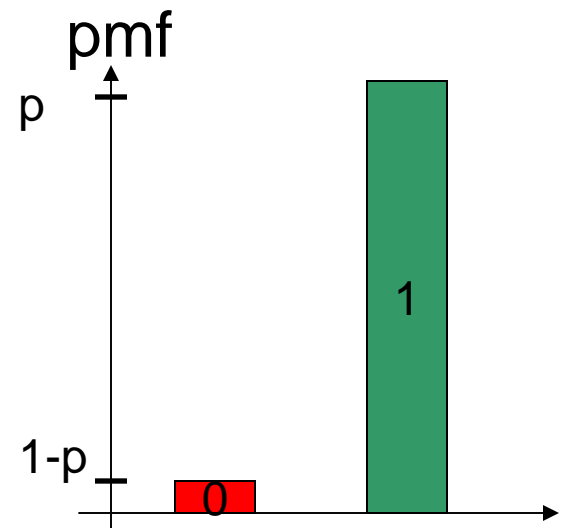


Bernoulli random variable

$$p_I(0) = 1 - \rho$$

$$p_I(1) = \rho$$

$$E[I] = 0 \cdot p(0) + 1 \cdot p(1) = \rho$$



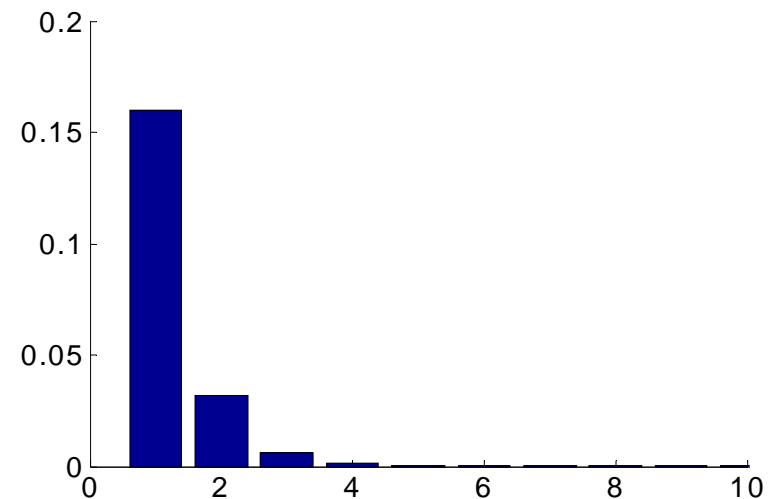
Geometric random variable

$$p_X(k) = (1-\rho)^{k-1} \cdot \rho, \quad k = 1, 2, \dots$$

$$E(X) = \sum_{k=1}^{\infty} k \cdot (1-\rho)^{k-1} \cdot \rho$$

$$= \rho \cdot \sum_{k=1}^{\infty} k \cdot (1-\rho)^{k-1}$$

$$= \frac{1}{\rho}$$



Sample mean vs. Expected value

$$\begin{aligned}\langle X \rangle_n &= \frac{x(1) + x(2) + x(3) + \dots + x(n)}{N} \\ &= \frac{x_1 N(1) + x_2 N(2) + x_3 N(3) + \dots + x_n N(n) + \dots}{N} \\ &= x_1 f_1(n) + x_2 f_2(n) + \dots + x_n f_n(n) + \dots \\ &= \sum_k x_k f_k(n)\end{aligned}$$

$$\lim_{n \rightarrow \infty} f_k(n) = p_x(x_k)$$

$$n \rightarrow \infty$$

$$E[X] = \sum_k x_k \cdot p_X(x_k)$$

Example 3

A betting game: toss a coin three times

Pay \$1.5

Rewards \$1, if 2 heads

\$8, if 3 heads

\$0, others

Expected reward ?

Expected gain ?

Expected value of functions of a random variable

X is discrete random variable.

$$Z = g(X);$$

$$E[Z] = E[g(X)] = \sum g(x) \cdot p_X(x_k)$$

- X is a noise voltage that is uniformly distributed in $Sx\{-3, -1, 1, 3\}$ with $p(x_k)=1/4$.
- $Z = X^2$

Pmf of Z : $p_z(1)=1/2$

$$p_z(9)=1/2$$

$$E[Z]=1 \cdot 1/2 + 9 \cdot 1/2 = 5$$

$$E[z]=\sum x^2 p(x_k) = 9 \cdot 1/4 + 1 \cdot 1/4 + 1 \cdot 1/4 + 9 \cdot 1/4 = 5$$

- X is random variable
- $Z = a \cdot g(X) + \beta \cdot h(X) + c$

$$E[Z] = a \cdot E[g(x)] + \beta \cdot E[h(x)] + c$$

$$E[a \cdot X] = a \cdot E[X]$$

$$E[g(x) + h(x)] = E[g(x)] + E[h(x)]$$

$$E[g(x) + c] = E[g(x)] + c$$